Finite Element Methods for Stochastic Structures and Conditional Simulation

by

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ABSTRACT

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This dissertation deals with the non-perturbative finite element methods for stochastic structures and conditional simulation techniques for random fields. Three different non-perturbative finite element schemes have been proposed to compute the first and second moments of displacement responses of stochastic structures. These three methods are based, respectively, on (i) the exact inverse of the global stiffness matrix for simple stochastic structures; (ii) the variational principles for statically-determinate beams; and (iii) the element-level flexibility for general stochastic statically indeterminate structures. The non-perturbative finite element method for stochastic structures possesses several advantages over the conventional perturbation-based finite element method for stochastic structures, including (i) applicability to large values of the coefficient of variation of random parameters; (ii) convergence to exact solutions when the finite element mesh is refined; (iii) requirement of less statistical information than that demanded by the high-order perturbatiobn methods.

Conditional simulation of random fields has been an extremely important research field in most recent years due to its application in urban earthquake monitoring systems. This study generalizes the available simulation technique for one-variate Gaussian random fields, conditioned by realizations of the fields, to multi-variate vector random field, conditioned by the realizations of the fields themselves as well as the realizations of the fields derivatives. Furthermore, a conditional simulation for non-Gaussian random fields is also proposed in this study by combining the unconditional simulation technique of non-Gaussian fields and the conditional simulation technique of Gaussian fields.

Finally, the dissertation incorporates the simulation technique of random field into the non-perturbation finite element method for stochastic structures, to handle the cases where only one-dimensional probability density function and the correlation function of the random parameters are available, the demanded two-dimensional probability density function is unavailable. Simulation technique is applied to generate the samples of random fields which are used to estimate the correlation between flexibilities over elements. The estimated correlation of flexibility is then used in finite element analysis for stochastic structures.

For each proposed approach, numerous examples and numerical results have been implemented.

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To My Parents

Chapter 1: Introduction

The finite element method has been proved to be one of the most powerful tools if not the most powerful tool in numerical analysis. It has been successfully applied to many engineering fields - aeronautical, civil, automobile, mechanical, ocean, etc. There are numerous commercial software packages available for its implementation. Indeed, since the term "Finite Element Method" was dubbed by Clough in the Sixties, the method has been developed so rapidly that even a person specializing in the finite element method itself finds it impossible to keep up with the entire field. Among many branches of the finite element method, the so-called "Stochastic Finite Element Method" has been developed since the Eighties. The stochastic finite element method deals with the structures which involve stochastic material and/or geometrical parameters and subjected to deterministic or stochastic external loads.

Due to inherent inhomogeneity of the material and many unestimated influences during processing and manufacturing of structures, the geometric and material parameters of the structures always deviate from their estimated or design values in more or less extent. This kind of deviation usually has a nature of uncertainty and often, not always, can be identified with the stochasticity. One example of material inhomogeneity has been displayed by Virkler et al (1978). In their repeated experimental study of fatigue crack growth on a set of carefully chosen specimen, which were made under precise control so that they are precisely the same material and are strictly manufactured to have the same geometrical dimensions, two kinds of randomness were observed in growth curves of the fatigue cracks of these structures, namely, the randomness among the growth curves for the group of specimen and the randomness in one growth curve for each specimen. In soil engineering, it is a common sense that the Young's modulus of the soil varies with the depth of the earth. The properties of concretes scatter spatially. In their experiments, Ikeda et al just demonstrated how the inhomogeneity in the concrete affects significantly the compressive behavior of the concrete. For the composite materials, the material properties change with the direction of the fibers, as well as the positions. For example, Beran et al (1996) recently discussed misorientation and statistical information of the elastic stiffness and compliance of an orthotropic polycrystal.

The uncertainty in geometrical parameters of the structures is even more common and its effect on the structural performance is more significant than the uncertainty in material properties. The uncertain initial imperfection of structural geometrical parameters, which has been widely investigated in the field of structural buckling, serves a good example of deviation of geometrical parameters from their design values. Actually, for the same scales of fluctuation of the plate thickness, the Young's modulus and the Possion ratio, the fluctuation of the plate thickness causes the biggest variation of the bending stress intensity factor for the bending of cracked plates (Ren, 1991).

The structures involving uncertain geometric and/or material parameters may be dubbed as uncertain structures or stochastic structures, a terminology contrary to deterministic structures in which the material and geometrical parameters are deterministic. The stochastic structures may be subject to deterministic or random applied loads. In the field of stochastic research, compared with the tremendous amount of literature devoted to deterministic structures under random loads, only extremely limited studies have been addressed to stochastic structures. Since the governing equations for stochastic structures are stochastic differential equations with random coefficients, possibly with the random initial and boundary conditions, the analytic solutions of the response for the stochastic structures are rarely obtainable, except for some extremely simple problems. The technique used to obtain the solution is merely the perturbation method.

In the recent decade, quite a few papers have been published which focused on the analysis of stochastic structures by numerical methods, mainly the finite element method. The so called "Stochastic Finite Element Method" has been widely quoted as a generalized version of the finite element method to stochastic structures. Apparently, the first applications of the finite element method for stochastic problems were by Su et al(Su, Wang and Stefanko 1969) and Cambou (Cambou 1975). Among numerous paper published since the Eighties, are three monographs (Nakagiri and Hisada 1985, Ghanem and Spanos 1991, and Kleiber and Hien 1993) and several review papers. Most recent review paper, of the critical nature, was written by Elishakoff et al (1996).

The existing finite element method for stochastic problems is basically a combination of the second-moment method and perturbation or series expansion technique (Nakagiri and Hisada 1985, Shinozuka and Yamazaki 1988, Ghanem and Spanos 1991). The perturbation-based or series expansion-based finite element method for stochastic problems, however, inherits the disadvantages of perturbation / series expansion techniques. For example, (a) only low-order (usually first-order) perturbation technique is practically implementable, since the recursive algorithm to obtain high-order perturbations is extremely time-consuming; (b) the accuracy of low-order perturbation method is acceptable only for the problems whose uncertain parameters have only small deviations, or, in the probabilistic terminology, small coefficients of variation;

(c) for dynamic response analysis, the perturbation-based method results in divergent solution when the time increases (Shinozuka and Yamakazi 1988). In fact, an efficient FEM for stochastic structures which can efficiently be applied to dynamic analysis is still beyond the scope of present day state of the art of stochastic mechanics.

In contrast to the overall applicability of the deterministic finite element method, the stochastic finite element method appears to be still in the state of infancy and seems incompetent. To develop more efficient and widely suitable stochastic finite element method has been one of main objectives among investigators in this area.

This research aims to propose several new finite element methods for stochastic structures. The new finite element formulations are proposed based on non-perturbative or non-series-expansion techniques and, therefore, do not possess the shortcomings of the conventional FEM for stochastic structures. The overall aim of the new methods is to develop efficient and effective finite element schemes for stochastic problems, which are applicable to *any* value of the coefficients of variation of stochastic parameters and converge to the exact solutions when the size of the finite elements tends to zero and require the probabilistic information on the stochastic parameters as less as possible.

In this research, we will first briefly review the existing finite element methods for stochastic structures (popularly known as stochastic FEM), which are based on either the perturbation technique or the series expansion methods. A considerable improvement of the first-order perturbation-based stochastic finite element method will be also proposed. We then propose three novel non-perturbative finite element methods for stochastic structures, based on (i) the exact inverse of the global stiffness matrix; (ii) the variational principles for the mean and variance-covariance functions; and (iii) element-level flexibility. We will also present novel conditional simulation techniques for multi-variate Gaussian random field and non-Gaussian random field, and incorporate the simulation technique into the finite element method for stochastic structures. Numerous examples are illustarted to show the advantages and efficiencies of the new proposed methods.

Chapter 2: Finite Element Method for Stochastic Structures -- Review and Improvement

2.1 FE Formulation by Perturbation Technique

The perturbation technique is the most widely used way to formulate finite element method for stochastic structures. The method has been systematically proposed by Nakagiri and some other researchers (Hisada and Nakagiri 1981, Nakagiri and Hisada 1985, Liu et al 1986, Kleiber and Hien 1993). Within the conventional procedure of finite element formulation, the system of global finite element equilibrium equations in displacement format can be written as

$$\boldsymbol{K}\boldsymbol{U}=\boldsymbol{F} \tag{1}$$

where $\boldsymbol{K}$=global stiffness matrix, $\boldsymbol{U}$= vector of unknown nodal displacements, and $\boldsymbol{F}$= vector of equivalent nodal forces. Assume that the structure involves stochastic material and/or geometrical parameters and is subjected to random external forces, the global stiffness matrix $\boldsymbol{K}$, the equivalent nodal force vector $\boldsymbol{F}$ and consequently the nodal displacement vector $\boldsymbol{U}$ are stochastic. By applying perturbation technique, each stochastic quantity is expanded into a series with respect to deviations of stochastic parameters, as follows

$$\boldsymbol{K}=\boldsymbol{K}^0+\sum_k \boldsymbol{K}_k^I\alpha_k+\sum_k\sum_l \boldsymbol{K}_{kl}^{II}\alpha_k\alpha_l+\cdots \tag{2}$$

$$\boldsymbol{F}=\boldsymbol{F}^0+\sum_k \boldsymbol{F}_k^I\alpha_k+\sum_k\sum_l \boldsymbol{F}_{kl}^{II}\alpha_k\alpha_l+\cdots \tag{3}$$

$$\boldsymbol{U}=\boldsymbol{U}^0+\sum_k \boldsymbol{U}_k^I\alpha_k+\sum_k\sum_l \boldsymbol{U}_{kl}^{II}\alpha_k\alpha_l+\cdots \tag{4}$$

where α_k's are fluctuations of stochastic parameters. It is worthy to mention that $\boldsymbol{K}^0, \boldsymbol{F}^0$ and $\boldsymbol{U}^0$ do not represent mathematical expectations of $\boldsymbol{K}, \boldsymbol{F}$ and $\boldsymbol{U}$, respectively. Rather, they are the deterministic values corresponding to mean values of stochastic parameters. Substituting expressions eqs.(2-4) into eq.(1), we can get a system of recursive equations. The first three equations corresponding to zero, first and second order perturbations are as follows

$$
\begin{aligned}
&\boldsymbol{K}^0 \boldsymbol{U}^0 = \boldsymbol{F}^0 \\
&\boldsymbol{K}_k^I \boldsymbol{U}^0 + \boldsymbol{K}^0 \boldsymbol{U}_k^I = \boldsymbol{F}_k^I \\
&\boldsymbol{K}_{kl}^{II} \boldsymbol{U}^0 + \boldsymbol{K}_k^I \boldsymbol{U}_l^I + \boldsymbol{K}_l^I \boldsymbol{U}_k^I + \boldsymbol{K}^0 \boldsymbol{U}_{kl}^{II} = \boldsymbol{F}_{kl}^{II}
\end{aligned} \tag{5}
$$

The solution of eq.(5) reads

$$
\begin{aligned}
&\boldsymbol{U}^0 = (\boldsymbol{K}^0)^{-1} \boldsymbol{F}^0 \\
&\boldsymbol{U}_k^I = (\boldsymbol{K}^0)^{-1} (\boldsymbol{F}_k^I - \boldsymbol{K}_k^I \boldsymbol{U}^0) \\
&\boldsymbol{U}_{kl}^{II} = (\boldsymbol{K}^0)^{-1} (\boldsymbol{F}_{kl}^{II} - \boldsymbol{K}_{kl}^{II} \boldsymbol{U}^0 - \boldsymbol{K}_k^I \boldsymbol{U}_l^I - \boldsymbol{K}_l^I \boldsymbol{U}_k^I)
\end{aligned} \tag{6}
$$

Eq.(6) consists of three recursive equations. The first equation gives the first-order approximation of the mathematical expectation of the displacement vector, which is just the deterministic response corresponding to mean values of uncertain parameters. The second equation gives the first-order perturbation of the displacement vector. The third equation gives the second-order perturbation of the displacement vector. It is seen that the higher-order perturbations of the displacement vector is determined recursively from the lower order perturbations of the displacement vector and lower-order or the same-order perturbation of stochastic input parameters.

By truncating the series expansion in eq.(2) after the first-order perturbation terms, we obtain the following first-order solutions for the mean and covariance of the displacement U, based on the first-order perturbation finite element method for stochastic structures

$$E[U] = U^0$$
$$Cov[U,U^T] = \sum_k \sum_l U_k^I (U_l^I)^T E[\alpha_k \alpha_l] \tag{7}$$

Analogously, if we truncate the series expansions in eq.(2) after the second-order perturbation terms, we get the second-order perturbation solutions for the mean and covariance of the displacement U based on second-order perturbation finite element method for stochastic structures

$$E[U] = U^0 + \sum_k \sum_l U_{kl}^{II} E[\alpha_k \alpha_l]$$
$$\begin{aligned} Cov[U,U^T] = & \sum_k \sum_l U_k^I (U_l^I)^T E[\alpha_k \alpha_l] + \sum_k \sum_l \sum_m [U_m^I (U_{kl}^{II})^T + U_{kl}^{II} (U_m^I)^T] E[\alpha_m \alpha_k \alpha_l] \\ & + \sum_k \sum_l \sum_m \sum_n U_{kl}^{II} (U_{mn}^{II})^T E[\alpha_k \alpha_l \alpha_m \alpha_n] \\ & - \sum_k \sum_l \sum_m \sum_n [U_{kl}^{II} (U_{mn}^{II})^T + U_{mn}^{II} (U_{kl}^{II})^T] E[\alpha_k \alpha_l] E[\alpha_m \alpha_n] \end{aligned} \tag{8}$$

It is seen that third and fourth moments of the stochastic parameters are required to obtain the second-order perturbation solutions. Conceptually, higher-order perturbation finite element method for stochastic structures can be formulated in a similar way, however, it is rarely applicable due to its huge computational effect and the unavailability of information on high-order moments of stochastic input parameters.

2.2 FE Formulation by Series Expansion

The series expansion technique can be also applied to formulate the finite element method for stochastic structures. The adoption of Taylor series expansion will result in exactly the same equations for the fluctuations of the displacement as those obtained by the perturbation technique (Handa and Anderson 1981, Zhu, Ren and Wu 1992). The introduction of the Neumann expansion is proposed by Shinozuka and Yamakazi (1988) to formulate the Neumann series based finite element method for stochastic structures. From eq.(1), we have

$$U = K^{-1} F \tag{9}$$

By using the Neumann expansion method, the stochastic global stiffness matrix is decomposed into two matrices

$$K = K_0 + \Delta K \tag{10}$$

where K_0 represents the mean stiffness matrix and ΔK is deviated part of the stiffness matrix K from K_0 : $\Delta K = K - K_0$. The Neumann expansion of K^{-1} takes the following form

$$K^{-1} = (K_0 + \Delta K)^{-1} = (I - P + P^2 - P^3 - \ldots) K_0^{-1} \tag{11}$$

where

$$P = K_0^{-1} \Delta K \tag{12}$$

Substitution of eq.(11) into eq.(9) gives rise to

$$U = (K_0 + \Delta K)^{-1} F = (I - P + P^2 - P^3 - \dots) K_0^{-1} F$$
$$= U_0 - U_1 + U_2 - U_3 + \dots \tag{13}$$

where

$$U_0 = K_0^{-1} F$$
$$K_0 U_i = \Delta K U_{i-1} \qquad (i = 1, 2, \dots) \tag{14}$$

The second equation of eq.(14) consists of a set of recursive equations. The first-order solutions for the mean and covariance of the displacement U, based on the first-order Neumann expansion finite element method for stochastic structures are obtained as

$$E[U] = U^0$$
$$Cov[U, U^T] = K_0^{-1} E[\Delta K U_0 U_0^T \Delta K] K_0^{-1} \tag{15}$$

where we have assumed that the applied forces are deterministic. Analogously, the second-order solutions for the mean and covariance of the displacement U based on second-order Neumann expansion finite element method for stochastic structures

$$E[U] = U^0 + K_0^{-1} E[\Delta K K_0^{-1} \Delta K] U_0$$
$$Cov[U, U^T] = K_0^{-1} E[\Delta K U_0 U_0^T \Delta K] K_0^{-1} - K_0^{-1} E[\Delta K K_0^{-1} \Delta K U_0 U_0^T \Delta K] K_0^{-1}$$
$$- K_0^{-1} E[\Delta K U_0 U_0^T \Delta K K_0^{-1} \Delta K] K_0^{-1} + K_0^{-1} E[\Delta K K_0^{-1} \Delta K U_0 U_0^T \Delta K K_0^{-1} \Delta K] K_0^{-1} \tag{16}$$
$$- K_0^{-1} E[\Delta K K_0^{-1} \Delta K] U_0 U_0^T E[\Delta K K_0^{-1} \Delta K] K_0^{-1}$$

Here we have used the symmetric property of the stiffness K and its deviation ΔK.

2.3 FE Formulation by Homogeneous Chaos

The finite element method for stochastic structures based on the Homogeneous chaos expansion was proposed by Ghanem and Spanos (1989,1991). By using Karhunen-Loeve expansion method, a spatial random field H can be expanded in the form

$$H(\boldsymbol{x}) = H_0(\boldsymbol{x}) + \sum_{i=1}^{M} \xi_i \sqrt{\lambda_i}\,\phi_i(\boldsymbol{x}) \tag{17}$$

where M=number of truncated terms in Karhunen-Loeve expansion. $H_0(\boldsymbol{x})$=mean value of the random field $H(\boldsymbol{x})$, ξ_i are random variables independent of $\boldsymbol{x}$, and λ_i and $\phi_i(\boldsymbol{x})$ are the eigenvalues and eigenfunctions of the covariance kernel, respectively,

$$\int_D C_{HH}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,\phi_i(\boldsymbol{x}_2)\,d\boldsymbol{x}_2 = \lambda_i \phi_i(\boldsymbol{x}_1) \tag{18}$$

where D=domain of integration and $C_{HH}(\boldsymbol{x}_1,\boldsymbol{x}_2)$=the covariance function of the random field. By using the Karhunen-Loeve expansion to expand the random field involved in the global stiffness matrix K, the finite element equilibrium equation (1) can be rewritten as

$$(\boldsymbol{K}_0 + \sum_{i=1}^{M} \xi_i \boldsymbol{K}_i)\boldsymbol{U} = \boldsymbol{F} \tag{19}$$

where $\boldsymbol{K}_0$=mean global stiffness matrix, $\boldsymbol{K}_i$=fluctuation of the global stiffness matrix associated with the random variable ξ_i. Multiplying with $\boldsymbol{K}_0^{-1}$, eq.(19) can be simplified to

$$(\boldsymbol{I}+\sum_{i=1}^{M}\xi_i\boldsymbol{P}_i)\boldsymbol{U}=\boldsymbol{K}_0^{-1}\boldsymbol{F} \tag{20}$$

where $\boldsymbol{P}_i = \boldsymbol{K}_0^{-1}\ \boldsymbol{K}_i$. The Karhunen-Loeve expansion can be also applied to the random displacement field $u(\boldsymbol{x})$. It reads

$$u(\boldsymbol{x})=\sum_{j=1}^{J}\chi_j\sqrt{\mu_j}\,\psi_j(\boldsymbol{x}) \tag{21}$$

where J=number of terms truncated in expansion, χ_j=random variables, μ_j and ψ_j are eigenvalues and eigenfunctions of the covariance kernel, respectively, of the displacement field. Since the covariance function C_{uu} of the displacement field $u(\boldsymbol{x})$ is unknown, eq.(21) is of no use in its current form.

A random variable χ can be expressed generally, through polynomial chaos, as

$$\begin{aligned}\chi &= a_0\Gamma_0+\sum_{i_1=1}^{\infty}a_{i_1}\Gamma_1(\xi_{i_1})\\ &+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}a_{i_1i_2}\Gamma_2(\xi_{i_1},\xi_{i_2})\\ &+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}a_{i_1i_2i_3}\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3})+\cdots\end{aligned} \tag{22}$$

where Γ's are successive polynomial chaos of their arguments and a's are deterministic constants. Truncating eq.(22) after the L-th polynomial and rewriting it, we get

$$\chi=\sum_{l=0}^{L}b_l\Psi_l[\{\xi_r\}] \tag{23}$$

where b's and Ψ's are similar to a's and Γ's, respectively. Applying expression (23) to each random variable in eq.(21), the random displacement field reads

$$u(x) = \sum_{j=1}^{J} \sum_{l=0}^{L} b_l^{(j)} \Psi_l[\{\xi_r\}] c_j(x) \tag{24}$$

where

$$c_j(x) = \sqrt{\mu_j}\, \psi_j(x) \tag{25}$$

Denoting

$$d_l(x) = \sum_{j=1}^{J} b_l^{(j)} c_j(x) \tag{26}$$

we obtain

$$u(x) = \sum_{l=0}^{L} d_l(x) \Psi_l[\{\xi_r\}] \tag{27}$$

Eq.(27) gives the random displacement field in terms of homogeneous chaos. Discretizing the displacement field into nodal displacement, the nodal displacement vector reads

$$U = \sum_{l=0}^{L} d_l \Psi_l[\{\xi_r\}] \tag{28}$$

where d_l is a vector of the same dimension as U. Substitution of eq.(28) into eq.(20) gives

$$\sum_{l=0}^{L} (I + \sum_{i=1}^{M} \xi_i P_i) d_l \Psi_l[\{\xi_r\}] = K_0^{-1} F \tag{29}$$

Multiplying eq.(29) with $\Psi_m[\{\xi_r\}]$, with m=0, 1, ... , L, and requiring the mean square error to be minimal, leads to

$$\sum_{l=0}^{L}\left[\sum_{i=0}^{M} E[\xi_i \Psi_l[\{\xi_r\}]\Psi_m[\{\xi_r\}]]\boldsymbol{P}_i\right]\boldsymbol{d}_l = E[\Psi_m[\{\xi_r\}]]\boldsymbol{K}_0^{-1}\boldsymbol{F} \tag{30}$$

where ξ_0=0 and $\boldsymbol{P}_0$=**I**. Eq.(30) consists of a set of algebraic equations to be solved for the vectors $\boldsymbol{d}_l$ after imposing the boundary conditions.

Since polynomial chaos have zero means, the mean displacement vector U is given by

$$E[\boldsymbol{U}] = \boldsymbol{d}_0 \tag{31}$$

and the orthogonality property of the polynomial chaos results in the following covariance matrix of the displacement vector

$$Cov(\boldsymbol{U},\boldsymbol{U}^T) = \sum_{l=0}^{L} E[\Psi_l[\{\xi_r\}]\,\Psi_l[\{\xi_r\}]]\boldsymbol{d}_l\boldsymbol{d}_l^T \tag{32}$$

2.4 Improved First-Order Perturbation FE Formulation

In the perturbation-based FEM for stochastic structures, the deterministic response U^0 is considered as first-order approximation of the expectation of the response, and is used to calculate second-order moment. It is seen that only the first-order moment (namely the mean) of the uncertain parameters is utilized to find $\boldsymbol{U}^0$, which obviously differs from the mean value of $\boldsymbol{U}$. To improve the solution, the second-order moment of stochastic parameters should be taken into account to determine the mean displacement. Rewriting each stochastic quantity as the sum of its mean and deviation, we have

$$\boldsymbol{K}=\bar{\boldsymbol{K}}+\delta\boldsymbol{K}\ , \qquad \boldsymbol{U}=\bar{\boldsymbol{U}}+\delta\boldsymbol{U}\ , \qquad \boldsymbol{F}=\bar{\boldsymbol{F}}+\delta\boldsymbol{F} \tag{33}$$

where upper bar represents mean value and δ indicates deviation. It is remarkable that $\bar{\boldsymbol{K}}$, $\bar{\boldsymbol{F}}$

and $\bar{\boldsymbol{U}}$ differ from $\boldsymbol{K}^0$, $\boldsymbol{F}^0$ and $\boldsymbol{U}^0$, respectively. Substitution of eq.(33) into eq.(1) gives

$$\bar{\boldsymbol{K}}\bar{\boldsymbol{U}}+\delta\boldsymbol{K}\bar{\boldsymbol{U}}+\bar{\boldsymbol{K}}\delta\boldsymbol{U}+\delta\boldsymbol{K}\delta\boldsymbol{U}=\bar{\boldsymbol{F}}+\delta\boldsymbol{F} \tag{34}$$

By taking expectation operator for two sides of above equation, we have

$$\bar{\boldsymbol{K}}\bar{\boldsymbol{U}}+E[\delta\boldsymbol{K}\delta\boldsymbol{U}]=\bar{\boldsymbol{F}} \tag{35}$$

Pre-multiplying both sides of eq.(35) by $\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}$ and taking expectation operator, we get

$$E[\delta\boldsymbol{K}\delta\boldsymbol{U}]+E[\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}\delta\boldsymbol{K}]\bar{\boldsymbol{U}}+E[\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}\delta\boldsymbol{K}\delta\boldsymbol{U}]=E[\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}\delta\boldsymbol{F}] \tag{36}$$

For second-order analysis of the stochastic structures, the first and second-order statistics (mean, variance and covariance) of the uncertain parameters are much more important than higher-order statistics. Furthermore, in most engineering problems, only the first and second-order moments of uncertain parameters are available, and at the same time only the first and second order moments of the response are of interest. Hence, in a first-order approximation, we ignore third-order quantity appearing in eq.(36) to obtain

$$E[\delta\boldsymbol{K}\delta\boldsymbol{U}]=E[\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}\delta\boldsymbol{F}]-E[\delta\boldsymbol{K}\bar{\boldsymbol{K}}^{-1}\delta\boldsymbol{K}]\,\bar{\boldsymbol{U}} \tag{37}$$

Substituting it back into eq.(35) yields

$$A\bar{U} = \bar{F} \tag{38}$$

where

$$A = \bar{K} + E[\delta K \bar{K}^{-1} \delta F] - E[\delta K \bar{K}^{-1} \delta K] \tag{39}$$

Eq.(38) is solved for the mean response. To obtain the variance and covariance of the response, we subtract eq.(35) from eq.(34) to reach

$$\delta K \bar{U} + \bar{K} \delta U + \delta K \delta U = \delta F + E[\delta K \delta U] \tag{40}$$

Solving out δU, we get

$$\delta U = \bar{K}^{-1} \{ \delta F + E[\delta K \delta U] - \delta K \bar{U} - \delta K \delta U \} \tag{41}$$

The variance-covariance matrix of the displacement is

$$\begin{aligned} Cov[U,U^T] &= E[\delta U \delta U^T] \\ &= \bar{K}^{-1} \{ E[\delta F \delta F^T] + E[\delta K \bar{U} \bar{U}^T \delta K] - E[\delta F \bar{U}^T \delta K] - E[\delta K \bar{U} \delta F^T] \\ &+ E[\delta K \delta U \bar{U}^T \delta K] + E[\delta K \bar{U} \delta K] - E[\delta F \delta U^T \delta K] - E[\delta K \delta U \delta F^T] \\ &+ E[\delta K \delta U \delta U^T \delta K] - E[\delta K \delta U] E[\delta U^T \delta K] \} \bar{K}^{-1} \end{aligned} \tag{42}$$

In order to be consistent with the mean displacement analysis, the third and fourth-order terms appearing in eq.(42) are ignored. Therefore, the variance and covariance matrix is simplified to

$$Cov[U,U^T] = \bar{K}^{-1} C \bar{K}^{-1} \tag{43}$$

where

$$C = E[\delta F \delta F^T] + E[\delta K \bar{U} \bar{U}^T \delta K] - E[\delta F \bar{U}^T \delta K] - E[\delta K \bar{U} \delta F^T] \tag{44}$$

It is noted that the mean displacement is obtained from eq.(38) by the improved first-order FEM, instead of from the first equation in eq.(6) within the conventional perturbation FEM. The difference is that A is comprised of the exact mean stiffness matrix $\bar{\boldsymbol{K}}$ and modifying terms due to second-order moments of uncertain parameters, while $\boldsymbol{K}^0$ in eq.(6) is just the deterministic stiffness matrix corresponding to mean parameters. The variance-covariance matrix of the displacement is obtained from eq.(43) by improved FEM, instead from eq.(7) within the conventional perturbation FEM.

2.5 Numerical Results for A Two-Bar Truss with Stochastic Young's Moduli

Consider a simple example of a two-bar truss structure as shown in Fig.1. Both bars have the same length L and cross-sectional area S, and Young's moduli E_1 and E_2, respectively. The global finite element equilibrium equation for the structure can be written as

$$\frac{S}{2L}\begin{bmatrix} E_1+E_2 & -E_1+E_2 \\ -E_1+E_2 & E_1+E_2 \end{bmatrix}\begin{Bmatrix} U \\ V \end{Bmatrix}=\begin{Bmatrix} -Q \\ 0 \end{Bmatrix} \tag{45}$$

where $\boldsymbol{U}=[U,V]^{\mathrm{T}}$=nodal displacement vector, $\boldsymbol{F}=[-Q,0]^{\mathrm{T}}$=equivalent nodal force vector. Assume that E_1 and E_2 are independent random variables with mean $\bar{E}$ and coefficient of variation r.

We have

$$\bar{\boldsymbol{K}}=\boldsymbol{K}^0=\frac{S\bar{E}}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{46}$$

$$\delta \boldsymbol{K} = \boldsymbol{K}_1^I \alpha_1 + \boldsymbol{K}_2^I \alpha_2 = \frac{S\bar{E}}{2L}\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \alpha_1 + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \alpha_2 \right\} \tag{47}$$

where α_1 and α_2 are normalized random variables as expressed in eq.(2).

Improved FEM Solutions - It is straightforward to obtain

$$E[\delta \boldsymbol{K} \bar{\boldsymbol{K}}^{-1} \delta \boldsymbol{K}] = \frac{S\bar{E} r^2}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

Substitution of eq.(48) into eq.(39) yields

$$\boldsymbol{A} = \frac{S\bar{E}}{L}\begin{bmatrix} 1-r^2 & 0 \\ 0 & 1-r^2 \end{bmatrix} \tag{49}$$

The mathematical expectations of the nodal displacements are then obtained from eq.(38)

$$\bar{U} = -\frac{QL}{S\bar{E}} \frac{1}{(1-r^2)} \qquad \bar{V} = 0 \tag{50}$$

Substitution of $\bar{U}$, $\bar{V}$ and δK into eq.(44) yields

$$C = \frac{1}{2}\left(\frac{rQ}{1-r^2}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{51}$$

The variances and covariance of the displacements are obtained from eq.(43)

Conventional Perturbation FEM Solution - Nakagiri and Hisada [8] solved this two-bar truss problem by means of perturbation-based FEM. Substituting eqs.(46,47) into eq.(6), we obtain the mean and variance of the displacements by conventional first-order perturbation FEM

$$Var[U] = Var[V] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2 \frac{r^2}{(1-r^2)^2} \tag{52}$$

$$Cov[U,V] = 0$$

$$\bar{U} = -\frac{QL}{S\bar{E}} \quad , \qquad \bar{V} = 0 \tag{53}$$

Substituting eqs.(46,47) into eq.(7) yields the mean and variance of the displacements obtained

$$Var[U] = Var[V] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2 r^2 \tag{54}$$

$$Cov[U,V] = 0$$

by the conventional second-order perturbation FEM

$$\bar{U} = -\frac{QL}{S\bar{E}}(1+r^2) \quad , \qquad \bar{V} = 0 \tag{55}$$

$$Var[U] = Var[V] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2 (r^2 + E[\alpha^4] - 2E[\alpha^3] - r^4) \tag{56}$$

$$Cov[U,V] = 0$$

Here it has been assumed that E_1 and E_2 share the same third and fourth moments $E[\alpha_1^3] = E[\alpha_2^3] = E[\alpha^3]$ and $E[\alpha_1^4] = E[\alpha_2^4] = E[\alpha^4]$, which are required in second-order perturbation solutions. It is worthy to note that the results obtained by the FEM based on first-order and second-order Neumann expansions are identical to those obtained by the FEM based on first-order and second-order perturbations, respectively.

Exact Solution - The exact solutions for the mean and variance of the displacements for this two-bar truss structures can be solved by computing the exact inverse of the stiffness matrix. The global stiffness matrix can be explicitly inverted to be

$$K^{-1} = \frac{L}{2S}\begin{bmatrix} C_1+C_2 & C_1-C_2 \\ C_1-C_2 & C_1+C_2 \end{bmatrix} \tag{57}$$

where $C_1=1/E_1$ and $C_2=1/E_2$. The means, variances and covariance of the displacements are, respectively

$$\bar{U} = -\frac{LQ\bar{C}}{S} \quad , \qquad \bar{V}=0 \tag{58}$$

$$\begin{aligned} &Var[U] = Var[V] = \frac{1}{2}(\frac{LQ}{S})^2\ Var[C] \\ &Cov[U,V] = 0 \end{aligned} \tag{59}$$

where $Var[C]=Var[C_1]=Var[C_2]$.

Firstly, assume that the random variables α_1 and α_2 possess uniformly distributed density function in the interval $[-a,a]$. Then

$$E[\alpha] = E[\alpha^3] = 0 \quad , \qquad E[\alpha^2] = \frac{a^2}{3} = r^2 \ , \qquad E[\alpha^4] = \frac{a^4}{5} = \frac{9r^4}{5} \tag{60}$$

and

$$\bar{C} = \frac{1}{2\sqrt{3}\bar{E}r}\ln\frac{1+\sqrt{3}r}{1-\sqrt{3}r} \quad , \qquad Var[C] = \frac{1}{(1-3r^2)\bar{E}^2} - \bar{C}^2 \tag{61}$$

where r is the coefficient of variation of random Young's moduli. Substitution of eq.(61) into eq.(58) and (59) results in the exact mean and variance of the displacement U

Substituting eq.(60) into eq.(56) gives variance of the displacement U by second-order

$$\bar{U} = -\frac{QL}{S\bar{E}}\frac{1}{2\sqrt{3}r}\ln\frac{1+\sqrt{3}r}{1-\sqrt{3}r}$$

$$Var[U] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2\left\{\frac{1}{1-3r^2} - \left[\frac{1}{2\sqrt{3}r}\ln\frac{1+\sqrt{3}r}{1-\sqrt{3}r}\right]^2\right\} \tag{62}$$

perturbation FEM

$$Var[U] = Var[V] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2\ (r^2+0.8r^4) \tag{63}$$

Secondly, if the random variables α_1 and α_2 are assumed to possess triangular density distribution, namely

$$f_E(x) = \begin{cases} \frac{1}{b}\left(1-\frac{|x|}{b}\right), & |x| \le b \\ 0, & else \end{cases} \tag{64}$$

then $E[\alpha]=E[\alpha^3]=0$, $Var[\alpha^2]=b^2/6=r^2$, $E[\alpha^4]=b^4/15=2.4r^4$. The second-order perturbation FEM gives

$$Var[U] = Var[V] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2\ (r^2+1.4r^4) \tag{65}$$

The exact mean and variance of the displacement U are, respectively

$$\bar{U} = -\frac{QL}{S\bar{E}}\left[\frac{1}{\sqrt{6}r}\ln\frac{1+\sqrt{6}r}{1-\sqrt{6}r} + \frac{1}{6r^2}\ln(1-6r^2)\right]$$

$$Var[U] = \frac{1}{2}\left(\frac{QL}{S\bar{E}}\right)^2\left\{\frac{1}{6r^2}\ln\frac{1}{1-6r^2} - \left[\frac{1}{\sqrt{6}r}\ln\frac{1+\sqrt{6}r}{1-\sqrt{6}r} + \frac{1}{6r^2}\ln(1-6r^2)\right]^2\right\} \tag{66}$$

For above two cases of uniform and triangular distributions, the mean and variance of the displacement have been numerically calculated by first-order and second-order perturbation FE methods, the improved FE method and the exact solution. The results are plotted in Fig.2.2-2.5. It is shown that the present improved solution, which is a first-order approximation solution in nature, is not only much better than the first-order perturbation solution, and but also better than the second-order perturbation solution. The results for the mean response obtained by the improved method and second-order perturbation FE method are very satisfactory in both cases. However, the result obtained by the first-order perturbation FE method is accompanied by a relatively large error when the coefficient of variation r increases. Indeed, as is seen in Fig.2.2, for r=0.3, the conventional first-order perturbation FEM results in about 10% error. The second order perturbation FEM is accompanied with about 1.7% error, whereas the error within the improved method is less than one percent. The results for the variance obtained by the improved method has less error than those obtained by the first-order or second-order perturbation methods. It implies that the improved method has an advantage to be applicable for larger range of coefficients of variation over the first-order or second-order perturbation methods. For example, in the uniform distribution case, if the percentage error of computed variance is required to be within 5%, the first-order perturbation method is accepted for $r \leq 0.11$, the second-order perturbation method is accepted for $r \leq 0.12$, but the improved method is acceptable for $r \leq 0.15$.

To conclude, we see that the proposed technique significantly expands the applicability of the perturbation techniques. In coming chapters, we will altogether abandon perturbation technique to fully confront the challenge that failed other investigators --- FEM valid for any value of coefficient of variation.

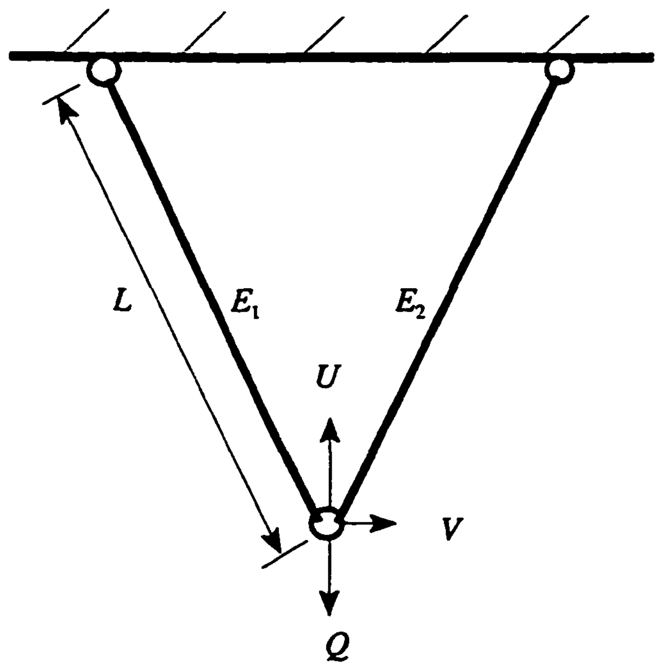

Fig.2.1. A two-bar truss structure with stochastic Young's moduli

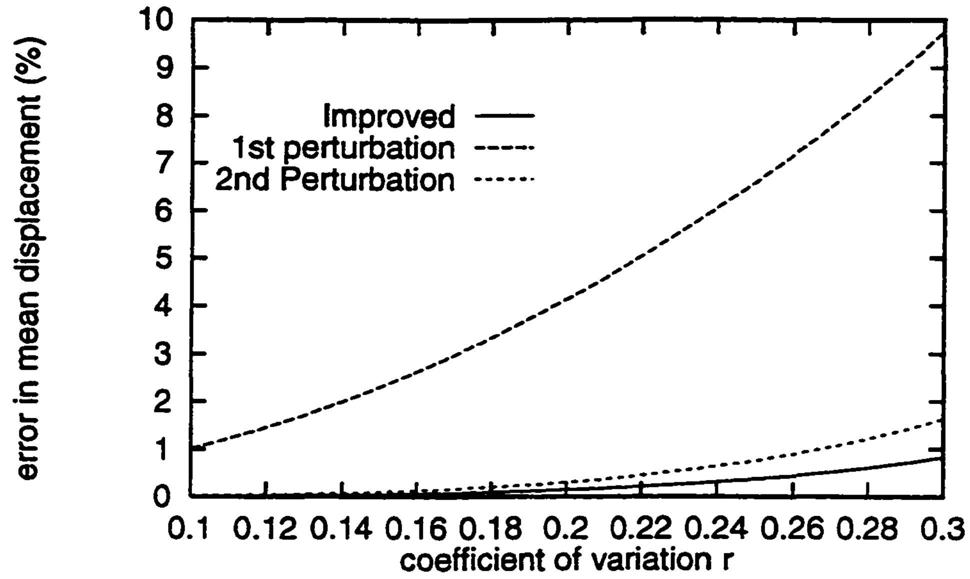

Fig.2.2. Error of mean displacement vs coefficient of variation of uniformly distributed Young's moduli

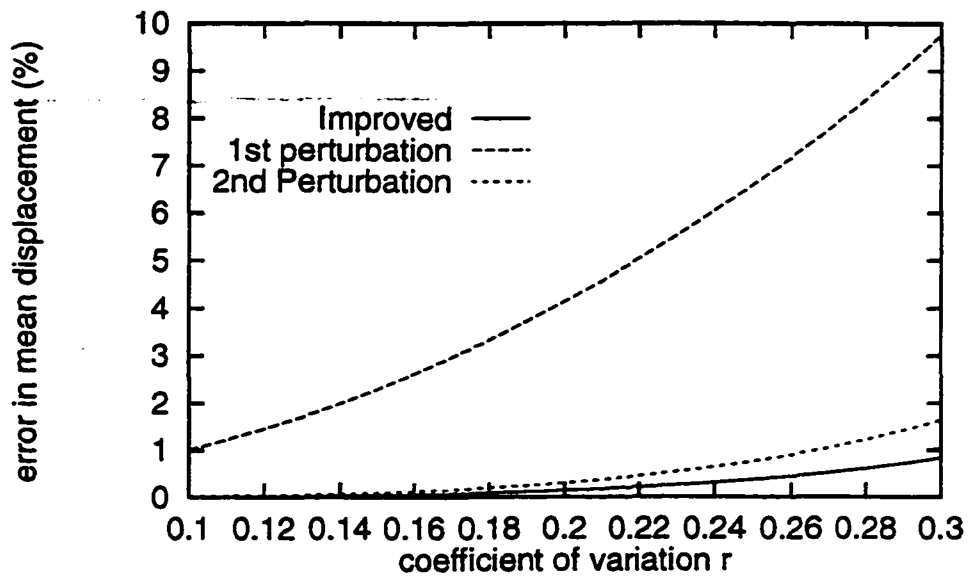

Fig.2.3. Error of variance of displacement vs coefficient of variation of uniformly distributed Young's moduli

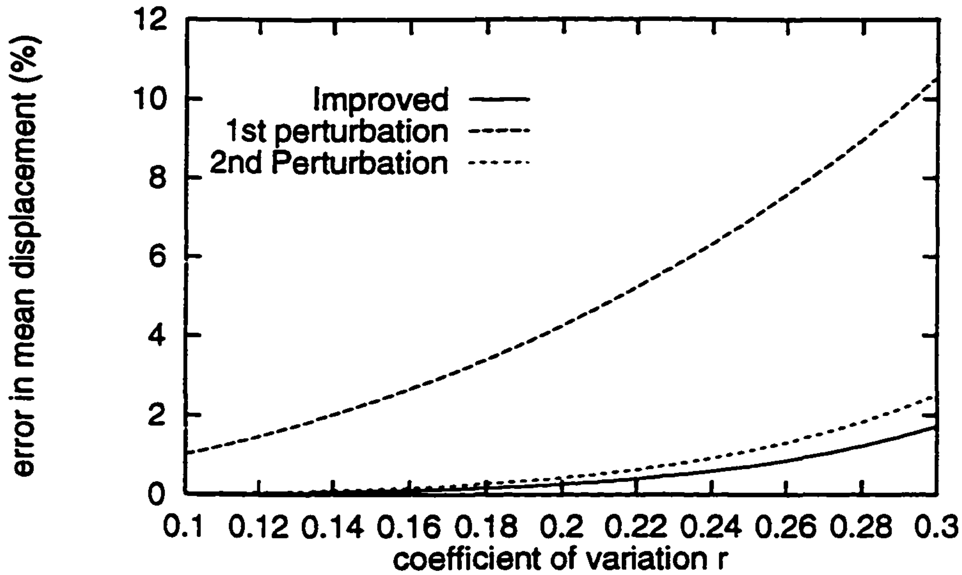

Fig.2.4. Error of mean displacement vs coefficient of variation of triangularly distributed Young's moduli

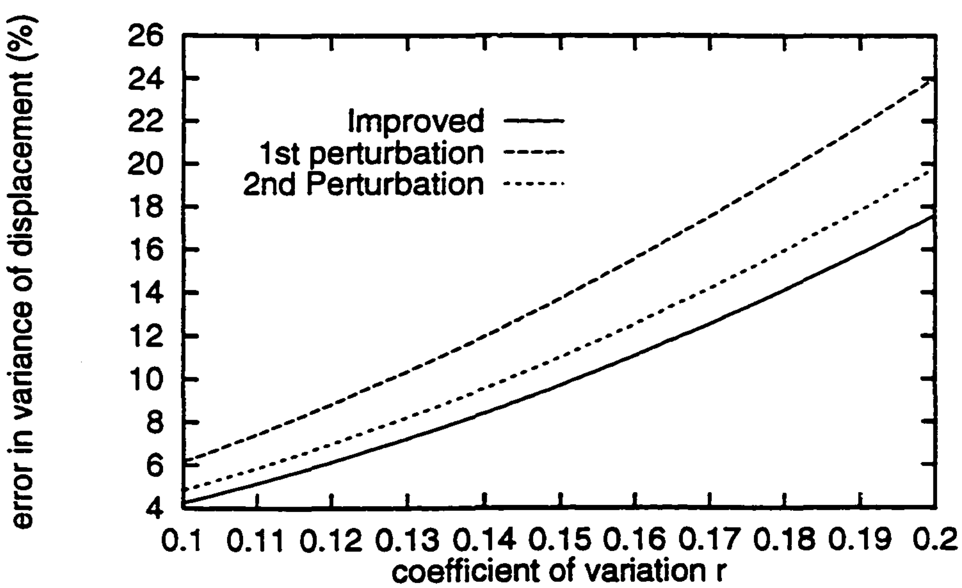

Fig.2.5. Error of variance of displacement vs coefficient of variation of triangularly distributed Young's moduli

Chapter 3: Finite Element Method for Stochastic Structures Based on Inverse of Stiffness Matrix

In the previous chapter, some existing approaches to formulate the finite element method for stochastic structures have been reviewed. These approaches include the perturbation method, series expansion methods. Chapter 2 also presented an improved first-order perturbation finite element method for stochastic structures.

In the next three chapters, we will present several new *non-perturbative* FEM for stochastic structures. As the first approach, this chapter discusses the finite element method for stochastic structures, which is based on the exact inverse of the stiffness matrix.

3.1 FEM Based on Direct Exact Inverse of Stiffness Matrix

3.1.1 Exact Inverse of Stiffness Matrix for Fixed-Free Bar Extension

Consider a fixed-free bar with length L, which is subjected to a deterministic distributed force $q(x)$ (Fig.3.1). We assume that the extensional stiffness $D(x)=EA$ of the bar constitutes a spatial random field, where $E(x)$=Young's modulus and $A(x)$=area of the cross-section. The governing equation of the bar is

$$\frac{d}{dx}[D(x)\frac{du}{dx}] = q(x) \tag{1}$$

where $u(x)$ is the axial displacement of the bar. Assume that the random extensional stiffness $D(x)$ can be expressed as a power function of another random field $G(x)$, namely $D(x) = bG^n(x)$,

where b and n are constants. For example, if Young's modulus $E(x)$ is a stochastic field and the cross-sectional area A is a deterministic constant, then $G(x)=E(x)$, $n=1$, $b=A$. If the radius of a circular cross-section of the bar is a stochastic field and the Young's modulus is a deterministic constant, then $G(x)=R(x)$, $n=2$ and $b=\pi E$. If both $E(x)$ and $R(x)$ are random fields, then $G(x)=E(x)R^2(x)$, $n=1$ and $b=\pi$. Integrating eq.(1) once and substituting $D(x)=bG^n(x)$ into eq.(1), we have

$$bG^n(x)\frac{du}{dx}=F(x) \tag{2}$$

where $F(x)=\int_x^L q(s)ds+F_0$ is the axial force acting on the cross-section x of the bar, and F_0 is the axial force acting at the end $x=L$. If there is any concentrated force F_1 acting at the cross-section x, it can be represented through use of Dirac-Delta function as $q(x)=F_1\delta(x)$. Here we have assumed that the bar is statically determinate, so that the closed-form solution exists and is used to verify the finite element solutions. If the bar is statically indeterminate, for example a fixed-fixed bar, the closed-form solution appears to be unobtainable. However, the finite element method based on exact inverse of the stiffness matrix is still applicable. Eq.(1) can also be rewritten as

$$\frac{du}{dx}=\frac{F(x)}{bG^n(x)}=\beta(x)Q^n(x) \tag{3}$$

where $\beta(x)=F(x)/b$, $Q(x)=1/G(x)$. Applying the finite element method, we divide the bar into N uniform elements with equal length $l=L/N$, and interpolate the displacement in j-th element linearly

$$u(x)=\frac{(x_{j+1}-x)}{l}U_j+\frac{(x-x_j)}{l}U_{j+1} \tag{4}$$

where x_j and U_j are longitudinal coordinate and displacement of j-th node, respectively, $x_j \leq x \leq x_{j+1}$. With principle of minimum potential energy, the element stiffness matrix k^e is obtained as follows

$$k^e=\frac{bG_j^n}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{5}$$

where G_j is the representative of random field $G(x)$ on j-th element. It should be remarked that the element stiffness matrix is derived based on eq.(1) rather than on eq.(2), it is applicable to both statically determinate and indeterminate bar problems. Assembling all element stiffness matrices and noting that U_1=0 (displacement at the left fixed end) , we get the following global equilibrium equation

$$\boldsymbol{KU}=\boldsymbol{P} \tag{6}$$

where $\boldsymbol{K}$ = global stiffness matrix, $\boldsymbol{U}=[U_2,U_3,\cdots,U_N,U_{N+1}]^T$=vector of unknown nodal displacements, and $\boldsymbol{P}=[P_2,P_3,\cdots,P_N,P_{N+1}]^T$=vector of equivalent nodal forces. Explicitly, the global stiffness matrix $\boldsymbol{K}$ reads

$$\boldsymbol{K}=\frac{b}{l}\begin{bmatrix} G_1^n+G_2^n & -G_2^n & 0 & 0 & \dots & 0 & 0 & 0 \\ -G_2^n & G_2^n+G_3^n & -G_3^n & 0 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -G_{N-1}^n & G_{N-1}^n+G_N^n & -G_N^n \\ 0 & 0 & 0 & 0 & \dots & 0 & -G_N^n & G_N^n \end{bmatrix} \tag{7}$$

The exact inverse $\boldsymbol{V}=\boldsymbol{K}^{-1}$ of the above global stiffness matrix $\boldsymbol{K}$ can be proved to be

$$V = \frac{l}{b}\begin{bmatrix} Q_1^n & Q_1^n & Q_1^n & \cdots & Q_1^n \\ Q_1^n & Q_1^n+Q_2^n & Q_1^n+Q_2^n & \cdots & Q_1^n+Q_2^n \\ Q_1^n & Q_1^n+Q_2^n & Q_1^n+Q_2^n+Q_3^n & \cdots & Q_1^n+Q_2^n+Q_3^n \\ \cdots & & & & \\ Q_1^n & Q_1^n+Q_2^n & Q_1^n+Q_2^n+Q_3^n & \cdots & Q_1^n+Q_2^n+\ldots+Q_N^n \end{bmatrix} \tag{8}$$

where $Q_i=1/G_i$. The (i,j)-th element, denoted by v_{ij}, of matrix V can be written as

$$v_{ij} = (K^{-1})_{ij} = \frac{l}{b}\sum_{r=1}^{\min(i,j)} Q_r^n \tag{9}$$

3.1.2 Solutions for Fixed-Free Bar Under End Force F_0

Assume that the bar is subjected to an end force F_0, i.e, $q(x)=0$. The equivalent nodal load vector is then $P=[0,0,\cdots,0,F_0]^T$. The displacement component is now obtained from eq.(6)

$$U_{j+1} = F_0 v_{jN} = \beta_0 l \sum_{r=1}^{j} Q_r^n \tag{10}$$

where $\beta_0=F_0/b$. Its mean-value and variance are, respectively,

$$E[U_{j+1}] = \beta_0 l \sum_{r=1}^{j} E[Q_r^n] = jl\beta_0 E[Q^n] \tag{11}$$

$$Var[U_{j+1}] = \beta_0^2 l^2 \sum_{r=1}^{j}\sum_{s=1}^{j} Cov(Q_r^n, Q_s^n) \tag{12}$$

It is worthy to mention that the coordinate at $(j+1)$-th node is $x=jl$. For comparison, in the following we derive the solutions by the conventional first-order perturbation FEM and the solutions obtained by direct integration.

Within first-order approximation, we have from eq.(2) of chapter 2

$$K_j^I = \frac{\partial K}{\partial G_j}\Big|_{G=\bar{G}} = \frac{nb\bar{G}^{n-1}}{l}\begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 1 & -1 & \dots & 0 \\ 0 & \dots & -1 & 1 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}, \qquad (j=2,3,\dots,n) \tag{13}$$

Also

$$K_1^I = \frac{nb\bar{G}^{n-1}}{l}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{14}$$

The mean displacement is obtained explicitly in the component form as

$$E[U_{j+1}] = U_{j+1}^0 = jl\beta_0\bar{G}^{-n} \tag{15}$$

From second equation of eq.(6) of chapter 2, we have

$$U_j^I = -(K^0)^{-1}K_j^I U^0 = -\frac{nb\bar{G}^{n-1}}{l}\bar{K}^{-1}\begin{Bmatrix} 0 \\ \cdot \\ -1 \\ 1 \\ \cdot \\ 0 \end{Bmatrix}(U_{j+1}^0 - U_j^0) \tag{16}$$

Due to eq.(15),

$$\bar{U}_{j+1}-\bar{U}_j=l\beta_0\bar{G}^{-n}=\bar{U}_2 \tag{17}$$

is constant. The matrix $\bar{K}^{-1}$ can be obtained from eq.(7)

$$(K^0)^{-1}=\frac{l\bar{G}^{-n}}{b}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \dots & & & & \\ 1 & 2 & 3 & \dots & n \end{bmatrix} \tag{18}$$

Substituting eq.(17) and eq.(18) into eq(16) and then substituting the result into the second equation of eq.(7) of chapter 2, we get the first-order perturbation solution for the displacement variance

$$Var[U_{j+1}]=n^2\beta_0^2l^2\bar{G}^{-2(n+1)}\sum_{r=1}^{j}\sum_{s=1}^{j}Cov(G_r,G_s) \tag{19}$$

To get the exact solution for this problem, one integrate eq.(3) to yield

$$u(x)=\int_0^x\beta(\xi)Q^n(\xi)d\xi \tag{20}$$

Note that $\beta(\xi)=\beta_0$ for the case where the bar is under the end force, the mean and variance of the displacement are, respectively

$$\bar{u}(x)=\beta_0\int_0^x E[Q^n(\xi)]d\xi=\beta_0xE[Q^n] \tag{21}$$

$$Var[u(x)]=\beta_0^2\int_0^x\int_0^x Cov[Q^n(\xi_1),Q^n(\xi_2)]d\xi_1d\xi_2 \tag{22}$$

3.1.3 Solutions for Fixed-Free Bar Under Uniform Distribution q_0

If the bar is under uniformly distributed extensional load $q(x)=q_0$, i.e., $F_0=0$ and $F(x)=q_0(L-x)$. The equivalent nodal load vector is then $P=q_0l[1,1,\cdots,1,0.5]^T$. The displacement component is again obtained from eq.(6)

$$U_{j+1}=q_0l(v_{j1}+v_{j2}+\cdots+v_{j(N-1)}+0.5v_{jN})=\beta_1l^2\sum_{r=1}^{j}(N+0.5-r)Q_r^n \tag{23}$$

where $\beta_1=q_0/b$. Its mean-value and variance are, respectively,

$$E[U_{j+1}]=\beta_1l^2\sum_{r=1}^{j}(N+0.5-r)E[Q_r^n]=j(N-0.5j)l^2\beta_1E[Q^n] \tag{24}$$

$$Var[U_{j+1}]=\beta_0^2l^4\sum_{r=1}^{j}\sum_{s=1}^{j}(N+0.5-r)(N+0.5-s)\,Cov(Q_r^n,Q_s^n) \tag{25}$$

The mean and the variance of the displacement obtained by the first-order perturbation FEM are, respectively

$$E[U_{j+1}]=\bar{U}_{j+1}=j(N-0.5j)l^2\beta_1\bar{G}^{-n} \tag{26}$$

$$Var[U_{j+1}]=n^2\beta_1^2l^4\bar{G}^{-2(n+1)}\sum_{r=1}^{j}\sum_{s=1}^{j}(N+0.5-r)(N+0.5-s)\,Cov(G_r,G_s) \tag{27}$$

The exact mean and variance of the displacement obtained by direct integration are, respectively

$$\bar{u}(x)=\beta_1\int_0^x(L-\xi)E[Q^n(\xi)]d\xi=\beta_1(Lx-0.5x^2)E[Q^n] \tag{28}$$

$$Var[u(x)]=\beta_1^2\int_0^x\int_0^x(L-\xi_1)(L-\xi_2)Cov[Q^n(\xi_1),Q^n(\xi_2)]d\xi_1d\xi_2 \tag{29}$$

Comparing three different solutions for the mean displacement and for variance of the displacement , one finds that the mean displacement obtained by the FEM based on exact inverse of stiffness coincides with the analytic solution, *regardless* of the values of the power order n and number of elements N if the random field $G(x)$ is homogeneous. The variance of displacement obtained by FEM based on exact inverse of stiffness matrix *converges* to the analytic solution when the number of elements increases. The truncation error, which accompanies the conventional finite element method for stochastic structures through truncating of the expansion series of the stiffness matrix, is nonexistent for finite element solutions with exact inverse approach, as it should. The mean displacement obtained by the conventional first-order perturbation FEM agrees well with the exact solution *only* when $E[Q^n]$ deviates insignificantly from $1/(E[G])^n$. The latter condition happens when both the parameter n and the coefficient of variation of the random field $G(x)$ are small.

3.1.4 Exact Inverse of Stiffness Matrix for Fixed-Fixed Bar Extension

The stiffness matrix for a fixed-fixed bar can be obtained from eq.(8) by cancelling its last row and last column as

$$K=\frac{b}{l}\begin{bmatrix} G_1^n+G_2^n & -G_2^n & 0 & 0 & \dots & 0 & 0 & 0 \\ -G_2^n & G_2^n+G_3^n & -G_3^n & 0 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -G_{N-2}^n & G_{N-2}^n+G_{N-1}^n & -G_{N-1}^n \\ 0 & 0 & 0 & 0 & \dots & 0 & -G_{N-1}^n & G_{N-1}^n+G_N^n \end{bmatrix} \tag{30}$$

Its inverse reads

$$
K^{-1} = \frac{l}{bH}\begin{bmatrix} g_1 f_1 & G_2 g_1 f_2 & G_2 G_3 g_1 f_3 & \cdots & G_2 G_2 \ldots G_{N-2} g_1 f_{N-2} & G_2 G_3 \ldots G_{N-1} g_1 f_{N-1} \\ & g_2 f_2 & G_3 g_2 f_3 & \cdots & G_3 \ldots G_{N-2} g_2 f_{N-2} & G_3 \ldots G_{N-1} g_2 f_{N-1} \\ & & & \cdots & & \\ & sym. & & & g_{N-2} f_{N-2} & G_{N-1} g_{N-2} f_{N-1} \\ & & & & & g_{N-1} f_{N-1} \end{bmatrix} \quad (31)
$$

where

$$
\begin{aligned}
H &= G_1 G_2 \cdots G_{N-1} + G_1 G_2 \cdots G_{N-2} G_N + \cdots + G_1 G_3 \cdots G_N + \cdots G_2 G_3 \cdots G_N \\
f_1 &= G_2 G_3 \cdots G_{N-1} + G_2 G_3 \cdots G_{N_2} G_N + \cdots G_2 G_4 \cdots G_N + G_3 G_4 \cdots G_N \\
f_2 &= G_3 G_4 \cdots G_{N-1} + G_3 G_4 \cdots G_{N-2} G_N + \cdots G_3 G_5 \cdots G_N + G_4 G_5 \cdots G_N \\
& \ldots\ldots \\
f_{N-2} &= G_{N-1} + G_N \\
f_{N-1} &= 1 \\
g_1 &= 1 \\
g_2 &= G_1 + G_2 \\
& \ldots\ldots \\
g_{N-2} &= G_1 G_2 \cdots G_{N-3} + G_1 G_2 \cdots G_{N-4} G_{N-2} + \cdots + G_1 G_3 \cdots G_{N-2} + G_2 G_3 \cdots G_{N-2} \\
g_{N-1} &= G_1 G_2 \cdots G_{N-2} + G_1 G_2 \cdots G_{N-3} G_{N-1} + \cdots + G_1 G_3 \cdots G_{N-1} + G_2 G_3 \cdots G_{N-1}
\end{aligned} \quad (32)
$$

3.1.5 Numerical Results for Bar with Stochastic Young's Modulus

Consider that Young's modulus $E(x)$ of the bar is a spatially random field, i.e., n=1, b=A and $G(x)$=$E(x)$, $Q(x)$=$1/E(x)$. Denote $E(x)$=$E_0[1+\alpha(x)]$, where E_0 is the mean and $\alpha(x)$ is a normalized random field possessing zero mean and the following two-dimensional Nataf's probability density function (Liu and Der Kiureghian 1986)

$$p_{\alpha(x_1)\alpha(x_2)}(u_1,u_2) = p_{\alpha(x_1)}(u_1)p_{\alpha(x_2)}(u_2)\,\frac{\varphi_2(z_1,z_2,\rho')}{\varphi(z_1)\varphi(z_2)} \tag{33}$$

where $p_{\alpha(x)}(u)$ is the marginal probability density function (PDF) which is assumed to be uniform in the interval $[-a,a]$, $\varphi(.)$ is the standard normal PDF, $\varphi_2(z_1,z_2,\rho')$ is the two-dimensional normal PDF with zero means, unit standard deviations and correlation coefficient ρ', which is related to the correlation coefficient ρ of $\alpha(x_1)$ and $\alpha(x_2)$ as follows

$$\rho' = (1.047 - 0.047\rho^2)\rho \tag{34}$$

where the correlation coefficient of $\alpha(x_1)$ and $\alpha(x_2)$ is assumed to be exponential

$$\rho(x_1,x_2) = \exp(-\frac{|x_1 - x_2|}{L}) \tag{35}$$

In eq.(33), z_1 and z_2 are marginal transformations of u_1 and u_2, respectively,

$$z_i = \Phi^{-1}[P_{\alpha(x)}(u_i)]\ , \qquad i=1,2 \tag{36}$$

where $P_{\alpha(x)}(.)$ is the cumulative probabilistic function of $\alpha(x)$, $\Phi(.)$ is the standard cumulative normal probability function.

As previously mentioned, the finite element solution of the mean displacement by exact inverse of the stiffness matrix is identical to analytic solution for both cases that the bar is subjected to end force or uniform distributed load, since the random Young's modulus $E(x)$ is a homogeneous random field. The mean displacement solution by the conventional first-order perturbation FEM is related to the exact solution as follows

$$E[U_{j+1}] = [\frac{1}{2\sqrt{3}\theta}\ln\frac{1+\sqrt{3}\theta}{1-\sqrt{3}\theta}]^{-1}\ \bar{u}(x) \tag{37}$$

where $\theta=\sqrt{Var[E(x)]}/E_0=a/\sqrt{3}$ is the coefficient of variation of the random field $E(x)$. Fig.3.2 shows the difference between the mean displacement solutions by the FEM based on exact inverse of the stiffness matrix and the first-order perturbation FEM. It is seen that the difference is less than 10 percents only when the coefficient of variation θ is less than 0.30.

The finite element solution for the variance of displacement differ from the analytic solution. The difference between the finite element solution by exact inverse depends on the scheme of finite element discretization only, whereas the difference between the finite element solution by first-order perturbation depends on the scheme of finite element discretization and the coefficient of variation of the random field $E(x)$. Fig.3.3 shows the variance of the end displacement obtained by the FEM based on exact inverse of the stiffness matrix and the first-order perturbation FEM. It is seen that the difference increases fast when the coefficient of variation r of the Young's modulus increases. The difference is greater than 10 % if the coefficient of variation of the field $E(x)$ is greater than 8.5 percents.

3.2 FEM through Decoupling of Shear and Bending Modes for Stochastic Beams

In the previous part, we have exemplified the finite element method for stochastic structures based the direct exact inverse of the global stiffness matrix. However, except for some simple cases like bar extension problems, the inverse of global stiffness matrix is usually unobtainable. The FEM for stochastic structures based the direct inverse can not be generalized for a wide application. In this section, we will apply the idea proposed by Fuchs (1992) to construct the element stiffness matrix and explicitly get the inverse of the global stiffness matrix for beam bending problems.

3.2.1 New Formulation of FE Stiffness Matrix

Consider an elastic beam with spatially varying bending stiffness $D(x)$. The beam is equally divided into N elements each with length a. The finite element equilibrium equation for the i-th element of the beam is derived as

$$\frac{D_i}{a^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} w_{i1} \\ a\theta_{i1} \\ w_{i2} \\ a\theta_{i2} \end{Bmatrix} = \begin{Bmatrix} Q_{i1} \\ M_{i1}/a \\ Q_{i2} \\ M_{i2}/a \end{Bmatrix} \tag{38}$$

where the subscripts " *1* " and " *2* " represent the left and right nodes of the element, respectively, w=displacement of the beam, $\theta=dw/dx$=slope of the beam, Q and M are generalized nodal forces representing shear force and bending moment, respectively, D_i is the representative of the bending stiffness $D(x)$ on the i-th element. Eq.(38) can be simply rewritten as

$$\frac{D_i}{a^3}\boldsymbol{K}_i\boldsymbol{q}_i = \boldsymbol{F}_i \tag{39}$$

where

$$\boldsymbol{K}_i = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \tag{40}$$

$$q_i = [w_{i1} \quad a\theta_{i1} \quad w_{i2} \quad a\theta_{i2}]^T \tag{41}$$

$$F_i = [Q_{i1} \quad M_{i1}/a \quad Q_{i2} \quad M_{i2}/a]^T \tag{42}$$

Since K_i is a full matrix, four simultaneous equations expressed by eq.(39) are coupled with each other. Orthogonal transformation of variables can be applied to decouple eq.(39). The eigenvalue matrix and eigenvector matrices of K_i are, respectively

$$\lambda = diag[0,0,2,30] \tag{43}$$

$$V = \begin{bmatrix} -1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \tag{44}$$

Let

$$q_i = V^T e_i \tag{45}$$

We pre-multiply eq.(39) by V to get

$$\frac{D_i}{a^3} diag[0,0,4,300] e_i = t_i \tag{46}$$

where

$$t_i = VF_i \tag{47}$$

Eq.(47) consists four equations. the first two equations read

The third and fourth equations are

$$t_{i1} = t_{i2} = 0 \tag{48}$$

$$\frac{D_i}{a^3}\begin{bmatrix} 4 & 0 \\ 0 & 300 \end{bmatrix}\begin{Bmatrix} e_{i3} \\ e_{i4} \end{Bmatrix} = \begin{Bmatrix} t_{i3} \\ t_{i4} \end{Bmatrix} \tag{49}$$

Equation (49) consists of two uncoupled equations, which, in actuality, constitute the generalized strain-stress law in terms of generalized strain $\boldsymbol{e}$ and generalized strain $\boldsymbol{t}$. Eq.(48) indicates that the element has two rigid-body displacement modes prior to imposition of displacement constraints. The advantage of uncoupledness of strain-stress law in eq.(49) will be used later to obtain the explicit expression of the displacement in terms of the varying bending stiffness vector with elements D_i (i=1,2,...,N). Bearing in mind that $t_{i1}=t_{i2}=0$, eq.(47) is simplified to

$$\boldsymbol{F}_i = \boldsymbol{V}^{-1}\boldsymbol{t}_i = \frac{1}{5}\begin{bmatrix} 0 & -\frac{5}{2} & 0 & \frac{5}{2} \\ 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}^T \begin{Bmatrix} t_{i3} \\ t_{i4} \end{Bmatrix} \tag{50}$$

From eq.(45), we deduce

$$\boldsymbol{e}_i = \boldsymbol{V}^{-T}\boldsymbol{q}_i \tag{51}$$

The third and fourth components of $\boldsymbol{e}_i$ can be explicitly obtained as

$$\begin{Bmatrix} e_{i3} \\ e_{i4} \end{Bmatrix} = \frac{1}{5} \begin{bmatrix} 0 & -\frac{5}{2} & 0 & \frac{5}{2} \\ 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} q_i \tag{52}$$

Eqs.(49,50,52) can be viewed as the generalized constitutive law, equilibrium condition and kinematic equation of the i-th element, respectively, if t and e are considered to be generalized stresses and strains. Assembling eq.(49), eq.(50) and eq.(52) over all elements, respectively, we get the global-level constitutive law, equilibrium condition and kinematic equation as follows

$$\boldsymbol{T} = \boldsymbol{S\epsilon} \ ; \qquad \boldsymbol{F} = \boldsymbol{QT} \ ; \qquad \boldsymbol{\epsilon} = \boldsymbol{Ru} \tag{53}$$

where

$$\begin{aligned} \boldsymbol{u} &= [w_1, a\theta_1, w_2, a\theta_2, \dots, w_{N+1}, a\theta_{N+1}]^T \\ \boldsymbol{F} &= [Q_1, M_1/a, Q_2, M_2/a, \dots, Q_{N+1}, M_{N+1}/a]^T \\ \boldsymbol{\epsilon} &= [e_{13}, e_{14}, e_{23}, e_{24}, \dots, e_{N3}, e_{N4}]^T \\ \boldsymbol{T} &= [t_{13}, t_{14}, t_{23}, t_{24}, \dots, t_{N3}, t_{N4}]^T \end{aligned} \tag{54}$$

and

Moreover,

$$\boldsymbol{S} = \frac{1}{a^3} diag[4D_1, 300D_1, 4D_2, 300D_2, \cdots, 4D_N, 300D_N] \tag{56}$$

Combining eqs.(53) gives the global finite element equilibrium equation

$$\boldsymbol{Ku} = \boldsymbol{F} \tag{57}$$

where $\boldsymbol{K}$ is the global finite element stiffness matrix

$$
Q = R^T = \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{5}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{5}{2} & \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ & & & \cdots & & \cdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \tag{55}
$$

$$
K = QSR \tag{58}
$$

represented as tne product of three matrices. We digress that Q is a $2(N+1)$ by $2N$ matrix of constants, and $R=Q^T$ is a $2N$ by $2(N+1)$ matrix of constants. The bending stiffness D_i (i=1,2, ... ,N) appears only in the $2N$ by $2N$ diagonal matrix S. This feature makes it possible to obtain the solution for the displacement explicitly with respect to the discretized bending stiffness of the beam.

It is worthy to remark that there is no requirement that the finite element mesh must be uniform. Analogous analysis can be carried out for the non-uniform mesh, with a more complicated transformation matrix V.

3.2.2 Imposition of Displacement Constraints

To obtain the explicit displacement vector u, one needs to invert the global stiffness matrix $\boldsymbol{K}$. However, by examining eqs.(57,58), the matrix $\boldsymbol{K}$ is singular without the incorporation of the displacement boundary conditions or constraints, moreover, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are non-square matrices.

We herein propose a way to impose the displacement boundary conditions or constraints. We assume that the beam is subjected to M displacement constraints including boundary conditions and/or intermediate supports. The number of constraints M equals two for statically determinate beams, whereas M>2 for statically indeterminate beams. We first apply only two displacement constraints onto eq.(57), as a result of that the stiffness matrix $\boldsymbol{K}$ becomes non-singular, with the attendant change of the matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ into square ones. It can be achieved simply by discarding two rows in the matrix $\boldsymbol{Q}$, which are related to the constrained nodal displacement components, together with two corresponding columns in the matrix $\boldsymbol{R}$. For example, if the beam is clamped at the left end, two displacement constraints first imposed are then $w_1=\theta_1=0$. To impose these two constraints, we simply eliminate the first and second rows of $\boldsymbol{Q}$ and the first and second columns of $\boldsymbol{R}$.

After the incorporation of the first two displacement constraints, eqs.(57,58) reduce to

$$\boldsymbol{K}_1 \boldsymbol{u}_1 = \boldsymbol{F}_1 \tag{59}$$

and

$$\boldsymbol{K}_1 = \boldsymbol{Q}_1 \boldsymbol{S} \boldsymbol{R}_1 \tag{60}$$

where $\boldsymbol{u}_1$ is obtained from $\boldsymbol{u}$ by canceling the two constrained displacements, $\boldsymbol{F}_1$ is obtained from $\boldsymbol{F}$ by canceling the corresponding two forces. Analogously, $\boldsymbol{Q}_1$ is obtained from $\boldsymbol{Q}$ without two rows corresponding to constrained displacements and $\boldsymbol{R}_1=\boldsymbol{Q}_1^T$. All matrices $\boldsymbol{Q}_1$, $\boldsymbol{R}_1$, $\boldsymbol{K}_1$ and $\boldsymbol{S}$ are $2N$ by $2N$ square matrices. The inverse matrix $\boldsymbol{Z}$ of $\boldsymbol{K}_1$ is then

$$\boldsymbol{Z}=\boldsymbol{K}_1^{-1}=\boldsymbol{R}_1^{-1}\boldsymbol{S}^{-1}\boldsymbol{Q}_1^{-1}=\boldsymbol{G}^T\boldsymbol{S}^{-1}\boldsymbol{G} \tag{61}$$

where $\boldsymbol{G}=\boldsymbol{Q}_1^{-1}$ (or $\boldsymbol{G}^T=\boldsymbol{Q}_1^{-T}=\boldsymbol{R}_1^{-T}$) can be numerically obtained since $\boldsymbol{Q}_1$ (or $\boldsymbol{R}_1$) is the matrix of constants. The inverse of $\boldsymbol{S}$ is straightforwardly obtained

$$\boldsymbol{S}^{-1}=a^3 diag[\frac{1}{4D_1},\frac{1}{300D_1},\frac{1}{4D_2},\frac{1}{300D_2},\cdots,\frac{1}{4D_N},\frac{1}{300D_N}] \tag{62}$$

For statically determinate beams, the displacement solution is obtained immediately from eq.(59)

$$\boldsymbol{u}_1=\boldsymbol{K}_1^{-1}\boldsymbol{F}_1=\boldsymbol{Q}_1^{-T}\boldsymbol{S}^{-1}\boldsymbol{Q}_1^{-1}\boldsymbol{F}_1 \tag{63}$$

For statically indeterminate beams, extra displacement constraints exist and should be incorporated into the equilibrium equation (59). Without loss of generality, suppose that $\boldsymbol{u}_1$ is divided into two parts, namely,

$$\boldsymbol{u}_1=\begin{Bmatrix}\boldsymbol{u}_1^c\\ \boldsymbol{u}_1^u\end{Bmatrix} \tag{64}$$

where $\boldsymbol{u}_1^c$=vector of constrained nodal displacements excluding two previously-imposed constraints, namely, $\boldsymbol{u}_1^c=0$; $\boldsymbol{u}_1^u$=vector of unconstrained nodal displacements. Eq.(63) becomes

$$\begin{Bmatrix}\boldsymbol{u}_1^c\\ \boldsymbol{u}_1^u\end{Bmatrix}=\begin{bmatrix}\boldsymbol{Z}_{cc} & \boldsymbol{Z}_{cu}\\ \boldsymbol{Z}_{uc} & \boldsymbol{Z}_{uu}\end{bmatrix}\begin{Bmatrix}\boldsymbol{F}_1^c\\ \boldsymbol{F}_1^u\end{Bmatrix} \tag{65}$$

where $\boldsymbol{F}_1^c$=vector of unknown nodal forces relevant to constrained nodal displacements and

F_1^c=vector of equivalent nodal forces of applied loads relevant to unconstrained nodal displacements. The matrices Z_{cc}, Z_{cu}, Z_{uc} and Z_{uu} are submatrices of Z. Solving out the unknown reactions F_1^c from the first equation of eq.(65) and applying the condition u_1^c=0, we have

$$F_1^c = -Z_{cc}^{-1} Z_{cu} F_1^u \tag{66}$$

Substituting eq.(66) back into the second equation of eq.(65), we arrive at the solution for the unknown nodal displacement vector u_1^u

$$u_1^u = (Z_{uu} - Z_{uc} Z_{cc}^{-1} Z_{cu})\, F_1^u \tag{67}$$

Rewriting $G = [G_c \;,\; G_u]$, where G_c and G_u are submatrices of G, and bearing in mind eq.(61), we have

$$u_1^u = [G_u^T S^{-1} G_u - G_u^T S^{-1} G_c (G_c^T S^{-1} G_c)^{-1} G_c^T S^{-1} G_u]\, F_1^u \tag{68}$$

3.2.3 Mean and Covariance of the Displacement

For statically determinate beams, the mean vector and variance-covariance matrix of the nodal displacement vector u_1 are, respectively,

$$E[u_1] = Q_1^{-T} E[S^{-1}] Q_1^{-1} F_1 \tag{69}$$

$$Cov[u_1, u_1^T] = Q_1^{-T} E[S^{-1} Q_1^{-1} F_1 F_1^T Q_1^{-T} S^{-1}] Q_1^{-1} - (E[u_1])^2 \tag{70}$$

where $E[\cdot]$ is the expectation operator. For statically undeterminate beams, the mean vector and variance-covariance matrix of the nodal displacement vector u_1^u are, respectively,

$$E[u_1^u] = G_u^T E[S^{-1} - H] G_u F_1^u \tag{71}$$

$$Cov[u_1^u, (u_1^u)^T] = G_u^T E[(S^{-1} - H) G_u F_1^u (F_1^u)^T G_u^T (S^{-1} - H)] G_u - (E[u_1^u])^2 \tag{72}$$

where

$$H = S^{-1} G_c (G_c^T S^{-1} G_c)^{-1} G_c^T S^{-1} \tag{73}$$

3.2.4 Numerical Results for 3-Segment Clamped-Hinged Beam

Consider a clamped-hinged beam subjected to a uniformly distributed deterministic load q (Fig.3.4). The beam is comprised of three equal length segments. The two side-segments have the stiffness $D_1 = D_0(1+k\alpha)$ and the mid-segment has the stiffness $D_2 = D_0(1+k\beta)$, where α and β are two random variables and k=constant. The normalized random variables α and β are assumed to possess the joint Pearson Type II distribution with the following density function (Johnson 1987)

$$p_{\alpha\beta}(x,y) = \frac{1}{\pi\sqrt{1-\rho^2}} \quad , \quad for \quad x^2 - 2\rho xy + y^2 \le 1 - \rho^2 \tag{74}$$

where ρ is the correlation coefficient between α and β and is fixed at $\rho=0.5$. The distribution in eq.(74) assures that the values of the bending stiffness are positive and bounded. in contrast to often used physically unjustifiable assumption of Gaussianity.

Fig.3.5 and Fig.3.6 portray the calculated results for the mean and auto-correlation of the displacement at the center of the beam, obtained via the new formulation, as well as by the first-order perturbation FEM, for various coefficients of variation $r=k/2$ of the stochastic stiffness. It is observed that the agreement between the new formulation solution and the first-order perturbation solution is good for small values of coefficient of variation. For larger values of coefficient of variation, the difference between two solutions increases. Fig.3.7 and Fig.3.8 depict the computed results of the mean value and auto-correlation of the displacement along the beam, for two different values of the coefficient of variation r=0.1 and r=0.3. For comparison, the results obtained by first-order perturbational FEM are also plotted. As we expect, the results obtained by the new FE formulation agree with the results obtained by first-order perturbation FEM for small coefficient of variation r=0.1. For a larger coefficient of variation r=0.3, the first-order perturbation based FEM solution underestimates the response autocorrelation by about 20%. In contrast to the perturbation analysis, the present approach can be applied for any coefficient of variation of the stochastic stiffness.

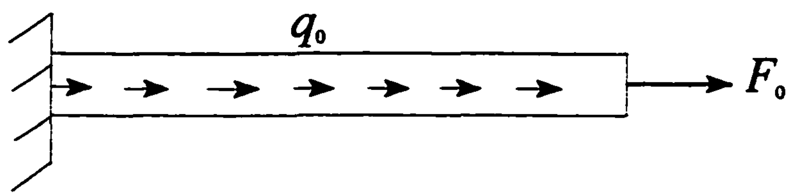

Fig.3.1. A bar in extension under end and uniform forces

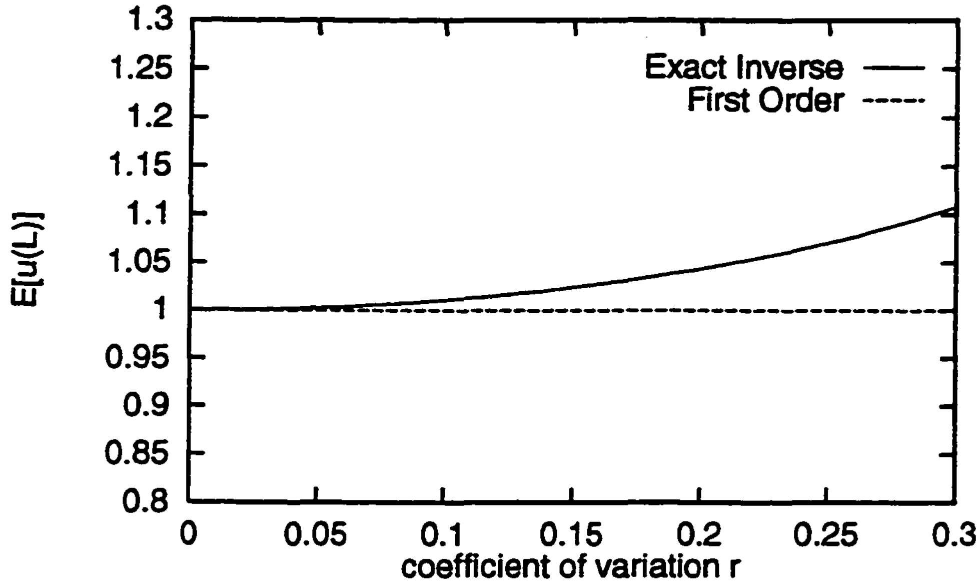

Fig.3.2. The mean of end displacement vs. the coefficient of variation of Young's modulus, for bar under end force

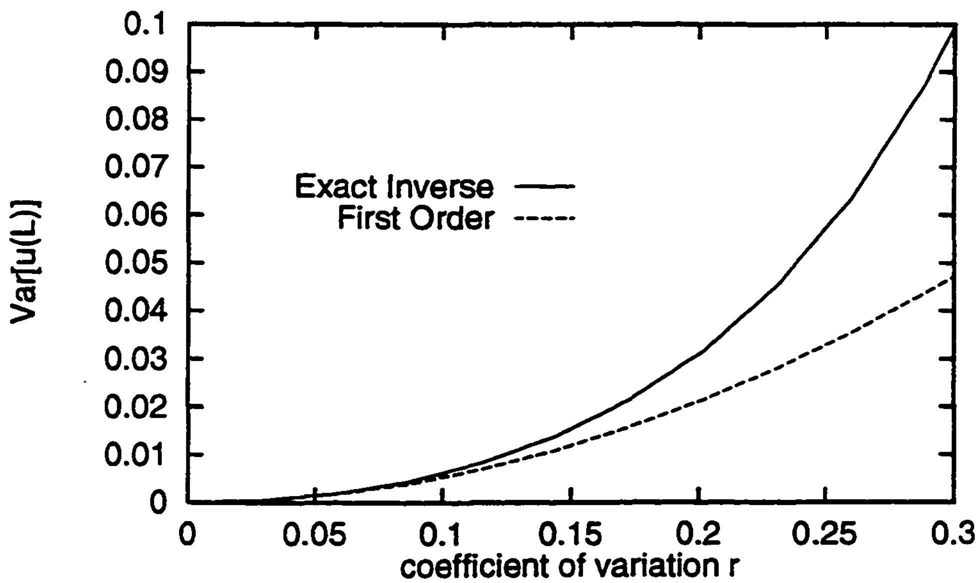

Fig.3.3. The variance of end displacement vs. the coefficient of variation of Young's modulus, for bar under end force

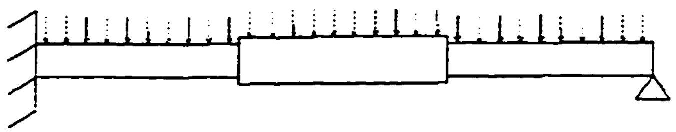

Fig.3.4. A clamped-hinged beam under uniform pressure

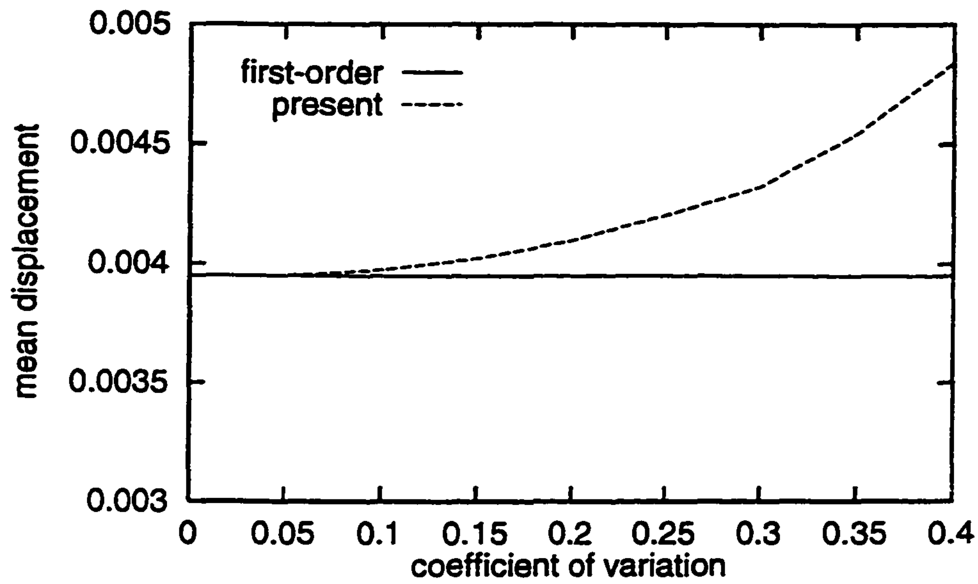

Fig.3.5 The mean value of mid-span displacement of clamped-hinged beam versus coefficient of variation

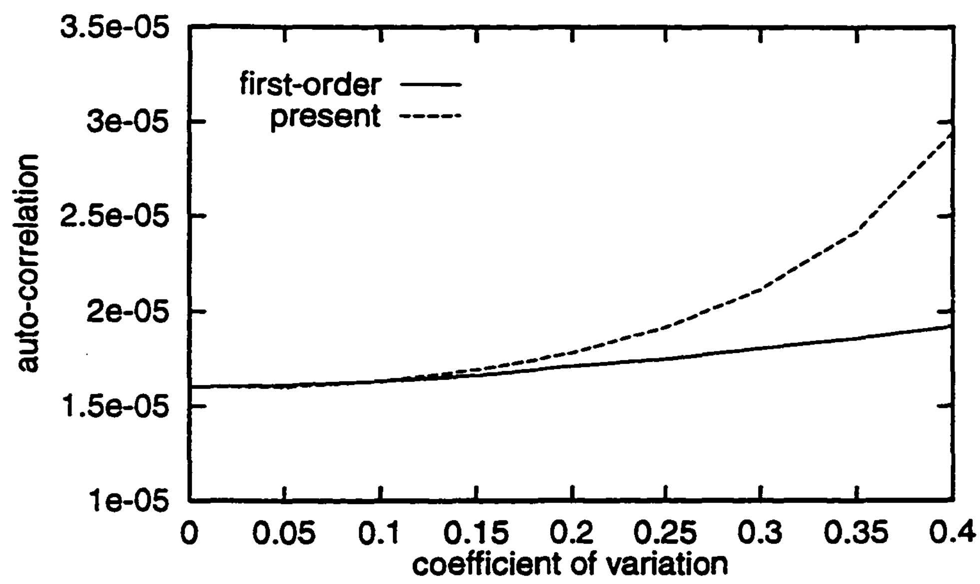

Fig.3.6. The auto-correlation of mid-span displacement of clamped- hinged beam versus coefficient of variation

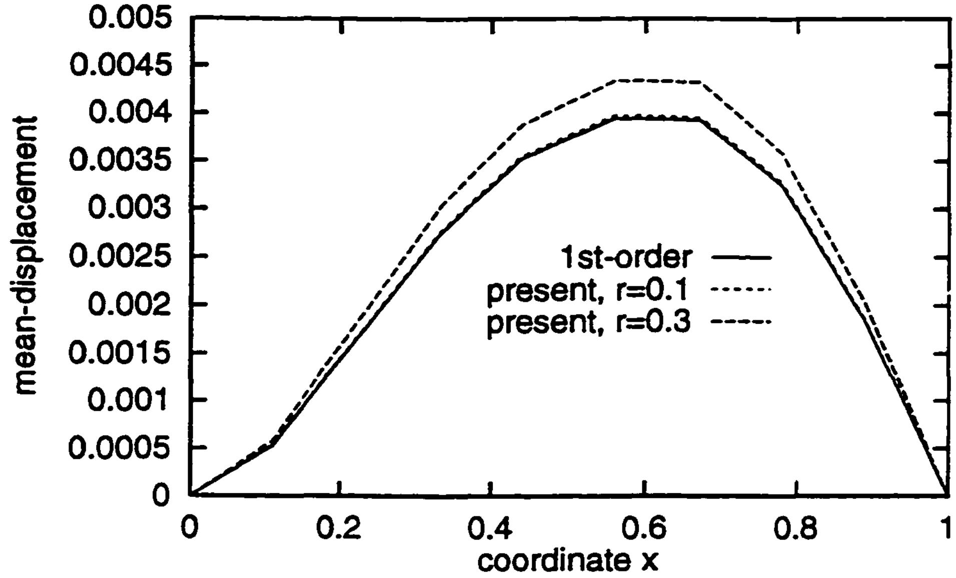

Fig.3.7 The mean displacement of clamped -hinged beam for different coefficients of variation r=0.1 and r=0.3

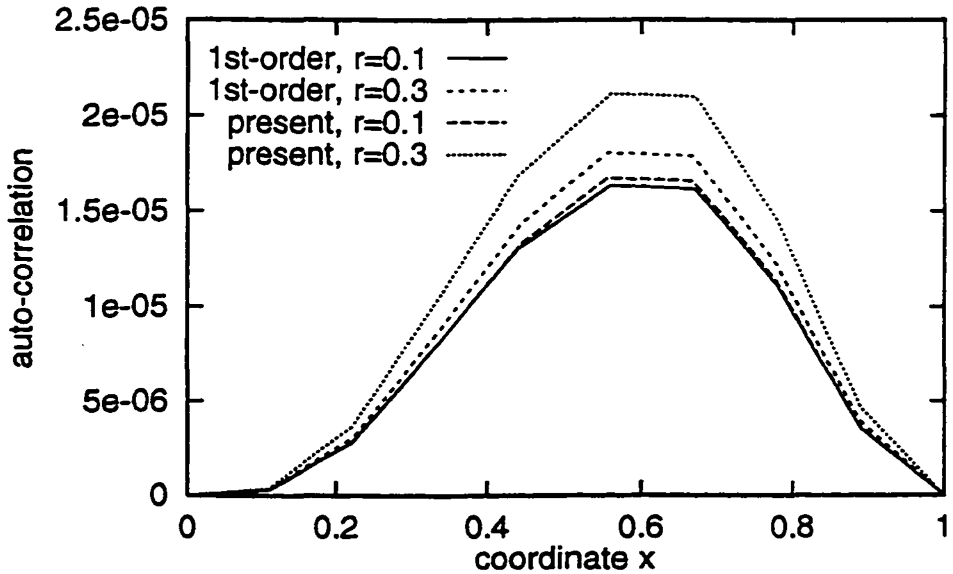

Fig.3.8. The auto-correlation of the displacement of clamped- hinged beam for r=0.1 and r=0.3

Chapter 4 Variational Principle Based FEM for Stochastic Beams

For beam bending problems, both spatially random material parameter (Young's modulus) or geometric parameters (dimension of the cross section) can be combined into one parameter (the bending stiffness). The governing equation of the beam bending is a stochastic differential equation with spatially varying random coefficient, along with random boundary conditions. The solution for mean and covariance function of the beam displacement with spatially random stiffness can be precisely obtained if the inner forces are statically determinate. This chapter, for the first time in the literature, derives the deterministic governing equations and boundary conditions for first and second moments (mean and covariance function) of displacements, which are uncoupled from each other.

The variational principles for the mean and covariance functions of the displacement can be constructed, respectively, from the derived governing equations and boundary conditions. The variational principles are then utilized to form the Galerkin method, Rayleigh-Ritz method and finite element method for the stochastic beams.

4.1 Basic Equations

The beam-bending problem with spatially stochastic stiffness is governed by the following equation

$$\frac{d^2}{dx^2}[EI(x)\frac{d^2w}{dx^2}] = q(x) \tag{1}$$

where $w(x)$=displacement, $q(x)$=transverse distributed force, $EI(x)$=the bending stiffness, which is assumed to be a spatially random field. Eq.(1) can be re-written as

$$\frac{d^2w}{dx^2} = \frac{m(x)}{EI(x)} \tag{2}$$

where

$$m(x) = \int_0^x \int_0^v q(u)\,du\,dv + Q_0 x + M_0 \tag{3}$$

is the bending moment in the cross-section of the beam, M_0 and Q_0 are constants of integration representing bending moment and shear force at the end x=0, respectively. Assume that moments and shear forces in the beam are statically determinate, namely that M_0 and Q_0 are independent from stochastic stiffness, but dependent upon the loading and boundary conditions. By taking expectation of eq.(2), we get

$$\frac{d^2\bar{w}}{dx^2} = \frac{m(x)}{D_0(x)} \tag{4}$$

where $\bar{w}(x)$ $=E[w(x)]$ is the mean of the displacement $w(x)$, $D_0(x)$ is defined as

$$\frac{1}{D_0(x)} = E[\frac{1}{EI(x)}] \tag{5}$$

where E(·) signifies mathematical expectation. Pre-multipying equation (4) by $D_0(x)$ and differentiating the result twice, we obtain the governing equation for the mean value of displacement response $\bar{w}(x)$

$$\frac{d^2}{dx^2}[D_0(x)\frac{d^2\bar{w}}{dx^2}] = q(x) \tag{6}$$

Eq.(6) is the governing equation for the mean of displacement. In form eq.(6) is identical to that of the beam with deterministic stiffness $D_0(x)$. Subtracting eq.(4) from eq.(2) and multiplying the resulting equation by itself but evaluated at the cross-section y, we get

$$\frac{d^2[w(x)-\bar{w}(x)]}{dx^2}\frac{d^2[w(y)-\bar{w}(y)]}{dy^2}=m(x)m(y)\left[\frac{1}{EI(x)}-\frac{1}{D_0(x)}\right]\left[\frac{1}{EI(y)}-\frac{1}{D_0(y)}\right] \tag{7}$$

Taking expectation of eq.(7) gives one form of the governing equation for the covariance function of the displacement

$$D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2\partial y^2}=m(x)m(y) \tag{8}$$

where $C(x,y)=E\{[w(x)-\bar{w}(x)][w(y)-\bar{w}(y)]\}$ is covariance function of displacements $w(x)$ at position x and $w(y)$ at position y, and

$$\frac{1}{D_1(x,y)}=E\left\{\left[\frac{1}{EI(x)}-\frac{1}{D_0(x)}\right]\left[\frac{1}{EI(y)}-\frac{1}{D_0(y)}\right]\right\} \tag{9}$$

Partially differentiating eq.(8) twice with respect to x and twice with respect to y, we arrive at an alternative form of governing equation for the covariance function of the displacement

$$\frac{\partial^4}{\partial x^2\partial y^2}\left[D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2\partial y^2}\right]=q(x)q(y) \tag{10}$$

The associated boundary conditions are proved in Appendix I. For the mean displacement $\bar{w}(x)$, the boundary conditions at x=0 and x=L read

$$\bar{w}=0 \quad or \quad \frac{d}{dx}[D_0(x)\frac{d^2\bar{w}}{dx^2}]=\bar{Q}$$
$$\frac{d\bar{w}}{dx}=0 \quad or \quad D_0\frac{d^2\bar{w}}{dx^2}=\bar{M} \tag{11}$$

where $\bar{M}$ and $\bar{Q}$ are prescribed moment and shear force at ends, respectively. The boundary conditions for the covariance function $C(x,y)$ are

$$\frac{\partial C}{\partial x}=0 \quad or \quad D_1\frac{\partial^4 C}{\partial x^2\partial y^2}=\bar{M}m(y)$$
$$C=0 \quad or \quad \frac{\partial}{\partial x}[D_1\frac{\partial^4 C}{\partial x^2\partial y^2}]=\bar{Q}m(y) \tag{12}$$

at x=0 and x=L, and

$$\frac{\partial C}{\partial y}=0 \quad or \quad D_0\frac{\partial^4 C}{\partial x^2\partial y^2}=\bar{M}m(x)$$
$$C=0 \quad or \quad \frac{\partial}{\partial y}[D_1\frac{\partial^4 C}{\partial x^2\partial y^2}]=\bar{Q}m(x) \tag{13}$$

at y=0 and y=L.

4.2 Variational Principles

The variational principle for the mean displacement $\bar{w}(x)$ corresponding to the governing equation (6) and boundary conditions in eq.(11) requires the minimizing of the following functional

$$\pi_1 = \int_0^L [\frac{1}{2} D_0(x) (\frac{d^2 \bar{w}}{dx^2})^2 - q(x) \bar{w}] dx - [M \frac{d\bar{w}}{dx} - Q\bar{w}]|_0^L \tag{14}$$

The above functional is identical to that of a deterministic beam (Hu 1981), which has an equivalent deterministic stiffness $D_0(x)$. The variational principle for the covariance function $C(x,y)$ corresponding to the governing equation (10) and boundary conditions in eqs.(12,13) requires the minimizing of the following functional

$$\begin{aligned} \pi_2 = & \int_0^L \int_0^L [\frac{1}{2} D_1(x,y) (\frac{\partial^4 C}{\partial x^2 \partial y^2})^2 - q(x) q(y) C] dx dy \\ & - [\int_0^L (\bar{M} \frac{\partial C}{\partial x} - \bar{Q} C) q(y) dy]|_{x=0}^{x=L} - [\int_0^L (\bar{M} \frac{\partial C}{\partial y} - \bar{Q} C) q(x) dx]|_{y=0}^{y=L} \\ & - [\bar{M}\bar{M} \frac{\partial^2 C}{\partial x \partial y} - \bar{M}\bar{Q} \frac{\partial C}{\partial x} - \bar{M}\bar{Q} \frac{\partial C}{\partial y} + \bar{Q}\bar{Q} C]|_{y=0}^{y=L}|_{x=0}^{x=L} \end{aligned} \tag{15}$$

The proof of this variational principle has been given in the study by Elishakoff, Ren and Shinozuka (1996). The functional π_2 was derived by first guessing the double integration term in the right side of the equation (15), from the equation (10), and then carrying out the variation of the term with respect to $C(x,y)$ to construct appropriate boundary terms. The physical implication of π_2 is not addressed here. The proof that the functional π_2 reaches its minimum value when the covariance function $C(x,y)$ is the exact one has been given in Appendix II.

It is seen that the function to be integrated in the first integral in the functional π_2 consists of mixed fourth-order derivative of correlation function $C(x,y)$ with respect to x and y. Hence, at least C^1-continuous interpolating functions are required to guarantee the convergence of the finite element formulation based on the variational principle in eq.(15). Furthermore, the very fact that only terms with at least x^2 and y^2 in interpolating polynomials contribute the stiffness matrix, implies that higher-order interpolating functions are required. Due to these two shortcomings, we propose an alternative form of the variational principle for the covariance function $C(x,y)$, which corresponds to the governing equation (8) and geometry boundary conditions in eqs.(12,13). The functional to be minimized reads

$$\pi_3 = \int_0^L\int_0^L [\frac{1}{2}(\frac{\partial^2 C}{\partial x \partial y})^2 - f_1(x,y)\, m(x)m(y)\, C]dxdy$$
$$+ \int_0^L m(x)H_1(x)C|_{y=L}dx + \int_0^L m(y)H_2(y)C|_{x=L}dy - GC|_{x=L,y=L} \quad (16)$$

where

$$H_1(x) = \int_0^L f_1(x,y)m(y)dy\ , \qquad H_2(y) = \int_0^L f_1(x,y)\, m(x)dx$$
$$G = \int_0^L\int_0^L f_1(x,y)m(x)m(y)dxdy \quad (17)$$

The variational principle in eq.(16) is applicable to the beams simply-supported at both ends x=0 and x=L or clamped at left end x=0, namely either w=dw/dx=0 at x=0 or w=0 at x=0,L. The functional π_3 requires C^0-continuous only and comparatively lower interpolating functions. Its proof is given in Appendix III.

4.3 Galerkin Method

The governing equations for the mean displacement $\bar{w}(x)$ and the covariance function

$C(x,y)$ can be solved through direct integration of eq.(4) and eq.(8), as was done by Elishakoff, Ren and Shinozuka (1995), for some specific cases. In most problems, they can be solved effectively by conventional approximate methods, such as Galerkin method or finite difference method. Since the governing equation for covariance function $C(x,y)$ is symmetric with respect to x and y, the product of one dimensional trial functions can be used for the covariance function $C(x,y)$. Assume that

$$\bar{w}(x) = \phi_0(x) + \sum_{i=1}^{N} A_i \ \phi_i(x)$$
$$C(x,y) = \psi_0(x)\psi_0(y) + \sum_{i=1}^{M} B_i \ \psi_i(x)\,\psi_i(y) \qquad (18)$$

where A_i and B_i are constants to be determined, $\phi_0(x)$ and $\psi_0(x)$ are particular functions satisfying non-homogeneous boundary conditions for mean displacement $\bar{w}(x)$ and covariance function $C(x,y)$, respectively, $\phi_i(x)$ (i=1,2,$\cdots$,N) and $\psi_i(x)$ (i=1,2,$\cdots$,M) are trial functions satisfying homogeneous boundary conditions. Galerkin's weighted residual formulae for the mean displacement $\bar{w}(x)$ governed by eq.(4) and for the covariance function $C(x,y)$ governed by eq.(8) are

$$\int_0^L \left[\frac{d^2\bar{w}(x)}{dx^2} - f_0(x)\,m(x)\right]\phi_i(x)dx = 0 \quad , \qquad i = 1,2,\cdots,N \qquad (19)$$

and

$$\int_0^L\int_0^L [\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2} - f_1(x,y)\, m(x) m(y)\,] \psi_i(x) \psi_i(y) dx dy = 0 \quad , \qquad i = 1, 2, \cdots, M \tag{20}$$

respectively. Substitution of expressions in eq.(18) into eqs.(19,20) yields the following two uncoupled systems of linear algebraic equations for constants A_i and B_i, respectively,

$$\begin{aligned} \sum_{j=1}^{N} K_{ij} A_j &= F_i \quad , \qquad i = 1, 2, \cdots, N \\ \sum_{j=1}^{M} S_{ij} B_j &= G_i \quad , \qquad i = 1, 1, \cdots, M \end{aligned} \tag{21}$$

where

$$K_{ij} = \int_0^L \phi_i(x)\, \phi_j''(x) dx \tag{22}$$

$$F_i = \int_0^L f_0(x)\, m(x) \phi_i(x) dx - \int_0^L \phi_i(x) \phi_0''(x) dx \tag{23}$$

$$S_{ij} = \int_0^L\int_0^L \psi_i(x)\,\psi_j''(x)\,\psi_i(y)\,\psi_j''(y)dxdy \tag{24}$$

$$G_i = \int_0^L\int_0^L f_1(x,y)\,m(x)m(y)\,\psi_i(x)\psi_i(y)dxdy - \int_0^L\int_0^L \psi_i(x)\psi_0''(x)\psi_i(y)\psi_0''(y)dxdy \tag{25}$$

Solutions of $\bar{w}(x)$ and $C(x,y)$ are obtained by determining constants A_i (i=1,2,$\cdots$,N) and B_i (i=1,2,$\cdots$,M) from eq.(21) and then substituting them back into eq.(18).

Alternatively, the Galerkin method can also be formulated by using eq.(6) and eq.(10) instead of eq.(4) and eq.(8), it reads

$$\int_0^L \{ \frac{d^2}{dx^2}[D_0(x)\frac{d^2\bar{w}(x)}{dx^2}] - q(x) \}\phi_i(x)dx = 0 \quad , \qquad i = 1,2,\cdots,N \tag{26}$$

$$\int_0^L\int_0^L \{ \frac{\partial^4}{\partial x^2\partial y^2}[D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2\partial y^2}] - q(x)q(y) \}\psi_i(x)\psi_i(y)dxdy = 0 \quad , \qquad i = 1,2,\cdots,M \tag{27}$$

Substitution of expressions in eq.(18) into eqs.(26,27) yields two sets of linear algebraic equations formally similar to eq.(21), provided the symbols in the equation are now given by

$$K_{ij}^{(1)} = \int_{0}^{L} \phi_i(x)\,[D_0(x)\phi_j''(x)]''\,dx \tag{28}$$

$$F_i^{(1)} = \int_0^L q(x)\phi_i(x)dx - \int_0^L \phi_i(x)[D_0(x)\phi_0''(x)]''\,dx \tag{29}$$

$$S_{ij}^{(1)} = \int_0^L\int_0^L \psi_i(x)\,\psi_i(y)\frac{\partial^4}{\partial x^2 \partial y^2}[D_1(x,y)\psi_j''(x)\,\psi_j''(y)]dxdy \tag{30}$$

$$\begin{aligned} G_i^{(1)} &= \int_0^L\int_0^L q(x)q(y)\psi_i(x)\psi_i(y)dxdy \\ &- \int_0^L\int_0^L \psi_i(x)\,\psi_i(y)\frac{\partial^4}{\partial x^2 \partial y^2}[D_1(x,y)\psi_0''(x)\psi_0''(y)]dxdy \end{aligned} \tag{31}$$

4.4 Rayleigh-Ritz Method

Rayleigh-Ritz method is an energy method based on variational principles. The trial functions in Rayleigh-Ritz method are not required to satisfy boundary conditions, since the variational principles imply the requirement of both the governing equation and boundary conditions. Hence, we cam simply express the mean and covariance functions of the displacement as follows

$$\begin{aligned} \bar{w}(x) &= \sum_{i=1}^{N} A_i\ \phi_i(x) \\ C(x,y) &= \sum_{i=1}^{M} B_i\ \psi_i(x)\,\psi_i(y) \end{aligned} \tag{32}$$

where no boundary conditions are imposed on the trial functions $\phi_i(x)$ and $\psi_i(x)$. Substituting them into variational principles in eq.(14) and eq.(15) and taking the minimum with respect to A_i and B_i, respectively, we obtain

$$\sum_{j=1}^{N} K_{ij}^{(2)} A_j = F_i^{(2)} \quad , \qquad i = 1,2,\cdots,N$$
$$\sum_{j=1}^{M} S_{ij}^{(2)} B_j = G_i^{(2)} \quad , \qquad i = 1,1,\cdots,M \tag{33}$$

where

$$K_{ij}^{(2)} = \int_0^L D_0(x)\phi_i''(x)\phi_j''(x)dx \tag{34}$$

$$F_i^{(2)} = \int_0^L q(x)\phi_i(x)dx - [\bar{Q}\phi_i(x) - \bar{M}\phi_i'(x)]|_0^L \tag{35}$$

$$S_{ij}^{(2)} = \int_0^L\int_0^L D_1(x,y)\psi_i''(x)\,\psi_j''(x)\,\psi_i''(y)\,\psi_j''(y)dxdy \tag{36}$$

$$\begin{aligned} G_i^{(2)} &= \int_0^L\int_0^L q(x)q(y)\psi_i(x)\psi_i(y)dxdy \\ &+ \int_0^L q(y)\psi_i(y)dy[\bar{M}\psi_i'(x) - \bar{Q}\psi_i(x)]|_{x=0}^{x=L} + \int_0^L q(x)\psi_i(x)dx[\bar{M}\psi_i'(y) - \bar{Q}\psi_i(y)]|_{y=0}^{y=L} \\ &+ [\bar{M}\bar{M}\psi_i'(x)\psi_i'(y) - \bar{M}\bar{Q}\psi_i'(x)\psi_i(y) - \bar{M}\bar{Q}\psi_i(x)\psi_i'(y) + \bar{Q}\bar{Q}\psi_i(x)\psi_i(y)]|_{y=0}^{y=L}|_{x=0}^{x=L} \end{aligned} \tag{37}$$

Rayleigh-Ritz method coincides with second formulation of Galerkin method, which is based on eq.(6) and eq.(10), if the trial functions used in Rayleigh-Ritz method satisfy the homogeneous boundary conditions and the beam does not have non-homogeneous boundary conditions. Generally, the displacement function of a deterministic beam, which has the same geometry and loads as the stochastic beam, but has a deterministic stiffness $D_0(x)$, can be adopted to be trial function for both the Galerkin and the Rayleigh-Ritz methods. It will be demonstrated later that the covariance function obtained by Galerkin or Rayleigh-Ritz method with such a choice of trial functions agrees with the exact solution extremely well.

4.5 Finite Element Formulation

4.5.1 Formulation for the Mean Displacement

The mean displacement $\bar{w}(x)$ is actually the response of a bending beam with varying bending stiffness $D_0(x)$, as we have mentioned in previous section. Therefore, the conventional finite element formulas for deterministic beams can be used, for example, the two-node cubic element. The interpolation of the mean displacement in the element $x_1 \leq x \leq x_2$ is given by

$$\bar{w} = \sum_{i=1}^{4} N_i \delta_i = \boldsymbol{N\delta} \tag{38}$$

where N is the vector of shape functions and

$$\boldsymbol{\delta} = [\delta_1, \delta_2, \delta_3, \delta_4]^T = [\bar{w}_1, (\frac{d\bar{w}}{dx})_1, \bar{w}_2, (\frac{d\bar{w}}{dx})_2]^T \tag{39}$$

is the nodal displacement vector. Discretizing the beam into n elements, substituting eq.(38) into eq.(14) and then minimizing the potential π_1, we get

$$\sum_{e=1}^{n} k_1^e \delta = \sum_{e=1}^{n} p_1^e \tag{40}$$

where the element stiffness matrix k_1^e and the element equivalent nodal force vector p_1^e are given, respectively, as follows

$$k_1^e = \int_{x_1}^{x_2} D_0(x) \frac{d^2 N^T}{dx^2} \frac{d^2 N}{dx^2} dx \tag{41}$$

$$p_1^e = \int_{x_1}^{x_2} N^T q(x) dx - [M \frac{dN}{dx} - QN]|_0^l \tag{42}$$

4.5.2 Formulation for the Covariance Function

To construct finite element formulas for the covariance function, we can apply either the variational principle in eq.(15) or the variational principle in eq.(16). Since the functional π_2 contains mixed fourth-order derivative of the covariance function with respect to x and y, C^1-continuous interpolating functions should be used to guarantee the convergence of the solution and higher-order interpolating polymonials must be assumed to contribute the stiffness matrix. On the other hand, the functional π_3 consists of only mixed second-order derivative of the covariance function $C(x,y)$ with respect to x and y, C^0-continuous interpolating functions can be used to reach the convergence requirement and lower-order interpolating polymonials can be adopted to satisfy the accuracy. Therefore, the functional π_3 is utilized hereby to construct the finite element equilibrium equations for the covariance function $C(x,y)$.

Due to the fact that the functional π_3 possesses symmetry in x and y, a four-node rectangular element, which is commonly used in plate-bending problems, is adopted here. The covariance function $C(x,y)$ in the element $x_1 \le x \le x_2$ and $y_1 \le y \le y_2$ is interpolated as follows

$$C = \sum_{i=1}^{4} N_i \boldsymbol{\delta}_i = N\boldsymbol{\delta} \tag{43}$$

where $\boldsymbol{\delta}$ is the vector of nodal degrees of freedom

$$\begin{aligned} \boldsymbol{\delta} &= [\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3, \boldsymbol{\delta}_4]^T \\ \boldsymbol{\delta}_i &= [C_i, (\frac{\partial C}{\partial x})_i, (\frac{\partial C}{\partial y})_i]^T \end{aligned} \tag{44}$$

and **N** is vector of shape functions

$$\begin{aligned} N &= [N_1, N_2, N_3] \ , \quad N_i = [N_{i1}, N_{i2}, N_{i3}] \\ N_{i1} &= \frac{\xi_i \eta_i}{8}(2 + \xi_i \xi + \eta_i \eta - \xi^2 - \eta^2)(\xi + \xi_i)(\eta + \eta_i) \\ N_{i2} &= \frac{\eta_i a}{8}(\xi^2 - 1)(\xi + \xi_i)(\eta + \eta_i) \\ N_{i3} &= \frac{\xi_i b}{8}(\eta^2 - 1)(\xi + \xi_i)(\eta + \eta_i) \end{aligned} \tag{45}$$

where $\xi = (x-x_1)/(x_2-x_1) - (x_2-x)/(x_2-x_1)$ and $\eta = (y-y_1)/(y_2-y_1) - (y_2-y)/(y_2-y_1)$ are local coordinates, $a=(x_2-x_1)/2$ and $b=(y_2-y_1)/2$ are side lengths of the rectangular element. Discretizing the domain into $n \times n$ elements, then substituting eq.(43) into eq.(16) and minimizing π_3, we get

$$\sum_{e=1}^{n \times n} K_2^e \boldsymbol{\delta} = \sum_{e=1}^{n \times n} F_2^e \tag{46}$$

where the element stiffness matrix K_2^e and the equivalent nodal force vector F_2^e are given, respectively, as follows

$$K_2^e = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial^2 N^T}{\partial x \partial y} \frac{\partial^2 N}{\partial x \partial y} dxdy \tag{47}$$

$$F_2^e = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_1(x,y) m(x) m(y) N^T dxdy \tag{48}$$

4.6 Numerical Examples

Example I: Cantilever Beam under Uniform Pressure

Consider a clamped-free beam subjected to uniformly distributed pressure (Fig.4.1). The beam is free at x=0 and clamped at x=L. The stiffness of the beam $D(x)$ is assumed to be a spatially random field. Let $1/D(x)=[1+\alpha(x)]/\bar{D}$, and suppose that the normalized random field $\alpha(x)$ possesses the uniform distribution with following probabilistic density

$$f_{\alpha(x)}(u) = \frac{1}{2a}\ , \qquad u \in [-a,a] \tag{49}$$

where a=constant. The correlation function $\rho(x,y)$ of the random field $\alpha(x)$ is assumed to be triangular

$$\rho(x,y) = 1 - \frac{|x-y|}{L}\ , \qquad |x-y| < L \tag{50}$$

From eq.(5) and eq.(9), we have

$$f_0 = \frac{1}{D_0} = \frac{1}{\bar{D}} \tag{51}$$

and

$$f_1(x,y) = \frac{\rho(x,y)}{\bar{D}_{1u}} \quad , \quad \frac{1}{\bar{D}_{1u}} = \frac{a^2}{3\bar{D}^2} \tag{52}$$

The following trial function corresponding to the displacement shape of a uniform beam under the uniform pressure is assumed

$$\psi_1(x) = x^4 - 4L^3x + 3L^4 \tag{53}$$

We will apply the first formulation of Galerkin method to solve this example. Substitute eq.(53) into eqs.(22-25), we obtain

$$K_{11} = \frac{12}{7}L^7 \ , \qquad F_1 = \frac{q}{14D_0}$$
$$S_{11} = \frac{144}{49}L^{14} \ , \qquad G_1 = \frac{1553}{97020}\frac{q^2L^{14}}{4\bar{D}_{1u}} \tag{54}$$

Hence

$$A_1 = \frac{q}{24\bar{D}_0} \ , \qquad B_1 = \frac{1553}{285120}\frac{q^2}{4\bar{D}_{1u}} \tag{55}$$

The mean displacement $\bar{w}(x)$ and covariance function $C(x,y)$ obtained by Galerkin method are, respectively,

$$\bar{w}(x) = \frac{q}{24\bar{D}_0}(x^4 - 4L^3x + 3L^4)$$

$$C(x,y) = \frac{0.00545q^2}{4\bar{D}_{1u}}(x^4 - 4L^3x + 3L^4)(y^4 - 4L^3y + 3L^4) \tag{56}$$

The mean displacement $\bar{w}(x)$ obtained by Galerkin method here coincides with the exact mean displacement, since the trial function $\phi_1(x)$ in eq.(53) is the exact bending mode of the beam. The exact covariance function $C(x,y)$ of the displacement for this cantilever beam has been derived to be (Appendix IV)

$$\begin{aligned} C(x,y) = \frac{q^2}{20160L\bar{D}_{1u}} &[259L^9 - 336L^8(x+y) + 440L^7xy + 21L^5(x^4+y^4) \\ &+ 63L^4(x^5+y^5) - 35L^4xy(x^3+y^3) - 84L^3xy(x^4+y^4) \\ &+ 35Lx^4y^4 + 21x^4y^4(x-y) - 7x^9] , \quad for \quad x \geq y \end{aligned} \tag{57}$$

Fig.4.2 portrays the approximate solution of variance of the displacement obtained above by Galerkin method and the exact solution in eq.(57). It is seen that both exact and approximate solutions are extremely close from each other.

Example 2: Simply Supported Beam under Uniform Pressure

Consider now a simply-supported beam subjected to uniformly distributed deterministic load q (Fig.4.3). The flexibility of the beam $1/D(x) = [1+\alpha(x)]/\bar{D}$ is assumed to be a spatially homogeneous random field. The normalized random field $\alpha(x)$ is assumed to satisfy the triangular distribution

$$f_{\alpha(x)}(u)=\frac{1}{b}\,(1-\frac{|u|}{b})\ ,\qquad u\in[-b,b] \tag{58}$$

where b=constant. The correlation function $\rho(x,y)$ is assumed to be exponential

$$\rho(x,y)=e^{-\frac{|x-y|}{L}} \tag{59}$$

Similar to eqs.(51,52), we can obtain for the triangular distribution

$$f_0=\frac{1}{\bar{D}} \tag{60}$$

and

$$f_1(x,y)=\frac{1}{\bar{D}_{1t}}\,\rho(x,y)\quad ,\quad \frac{1}{\bar{D}_{1t}}=\frac{b^2}{6\bar{D}^2} \tag{61}$$

The displacement shape of the beam with uniform stiffness under the same load is selected to be the trial function, namely

$$\phi(x)=\psi(x)=x^4-2Lx^3+x \tag{62}$$

We apply the Rayleigh-Ritz method to solve this example. It is easily verified that the above trial function yields the exact mean displacement by Rayleigh-Ritz method. For covariance function, substitution of eq.(62) into eqs.(36,37) yields

$$S_{11}^{(2)}=28.7\bar{D}_{1t}L^{10}\ ,\qquad G_1^{(2)}=\frac{0.04q^2L^{10}}{4} \tag{63}$$

Hence B_1=0.00139q^2/4 $\bar{D}_{1t}$. The covariance function $C(x,y)$ of the beam displacement then become

$$C(x,y)=\frac{0.00139q^2}{4\bar{D}_{1t}}(x^4-2Lx^3+x)(y^4-2Ly^3+y) \tag{64}$$

The exact solution of covariance function $C(x,y)$ of the displacement for this simply-supported beam has been also derived in Appendix V to be

$$C(x,y)=p(x,y)+\frac{xy}{L}p(L,L)-\frac{1}{L}[xp(L,y)+yp(L,x)]\ , \qquad for \quad x\geq y \tag{65}$$

where the particular solution is

$$p(u,v)=\frac{q^2}{4\bar{D}_{1t}}[(8L+3u)(4L-v)L^6-\frac{2}{3}(L+u)L^4v^3+\frac{1}{6}(8L+3u)L^3v^4-\frac{1}{10}(7L+2u)L^2v^5$$
$$+\frac{1}{15}(2L+u)Lv^6-\frac{1}{21}Lv^7+L^4(4L^2+3Lu+u^2)(8L^2-5Lv+v^2)e^{-\frac{(u-v)}{L}} \tag{66}$$
$$-L^5(8L+3v)(4L^2+3Lu+u^2)e^{-\frac{u}{L}}-L^5(8L+3u)(4L^2+3Lv+v^2)e^{-\frac{v}{L}}]$$

Fig.4.4 shows that comparison of approximate solution of covariance function obtained by the single term Rayleigh-Ritz method and the exact solution given in eq.(65). The agreement is excellent.

Example 3: Simply-Supported Beam under Uniform Pressure - FEM Solution

Reconsider the simply supported beam in the example 2. The stiffness $D(x)$ of the beam is again assumed to be a spatially homogeneous random field. Now let $D(x)=D_0[1+k\,\alpha(x)]$, where

k=constant and $\alpha(x)$ is a normalized random field. It is assumed that $\alpha(x)$ possesses a uniform two-dimensional Pearson Type II probability distribution (Johnson 1987)

$$p_{\alpha(x)\alpha(y)}(u,v) = \begin{cases} \dfrac{1}{\pi\sqrt{1-\rho^2(x,y)}} \ , & for \quad (u,v)\in\Omega:\ u^2-2\rho uv+v^2\leq 1-\rho^2 \\ 0 & , \ elsewhere \end{cases} \tag{67}$$

where $\rho(x,y)$ is the function characterizing the correlation between $\alpha(x)$ and $\alpha(y)$, and is assumed to be exponential, namely

$$\rho(x,y) = \exp(-\frac{|x-y|}{d}) \ , \quad |x-y|\leq L \tag{68}$$

where d=scale of fluctuation. The normalized random field $\alpha(x)$ has zero mean and correlation function of $\rho(x,y)/4$. Its one-dimensional marginal distribution reads

$$p_{\alpha(x)}(u) = \frac{2}{\pi}\sqrt{(1-u^2)} \quad , \quad u\in[-1,1] \tag{69}$$

Therefore, the mean and correlation function of the flexibility $f(x)=1/D(x)$ can be obtained, respectively, as

$$f_0 = E[f] = \int_{-1}^{1} \frac{1}{D_0(1+ku)} p_{\alpha(x)}(u)du = \frac{2}{\pi D_0}\int_{-1}^{1}\frac{\sqrt{1-u^2}}{1+ku}du \tag{70}$$

and

where

$$C_1 = 1+\frac{k\sqrt{1+\rho}}{2}u \quad , \quad C_2 = \frac{k\sqrt{1+\rho}}{2}u \tag{72}$$

The covariance function between the flexibilities $f(x)$ and $f(y)$ is $Cov[f(x),f(y)]=R[f(x),f(y)]$-

$$f_1(x,y) = Cov[f(x),f(y)] = \int\int_{\Omega} \frac{1}{D_0^2(1+ku)(1+kv)} p_{\alpha(x)\alpha(y)}(u,v)dudv$$

$$= \frac{1}{\pi D_0^2} \int_{-1}^{1} \frac{1}{C_1 C_2} \ln\frac{C_1 + C_2\sqrt{1-u^2}}{C_1 - C_2\sqrt{1-u^2}} du \qquad (71)$$

$E[f(x)]E[f(y)]$.

To illustrate the accuracy and efficiency of the variational principle based finite element method presented in this study, we calculate the mean and covariance functions of the displacement of the simply-supported beam by both the first-order perturbation finite element method and the present finite element method. The scale of fluctuation in eq.(68) is taken to be d=0.5. The results are depicted in Figs. 4.5-4.8. Fig.4.5 portrays the mean displacements at mid-span of the simply-supported beam for different values of the coefficient of variation of the stochastic bending stiffness. The results have been normalized by the perturbation solution w_c=0.01302/D_0. The exact solution of the mid-displacement of the simply-supported beam is obtainable to be

$$\bar{w}(x=L/2) = \frac{5qL^4\bar{f}}{384} \qquad (73)$$

The variation of the exact mid-displacement with respect to the coefficient of variation r of the random stiffness can be also plotted in Fig.4.5 and coincides with that of the new finite element result. Thus we reach a remarkable conclusion that the present solution coincides with the exact solution for any value of coefficient of variation of the stiffness. For small values of coefficient of variation of the bending stiffness, the solution obtained by the first-order perturbation method also agrees with the exact solution or the new finite element solution. However, the difference between two solutions increases when the coefficient of variation of the bending stiffness increases, as expected. This observation clearly demonstrates the superiority of the proposed method over the perturbation-based method.

Fig.4.6 portrays the variances of the mid-displacement, obtained by the present finite element method and the first-order perturbative finite element method, for different values of coefficient of variation of the stochastic bending stiffness. Again, it is seen that the result obtained by the present method agrees with the results obtained by the first-order perturbative finite element method for small values of coefficient of variation of the stiffness. However, for larger values of the coefficient of variation of the stiffness, the difference between the perturbation solution and the new variational principle based solution increases. For example, the differences are about 4 percents, 8 percents and 35 percents, respectively, for the coefficient of variation r=0.1, 0.15, and 0.3.

Fig.4.7 and Fig.4.8 compare more illustratatively the results obtained by the first-order perturbation method and the present variational principle based finite element method. Fig.4.7 shows the changes of the variance of the displacement along the beam cross-section for the coefficient of variation of the stiffness r=0.15. Fig.4.8 shows the changes of the variance of the displacement for the coefficient of variation of the stiffness r=0.3. It is seen that in the case of the small value of the coefficient of variation of the stiffness (r=0.15) , two solutions are close to each other. For the case of r=0.3, however, the first-order perturbation solution is much less those by the variational principle based finite element method. This confirms the conclusion that the perturbation-based solutions are acceptable only for the small values of the coefficient of variation.

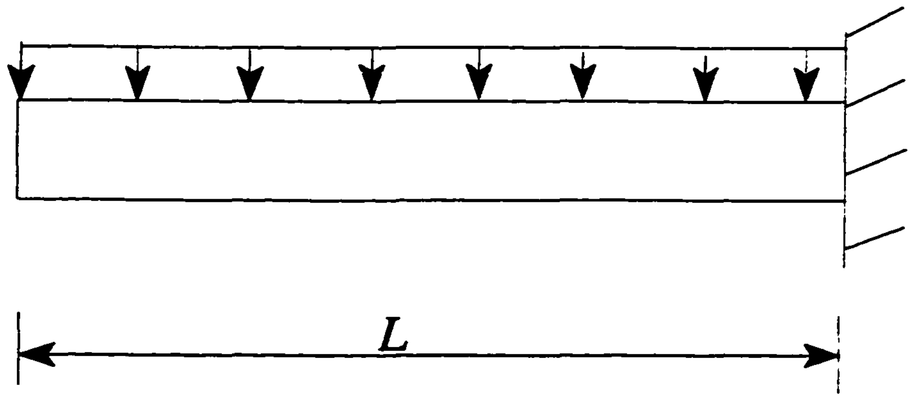

Fig.4.1. Cantilever beam under uniform pressure

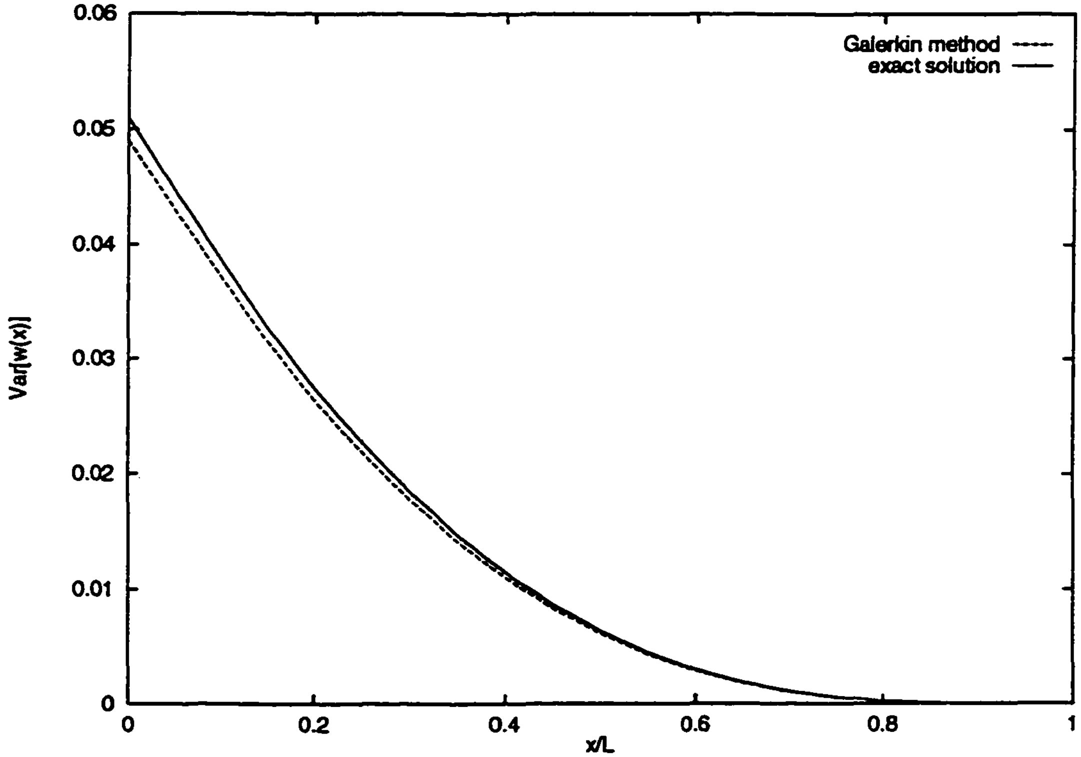

Fig.4.2. Comparison of variances of displacement for cantilever beam under uniform pressure, obtained by Galerkin method and exact solution. Variances are normalized by $q^2L^8/4D_{1u}$

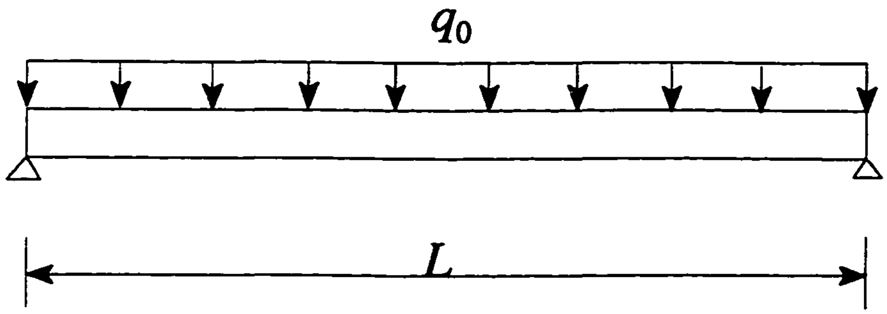

Fig.4.3. Simply-supported beam under uniform pressure

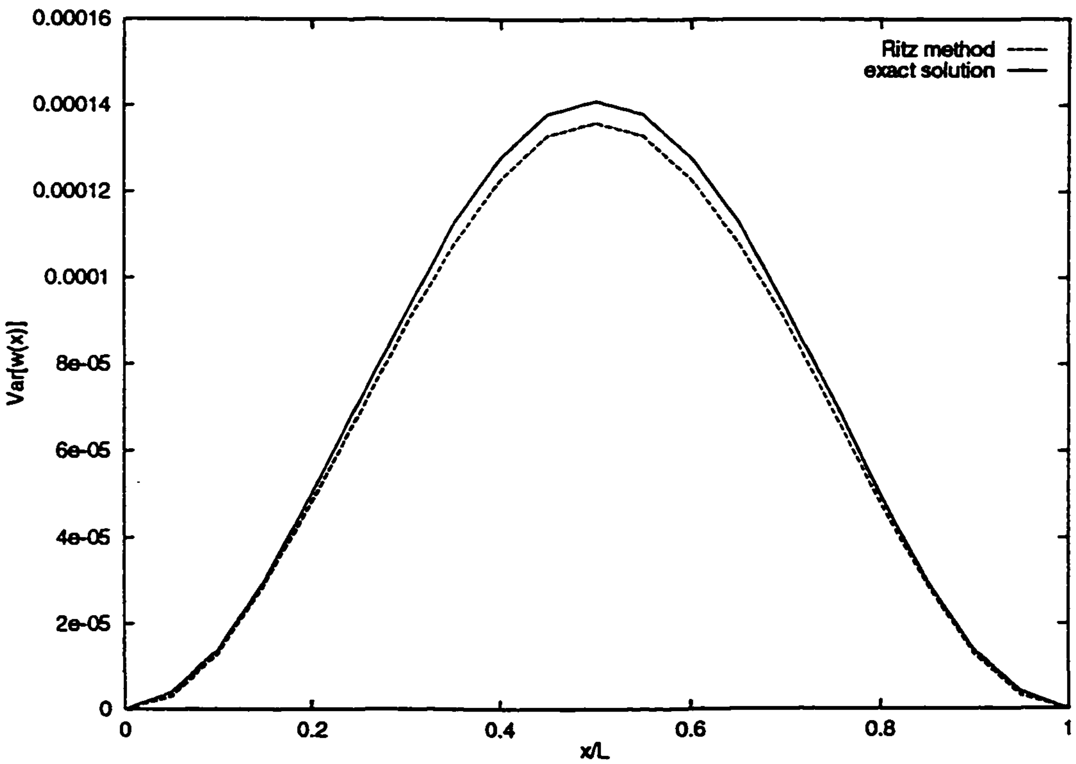

Fig.4.4. Comparison of variances of displacement for simply-supported beam under uniform pressure, obtained by Rayleigh-Ritz method and exact solution. Variances are normalized by $q^2L^8/4D_{1t}$

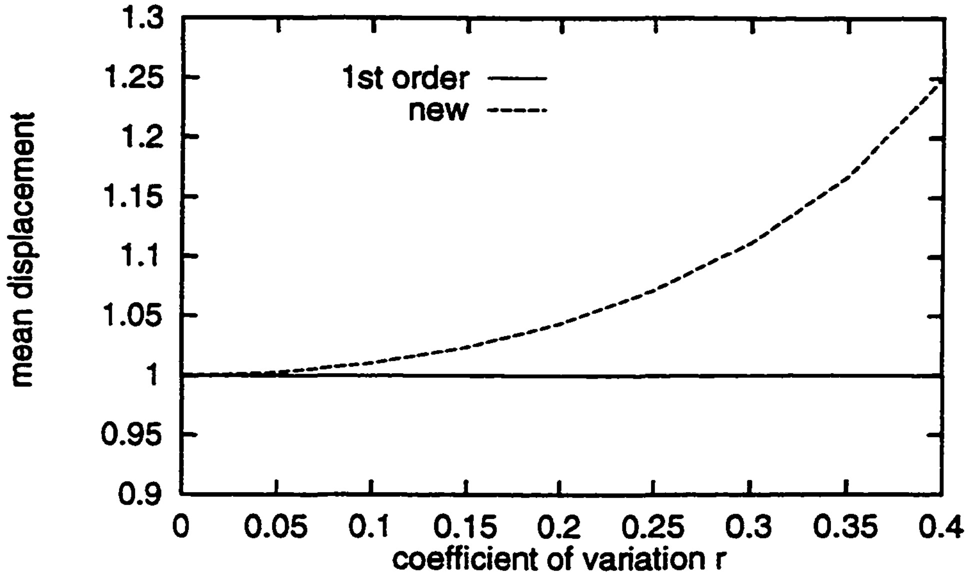

Fig.4.5. The mean displacement at the middle of the simply-supported beam vs coefficient of variation of random stiffness r

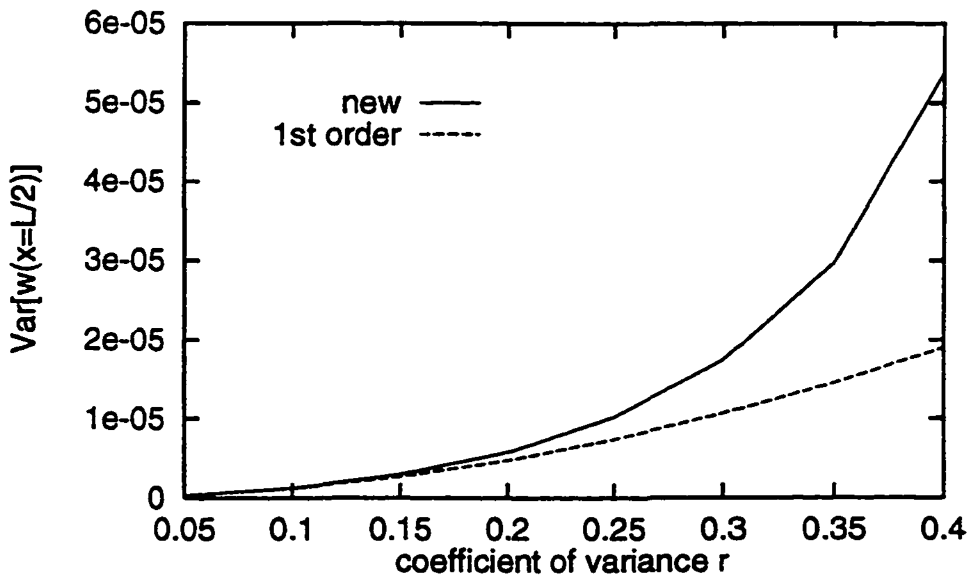

Fig.4.6. The variance of the mid-displacement of simply supported beam vs coefficient of variation of random stiffness r

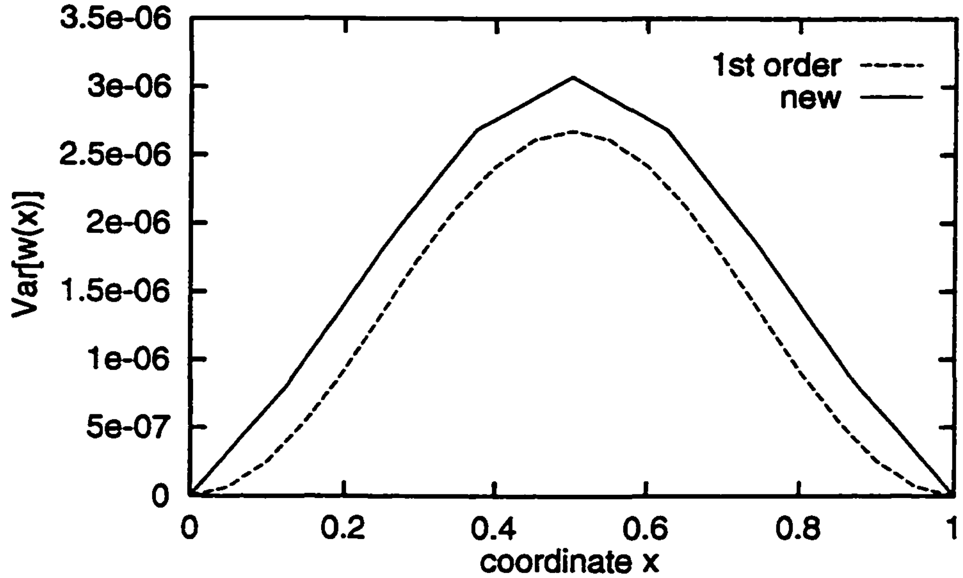

Fig.4.7. The variances of the displacement along the beam's axis (coefficient of variation of random stiffness r=0.15)

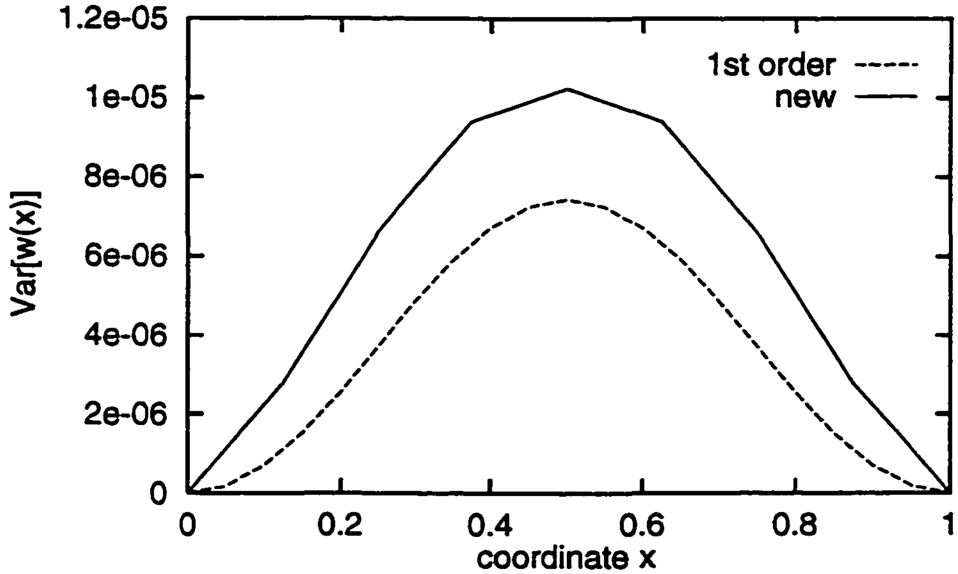

Fig.4.8. The variance of the displacement along the beam's axis (coefficient of variation of ranomd stiffness r=0.3)

Chapter 5: Element-Level Flexibility-Based Finite Element Method for Stochastic Structures

In Chapters 3 and 4, we have proposed two new non-perturbative finite element methods for stochastic structures, namely, (a) the exact inverse based FEM for simple stochastic structures and, (b) the variational principle based FEM for statically determinate stochastic beams. However, these two approaches encounter difficulties to be generalized as a general tool for two- or three-dimensional structures. This chapter presents a new, element-level flexibility-based finite element method for stochastic problems. The unconventional step in the new formulation is to divide the element-level finite element equilibrium equation by the element stiffness so that the reciprocal of the stiffness (the flexibility) would appear at the right side of the equation, and thus becomes uncoupled from the unknown displacement. The mean and covariance of the displacement are then obtained in terms of the mean and covariance of the flexibility.

The concept and formulation of the element-level flexibility-based finite element for stochastic structures will be first elucidated through the two-node cubic element for beam bending analysis. The general formulation for arbitrary structures will be demonstrated with the emphasis on the plane stress and strain problems.

5.1 Formulation for Beam Bending Analysis

Consider a beam with length L and spatially varying stochastic bending stiffness $D(x)$. Following conventional procedure of the finite element method, we discretize the beam into N elements. The FE equilibrium equation for arbitrary j-th element can be obtained from the principle of virtual work, as follow

$$D_j \boldsymbol{k}_j \boldsymbol{u}_j = \boldsymbol{P}_j^i + \boldsymbol{P}_j^o \qquad (j=1,2,\cdots,N) \tag{1}$$

where D_j=bending stiffness of j-th element, which is a random variable representing the spatially random stiffness field $D(x)$ on j-th element, $\boldsymbol{u}_j$=unknown nodal displacement vector, which is also random due to random stiffness, $\boldsymbol{P}_j^i$=nodal internal force vector, $\boldsymbol{P}_j^o$=equivalent nodal force vector caused by external loads, $\boldsymbol{k}_j$=modified element stiffness matrix, which equals to the element stiffness matrix divided by the bending stiffness D_j. Suppose that the two nodes of j-th element are denoted by x_j and x_{j+1}, respectively, then we have

$$\begin{aligned} \boldsymbol{u}_j &= [w_j, w'_j, w_{j+1}, w'_{j+1}]^T = [\boldsymbol{\delta}_j^T, \boldsymbol{\delta}_{j+1}^T]^T \\ \boldsymbol{P}_j &= [Q_{j1}, M_{j1}, Q_{j2}, M_{j2}]^T = [\boldsymbol{p}_{j1}^T, \boldsymbol{p}_{j2}^T]^T \end{aligned} \tag{2}$$

$$\boldsymbol{k}_j = \frac{1}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}_{11}^j & \boldsymbol{k}_{12}^j \\ \boldsymbol{k}_{21}^j & \boldsymbol{k}_{22}^j \end{bmatrix} \tag{3}$$

where a=x_{j-1}-x_j=length of the element, $\boldsymbol{p}_{j1}$ and $\boldsymbol{p}_{j2}$ are nodal forces on left and right nodes, respectively, of the j-th element, $\boldsymbol{k}_{rs}$'s (r,s=1,2) are two by two submatrices of the modified element stiffness matrix $\boldsymbol{k}_j$. Dividing eq.(1) by the bending stiffness D_j yields

$$\boldsymbol{k}_j \boldsymbol{u}_j = f_j(\boldsymbol{P}_j^i + \boldsymbol{P}_j^o) \tag{4}$$

where f_j=1/D_j=bending flexibility of j-th element. Noting that k_j is a deterministic matrix and assuming that the beam is subjected to deterministic external loads, we reach after taking mathematical expectation on both sides of eq.(4)

$$k_j \bar{u}_j = \bar{f}_j P_j^o + E[f_j P_j^i] \tag{5}$$

Eq.(4) can be treated as two sets of equations, based on eqs.(2,3), as

$$\begin{aligned} k_{11}^j \bar{\delta}_j + k_{12}^j \bar{\delta}_{j+1} &= \bar{f}_j p_{j1}^o + E[f_j p_{j1}^i] \\ k_{21}^j \bar{\delta}_j + k_{22}^j \bar{\delta}_{j+1} &= \bar{f}_j p_{j2}^o + E[f_j p_{j2}^i] \\ (j=1,2,&\cdots,N) \end{aligned} \tag{6}$$

Similarly, for (j+1)-th element, we have

$$\begin{aligned} k_{11}^{j+1} \bar{\delta}_{j+1} + k_{12}^{j+1} \bar{\delta}_{j+2} &= \bar{f}_{j+1} p_{j+1,1}^o + E[f_{j+1} p_{j+1,1}^i] \\ k_{21}^{j+1} \bar{\delta}_{j+1} + k_{22}^{j+1} \bar{\delta}_{j+2} &= \bar{f}_{j+1} p_{j+1,2}^o + E[f_{j+1} p_{j+1,2}^i] \\ (j=1,2,&\cdots,N-1) \end{aligned} \tag{7}$$

Assembling the second set equations of eq.(6) and the first set equations of eq.(7), we obtain

$$k_{21}^j \bar{\delta}_j + (k_{22}^j + k_{11}^{j+1}) \bar{\delta}_{j+1} + k_{12}^{j+1} \bar{\delta}_{j+2} = \bar{f}_j p_{j2}^o + \bar{f}_{j+1} p_{j+1,1}^o + E[f_j p_{j2}^i + f_{j+1} p_{j+1,1}^i] \tag{8}$$

Note that the external equivalent nodal force vectors are independent from the stochastic stiffness and can be calculated from the applied loads, whereas the internal forces between two adjacent elements, namely the internal force acting on node 2 of j-th element and the internal force acting on node 1 of (j+1)-th element, have the same magnitude and opposite signs due to the principle of action and reaction

$$p_{j2}^i = -p_{j+1,1}^i = p_{j+1}^i \tag{9}$$

Substitution of eq.(9) into eq.(8) gives

$$k_{21}^{j}\bar{\delta}_j + (k_{22}^{j} + k_{11}^{j+1})\bar{\delta}_{j+1} + k_{12}^{j+1}\bar{\delta}_{j+2} = \bar{f}_j p_{j2}^{o} + \bar{f}_{j+1} p_{j+1,1}^{o} + E[(f_j - f_{j+1}) p_{j+1}^{i}] \tag{10}$$

$$(j=1,2,\cdots,N-1)$$

For statically determinate beams, the internal forces are determined by the applied loads, irrespective of the bending stiffness of the beam. Therefore, p_{j+1}^{i} can be taken out from the expectation operator in eq.(10) if the external forces are deterministic. If the stochastic bending stiffness of the beam is weakly homogeneous, the term with expectation operator in eq.(10) vanishes, since $E[f_j] = E[f_{j+1}] = \bar{f}$. Eq.(10) then reduces to

$$k_{21}^{j}\bar{\delta}_j + (k_{22}^{j} + k_{11}^{j+1})\bar{\delta}_{j+1} + k_{12}^{j+1}\bar{\delta}_{j+2} = \bar{f}_j p_{j2}^{o} + \bar{f}_{j+1} p_{j+1,1}^{o} \qquad (j=1,2,\cdots,N-1) \tag{11}$$

For statically indeterminate beams, the third term of the right side of eq.(10) is negligible compared to first two terms, if a fine finite element mesh is adopted. Indeed, the third term should approach zero when the elements are extremely small, since f_j approaches f_{j-1}. Hence, eq.(11) can be applied for both statically determinate and indeterminate beams, on the remark that it is approximate for statically indeterminate beams.

By examining eq.(11), one can find that the internal force p_{j-1} (j=1,2, · · ·,N-1) on (j+1)-th junction nodes no longer appears after assembling j-th and (j+1)-th elements. However, at the two ends of the beam, there is no assembling of elements. At the left end of the beam, we have from eq.(6)

$$k_{11}^{1}\bar{\delta}_1 + k_{12}^{1}\bar{\delta}_2 = \bar{f}_1 p_{11}^{o} + E[f_1 p_{11}^{i}] \tag{12}$$

and at the right end of the beam

$$k_{21}^{N}\bar{\delta}_N + k_{22}^{N}\bar{\delta}_{N+1} = \bar{f}_N p_{N2}^{o} + E[f_N p_{N2}^{i}] \tag{13}$$

The internal forces in eqs.(12,13) are reaction forces of the supports and will be determined in the same way as in the conventional finite element analysis. Therefore, by assembling all elements together, the global finite element equilibrium equation will has the form of

$$K\bar{U} = F \tag{14}$$

where K=modified global stiffness matrix, U=vector of global unknown nodal displacements, F=global vector of modified equivalent nodal forces consisting of $\bar{f}_1 p_{11}^{o}$, $\bar{f}_j p_{j2}^{o} + \bar{f}_{j+1} p_{j+1,1}^{o}$

(j=1,2,•••, N-1) and $\bar{f}_N p_{N2}^{o}$ as its 2-element sub-vectors. In case that the stochastic stiffness

field is weakly homogenous, $\bar{f}_j = \bar{f}_{j+1} = \bar{f}$. Eq.(14) becomes

$$K\bar{U} = \bar{f}P \tag{15}$$

where P=global vector of equivalent nodal forces. The mean displacement is then

$$\bar{U} = K^{-1}F = \bar{f}K^{-1}P \tag{16}$$

To obtain the covariance matrix of the displacement, one subtracts eq.(5) from eq.(4) and get

$$k_j \Delta u_j = \Delta f_j P_j^o + f_j P_j^i - E[f_j P_j^i] \tag{17}$$

where $\Delta(\cdot) = (\cdot) - (\bar{\cdot})$ is deviation of $(\cdot)$. Analogous to eqs.(6,7), we write

$$\begin{aligned} k_{11}^j \Delta\delta_j + k_{12}^j \Delta\delta_{j+1} &= \Delta f_j p_{j1}^o + f_j p_{j1}^i - E[f_j p_{j1}^i] \\ k_{21}^j \Delta\delta_j + k_{22}^j \Delta\delta_{j+1} &= \Delta f_j p_{j2}^o + f_j p_{j2}^i - E[f_j p_{j2}^i] \\ (j=1,2,\cdots,N) \end{aligned} \tag{18}$$

and

$$\begin{aligned} k_{11}^{j+1} \Delta\delta_{j+1} + k_{12}^{j+1} \Delta\delta_{j+2} &= \Delta f_{j+1} p_{j+1,1}^o + f_{j+1} p_{j+1,1}^i - E[f_{j+1} p_{j+1,1}^i] \\ k_{21}^{j+1} \Delta\delta_{j+1} + k_{22}^{j+1} \Delta\delta_{j+2} &= \Delta f_{j+1} p_{j+1,2}^o + f_{j+1} p_{j+1,2}^i - E[f_{j+1} p_{j+1,2}^i] \\ (j=1,2,\cdots,N-1) \end{aligned} \tag{19}$$

The assembly of the second equation of eq.(18) and the first equation of eq.(19) gives rise to

$$k_{21}^j \Delta\delta_j + (k_{22}^j + k_{11}^{j+1}) \Delta\delta_{j+1} + k_{12}^{j+1} \Delta\delta_{j+2} = \Delta F_{j+1} \qquad (j=1,2,\cdots,N-1) \tag{20}$$

where

$$\Delta F_{j+1} = \Delta f_j p_{j2}^o + \Delta f_{j+1} p_{j+1,1}^o + f_j p_{j2}^i + f_{j+1} p_{j+1,1}^i - E[f_j p_{j2}^i + f_{j+1} p_{j+1,1}^i] \tag{21}$$

Due to eq.(9), eq.(21) reduces to

$$\Delta F_{j+1} = \Delta f_j p_{j2}^o + \Delta f_{j+1} p_{j+1,1}^o + (f_j - f_{j+1}) p_{j+1}^i - E[(f_j - f_{j+1}) p_{j+1}^i] \tag{22}$$

Analogous to eqs.(12,13), at the left end of the beam we have

$$k_{11}^1 \Delta\delta_1 + k_{12}^1 \Delta\delta_2 = \Delta f_1 p_{11}^o + f_1 p_{11}^i - E[f_1 p_{11}^i] \tag{23}$$

and at the right end of the beam

$$k_{21}^{N}\Delta\delta_{N}+k_{22}^{N}\Delta\delta_{N+1}=\Delta f_{N}p_{N2}^{o}+f_{N}p_{N2}^{i}-E[f_{N}p_{N2}^{i}] \tag{24}$$

The global finite element equilibrium equation for the displacement deviation can then be written, after assembling all elements, as

$$K\Delta U=\Delta F \tag{25}$$

and the covariance matrix of the displacement is

$$Cov(U,U^{T})=K^{-1}Cov(\Delta F,\Delta F^{T})K^{-T} \tag{26}$$

where the jk-th element of the covariance matrix $Cov(\Delta F,\Delta F^{T})$ is obtained from eq.(22-24)

$$\begin{aligned}E[\Delta F_{j},\Delta F_{k}^{T}] &= Cov(f_{j},f_{k})p_{j2}^{o}(p_{k2}^{o})^{T}+Cov(f_{j},f_{k+1})p_{j2}^{o}(p_{k+1,1}^{o})^{T}+Cov(f_{j+1},f_{k}\\ &+Cov(f_{j+1},f_{k+1})p_{j+1,1}^{0}(p_{k+1,1}^{0})^{T}+p_{j2}^{o}Cov[f_{j},(f_{k}-f_{k+1})(p_{k+1}^{i})^{T}]+p_{j+1,1}^{o}Cov[f_{j+1},(f\\ &+Cov[(f_{j}-f_{j+1})p_{j+1}^{i},f_{k}](p_{k2}^{o})^{T}+Cov[(f_{j}-f_{j+1})p_{j+1}^{i},f_{k+1}](p_{k+1,1}^{o})^{T}\\ &+Cov[(f_{j}-f_{j+1})p_{j+1}^{i},(f_{k}-f_{k+1})(p_{k+1}^{i})^{T}]-E[(f_{j}-f_{j+1})p_{j+1}^{i}]E[(f_{k}-f_{k+1})(p_{k+1}^{i})^{T}\end{aligned} \tag{27}$$

For statically determinate structures, the internal forces are independent of the stochastic bending stiffness or flexibility. In this case, eq.(27) reduces to

$$
\begin{aligned}
E[\Delta F_j, \Delta F_k^T] = {} & Cov(f_j, f_k) p_{j2}^o (p_{k2}^o)^T + Cov(f_j, f_{k+1}) p_{j2}^o (p_{k+1,1}^o)^T + Cov(f_{j+1}, f_k \\
& + Cov(f_{j+1}, f_{k+1}) p_{j+1,1}^0 (p_{k+1,1}^0)^T + p_{j2}^o Cov[f_j, (f_k - f_{k+1})] (p_{k+1}^i)^T + p_{j+1,1}^o Cov[f_{j+1}, \\
& + p_{j+1}^i Cov[(f_j - f_{j+1}), f_k] (p_{k2}^o)^T + p_{j+1}^i Cov[(f_j - f_{j+1}), f_{k+1}] (p_{k+1,1}^o \\
& + p_{j+1}^i Cov[(f_j - f_{j+1}), (f_k - f_{k+1})] (p_{k+1}^i)^T
\end{aligned}
\tag{28}
$$

Moreover, when the size of elements tends to zero, the terms with internal forces become smaller and smaller, since the terms like $Cov[f_j, (f_k - f_{k+1})]$ tend to zero. Therefore, as the result of approximation, we can simply neglect the terms with internal forces in eq.(28). Then,

$$
\begin{aligned}
E[\Delta F_j, \Delta F_k^T] = {} & Cov(f_j, f_k) p_{j2}^o (p_{k2}^o)^T + Cov(f_j, f_{k+1}) p_{j2}^o (p_{k+1,1}^o)^T \\
& + Cov(f_{j+1}, f_k) p_{j+1,1}^o (p_{k2}^o)^T + Cov(f_{j+1}, f_{k+1}) p_{j+1,1}^0 (p_{k+1,1}^0)^T
\end{aligned}
\tag{29}
$$

The above equation can also be applied approximately for statically indeterminate structures, based on the same reasoning pertinent to the mean displacement.

5.2 Formulation for Plane Stress and Strain Analysis

Suppose a two-dimensional structure, which is in the plane stress/strain state. The structure occupies an area Ω with the boundary $\partial\Omega = S_u + S_p$, where the displacements are constrained on the boundary S_u and the traction forces are given on the boundary S_p. The Young's modulus $E(x)$ of the structure is assumed to be a spatially varying stochastic field.

The strain-displacement relationship, constitutional law and equilibrium equation for the plane stress or strain elasticity are, respectively,

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \tag{30}$$

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} E & -\mu E & 0 \\ -\mu E & E & 0 \\ 0 & 0 & \frac{1+\mu}{2}E \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{Bmatrix} = D\epsilon \tag{31}$$

and

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + q_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + q_y = 0 \end{aligned} \tag{32}$$

where u is the displacement in the x-direction and v is the displacement in the y-direction, q_x and q_y are distributed body forces in x and y directions, respectively; ϵ_x and ε_y are extensional strains in x and y directions and γ_{xy} is the shear strain; σ_x and σ_y are normal stresses in x and y directions and τ_{xy} is the shear stress; E is the Young's modulus and ν is the Possion ratio. The attendant boundary conditions can be described as follows

On the displacement prescribed boundary Su: $u=\bar{u}$, $v=\bar{v}$

On the force prescribed boundary Sp: $t_n=\bar{t}_n$, $t_s=\bar{t}_s$

where t_n and t_s are tractions in the normal and tangent directions of the boundary, respectively, and a overbar represents the prescribed value.

Here we will only derive a four-node quadrilateral element for plane stress or strain analysis (Fig.4.1). The geometrical domain of the element is defined in terms of shape functions and its nodal coordinates, as follows

$$\boldsymbol{x}=\begin{Bmatrix} x \\ y \end{Bmatrix}=\sum_{i=1}^{4} N_i(\xi,\eta)\begin{Bmatrix} x_i \\ y_i \end{Bmatrix}=\boldsymbol{N}(\xi,\eta)\boldsymbol{X} \tag{33}$$

where $\boldsymbol{x}=(x,y)^T$ is the coordinate system, (x_i, y_i) (i=1,2,3,4) is the coordinate of the i-th node of the element, $\boldsymbol{X}=[x_1 , y_1 , x_2 , y_2 , x_3 , y_3 . x_4 , y_4]^T$ is the vector of element nodal coordinates, $\boldsymbol{N}(\xi, \eta)$ is the shape function matrix defined as

$$N(\xi,\eta)=\begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{bmatrix} \quad (34)$$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \qquad i=(1,2,3,4)$$

where ξ and η consist of the local coordinate system associated with the element, (ξ_i, η_i) is the local coordinate of the i-th node of the element. The displacements in the element are interpolated by the same expressions as for the coordinates

$$\boldsymbol{u}=\begin{Bmatrix} u \\ v \end{Bmatrix}=\sum_{i=1}^{4} N_i(\xi,\eta)\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}=N(\xi,\eta)\boldsymbol{d} \quad (35)$$

where (u_i, v_i) $(i=1,2,3,4)$ is the displacement of the i-th node of the element, and

$$\boldsymbol{d}=[u_1,v_1,u_2,v_2,u_3,v_3,u_4,v_4]^T \quad (36)$$

is the vector of the element nodal degrees of freedom. The element with the same interpolation expressions for the displacements and the coordinates is called "isoparametric element".

Substituting eq.(35) into eq.(30) gives the strain vector in terms of the shape functions and the nodal degrees of freedom

$$\epsilon=\boldsymbol{B}\boldsymbol{d} \quad (37)$$

where

$$
\boldsymbol{B}=[\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{B}_3,\boldsymbol{B}_4] \ , \qquad \boldsymbol{B}_i=\begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial y} \end{bmatrix} \quad (i=1,2,3,4) \tag{38}
$$

$\boldsymbol{B}$ is called the strain matrix. The principle of minimum potential energy applying to the element requires

$$
\pi=\int_A \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{D}\boldsymbol{\epsilon}dxdy-\int_A \boldsymbol{q}^T\boldsymbol{u}dxdy-\int_{\partial A}(\boldsymbol{t}^o+\boldsymbol{t}^i)^T\boldsymbol{u}ds=\min \tag{39}
$$

where A is the area of the element, ∂A is the boundary of A; $\boldsymbol{q}=(q_x, q_y)^T$ is the vector of body forces, $\boldsymbol{t}^o$ and t^i are boundary tractions due to the externally applied load and the inner forces between adjacent elements. Substituting eqs.(35,37) into eq.(39) gives

$$
\pi=\frac{1}{2}\boldsymbol{d}^T\boldsymbol{k}\boldsymbol{d}-(\boldsymbol{p}^o+\boldsymbol{p}^i)^T\boldsymbol{d} \tag{40}
$$

where $\boldsymbol{k}$ is the element stiffness matrix, $\boldsymbol{p}^o$ and $\boldsymbol{p}^i$ are equivalent nodal forces of externally applied load and boundary tractions between adjacent elements, respectively. They read

$$k = \int_A B^T D B\,dxdy$$

$$p^o = \int_A N^T q\,dxdy + \int_{\partial A} N^T t^o ds \tag{41}$$

$$p^i = \int_{\partial A} N^T t^i ds$$

Minimizing eq.(40) results in the following element-level finite element equilibrium equation

$$kd = p^o + p^i \tag{42}$$

Bearing in mind that the stochasticity in the element stiffness matrix is the Young's modulus of that element , we rewrite eq.(42) for j-th element as follows

$$E_j k_j d = p_j^i + p_j^o \tag{43}$$

where k_j=the modified stiffness matrix for j-th element, which equals to the element stiffness matrix divided by the Young's modulus of that element. As done for beam bending cases in previous section, we divide eq.(43) by E_j to reach

$$k_j d = f_j(p_j^i + p_j^o) \tag{44}$$

where f_j=$1/E_j$ is the element flexibility. Taking expectation operator to eq.(44) yields

$$k_j d = E[f_j p_j^i] + \bar{f} p_j^o \tag{45}$$

Here we have assumed that the Young's modulus is a weekly homogeneous field and $\bar{f}$ is the mean value of the flexibility. The inner force p_j^i in eq.(45) is related with the stochastic flexibility in general cases. However, as we discussed in previous section, two different schemes can be used to handle the inner force. The first one is simply to ignore the term associated with the inner force in eq.(45), since the inner force appears in pair due to the action and reaction principle and the assembly of the pair terms associated with the same inner force will be like $E[(f_j\text{-}f_{j-1})p_j^i]$, which decreases to zero if the finite element mesh is extremely fine. The second scheme is to apply the inner force corresponding to the mean Young's modulus as an approximation, then the assembly of the pair terms associated with the same inner force is $E[(f_j\text{-}f_{j-1})p_j^i]=0$. Hence, after assembling eq.(45) over all elements, we get the following global finite element equilibrium equation for the mean displacement vector

$$\boldsymbol{K}\bar{\boldsymbol{U}}=\bar{f}\boldsymbol{P} \tag{47}$$

where $\boldsymbol{K}$ is the modified global stiffness matrix, $\boldsymbol{U}$ is the global nodal displacement vector and $\boldsymbol{P}$ is the global nodal equivalent external force vector. The mean nodal displacement vector is then solved by

$$\bar{\boldsymbol{U}}=\bar{f}\boldsymbol{K}^{-1}\boldsymbol{P} \tag{48}$$

The deviation of the nodal displacement vector can be obtained, analogous to the beam bending cases. Subtracting eq.(45) from eq.(44), we get

$$k_j \Delta d = \Delta f_j p_j^o + f_j p_j^i - E[f_j p_j^i] \tag{49}$$

Assembling eq.(48) over all elements gives

$$K\Delta U = \Delta P = \sum_{j=1}^{N} \Delta P_j \tag{50}$$

where ΔU=deviation of the global displacement vector from the mean value, and ΔP_j=expanded vector of the deviation of the element equivalent nodal force vector

$$\Delta p_j = \Delta f_j p_j^o + f_j p_j^i - E[f_j p_j^i] \tag{51}$$

Based on the discussion above for the mean displacement, we approximate the flexibility-related inner force by the inner force corresponding to the mean Young's modulus, namely

$$p_j^i \approx \bar{p}_j^i = \bar{k}_j \bar{d} - p_j^o \tag{52}$$

Eq.(50) reduces to

$$\Delta p_j = \Delta f_j \bar{k} \bar{d} \tag{53}$$

The covariance function of the nodal displacement vector is obtained by

$$Cov[U, U^T] = K^{-1} Cov[\Delta P, \Delta P^T] K^{-1} \tag{54}$$

where the covariance matrix $Cov[\Delta P, \Delta P^T]$ is obtained through assembling the following submatrices

$$Cov[\Delta p_i, \Delta p_j] = cov(\Delta f_i, \Delta f_j) \bar{k}_i \bar{d}_i \bar{d}_j^T \bar{k}_j \tag{55}$$

If we further approximate the inner force by the proposed first scheme, namely by simply discarding the terms associated with the inner force in eq.(48), we reach

$$Cov[\Delta p_i, \Delta p_j] = cov(\Delta f_i, \Delta f_j) p_i^o (p_j^o)^T \tag{56}$$

5.3 Examples

5.3.1 Clamped Beam Subjected to Uniform Load q

Consider a statically undeterminate beam -- a beam clamped at both ends and subjected to the uniform pressure q=1 (Fig.5.2). The length of the beam is assumed to be unit. The bending stiffness of the beam is assumed to be a spatially stochastic field. Let $D(x) = D_0[1 + k\alpha(x)]$ and assume that the normalized field α(x) possesses a uniform two-dimensional Pearson Type II probability distribution as shown in eq.(67) of chapter 4, where the correlation function $\rho(x,y)$ is assumed to be exponential as in eq.(68) of chapter 4.

Fig.5.3 depicts the mean displacement of the clamped beam for the coefficient variation of the stochastic stiffness r=0.1 and r=0.3. The first-order perturbation solution, which is the solution corresponding to the mean stiffness, is independent of the coefficient of variation. The present flexibility based finite element method, with or without internal forces, give the same result, since the stochastic bending field is assumed to be weakly homogeneous and the internal forces corresponding to the mean stiffness are used, instead of unknown exact internal forces. One finds that for the case r=0.1, the perturbation solution is close to the flexibility-based solution; while in the case r=0.3, the difference between two solutions for mean displacement is about 11%. Fig.5.4 shows the standard deviation of the mid-displacement of the clamped beam, normalized by the perturbation-based solution σ_w=0.0019k/D_0. As in the simple-supported beam case, the perturbation solution turns out to be close to the solution of the flexibility-based method only for small values of coefficient of variation of the stiffness, r<=0.1. For larger values of the coefficient of variation of the stiffness, the difference between the perturbation solution and the flexibility-based solution increases. For example, the difference constitutes about 18 percents if the coefficient of variation is 0.3.

Fig.5.5 and Fig.5.6 contrast the results, obtained by the first-order perturbation method and the present flexibility-based method with/without consideration of internal forces, for the clamped beam. The coefficient of variation of the stiffness is taken to be 0.1 or 0.3. In perfect analogy with the simply-supported beam case, we deduce that in the case of the small value (r=0.1) of the coefficient of variation of the stiffness, three solutions are quite close to each other. For the case r=0.3, however, the first-order perturbation solution is much less than the results obtained by the flexibility-based method. This finding once again confirms that the perturbation-based solutions are acceptable only for the small values of the coefficient of variation. Such a result can certainly be expected. However this work appears to be the first one which directly evaluates the errors associated with the perturbation method for the statically indeterminate problems for which the exact solution is unavailable. Moreover, the present work allows to obtain accurate solution for arbitrary value of coefficient of variation.

5.3.2 Square in Uniform Pressure

In the second example, we consider a two-dimensional problem - a square under uniform pressure on one side and simply-supported on the opposite side, the two other sides are free (Fig.5.7). The Young's modulus of the square is again assumed to be a spatial random field,

$E(x,y)=E_o[\ 1+k\alpha(x,y)]$, where E_o is the mean Young's modulus, k is a constant and $\alpha(x,y)$ is a normalized two-dimensional stochastic field with zero mean and Pearson type II probability density function, which is given in eq.(67) of chapter 4, and the following exponential correlation function

$$\rho(x_1,y_1;x_2,y_2) = \exp\{-\frac{[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}}{\theta}\} \quad (57)$$

where θ=constant. A uniform 4-node element mesh is adopted to discretize the square and the mean and variance-covariances of the displacement u and v are computed for various coefficient of variation of the stochastic Young's modulus and different number of elements used. In calculation, the inner forces are substituted by the inner forces corresponding to the mean Young's modulus, the constants are taken as follows: θ=0.5, the side length of the square=1, the mean Young's modulus=$1.0\text{x}10^7$. The computed results have been plotted in Fig.5.8 - Fig.5.11. Fig.5.8 shows the mean values of the displacement in y direction at corner A for different values of the coefficient of variation of the Young's modulus, obtained by the present element-level flexibility based finite element method and the stiffness based first-order perturbation finite element method. As we expect, the solution by the flexibility based FEM agrees well with the solution by the perturbation based FEM for small values of the coefficient of variation of the Young's modulus. However, the difference between both solutions increases when the value of coefficient of variation increases. It should be emphasized that the flexibility based FEM requires the same computational time as the first-order perturbation method, hence, its advantages of wide applicability does not rely on the sacrifice of the computing cost.

Fig.5.9 gives distribution of the covariance function of the y-direction displacements at corner A and at the arbitrary location (x,y). It is seen that the covariance function reaches its maximum value at (1,1), as it should be, where the covariance coincides the variance. At simply-supported side x=0, the covariance equals to zero due to the boundary condition. Fig.5.10 illustrates the variance of the displacements in both x and y directions at corner A for various values of coefficient of variation of the Young's modulus, obtained by the present element-level flexibility based finite element method and the stiffness based first-order perturbation finite element method. As we have pointed out for the case of the mean displacement, again it is seen that the solution by the flexibility based FEM agrees well with the solution by the perturbation based FEM for small values of the coefficient of variation of the Young's modulus. However, the both solutions diverge from each other when the value of coefficient of variation increases. The conclusion is that the perturbation based method is suitable to small values of the coefficient of variation, however, it underestimates the solution remarkably and therefore is not applicable in the cases where the coefficient of variation of the Young's modulus is large.

Fig.5.11 demonstrates the solutions of the variance of the displacement at corner A for different finite element discretization. The two solutions do not converge to the same value when the number of elements in meshing increases. It implies that the difference is not caused by the roughness or fineness of the FE mesh, but a result of the methods applied. The disadvantages of the perturbation method will not be overcome by the increasing of the elements in discretization.

5.3.3 A Rectangle under a Concentrated Load

Our third example is a rectangle subjected to a concentrated force acting at upper right corner (Fig.5.12). The left side of the rectangle is clamped and the width of the rectangle is twice as its height. We will assume that the Young's modulus of the rectangle is a two-dimensional stochastic field. Denote $E(x,y)=E_0[1+k\alpha(x,y)]$, where, as in previous example, E_0 is the mean Young's modulus, k is a constant and $\alpha(x,y)$ is a normalized two-dimensional stochastic field with zero mean and again Pearson Type II probability density function and the exponential correlation function shown in eq.(57). Analogous to the previous example, a uniform 4-node element mesh is adopted to discretize the rectangle and the mean and variance-covariances of the displacement u and v are computed for various coefficient of variation of the stochastic Young's modulus. In calculation, the inner forces are substituted by the inner forces corresponding to the mean Young's modulus, the constants are taken as follows: $\theta=0.5$, the width of the rectangle=2 and the height=1, the mean Young's modulus=1.0×10^7.

Fig.5.13 shows the covariance function between the vertical displacements at the upper-right corner, where the concentrated force acts, and at the arbitrary location (x,y). Since the boundary condition at the clamped side x=0, the covariance function equals to zero at x=0. For (x,y) coincides the upper-right corner, the covariance function reduces to the variance, which of course is the maximum value of the covariance function. Fig.5.14 and Fig.5.15 portray the variance of the vertical displacement and horizontal displacement at the upper-right corner, respectively. This example shows again that the perturbation solution is acceptable only for small values of coefficient of variation of the Young's modulus.

To summarize, the element-level flexibility-based finite element method decouples the correlation of the stochastic response (displacement) and the stochastic input (bending stiffness) by dividing the finite element equilibrium equation with the stiffness. The new method is not only applicable to the one-dimensional problems, but two- or three-dimensional problems. As the non-perturbative finite element method proposed in previous Chapters, the flexibility-based finite element method is also a general approach applicable to any value of coefficient of variation of the stochastic stiffness.

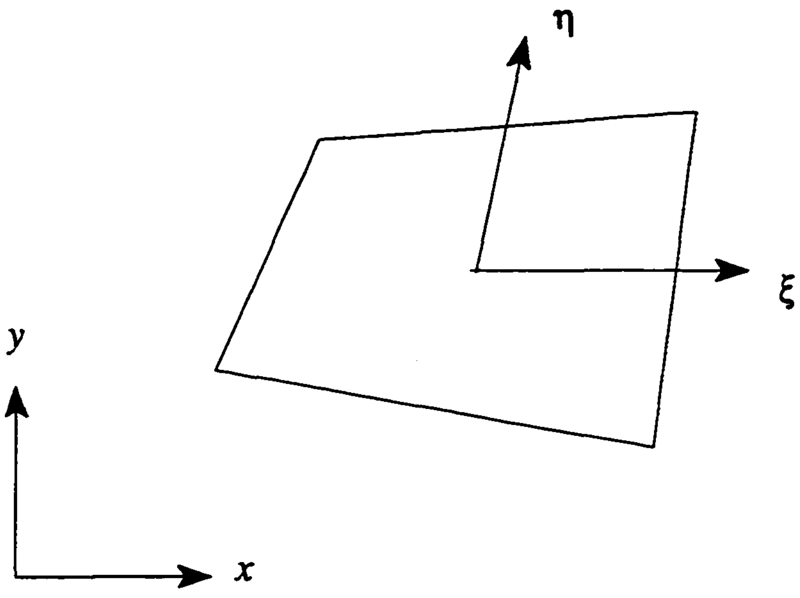

Fig.5.1 4-node rectangular element for plane stress/strain analysis

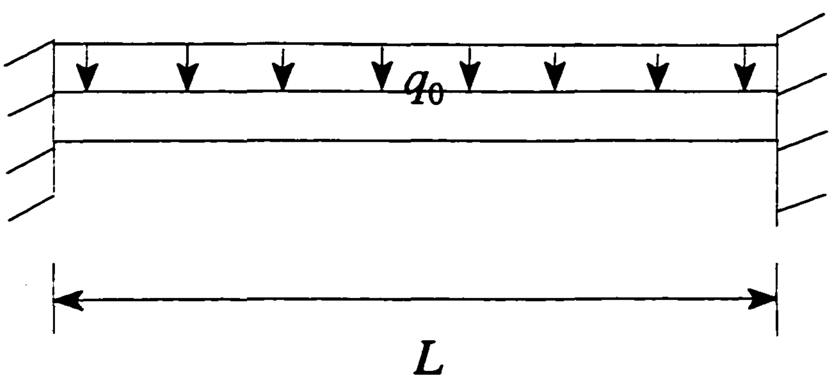

Fig.5.2 Clamped beam under uniform pressure

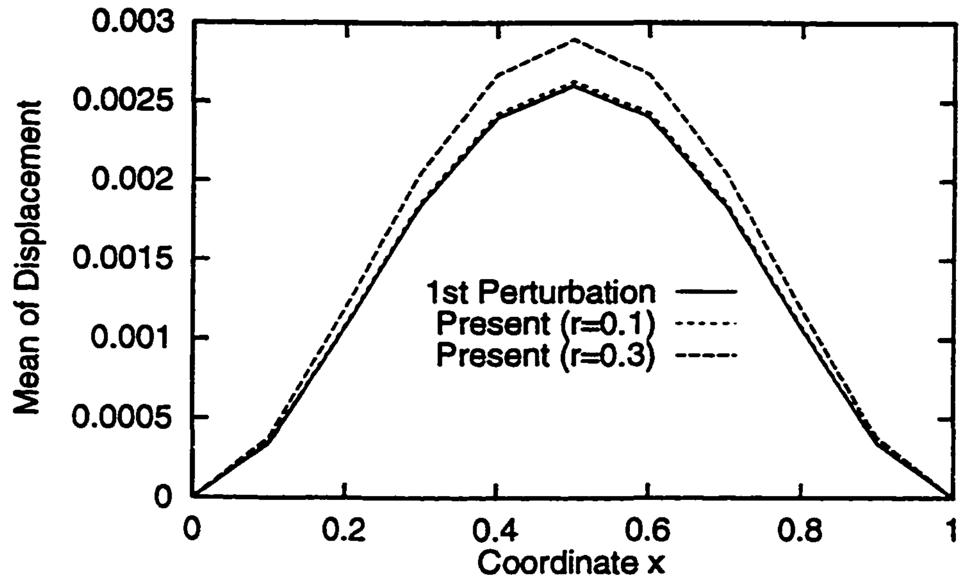

Fig.5.3. Mean displacement of clamped beam under uniform pressure vs axial coordinate (coefficent of variation of stochastic bending stiffness r=0.1 or r=0.3)

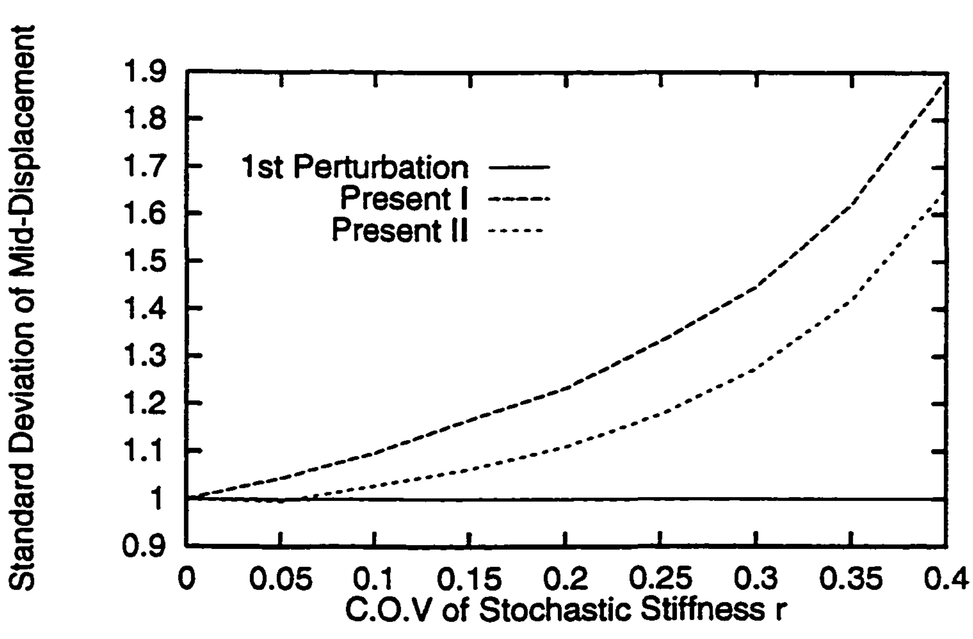

Fig.5.4. Normalized standard deviation of mid-displacement of clamped beam under uniform pressure vs coefficent of variation of stochastic bending stiffness

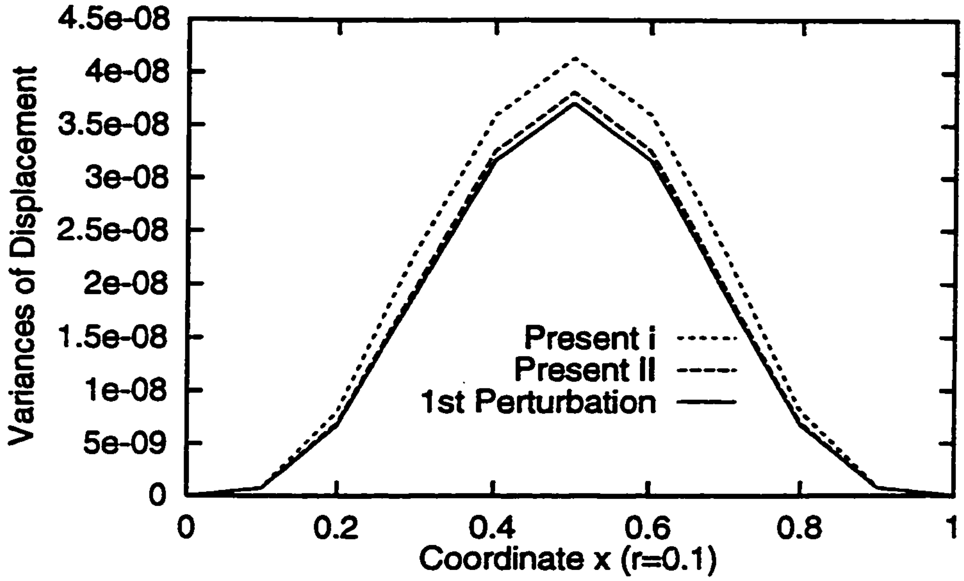

Fig.5.5. Displacement variance of clamped beam under uniform pressure vs axial coordinate (coefficent of variation of stochastic bending stiffness r=0.1)

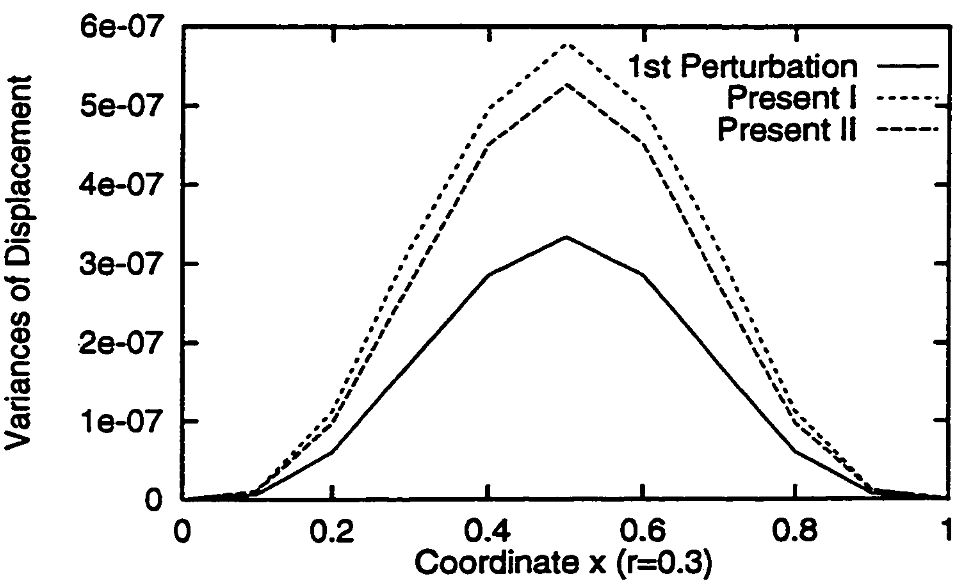

Fig.5.6. Displacement variance of clamped beam under uniform pressure vs axial coordinate (coefficent of variation of stochastic bending stiffness r=0.3)

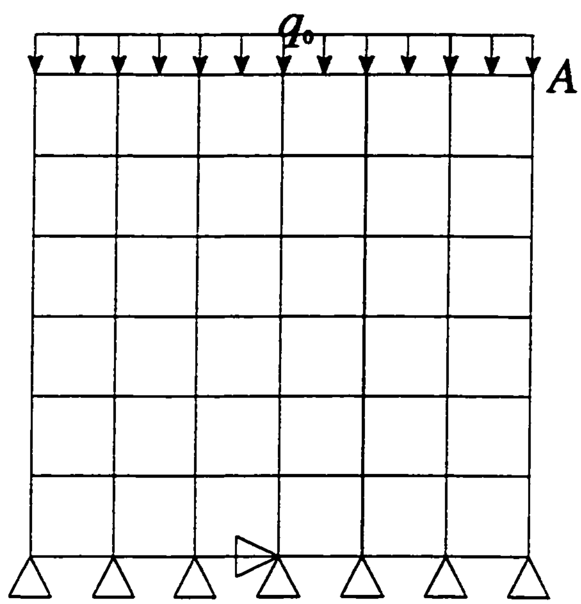

Fig.5.7 A Square under uniform pressure

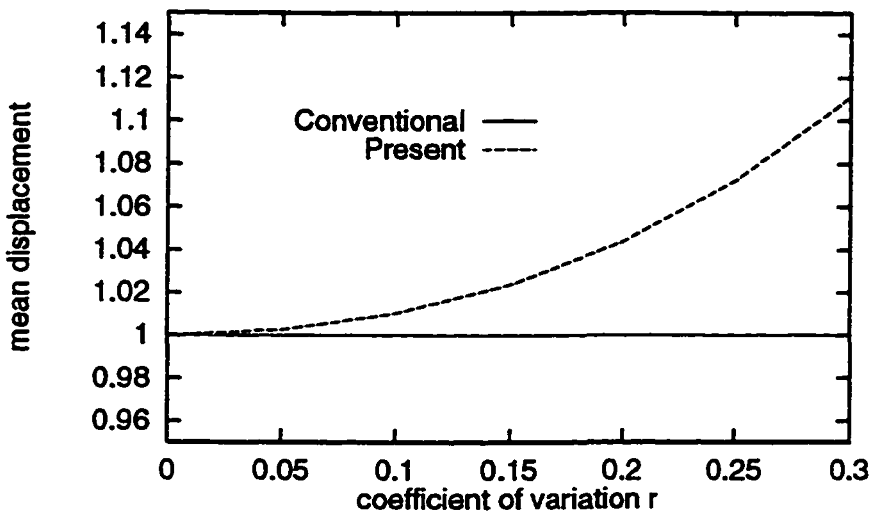

Fig.5.8. Mean displacement at corner A vs coefficient of variation of Young's modulus

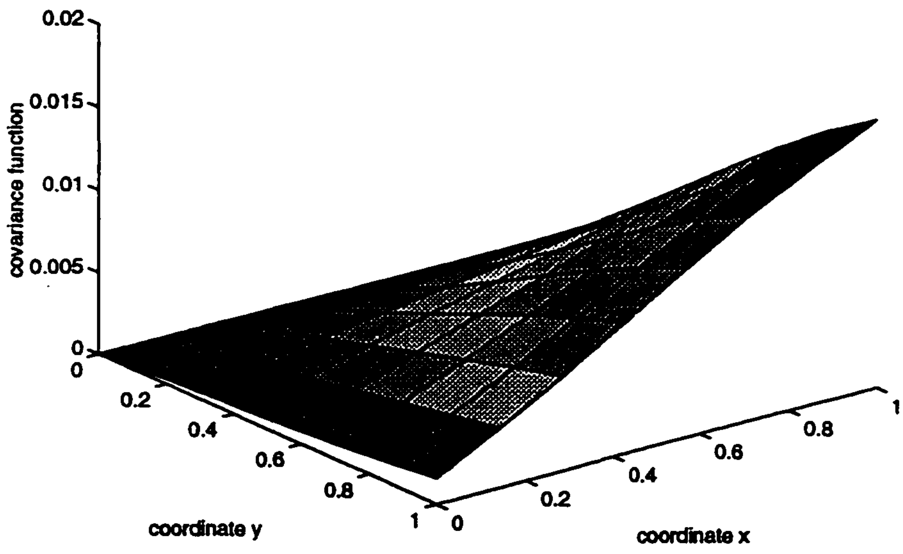

Fig.5.9. Distribution of covariance function of displacement at corner A (r=0.15)

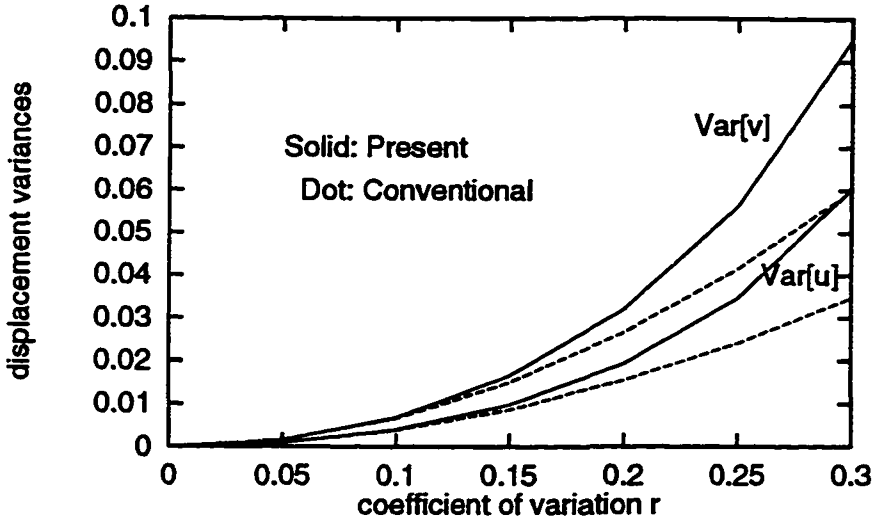

Fig.5.10. variance of vertical displacement at corner A vs coefficient of variation of Young's modulus

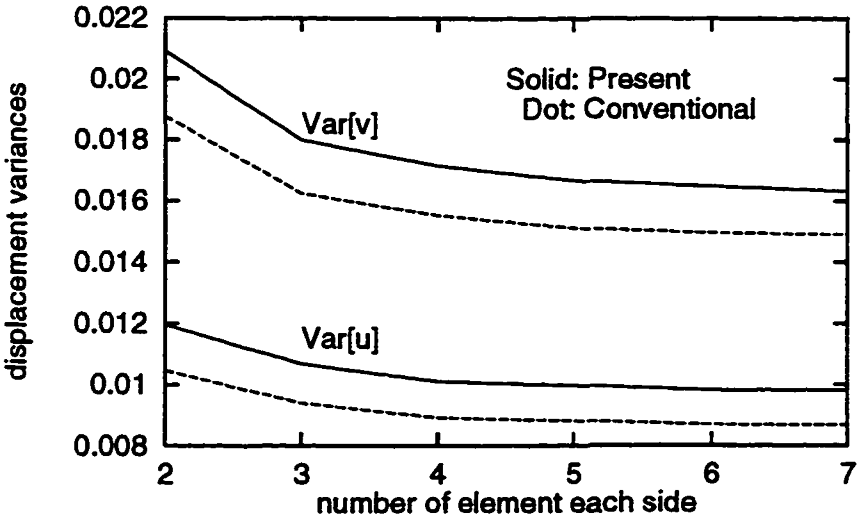

Fig.5.11. Variances of displacement at corner A vs number of element at side

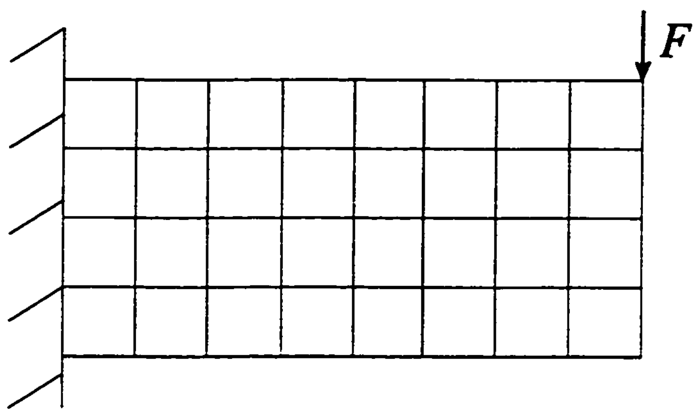

Fig.5.12. A rectangle under concentrated force

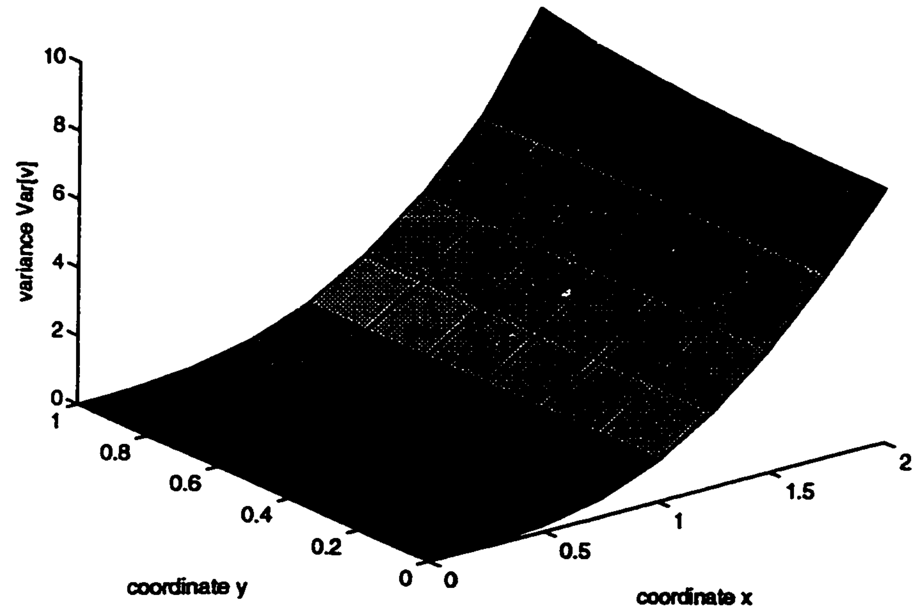

Fig.5.13. Covariance function of the vertical displacement at corner

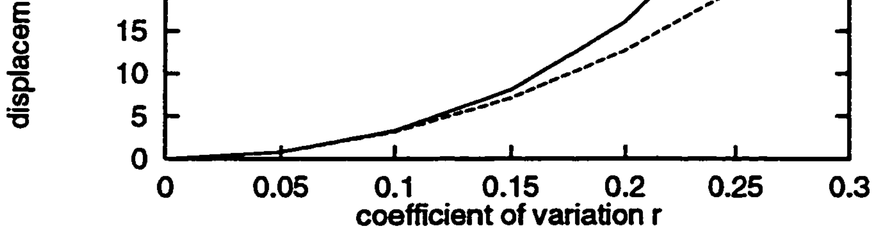

Fig.5.14. Variance of vertical displacement at corner vs coefficient of variation of Young's modulus

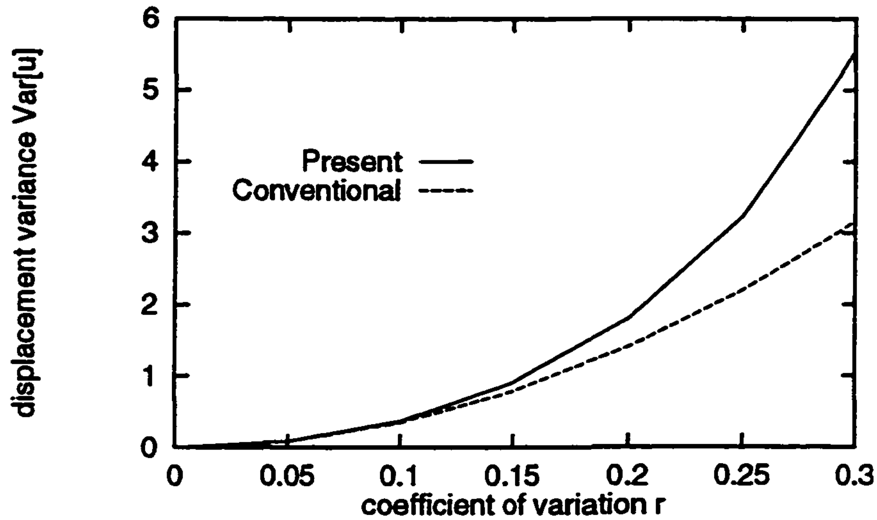

Fig.5.15. Variance of horinzontal displacement at corner vs coefficient of variation of Young's modulus

Chapter 6: Conditional Simulation of Random Fields and Its Combination with FEM for Stochastic Structures

In the previous three Chapters, we have proposed several novel non-perturbative finite element methods for stochastic structures. These non-perturbative methods have the advantages of applicability to the any value range of the coefficient of variation of random parameters. Moreover, the solution converges to the exact solution when the element size of the finite element mesh tends to zero, while such a property is locked by the conventional perturbative finite element method. However, it is seen that most non-perturbative FE methods require the two-dimensional probability density function of the random Young's modulus or the stiffness. From the practical point of view, it is a common case that the one dimensional probability density function as well as the correlation function of random field are provided. The requirement of the two-dimensional probability density function may exceed the available data information for an engineering problem. Furthermore, some measurements may have been conducted on the random Young's modulus or the stiffness, therefore the random field is, in fact, a conditional field conditioned by the measured data. In this Chapter, we deal with the conditional simulation technique of the random fields, both Gaussian and non-Gaussian. The proposed new technique is applied to produce a set of samples of random fields, conditioned or unconditioned, based on the available one-dimensional probability density function and the correlation function. The produced samples are then used to calculate the correlation of the flexibility, which is required in non-perturbative finite element method for stochastic structures. The incorporation of the simulation technique and the proposed non-perturbative FEM for stochastic structures enables us to analyze the engineering problems, in which only information on one-dimensional probability density function and correlation function of random parameters have been given.

6.1. Conditional Simulation of Multi-Variate Gaussian Fields

6.1.1 Basic Formulation

Assume that $G(x)=[G_1(x),G_2(x),\dots, G_n(x)]^T$ is a homogeneous n-variate Gaussian random vector field with zero-mean and cross-covariance matrix $C[G_k(x_i),G_l(x_j)] = E[G_k(x_i)G_l(x_j)]$ $(k,l=1,2,\dots,n)$. Let $G_l=[G_l(x_1),G_l(x_2),\dots,G_l(x_N)]^T$ be a vector of random variables of l-th component of $G(x)$ at locations x_i ($i=1,2,\dots, N$) and $G'_l=[G'_l(x_1),G'_l(x_2),\dots,G'_l(x_M)]^T$ be a vector of random variables of l-th component of the derivative field $G'(x)=dG(x)/dx$ at locations x_j ($j=1,2,\dots, M$). Let g_l be a realization of the vector G_l and g'_l be a realization of G'_l. We approximate the actual field $G(x)$ by its simulated counterpart $G^{(s)}(x)$. In component form, we have

$$\begin{aligned} G_l^{(s)}(x) &= G_l^{(k)}(x) + [G_l(x) - G_l^{(k)}(x)] \\ &= G_l^{(k)}(x) + \epsilon_l(x) \quad , \qquad l=1,2,\dots,n \end{aligned} \tag{1}$$

where the following expression is postulated for $G_l^{(k)}(x)$

$$G_l^{(k)}(x) = \sum_{m=1}^{n} G_m^T \lambda_{ml}(x) + \sum_{m=1}^{n} G'^T_m \mu_{ml}(x) \tag{2}$$

Eq.(2) is proposed here as the kriging estimate of l-th component $G_l(x)$ of the multi-variate random field $G(x)$, and $\lambda_{ml}(x)=[\lambda_{ml1}(x),\lambda_{ml2}(x),\ldots,\lambda_{mlN}(x)]^T$ and $\mu_{ml}(x)=[\mu_{ml1}(x),\mu_{ml2}(x),\ldots,\mu_{mlM}(x)]^T$ are vectors of kriging weight functions. Eq.(2) takes into account both the random field and its derivative for the stochastic interpolation of the random field. It represents a generalization of existing kriging estimate proposed for uni-variate random fields (Hoshiya 1994), which did not include information on realizations of the field's derivatives. It will be shown that eq.(2) reduces to kriging estimate of uni-variate fields for the particular case when the components of the field are mutually uncorrelated and the derivative of the random field is not recorded. In eq.(1), $\varepsilon_l(x)$ is the error between the exact field $G_l(x)$ and its kriging estimate $G_l^{(k)}(x)$

$$\epsilon_l(x) = G_l(x) - \sum_{m=1}^{n} G_m^T \lambda_{ml}(x) - \sum_{m=1}^{n} G'^T_m \mu_{ml}(x) \tag{3}$$

In conventional kriging technique as well as in conventional regression method, it is required that the variance of the error $\epsilon_l(x)$ attain a minimum. Note that the kriging estimate in eq.(3) consists of both the random variables of the field and its derivative at specified locations. The expression in eq.(3), in fact, forms an interpolation of random variables. It is instructive to recall some facts from the interpolation theory of deterministic functions: the interpolating coefficients are determined based on the conditions that at locations where the function is specified, the interpolated function must be equal to the specified values of the function; moreover, at locations where the derivative of the function is specified, the derivative of the interpolated function must be equal to the specified values of the derivative. Following this line of thought, we impose special conditions on the kriging estimate $G_l^{(k)}(x)$ instead of requiring the variance of the error $\epsilon_l(x)$ to attain a minimum. Generalizing the idea used for interpolation of a deterministic function, we require simultaneously that, in the probabilistic sense, the kriging estimate must be equivalent to the actual field where the realizations of the field are measured, and the derivative of the kriging estimate must be equivalent to the derivative of the actual field where the realizations of the derivative of the field are measured. Explicitly, the following four conditions should be satisfied:

(1)

$$E[G_k G_l(x)] = E[G_k G_l^{(k)}(x)] \ , \quad k=1,2,...,n \tag{4}$$

Substitution of eq.(2) yields

$$\sum_{m=1}^{n} C[G_k, G_m^T]\, \lambda_{ml} + \sum_{m=1}^{n} C[G_k, G_m'^T]\, \mu_{ml} = C[G_k, G_l(x)] \qquad k=1,2,...,n \tag{5}$$

where $\lambda_{ml}=\lambda_{ml}(x)$ and $\mu_{ml}=\mu_{ml}(x)$ for simplicity.

(2)

$$E[G_k G_l'(x)] = E[G_k G_l'^{(k)}(x)] \ , \quad k=1,2,...,n \tag{6}$$

From eq.(2), we have

$$G_l'^{(k)}(x) = \sum_{m=1}^{n} G_m^T \lambda'_{ml}(x) + \sum_{m=1}^{n} G_m'^T \mu'_{ml}(x) \tag{7}$$

Hence

$$\sum_{m=1}^{n} C[G_k, G_m^T]\,\lambda'_{ml} + \sum_{m=1}^{n} C[G_k, G'^T_m]\,\mu'_{ml} = C[G_k, G'_l(x)] \tag{8}$$

$$k = 1, 2, \ldots, n$$

Integrating eq.(8) once yields

$$\sum_{m=1}^{n} C[G_k, G_m^T]\,\lambda_{ml} + \sum_{m=1}^{n} C[G_k, G_m'^T]\,\mu_{ml} = C[G_k, G_l(x)] + A_l \tag{9}$$

$$k = 1, 2, \ldots, n$$

where A_l=vector of constants. Comparison of eq.(5) and eq.(9) results in A_l=0.

(3)

$$E[G_k' G_l(x)] = E[G_k' G_l^{(k)}(x)] \;,\quad k=1,2,\ldots,n \tag{10}$$

Substitution of eq.(2) yields

$$\sum_{m=1}^{n} C[G_k', G_m^T]\,\lambda_{ml} + \sum_{m=1}^{n} C[G_k', G_m'^T]\,\mu_{ml} = C[G_k', G_l(x)] \tag{11}$$

$$k = 1, 2, \ldots, n$$

(4)

$$E[G_k' G_l'(x)] = E[G_k' G_l'^{(k)}(x)] \;,\quad k=1,2,\ldots,n \tag{12}$$

Substitution of eq.(2) and eq.(7) gives

$$\sum_{m=1}^{n} C[G_k', G_m^T]\, \lambda'_{ml} + \sum_{m=1}^{n} C[G_k', G_m'^T]\, \mu'_{ml} = C[G_k', G_l'(x)] \tag{13}$$

$$k = 1, 2, \ldots, n$$

Integrating eq.(13) once results in

$$\sum_{m=1}^{n} C[G_k', G_m^T]\, \lambda_{ml} + \sum_{m=1}^{n} C[G_k', G_m'^T]\, \mu_{ml} = C[G_k', G_l(x)] + B_l \tag{14}$$

$$k = 1, 2, \ldots, n$$

where B_l=vector of constants. Comparison of eq.(11) and (14) also yields B_l=0.

It is remarkable that the first and third conditions are equivalent to the requirement of minimizing the variance of the error $\epsilon_l(x)$, whereas the second and fourth conditions are equivalent to the requirement of minimizing the variance Var[$\epsilon_l'(x)$] of the error's derivative $\epsilon_l'(x)$. Whereas the former requirement, namely to minimize Var[$\epsilon_l(x)$], is obvious, the latter one, namely to minimize Var[$\epsilon_l'(x)$], appears to be less transparent at the first glance.

Note that once we obtain B_l=0, eq.(14) coincides with eq.(11). Analogously, since A_l=0, eq.(9) coincides with eq.(5). Indeed, one may show that the condition Var[$\epsilon_l'(x)$]=min is a consequence of the condition Var[$\epsilon_l(x)$]=min. One may however visualize the situation in which these conditions are independent. In present formulation, four requirements reduce to two independent conditions. Combination of requirements (1)-(4) yields the following linear equations for the kriging weight functions $\lambda_{ml}(x)$ and $\mu_{ml}(x)$

Eq.(15) consists of $n\times(N+M)$ equations, from which $n\times(N+M)$ unknown kriging weights λ_{ml} and μ_{ml} for the component $G_l(x)$ can be determined.

It is appropriate to mention that the classical regression method in statistics can be

$$\sum_{m=1}^{n} C[G_k, G_m^T]\,\lambda_{ml} + \sum_{m=1}^{n} C[G_k, G'^T_m]\,\mu_{ml} = C[G_l(x), G_k]$$

$$\sum_{m=1}^{n} C[G'_k, G_m^T]\,\lambda_{ml} + \sum_{m=1}^{n} C[G'_k, G'^T_m]\,\mu_{ml} = C[G_l(x), G'_k] \tag{15}$$

$$k=1,2,\ldots,n$$

generalized in a way similar to present formulation, in order to estimate multi-variate random fields. Both the generalized classical regression method and generalized kriging method will then coincide with each other provided that the same set of requirements are imposed, and Gaussian fields are considered.

The error $\epsilon_l(x)$ possesses following properties

(i) The mean of the error vector vanishes

$$E[\epsilon_l(x)] = E[G_l(x) - \sum_{m=1}^{n} G_m^T\lambda_{ml} - \sum_{m=1}^{n} G'^T_m\mu_{ml}]$$

$$= E[G_l(x)] - \sum_{m=1}^{n} E[G_m^T]\lambda_{ml} - \sum_{m=1}^{n} E[G'^T_m]\mu_{ml} = 0 \tag{16}$$

(ii) The error vector and the random vector G_k are uncorrelated

$$E[\epsilon_l(x)G_k] = E\{\, G_k[G_l(x) - \sum_{m=1}^{n} G_m^T\lambda_{ml} - \sum_{m=1}^{n} G'^T_m\mu_{ml}]\,\}$$

$$= C[G_k, G_l(x)] - \sum_{m=1}^{n} C[G_k, G_m^T]\lambda_{ml} - \sum_{m=1}^{n} C[G_k, G'^T_m]\mu_{ml} = 0 \tag{17}$$

$$k = 1,2,\ldots,n$$

(iii) The error vector and the random vector G'_k are uncorrelated

(iv) At any recorded location x_i, eq.(15) becomes an identity by simply letting $\lambda_{mli}=1$ and

$$E[\epsilon_l(x)G'_k] = E\{ G'_k[G_l(x) - \sum_{m=1}^{n} G_m^T\lambda_{ml} - \sum_{m=1}^{n} G'^T_m \mu_{ml}] \}$$

$$= C[G'_k, G_l(x)] - \sum_{m=1}^{n} C[G'_k, G_m^T]\lambda_{ml} - \sum_{m=1}^{n} C[G'_k, G'^T_m]\mu_{ml} = 0 \tag{18}$$

$$k = 1, 2, \ldots, n$$

taking other kriging weights equal to zero, i.e., $\lambda_{mlj}=0$ ($j \neq i$ or $m \neq l$) and every $\mu_{mkj}=0$. We then have

$$\epsilon_l(x_i) = G_l(x_i) - \sum_{m=1}^{n} G_m^T\lambda_{ml} - \sum_{m=1}^{n} G'^T_m\mu_{ml}$$
$$= G_l(x_i) - G_l(x_i) = 0 \tag{19}$$

Subsequently,

$$E[\epsilon_l(x)\,\epsilon_k(x_i)] = 0 \tag{20}$$

(v) The error vector and the kriging estimate are uncorrelated

$$E[\epsilon_l(x)\, G_k^{(k)}(x)] = E\{[\sum_{s=1}^{n} \lambda_{sk}^T G_s + \sum_{s=1}^{n} \mu_{sk}^T G'_s]\,[G_l(x) - \sum_{m=1}^{n} G_m^T\lambda_{ml} - \sum_{m=1}^{n} G'^T_m\mu_{ml}]\}$$

$$= \sum_{s=1}^{n} \lambda_{sk}^T \{ C[G_s, G_l(x)] - \sum_{m=1}^{n} C[G_s, G_m^T]\,\lambda_{ml} - \sum_{m=1}^{n} C[G_s, G'_m]\,\mu_{ml}\}$$
$$+ \sum_{s=1}^{n} \mu_{sk}^T \{ C[G'_s, G_l(x)] - \sum_{m=1}^{n} C[G'_s, G_m]\,\lambda_{ml} - \sum_{m=1}^{n} C[G'_s, G'_m]\,\mu_{ml}\} = 0 \tag{21}$$

$$k = 1, 2, \ldots, n$$

(vi) Correlation matrix of the error vector is obtainable as

$$E[\epsilon_l(x)\epsilon_k(x)] = E\{\epsilon_l(x)[G_k(x) - G_k^{(k)}(x)]\} = E[\epsilon_l(x)G_k(x)]$$

$$= C[G_l(x), G_k(x)] - \sum_{m=1}^{n} \boldsymbol{\lambda}_{ml}^T C[\mathbf{G}_m, G_k(x)] - \sum_{m=1}^{n} \boldsymbol{\mu}_{ml}^T C[\mathbf{G'}_m, G_k(x)] \tag{22}$$

In particular, the variance of the error reads

$$Var[\epsilon_l(x)] = Var[G_l(x)] - \sum_{m=1}^{n} \boldsymbol{\lambda}_{ml}^T C[\mathbf{G}_m, G_l(x)] - \sum_{m=1}^{n} \boldsymbol{\mu}_{ml}^T C[\mathbf{G'}_m, G_l(x)] \tag{23}$$

We arrive at the following

Theorem The variance of the error $\varepsilon_l(x)$ reaches its minimum, given in eq.(23), if and only if the weights λ_{ml} and μ_{ml} satisfy eq.(15).

Proof: Assume that λ_{ml}^* and μ_{ml}^* form an arbitrary set of kriging weights and $\varepsilon_l^*(x)$ is the associated error. Denote $\boldsymbol{\lambda}_{ml}^* = \boldsymbol{\lambda}_{ml} + \delta\boldsymbol{\lambda}_{ml}$ and $\boldsymbol{\mu}_{ml}^* = \boldsymbol{\mu}_{ml} + \delta\boldsymbol{\mu}_{ml}$, where λ_{ml} and μ_{ml} are the kriging weights satisfying eq.(11), and $\delta\lambda_{ml}$ and $\delta\mu_{ml}$ are deviations. From eq.(3), we have

$$Var[\epsilon_l^*(x)] = Var[G_l(x)] - 2\sum_{k=1}^{n} \lambda_{kl}^{*T} C[G_k, G_l(x)] - 2\sum_{k=1}^{n} \mu_{kl}^{*T} C[G_k', G_l(x)]$$

$$+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \lambda_{kl}^{*T} C[G_k, G'^T_m]\mu_{ml}^* + \sum_{k=1}^{n}\sum_{m=1}^{n} \lambda_{kl}^{*T} C[G_k, G_m^T]\lambda_{ml}^* + \sum_{k=1}^{n}\sum_{m=1}^{n} \mu_{kl}^{*T} C[G_k', G'^T_m]\mu_{ml}^*$$

$$= Var[G_l(x)] - 2\sum_{k=1}^{n} \lambda_{kl}^{T} C[G_k, G_l(x)] - 2\sum_{k=1}^{n} \mu_{kl}^{T} C[G_k', G_l(x)]$$

$$+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \lambda_{kl}^{T} C[G_k, G'^T_m]\mu_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \lambda_{kl}^{T} C[G_k, G_m^T]\lambda_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \mu_{kl}^{T} C[G_k', G'^T_m]\mu_{ml}$$

$$- 2\sum_{k=1}^{n} \delta\lambda_{kl}^{T} C[G_k, G_l(x)] - 2\sum_{k=1}^{n} \delta\mu_{kl}^{T} C[G_k', G_l(x)]$$

$$+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^{T} C[G_k, G'^T_m]\mu_{ml} + 2\sum_{k=1}^{n}\sum_{m=1}^{n} \delta\mu_{kl}^{T} C[G_k', G_m^T]\lambda_{ml}$$

$$+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^{T} C[G_k, G_m^T]\lambda_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \delta\mu_{kl}^{T} C[G_k', G'^T_m]\mu_{ml}$$

$$+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^{T} C[G_k, G'^T_m]\delta\mu_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^{T} C[G_k, G'^T_m]\delta\lambda_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \delta\mu_{kl}^{T} C[G_k', G'^{T}_{m}$$

(24)

Bearing in mind eq.(15), eq.(24) becomes

$$
\begin{aligned}
Var[\epsilon_l^*(x)] &= Var[G(x)] - \sum_{k=1}^{n} \lambda_{kl}^T C[G_k, G_l(x)] - \sum_{k=1}^{n} \mu_{kl}^T C[G_k', G_l(x)] \\
&+ 2\sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^T C[G_k, G'^T_m]\delta\mu_{ml} + \sum_{k=1}^{n}\sum_{m=1}^{n} \delta\lambda_{kl}^T C[G_k, G'^T_m]\delta\lambda_{ml} \\
&+ \sum_{k=1}^{n}\sum_{m=1}^{n} \delta\mu_{kl}^T C[G_k', G'^T_m]\delta\mu_{ml} \\
&= Var[\epsilon_l(x)] + E[(\sum_{m=1}^{n} \delta\lambda_{ml}^T G_m + \sum_{m=1}^{n} \delta\mu_{ml}^T G'_m)^2] \geq Var[\epsilon_l(x)]
\end{aligned}
\tag{25}
$$

Necessity: Let $Var[\epsilon_l^*(x)] = Var[\epsilon_l(x)]$, then by virtue of eq.(25), $\sum_{m=1}^{n} \delta\lambda_{ml}^T G_m + \sum_{m=1}^{n} \delta\mu_{ml}^T G'_m = 0$. We multiply it by G_k and G'_k (k=1,2,...,n), respectively, to obtain

$$
\begin{aligned}
&\sum_{m=1}^{n} C[G_k, G_m^T]\,\delta\lambda_{ml} + \sum_{m=1}^{n} C[G_k, G'^T_m]\,\delta\mu_{ml} = 0 \\
&\sum_{m=1}^{n} C[G'_k, G_m^T]\,\delta\lambda_{ml} + \sum_{m=1}^{n} C[G'_k, G'^T_m]\,\delta\mu_{ml} = 0 \\
&k = 1,2,\ldots,n
\end{aligned}
\tag{26}
$$

The solution of eq.(26) is $\delta\lambda_{ml}=\delta\mu_{ml}=0$ provided the cross-covariance matrix of $G(x)$ is non-singular. Hence, $\lambda_{ml}^*=\lambda_{ml}$ and $\mu_{ml}^*=\mu_{ml}$.

Sufficiency: Let $\lambda_{ml}^*=\lambda_{ml}$ and $\mu_{ml}^*=\mu_{ml}$. Then the deviations vanish identically, $\delta\lambda_{ml}=\delta\mu_{ml}=0$. Immediately, this results in $Var[\epsilon_l^*(x)] = Var[\epsilon_l(x)]$. The proof is completed.

Properties (i)-(vi) state that the error vector $\boldsymbol{\epsilon}(x_r) = [\epsilon_1(x_r), \epsilon_2(x_r), \dots, \epsilon_n(x_r)]^T$ of $\boldsymbol{G}(x)$ at any location x is independent from the random vector $\boldsymbol{G}_l$ and the random derivative vector $\boldsymbol{G}'_l$ (l=1,2,...,n), which represent the random field $\boldsymbol{G}(x)$ at recorded locations x_i (i=1,2,...,N), and its derivative $\boldsymbol{G}'(x)$ at recorded locations x_j (j=1,2,...,M). The error vector $\epsilon(x)$ is uncorrelated with error vectors $\epsilon(x_i)$ at recorded locations x_i (i=1,2,...,N). In addition, it is also uncorrelated with the kriging estimate $G_l^{(k)}(x)$ (l=1,2,...,n) of the random field at the same location x. However, different error components at unrecorded location x are correlated with each other. These important properties of $\epsilon(x)$ in eqs.(16-23) guarantee that the error vector $\epsilon(x)$ can be simulated separately from the kriging estimate. Hence, to simulate Gaussian vector random field $\boldsymbol{G}(x)$ at an unrecorded location x under the condition of known realizations $\boldsymbol{g}_l$ and $\boldsymbol{g}'_l$ (l=1,2,...,n), we can calculate the kriging estimate of each component $g_l^{(k)}(x)$ (l=1,2,...,n) and simulate its error vector $\epsilon(x)$ separately, and subsequently formulate their sum. The l-th component of the conditioned multi-variate random field $\boldsymbol{G}(x)$ is given by

$$G_l(x) = \sum_{m=1}^{n} \boldsymbol{\lambda}_{ml}^T \boldsymbol{g}_m + \sum_{m=1}^{n} \boldsymbol{\mu}_{ml}^T \boldsymbol{g}'_m + \epsilon_l(x) \qquad l=1,2,\dots,n \tag{27}$$

The error vector $\epsilon(x)$ is a n-component vector random field with zero-mean and covariance matrix given eq.(22). Its simulation techniques are widely available. If the multi-variate random field needs to be simulated at several unrecorded locations x_r (r=1,2,...,R), a location-by-location recursive procedure can be applied, in analogy to uni-variate random field case (Hoshiya, 1993).

5.1.2. Multi-Variate Field with Uncorrelated Components

In the particular case that components of the multi-variate random field $G(x)$ are mutually uncorrelated, namely $C[G_k(x_i), G_m(x_j)] = 0$ for $k \neq m$, eq.(11) reduces to

$$\begin{bmatrix} C[G_k, G_k^T] & C[G_k, G'^T_k] \\ C[G'_k, G_k^T] & C[G'_k, G'^T_k] \end{bmatrix} \begin{Bmatrix} \lambda_{kl} \\ \mu_{kl} \end{Bmatrix} = \begin{Bmatrix} C[G_l(x), G_k] \\ C[G_l(x), G'_k] \end{Bmatrix} \tag{28}$$

$$k = 1, 2, \ldots, n$$

If $k \neq l$, eq.(28) becomes

$$\begin{bmatrix} C[G_k, G_k^T] & C[G_k, G'^T_k] \\ C[G'_k, G_k^T] & C[G'_k, G'^T_k] \end{bmatrix} \begin{Bmatrix} \lambda_{kl} \\ \mu_{kl} \end{Bmatrix} = \{ 0 \} \tag{29}$$

$$k = 1, 2, \ldots, n \quad ; \quad k \neq l$$

Due to non-singularity of auto-covariance matrix $C[G_k(x_i), G_k(x_j)]$ of k-th component $G_k(x)$, only trivial solution exists for eq.(29), namely

$$\lambda_{kl} = 0 \quad , \qquad \mu_{kl} = 0 \quad (k = 1, 2, \ldots, n; \quad k \neq l) \tag{30}$$

For $k = l$, eq.(28) becomes

$$\begin{bmatrix} C[G_l, G_l^T] & C[G_l, G'^T_l] \\ C[G'_l, G_l^T] & C[G'_l, G'^T_l] \end{bmatrix} \begin{Bmatrix} \lambda_l \\ \mu_l \end{Bmatrix} = \begin{Bmatrix} C[G_l(x), G_l] \\ C[G_l(x), G'_l] \end{Bmatrix} \tag{31}$$

where $\lambda_l = \lambda_{ll}$ and $\mu_l = \mu_{ll}$ for simplicity. Eq.(31) has a unique solution unless the covariance matrix $C[G_l(x_i), G_l(x_j)]$ of l-th component $G_l(x)$ is singular. The latter case takes place, for example, when the correlation coefficient between $G_l(x_i)$ and $G_l(x_j)$ equals unity in its absolute value. Eq.(2) and eq.(23) then become, respectively,

$$G_l^{(k)}(x) = \boldsymbol{G}_l^T \boldsymbol{\lambda}_l + \boldsymbol{G'}_l^T \boldsymbol{\mu}_l \tag{32}$$

$$Var[\epsilon_l(x)] = Var[G_l(x)] - \boldsymbol{\lambda}_l^T C[\boldsymbol{G}_p G_l(x)] - \boldsymbol{\mu}_l^T C[\boldsymbol{G'}_l^T, G_l(x)] \tag{33}$$

The different error components $\varepsilon_l(x)$ and $\varepsilon_k(x)$ for $k \neq l$ are uncorrelated, namely $E[\varepsilon_l(x)\varepsilon_k(x)]=0$. It is seen that in the case when components of a multi-variate random field $G(x)$ are uncorrelated, each component $G_l(x)$ of $G(x)$ can be simulated separately. In fact, eqs.(31-33) consist of collection of uncoupled expressions for conditional simulation of uni-variate random field under realizations of the field and realizations of its derivative. Furthermore, if the realizations of the derivative field are not specified or are not considered, we simply take all $\mu_l=0$ to obtain from eqs.(31-33)

$$C[\boldsymbol{G}_p \boldsymbol{G}_l^T] \boldsymbol{\lambda}_l = C[G_l(x), \boldsymbol{G}_l] \tag{34}$$

$$G_l^{(k)}(x) = \boldsymbol{G}_l^T \boldsymbol{\lambda}_l \tag{35}$$

$$Var[\epsilon_l(x)] = Var[G_l(x)] - \boldsymbol{\lambda}_l^T C[\boldsymbol{G}_p G_l(x)] \tag{36}$$

The above three equations are identical to those proposed by Hoshiya (1994) for uni-variate random field under realizations of the random field.

6.2. Conditional Simulation of Non-Gaussian Fields

Now let $N(\boldsymbol{x})$ be a homogeneous time-independent univariate non-Gaussian random field with zero-mean and targeted autocorrelation function $R_{NN}^{(T)}(\boldsymbol{\xi})=\mathrm{E}[N(\boldsymbol{x}_1)N(\boldsymbol{x}_2)]$, where $\boldsymbol{\xi}=\boldsymbol{x}_1-\boldsymbol{x}_2$. Let B be a random variable with zero mean, variance $\sigma_B^2=R_{NN}(0)$ and given non-Gaussian distribution function $F_B(b)$, which is also the (one-dimensional) distribution function of non-Gaussian random field $N(\boldsymbol{x})$. Assume that $G(\boldsymbol{x})$ is a homogeneous Gaussian random field having same first and second moments as non-Gaussian field $N(\boldsymbol{x})$, namely zero-mean and autocorrelation function $R_{GG}(\xi)=R_{NN}^{(T)}(\xi)$. Denoting by H a Gaussian random variable with zero-mean, variance $\sigma_H^2=R_{GG}(0)$ and Gaussian distribution function $F_H(h)$. If one has a set of realizations $n(\boldsymbol{x}_i)$ of the non-Gaussian field $N(\boldsymbol{x})$ at recorded locations $\boldsymbol{x}_i$ (i=1,2,...,N), the problem consists of simulating $N(\boldsymbol{x})$ at other locations, namely at $\boldsymbol{x}_r$ (r=1,2,...,M).

In order to utilize the existing technique for conditional simulation of Gaussian random field (Hoshiya and Maruyama 1994), following the technique for non-conditional simulation of non-Gaussian random fields (Yamazaki and Shinozuka 1988), one first should map the set of non-Gaussian realizations $n(\boldsymbol{x}_i)$ into a set of Gaussian realizations $g(\boldsymbol{x}_i)$ by means of following transformation

$$g(\boldsymbol{x}_i) = F_H^{-1}\{F_B[n(\boldsymbol{x}_i)]\} \qquad (i=1,2,...,N) \tag{37}$$

where $g(x_i)$ (i=1,2,...,N) can be considered as a set of realizations of Gaussian random field $G(\boldsymbol{x})$. With the simulation method proposed in previous section, one can obtain a set of estimates $g(\boldsymbol{x}_r)$ of $G(\boldsymbol{x})$ at unknown locations $\boldsymbol{x}_r$, then one should map them "back" into non-Gaussian ones $n(\boldsymbol{x}_r)$ in terms of inverse mapping of eq.(37)

$$n(\boldsymbol{x}_r) = F_B^{-1}\{F_H[g(\boldsymbol{x}_r)]\} \qquad (r=1,2,...,M) \tag{38}$$

$n(\boldsymbol{x}_r)$ (r=1,2,...,M) consist of a set of approximate samples of non-Gaussian random field $N(\boldsymbol{x})$ under the condition of $N(\boldsymbol{x}_i)=n(\boldsymbol{x}_i)$. Because of the nonlinearity of the mapping, M estimates $n(\boldsymbol{x}_r)$ and N realizations $n(\boldsymbol{x}_i)$ may not match in general cases the targeted autocorrelation function

$R_{NN}^{(T)}(\xi)$. When the targeted autocorrelationship is not achieved, an iterative procedure should be applied to improve $n(x_r)$ as follows. Suppose that $R_{GG}^{(S)}(\xi)$ is the estimated autocorrelation function of Gaussian random field $G(x)$ calculated by an amount of unconditional simulations and $R_{NN}^{(S)}(\xi)$ is the estimated autocorrelation function of non-Gaussian random field $N(x)$ calculated from corresponding simulations by inversely mapping Gaussian simulations into non-Gaussian simulations. The error between the estimated autocorrelation function $R_{NN}^{(S)}(\xi)$ and the targeted autocorrelation function $R_{NN}^{(T)}(\xi)$ generally stems from two sources, namely from the process of simulation and the process of mapping. Simulation error is caused by the simulation of Gaussian random field itself. The error should be reduced by simulation technique, for instance, by increasing number of samples. Mapping error is caused by nonlinearity of the mapping. The mapping error can be reduced by adjusting the assumed correlation function of Gaussian random field $G(x)$.

The total error of simulation procedure is the difference between the targeted correlation function $R_{NN}^{(T)}(\xi)$ and the simulated correlation function $R_{NN}^{(S)}(\xi)$ of non-Gaussian fields. It reads

$$e(\xi)=R_{NN}^{(T)}(\xi)-R_{NN}^{(S)}(\xi) \tag{39}$$

The error of simulation of Gaussian fields, denoted by $e^S(\xi)$, can be written as

$$e^S(\xi)=R_{GG}(\xi)-R_{GG}^{(S)}(\xi) \tag{40}$$

where $R_{GG}(\xi)$ is the correlation function of the Gaussian field. In first iteration, $R_{GG}(\xi)$ is assumed to be identical to the targeted correlation function of the non-Gaussian field, and in later iterations it is obtained through the iteration equation. $R_{GG}^{(S)}(\xi)$ is the simulated correlation function of the Gaussian field. the second part of error, denoted by $e^M(\xi)$, is then

$$e^{M}(\xi) = e(\xi) - e^{S}(\xi) = R_{NN}^{(T)}(\xi) - R_{NN}^{(S)}(\xi) - e^{S}(\xi) \tag{41}$$

the superscript "M" indicating that $e^{M}(\xi)$ is the mapping error. It is natural to require that $|e^{M}(\xi)|$ should be sufficiently small. the following mapping accuracy criterion is adopted herein: if the estimated autocorrelation function satisfies the requirement $|e^{M}(\xi)| \le \varepsilon$ (where ε is an allowable error), one stops the iteration. If, however, the criterion is violated, i.e., $|e^{M}(\xi)| > \varepsilon$, one should update the assumed autocorrelation function of Gaussian random field as follows

$$R_{GG}^{1}(\xi) = R_{GG}(\xi)\,[\,1 + k\frac{R_{NN}^{(T)}(\xi) - R_{NN}^{(S)}(\xi)}{R_{NN}^{(T)}(0)}\,] \tag{42}$$

where k is a factor analogous to relaxation factor in iteration methods of solution of linear equations. It affects the convergence of the iteration. In most cases, k=1. In some cases, k should be taken to be less than unity in order to avoid divergence of the iteration. the algorithm then proceeds to eq.(37) to start next iteration until the condition $|e^{M}(\xi)| \le \varepsilon$ is satisfied. The iteration equation (42) is a natural generalization from the equation utilized by Yamazaki and Shinozuka (1988) in the context of unconditional simulation

$$S_{GG}^{1}(\kappa) = \frac{S_{GG}(\kappa)}{S_{NN}^{(S)}(\kappa)}\,S_{NN}^{(T)}(\kappa) \tag{43}$$

where $S(\kappa)$ is spectral density of random fields.

If $N(x)$ is a non-Gaussian random field (homogeneous or non-homogeneous) with nonzero mathematical expectation $m(x)$, probability distribution function $F_{N(x)}[n(x)]$ and autocorrelation function $R_{NN}(x_1,x_2)$, one defines a "centered" field $N_1(x)=N(x)-m(x)$. $N_1(x)$ is a random field with zero mean, probability distribution function $F_{N_1(x)}[n_1(x)] = F_{N(x)}[n_1(x)+m(x)]$

and autocorrelation function $R_{N_1N_1}(x_1,x_2) = R_{NN}(x_1,x_2) - m(x_1)m(x_2)$. The conditional simulation

of $N(x)$ can be achieved through conditional simulation of $N_1(x)$ and then letting $N(x)=N_1(x)+m(x)$.

6.3. Incorporation of FEM and Simulation Technique

In the sections 6.1 and 6.2, we have discussed the simulation technique for multi-variate Gaussian random fields and non-Gaussian random fields whose one-dimensional probability density function and correlation function are given, together with possible conditioning measurements. A set of samples of the simulated random field can be generated. This set of samples can then be used to calculate the probability information required for the non-perturbative finite element methods for stochastic structures.

Assume that $\boldsymbol{D}(x)$ =[$D(x_1)$, $D(x_2)$, $D(x_3)$, $\cdots$, $D(x_N)$] ^{T} is a vector consisting of the element stiffness over finite elements j=1, 2, $\cdots$, N, and $\boldsymbol{D}_i(x)$ =[$D_i(x_1)$, $D_i(x_2)$, $D_i(x_3)$, $\cdots$, $D_i(x_N)$] ^{T} (i=1, 2, $\cdots$, M) are a set of samples of $\boldsymbol{D}(x)$. The mean of the element stiffness $D(x_j)$ is calculated as follows

$$E[D(x_j)] = \frac{1}{M}\sum_{i=1}^{M} D_i(x_j) \qquad \textbf{(44)}$$

The correlation between $D(x_j)$ and $D(x_k)$ is calculated as

$$C[D(x_j),D(x_k)] = \frac{1}{M}\sum_{i=1}^{M} D_i(x_j)\,D_i(x_k) \tag{45}$$

Similarly, the mean and the correlation function of the flexibility, the reciprocal of the stiffness, are calculated as follows

$$E[f(x_j)] = E[\frac{1}{D(x_j)}] = \frac{1}{M}\sum_{i=1}^{M} \frac{1}{D_i(x_j)} \tag{46}$$

$$C[f(x_j),f(x_k)] = C[\frac{1}{D(x_j)},\frac{1}{D(x_k)}] = \frac{1}{M}\sum_{i=1}^{M} \frac{1}{D_i(x_j)}\frac{1}{D_i(x_k)} \tag{47}$$

6.4. Examples

Example 1: First, consider an one-dimensional spatially random field $G(x)$ with following unconditional correlation function

$$C_{GG}(x_1,x_2) = \sigma_0^2 \exp[-(x_1-x_2)^2] \tag{48}$$

Realizations of the field and/or the derivative of the field are measured at locations x=0 and x=1. The following four sub-cases of realizations are considered:

(i) Only the realizations of the field are measured at x=0 and x=1, i.e., $g(0)=\alpha$, $g(1)=\beta$. From eq.(34), we have

$$\begin{bmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1(x) \\ \lambda_2(x) \end{Bmatrix} = \begin{Bmatrix} e^{-x^2} \\ e^{-(x-1)^2} \end{Bmatrix} \tag{49}$$

which leads to

$$\lambda_1(x)=\frac{e^{-x^2}-e^{-1}e^{-(x-1)^2}}{1-e^{-2}}\ ,\quad \lambda_2(x)=\frac{-e^{-1}e^{-x^2}+e^{-(x-1)^2}}{1-e^{-2}} \tag{50}$$

The conditional mean and variance of the field are obtained, respectively, as

$$\begin{aligned} E[G(x)|g(0)=\alpha;g(1)=\beta] &= \frac{(\alpha-\beta e^{-1})e^{-x^2}+(\beta-\alpha e^{-1})e^{-(x-1)^2}}{1-e^{-2}} \\ Var[G(x)|g(0)=\alpha;g(1)=\beta] &= \sigma_0^2[1-\frac{e^{-2x^2}-2e^{-2(x^2-x+1)}+e^{-2(x-1)^2}}{1-e^{-2}}] \end{aligned} \tag{51}$$

(ii) Only the realizations of the derivative of the field are measured at x=0 and x=1, i.e., $g'(0)=\gamma$, $g'(1)=\delta$. From eq.(15), we have

$$\begin{bmatrix} 2 & -2e^{-1} \\ -2e^{-1} & 2 \end{bmatrix} \begin{Bmatrix} \mu_1(x) \\ \mu_2(x) \end{Bmatrix} = \begin{Bmatrix} 2xe^{-x^2} \\ 2(x-1)e^{-(x-1)^2} \end{Bmatrix} \tag{52}$$

The solution of the above equation is

$$\mu_1(x)=\frac{xe^{-x^2}+(x-1)e^{-1}e^{-(x-1)^2}}{1-e^{-2}}\ ,\quad \mu_2(x)=\frac{xe^{-1}e^{-x^2}+(x-1)e^{-(x-1)^2}}{1-e^{-2}} \tag{53}$$

The conditional mean and variance of the field are then

$$\begin{aligned} &E[G(x)|g'(0)=\gamma;g'(1)=\delta] \\ &=\frac{x(\gamma+\delta e^{-1})e^{-x^2}+(x-1)(\delta+\gamma e^{-1})e^{-(x-1)^2}}{1-e^{-2}} \\ &Var[G(x)|g'(0)=\gamma;g'(1)=\delta] \\ &=\sigma_0^2[1-\frac{2x^2e^{-2x^2}-4x(x-1)e^{-2(x^2-x+1)}+2(x-1)^2e^{-2(x-1)^2}}{1-e^{-2}}] \end{aligned} \tag{54}$$

(iii) The realization of the field at x=0 and the realization of the derivative of the field at x=1 are measured, i.e., $g(0)=\alpha$, $g'(1)=\delta$. Eq.(15) reads

$$\begin{bmatrix} 1 & -2e^{-1} \\ -2e^{-1} & 2 \end{bmatrix} \begin{Bmatrix} \lambda_1(x) \\ \mu_2(x) \end{Bmatrix} = \begin{Bmatrix} e^{-x^2} \\ 2(x-1)e^{-(x-1)^2} \end{Bmatrix} \tag{55}$$

The solution is

$$\lambda_1(x) = \frac{e^{-x^2} + 2(x-1)e^{-1}e^{-(x-1)^2}}{1-2e^{-2}}, \quad \mu_2(x) = \frac{e^{-1}e^{-x^2} + (x-1)e^{-(x-1)^2}}{1-2e^{-2}} \tag{56}$$

The conditional mean and variance of the field are then

$$\begin{aligned} & E[G(x)|g(0)=\alpha; g'(1)=\delta] \\ & = \frac{(\alpha+\delta e^{-1})e^{-x^2} + (x-1)(\delta+2\alpha e^{-1})e^{-(x-1)^2}}{1-2e^{-2}} \\ & Var[G(x)|g(0)=\alpha; g'(1)=\delta] \\ & = \sigma_0^2 [1 - \frac{e^{-2x^2} + 4(x-1)e^{-2(x^2-x+1)} + 2(x-1)^2 e^{-2(x-1)^2}}{1-2e^{-2}}] \end{aligned} \tag{57}$$

(iv) The realizations of both the field and the derivative of the field are measured at $x=0$ and $x=1$, $g(0)=\alpha$, $g(1)=\beta$, $g'(0)=\gamma$ and $g'(1)=\delta$. The governing equations for kriging weights now become

$$\begin{bmatrix} 1 & e^{-1} & 0 & -2e^{-1} \\ e^{-1} & 1 & 2e^{-1} & 0 \\ 0 & 2e^{-1} & 2 & -2e^{-1} \\ -2e^{-1} & 0 & -2e^{-1} & 2 \end{bmatrix} \begin{Bmatrix} \lambda_1(x) \\ \lambda_2(x) \\ \mu_1(x) \\ \mu_2(x) \end{Bmatrix} = \begin{Bmatrix} e^{-x^2} \\ e^{-(x-1)^2} \\ 2xe^{-x^2} \\ 2(x-1)e^{-(x-1)^2} \end{Bmatrix} \tag{58}$$

The solution of the above equations is

$$\lambda_1(x) = \frac{(1-3e^{-2}+4xe^{-2})e^{-x^2} + e^{-1}(-3+e^{-2}+2x-2xe^{-2})e^{-(x-1)^2}}{1-6e^{-2}+e^{-4}}$$

$$\lambda_2(x) = \frac{e^{-1}(-1-e^{-2}-2x+2xe^{-2})e^{-x^2} + (1+e^{-2}-4xe^{-2})e^{-(x-1)^2}}{1-6e^{-2}+e^{-4}}$$

$$\mu_1(x) = \frac{(2e^{-2}+x-3xe^{-2})e^{-x^2} + e^{-1}(-2+x+xe^{-2})e^{-(x-1)^2}}{1-6e^{-2}+e^{-4}} \tag{59}$$

$$\mu_2(x) = \frac{e^{-1}(1-e^{-2}+x+xe^{-2})e^{-x^2} + (-1+e^{-2}+x-3xe^{-2})e^{-(x-1)^2}}{1-6e^{-2}+e^{-4}}$$

The conditional mean and variance of the actual field are then

$$\begin{aligned} &E[G(x)|g(0)=\alpha; g(1)=\beta; g'(0)=\gamma; g'(1)=\delta] = \alpha\lambda_1(x) + \beta\lambda_2(x) + \gamma\mu_1(x) + \delta\mu_2(x) \\ &Var[G(x)|g(0)=\alpha; g(1)=\beta; g'(0)=\gamma; g'(1)=\delta] = \sigma_0^2[1 + \frac{f(x)}{1-6e^{-2}+e^{-4}}] \end{aligned} \tag{60}$$

where

$$\begin{aligned} f(x) = &(-3+e^{-2}+4x-4xe^{-2}-2x^2+6x^2e^{-2})e^{-2(x-1)^2} \\ &+(-1+3e^{-2}-8xe^{-2}-2x^2+6x^2e^{-2})e^{-2x^2} \\ &+(6-2e^{-2}+4x+4xe^{-2}-4x^2-4x^2e^{-2})e^{-2(x^2-x+1)} \end{aligned} \tag{61}$$

Fig.6.1 portray the conditional variances of the field for above four cases. Fig.6.2 gives amplified conditional variance for case 4 in the interval [0,1]. It is seen that the conditional variance vanishes at those locations where the realizations of the field are measured; moreover, the slope of the conditional variance vanishes where the realizations of the derivative of the field are measured. As expected, both the conditional variance and its slope vanish where the realizations of the field and its derivative are measured. A peak appears at the location where the realization of the derivative of the field vanishes. Far from the measured locations, the conditional variance tends to unconditional variance. For cases 1,2 and 4, the conditional variance is symmetric with respect to x=0.5, since the realizations are symmetric and the field is homogeneous. It is seen that the realizations of the derivative of the field have significant effect on the conditional variance. The examination of Fig.6.1 leads to a remarkable conclusion that conditioning of the derivative of the field significantly alters its properties of the field in the vicinity where the measurements have been made. Therefore, for sufficiently accurate conditional simulation, maximum information should be sought and utilized. It is suggested that realizations of the field as well as of its derivative should be measured wherever possible, in order to obtain more accurate estimation of the field at unmeasured locations.

Example 2: Consider a 3-component random field $G(x)=[G_1(x),G_2(x),G_3(x)]^T$ with zero mean vector. The cross-covariance matrix of $G(x)$ is assumed to be

$$C_{GG}(x_1,x_2) = \begin{bmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 & r_{13}\sigma_1\sigma_3 \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 & r_{23}\sigma_2\sigma_3 \\ r_{13}\sigma_1\sigma_3 & r_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix} \exp(-\xi^2) \tag{62}$$

where $\xi=x_2-x_1$, σ_1, σ_2 and σ_3 are standard deviations, and r_{12}, r_{13} and r_{23} are correlation coefficients. Assume that a set of realizations of $G(x)$ and $G'(x)$ at x=0 are measured to be

$$\begin{aligned} &g_1(0)=1\ , \quad g'_1(0)=0\ , \quad g_2(0)=0.5 \\ &g'_2(0)=1\ , \quad g_3(0)=0.5\ , \quad g'_3(0)=0 \end{aligned} \tag{63}$$

With n=3 and N=M=1, eq.(15) is comprised of six equations governing six kriging weights λ_{ml} and μ_{ml} (m=1,2,3) for each component $G_l(x)$ of $G(x)$. After solving out kriging weights λ_{ml}

and μ_{ml}, the covariance matrix of the error vector is obtained from eq.(22). The conditional means, variances and covariances of the field are then obtained by using eq.(27) and properties of the error vector. For the case $\sigma_1=2$, $\sigma_2=1.5$, $\sigma_3=1$, $r_{12}=0.75$, $r_{13}=0.25$ and $r_{23}=0.5$, the conditional means, variances and covariances of and between components of the field are illustrated in Figs.6.3-6.5. It is seen that the conditional mean of each component starts from the realization value at $x=0$ with the slope equal to the realization value of the derivative of the component. Comparison of the conditional means of $G_1(x)$ and $G_2(x)$, which have the same realization of the field but different realizations of the derivative, shows that the realizations of the derivatives of the field have significant effects on the conditional means. The conditional variances and covariances, however, are independent of the values of the realizations of the field, a conclusion which is identical to one obtained by Hoshiya (1994), and Kameda and Morikawa (1994). Here, however, we arrive at an additional feature: the conditional variances and covariances are also independent of the values of the realizations of the derivatives. The conditional variances and covariances are zero where the realizations of the field are given. As expected, the conditional means and conditional variances and covariances converge to their unconditional counterparts far from the locations the realization of the field or its derivative being measured.

In this connection, some natural questions arise: In the absence of measured derivatives, can one use numerical values of the field to calculate the derivatives? The reply is negative. Indeed, one can use numerical values derived from the measured values of the field, but it would not improve the result. Consider, for example, one-dimensional field $G(x)$. If we know two realizations $g(x_1)$ and $g(x_2)$, the realization of the derivative $G'(x_1)$ can be approximated as

$g'(x_1)=[g(x_2)-g(x_1)]/(x_2-x_1)$. However, the realizations $g(x_1)$ and $g(x_2)$ have already been taken into account to simulation the field. $G(x)$ conditioned by $g(x_1)$, $g(x_2)$ and $g'(x_1)$ and $G(x)$ conditioned by $g(x_1)$ and $g(x_2)$ coincide when x_2 moves to x_1. One practical way to monitor the derivatives, as we have mentioned introduction of this study, is to use the strain/stress sensors as well as the displacement sensors.

As we have demonstrated above, the inclusion of information on derivatives may considerably alter the behavior of simulated results. In the example considered, the conditional variance is reduced when conditioned on the field and its derivatives. The arising pertinent question reads: Is this dependent on the problem considered? We must digress that the conditional covariance and variance are independent of the realizations of both the field and its derivatives. The conditional mean , however, is dependent on the realizations. The conditional variance is reduced when conditioned on the field and its derivatives. As far as the scale of the random field is concerned, it does not alter the property of the conditioned field.

Example 3: To verify effectiveness and accuracy of the proposed procedure for conditional simulation of non-Gaussian random fields, we hereby apply this procedure to simulate beta-distributed non-Gaussian random field. The conditional simulation procedure is utilized to simulate unconditional autocorrelation function and probability distribution function through conditionally generating a number of samples of the random field; each sample is obtained by generating a realization of the random field at a certain location randomly, and then generating realizations at other locations based on the realization at that location and given correlation function by using present conditional simulation procedure, and the results are compared with the targeted ones. The conditional mathematical expectation and variances are also calculated when realizations of the random fields are specified at several locations.

Let us assume that $N(x)$ is a one-dimensional random field and its N realizations $n(x_i)$ are given at locations x_1, x_2, ..., and x_n. $N(x)$ possesses a lognormal distribution, its probability density function has the form of

$$F_{N(x)}(u) = \frac{1}{u\sigma\sqrt{2\pi}}\exp[-\frac{(\log u - a)^2}{2\sigma^2}] \tag{64}$$

where a and σ are two constants. In numerical implementation, a=0 and σ=0.5 or 1, which gives a highly skewed lognormal distribution. $N(x)$ has mean value and variance as follows, respectively,

$$\begin{aligned} m &= E[N(x)] = \exp[a^2 + \frac{\sigma^2}{2}] \\ \sigma_N^2 &= Var[N(x)] = \exp[2(a^2+\sigma^2)] - \exp[2a^2+\sigma^2] \end{aligned} \tag{65}$$

The correlation function of $N(x)$ is assumed as

$$R_{NN}(\xi) = \sigma_N^2 e^{-|\xi|} \tag{66}$$

where ξ is the distance between two locations. For lognormal distribution case, the mapping between non-Gaussian and Gaussian fields of eq(37) and eq.(38) have explicit expressions as

$$\begin{aligned} g(x_i) &= \frac{\sigma_H}{\sigma}\{\log[n(x_i)] - a\} \\ n(x_i) &= \exp[\frac{\sigma}{\sigma_H}g(x_i) + a] \end{aligned} \tag{67}$$

Fig.6.6 shows a line divided into 12 equal segments with length 0.5. To verify accuracy of the proposed conditional simulation procedure, we first simulate unconditional probability distribution function and autocorrelation function of the random field by conditional simulation technique. We first simulate random field at one location $x_l=x_1$, namely, $N(x_l)$, which is a random variable with given probability distribution function $F_{N(x)}(u)$. Then we apply proposed conditional simulation procedure to simulate a set of realizations $n_k(x_r)$ of $N(x)$ at all other locations, for each realization $n_k(x_1)$ of $N(x_1)$. Based on simulated samples, probability distribution function and autocorrelation function can be estimated. If the simulated autocorrelation function does not match the targeted one, the autocorrelation function of assumed Gaussian field should be updated by eq.(42) to re-generate samples of Gaussian random field $G(x)$ until the simulated autocorrelation function of non-Gaussian field matches the targeted one. Fig.6.7 illustrates simulated and targeted probability density functions of the lognormal random field $N(x)$ at location x_5, with 3,000 samples. The simulated result has a good agreement with the targeted probability distribution function. Fig.6.8 illustrates the simulated and targeted autovariance functions. It is seen that the simulated function agrees with the target function very well after a single iteration for the case σ=0.5. However, for the case σ=1, ten iterations turn out to be needed for convergence.

Fig.6.9 and Fig.6.10 give the (simulated) conditional mathematical expectations and conditional standard deviations of lognormal random field $N(x)$ for cases σ=0.5 and 1, respectively. Two conditioning realizations at recorded locations x_1=0 and x_7=3 are specified to be $n(x_1)$=1.64 and $n(x_7)$=0.483 for σ=0.5; $n(x_1)$=0.525 and $n(x_7)$=1.38 for σ=1, respectively. It is seen that in the vicinity of recorded locations (at which realizations are given) the conditional mean approaches the value of the realization itself, whereas the conditional variance tends to zero. Far apart from recorded locations, the conditional mean and variance converge to, respectively, unconditional mean and variance, as it should be. It is apparent that the conditional mean depends on values of given realizations since all the realizations should pass through the realization values. This feature agrees with the conclusion obtained by Kameda and Morikawa (1993) obtained analytically, for Gaussian random fields.

in order to see the effectiveness of the constant k in eq.(42), several values of k are chosen to check the convergence of iteration procedure. Table.1 gives number of iterations needed to satisfy the requirement of the mapping accuracy criterion, namely $e^M(\xi)/R_{NN}^{(T)}(0) \leq 5\%$. Based on our numerical experience, for most non-Gaussian fields (for examples, lognormal distribution with $\sigma=0.5$ as well as beta, exponential, truncated Gaussian distributions), $k=1$ gives satisfactory results. However, in some cases (for example lognormal distribution with $\sigma=1$), $k=1$ causes divergence of iteration procedure.

Table.1 Number of iterations needed to meet requirement of the mapping error

($e^M(\xi)/R_{NN}^{(T)}(0) \leq 5\%$, "-" represents divergence)

k	0.4	0.6	1
$\sigma=0.5$	1	1	1
$\sigma=1$	10	-	-

Example 4: The rectangle plate under concentrated force

Let us re-analyze example 3 in Chapter 5 - a rectangle under a concentrated force acting on the right-upper corner. The rectangle has the width of 2 and the height of 1. The Young's modulus of the material is assumed to be a random field, $E(x)=E_0\ [1+\ \alpha(x)\ ]$, where E_0 is the mean value of $E(x)$. The normalized random field $\alpha(x)$ has the zero mean. Differently from the example 3 in Chapter 5, the field $\alpha(x)$ is assumed to have only the one-dimensional probability density function and the correlation function available, and its two-dimensional probability density function is unknown. In calculation, $\alpha(x)$ is assumed to vary uniformly in interval [-a,a] and the correlation function of $\alpha(x)$ is assumed to be exponential, as given in eq.(67) in Chapter 5. The mean value of the Young's modulus E_0 is taken to be $1.0\text{x}10^4$.

We use three different approaches to calculate the mean and the variance-covariance of the displacements, namely, (i) the simulation technique incorporated with the element-level flexibility based FEM for stochastic structures, developed in Chapter 5; (ii) the simulation technique and the conventional deterministic FEM; and (iii) the first-order perturbation FEM for stochastic structures. In first approach, we generate a set of samples of the random field $\alpha(x)$, based on the simulation technique for non-Gaussian random fields proposed in section 2. The generated samples are used to calculate the mean and variance-covariance of element-level flexibilities, which will then be used in the element-level flexibility based FEM for stochastic structures to obtain the mean and variance-covariance of the displacement. There is only one finite element analysis. In second approach, for each generated sample of the random field $\alpha(x)$, the deterministic finite element method is used to obtain one solution of the displacement. The mean and variance-covariance of the displacement are then obtained by averaging all set of the solutions, corresponding the whole set of the samples. The number of finite element analysis conducted equals to the number of the samples. The third approach applies the conventional first-order perturbation-based FEM for stochastic structures to get the first approximation of the mean and variance-covariance of the displacement.

Fig.6.11 and 6.12 portray the mean and variance solutions of the vertical displacement at the right-upper corner, respectively, obtained by the three approaches. One can see that the solution obtained by the first approach and the solution obtained by the second approach coincide pretty well, for any value of the coefficient of variation, while the perturbation based solution is acceptable only for small values of the coefficient of variation, a conclusion we have achieved several times in this study. It can also be concluded from Fig.6.11 and 6.12 that the element-level flexibility based FEM for stochastic structures is proved to be very efficient and accuracy, even though the inner-forces corresponding to the mean Young's modulus is applied as the real inner forces. The incorporation of the simulation technique can reduce the requirement on data information of the stochastic parameters by the non-perturbative FEM for stochastic, which usually needs to know the two-dimensional probability density function of the stochastic parameters. Also, compared to the conventional simulation based deterministic FEM, the approach of incorporating the simulation technique and the non-perturbative FEM for stochastic structures consumes much less computational time. For example, in this calculation where 6000 samples and 3 by 5 mesh were adopted and the computing was conducted in VAX 6000-320 machine, the incorporation approach spent 2 minute 50 second CPU time, while the conventional deterministic FEM and simulation approach spent 24 minute 18 second CPU time. For a finer mesh, their difference in CPU time consuming becomes even more significant.

To conclude, the conditional simulation developed in this Chaptercan be applied to multi-variate Gaussian random field and non-Gaussian random field. The incorporation of conditional simulation technique and non-perturbative finite element method can solve the conditional stochastic structural problems which possess only limited statistical information (first and second moments) on the stochastic parameters.

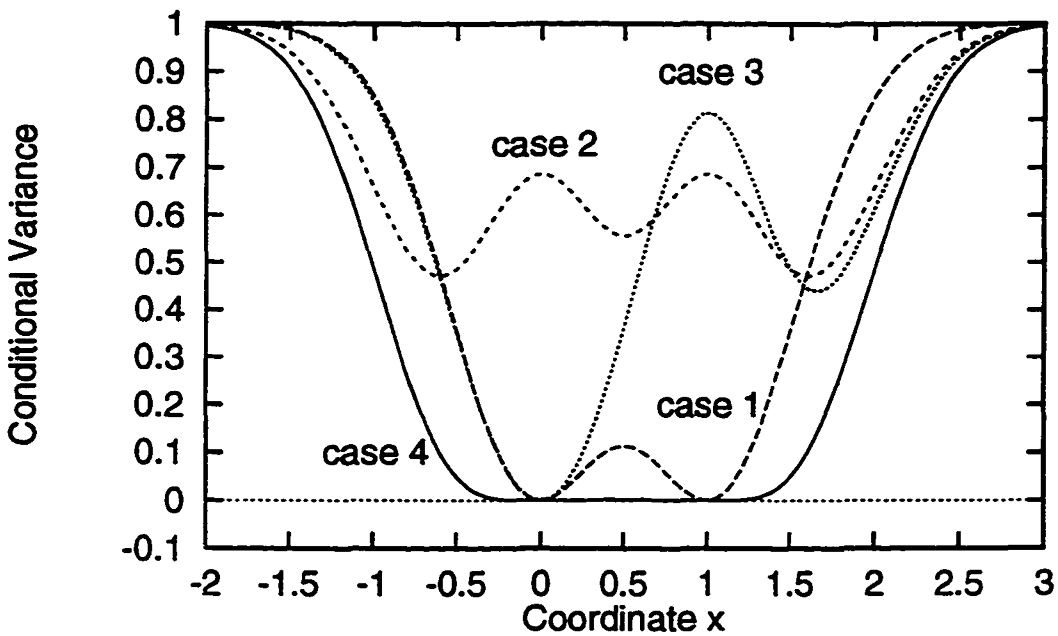

Fig.6.1: The conditional variance of the field $G(x)$, normalized by the unconditional variance σ_0^2.

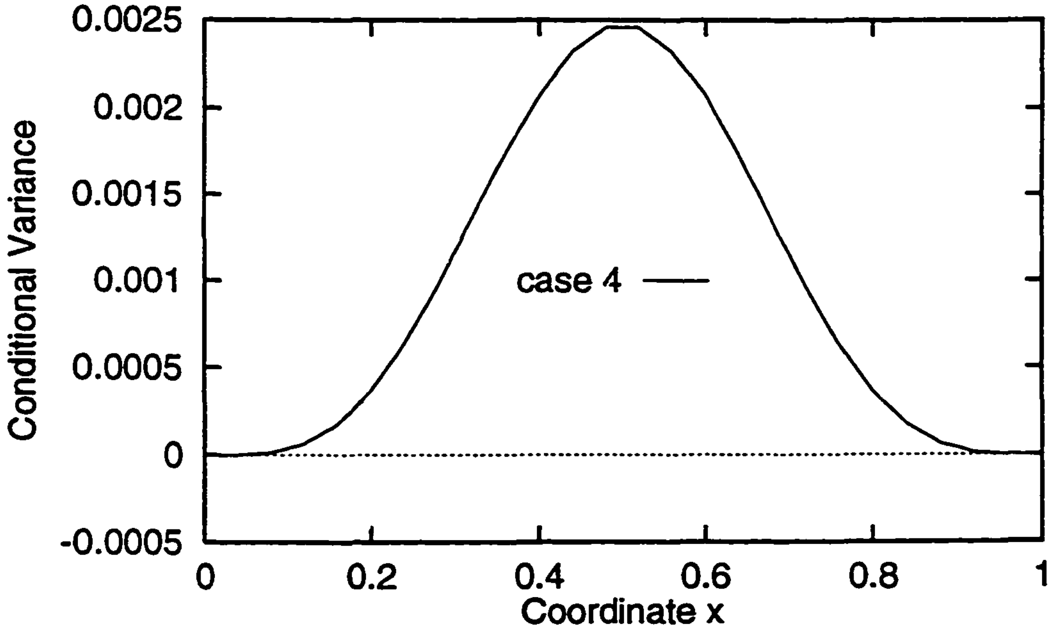

Fig.6.2: Magnification of the conditional variance of the field $G(x)$ for case 4 in interval [0,1], normalized by the unconditional variance σ_0^2

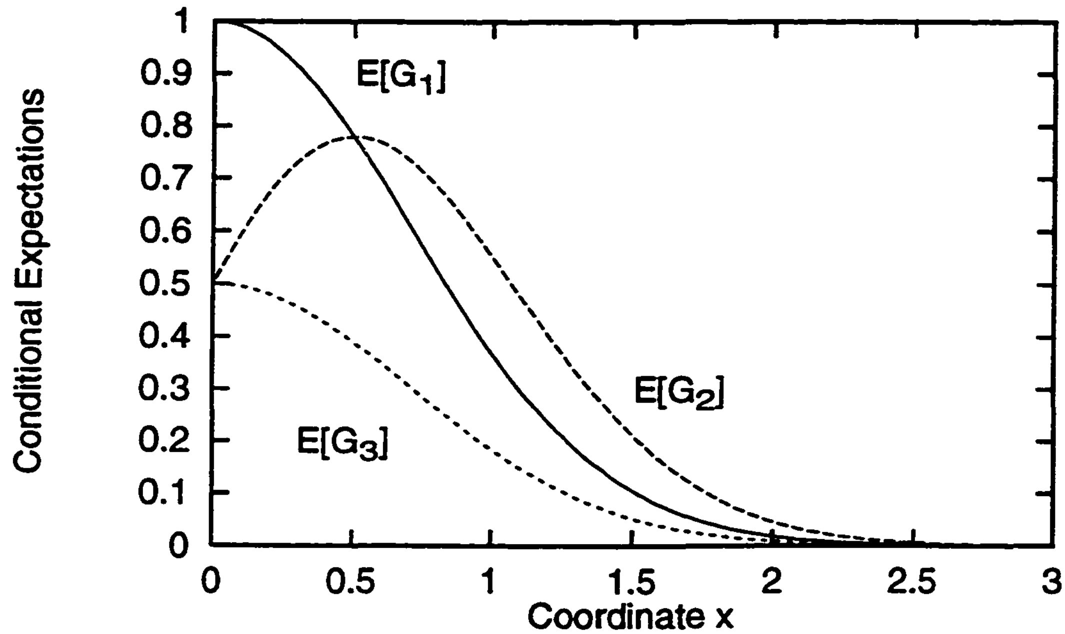

Fig.6.3: Conditional means of the 3-variate field with realizations specified at x=0

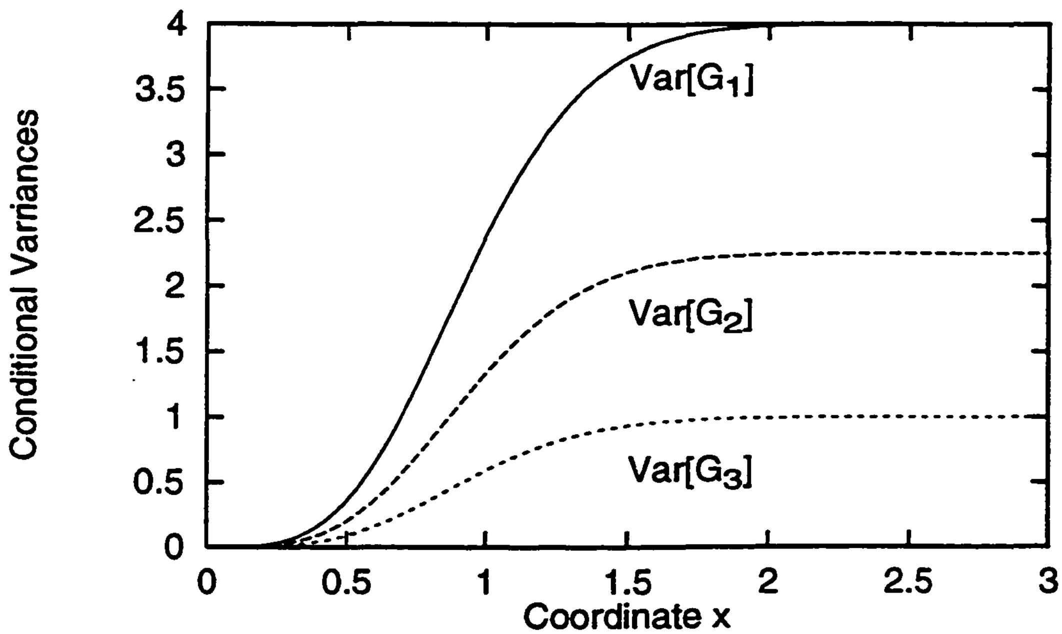

Fig.6.4: Conditional variances of the 3-variate field with realizations specified at x=0

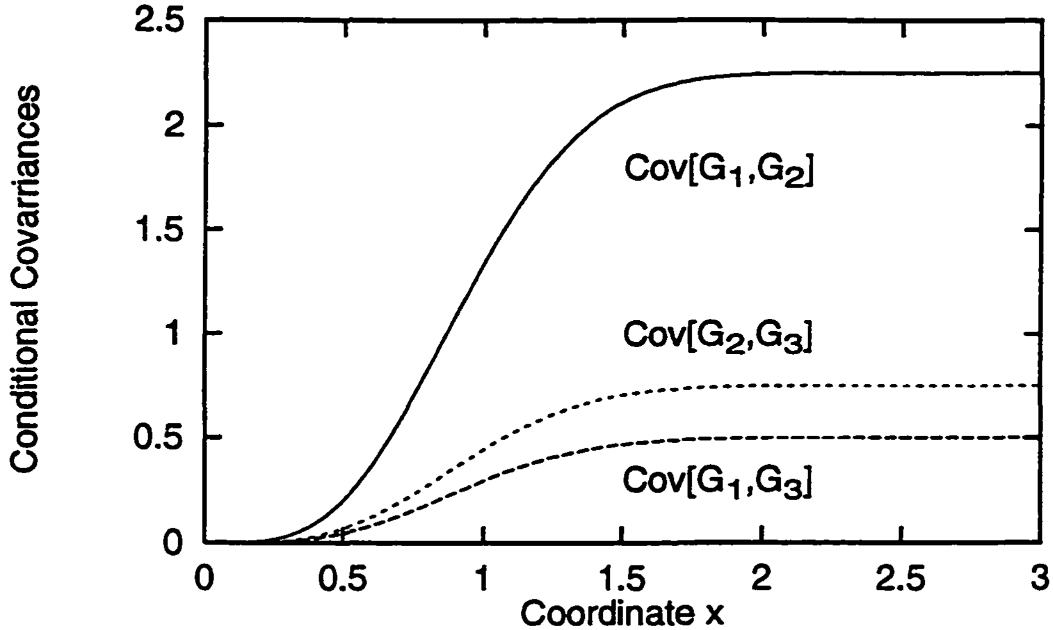

Fig.6.5: Conditional covariances of the 3-variate field with realizations specified at x=0

Fig.6.6: A line divided into 12 equal segments with length 0.5

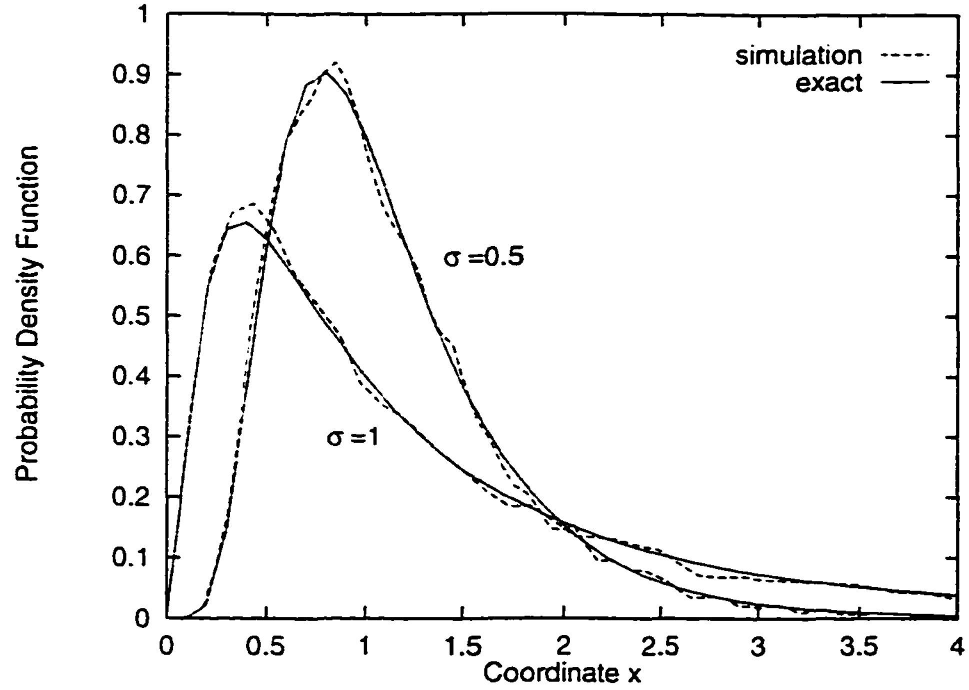

Fig.6.7: Comparison of simulated and exact probability density functions of lognormal random field (3000 samples)

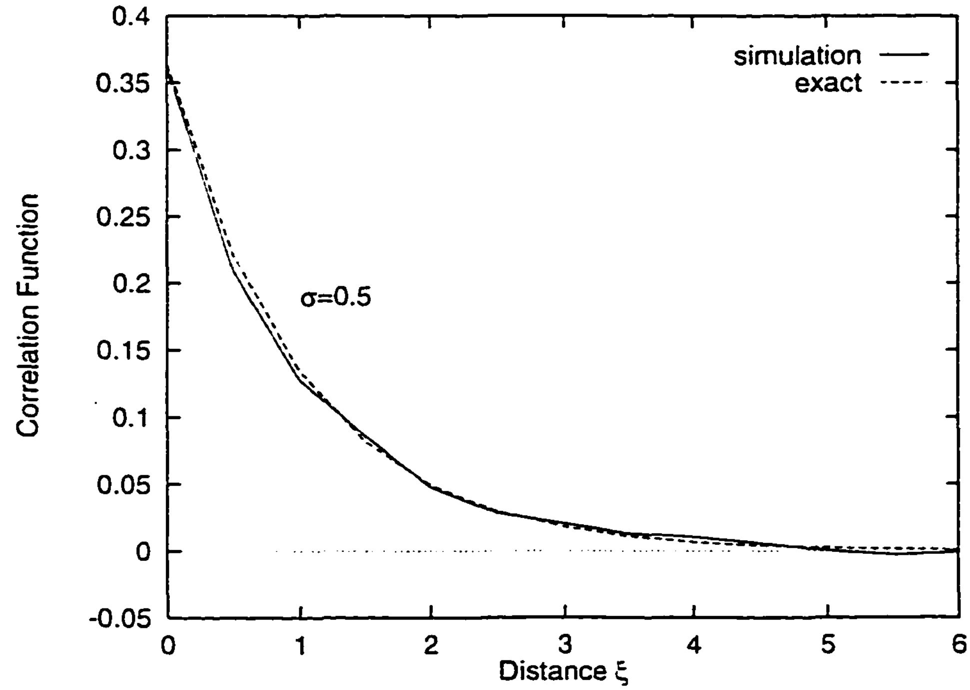

Fig.6.8(a): Comparison of simulated and exact correlation functions of lognormal random field with σ=0.5 (3000 samples)

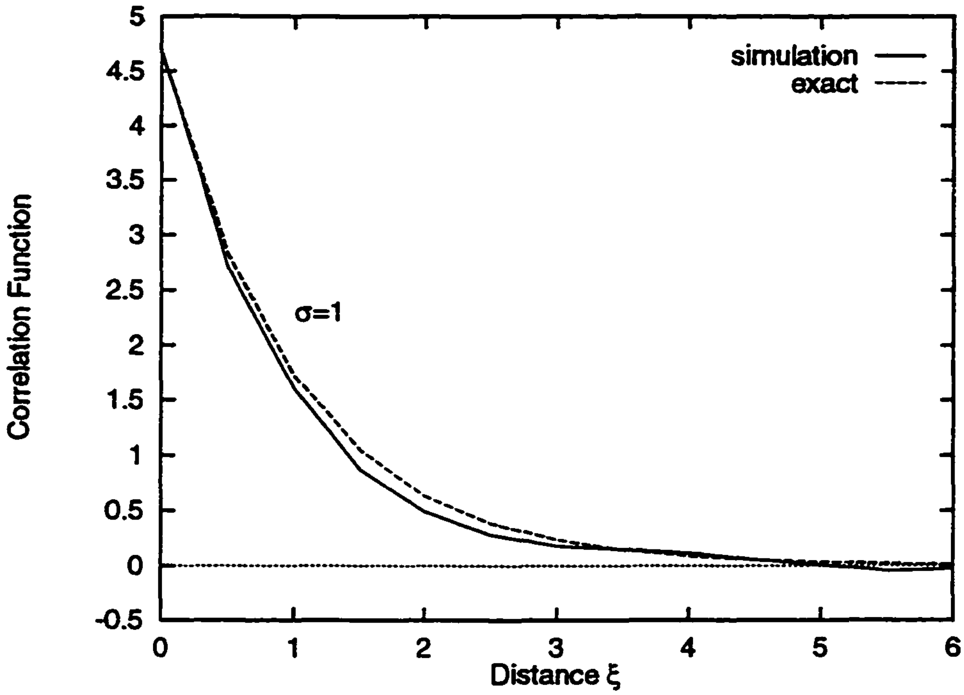

Fig.6.8(b): Comparison of simulated and exact correlation functions of lognormal random field with σ=1 (3000 samples)

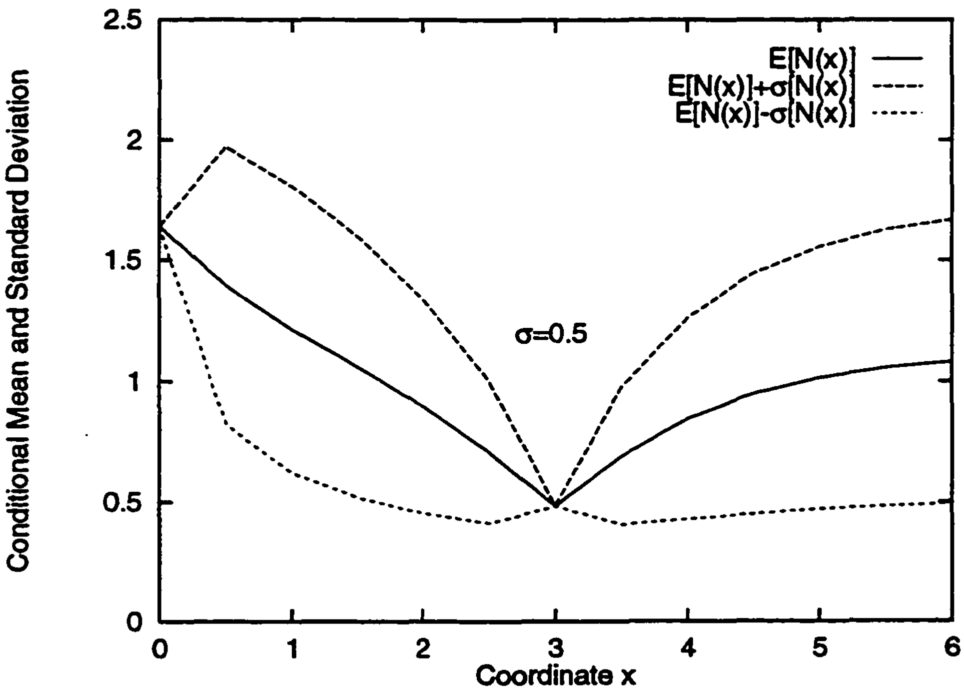

Fig.6.9: Variation of conditional probabilistis characteristics E[N(x)], E[N(x)]±σ[N(x)] (σ=0.5, E[N(x)]=conditional mean; σ[N(x] =conditional standard deviation)

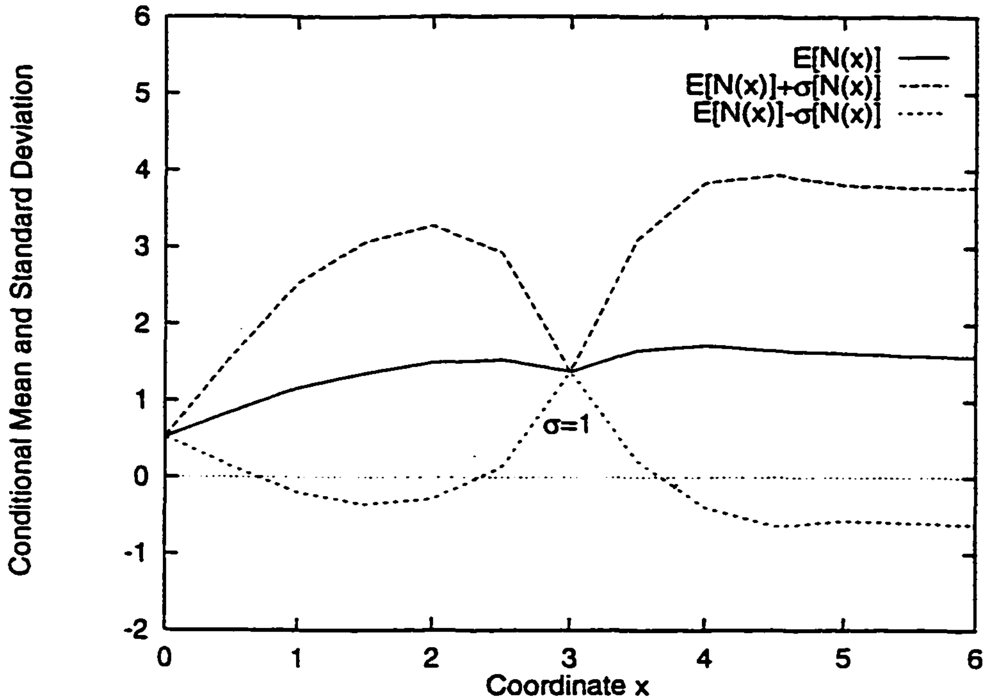

Fig.6.10: Variation of conditional probabilistic characteristics E{N(x)], E[N(x)]±σ[N(x)] (σ=1, E[N(x)]=conditional mean: σ[N(x)]] =conditional standard deviation)

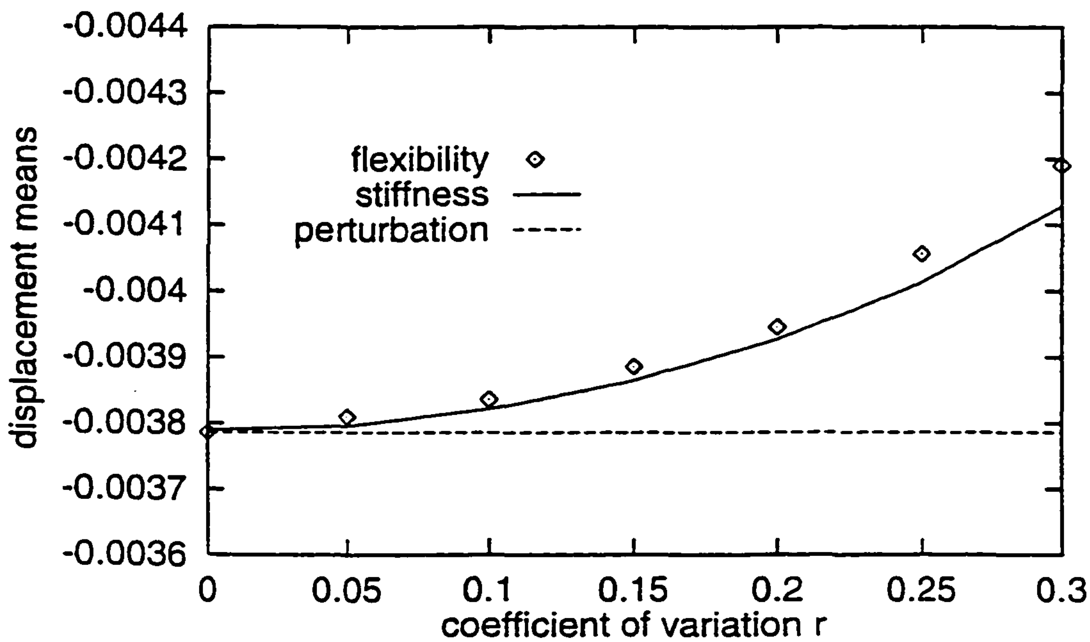

Fig.6.11: Mean of vertical displacement at right-upper corner of the rectangle vs. coefficient of variation

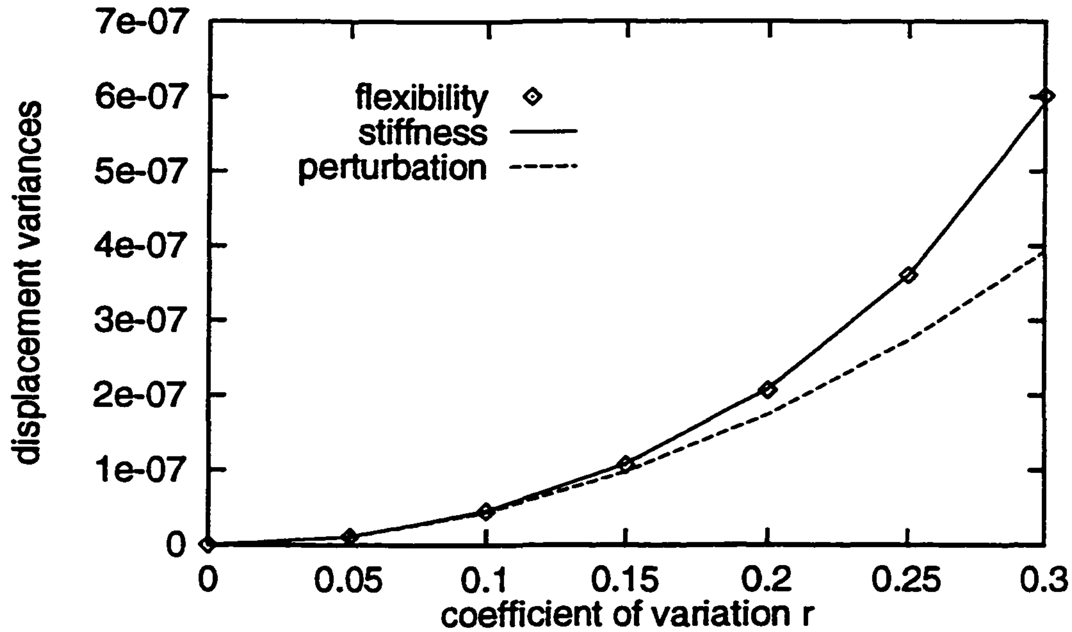

Fig.6.12: Vraiance of vertical displacement at right-upper corner of the rectangle vs. coefficient of variation

Chapter 7: Conclusions

In this dissertation, we have first provided with a reviewof the existing finite element methods for stochastic structures. The exisitng stochastic FEMs, as mentioned already, are mainly based on either the perturbation technique or the series expansion methods. Chapter 2 reviewed the first-order and second perturbation based stochastic finite element method, the first-order and second order Neumann expansion based stochastic finite element method and the stochastic finite element method based on the homogeneous chaos. We have noted significant drawbacks of conventional approaches. Chapter 2 also provided with the improved first-order perturbation-based stochastic finite element method. The improved method dramatically increases the accuracy of the first-order perturbation method, while at the same time it requires the minimum probabilistic information of the stochastic parameters, namely the first and second-order moments.

The finite element method for stochastic structures based on the exact inverse of the stiffness matrix has been presented in Chapter 3. For simple structures such as bars in extension and shear beams were considered. The fact that the exact inverse of the global stiffness matrix is directly obtainable gives possibility of solving the displacement explicitly in terms of the stochastic parameters and applied loads, and therefore obtaining the mean, variance and covariances of the displacement. The difficulty of the stochastic finite element method based on direct exact inverse of the stiffness matrix is the nonuniform obtainability of the direct inverse of the stiffness matrix.

The application of an alternative approach to construct the stiffness matrix for bending of general beams can partially overcome the difficulty of the exact inverse of the stiffness matrix. Chapter 3 also presentsed the stochastic finite element method based on the diagonalization of the element stiffness matrix, which is applicable to the bending problems of both statically determinate and statically indeterminate beams. The diagonalization of the element stiffness matrix (due to Fuchs for the deterministic setting, not in the FEM context) enables to formulate the generalized strain-displacement relationship, constitutional law and equilibrium equations, the global stiffness matrix is then constructed as the product of two constant matrices and one diagonal matrix containing stochastic parameters. The global stiffness matrix as the product of three matrices is easily inverted. Therefore, the mean, variance and covariances of the displacement can be calculated.

For statically determinate beams, we derived the uncoupled governing equations for the mean displacement function and covariance function of the displacement and attendant boundary conditions. Furthermore, we proposed two separate variational principles for the mean displacement and covariance function of the displacement, respectively. Based on the established variational principles, the stochastic-version Galerkin method and Ritz method are proposed to obtain the approximate solutions for the mean displacement and covariance. Chapter 4 derived the uncoupled governing equations directly for the mean and covariance functions of the displacement and their corresponding variational principles. As a result, stochastic Galerkin method and Ritz method have been proposed based on either the variational principles or the governing equations. Both the variational principles and stochastic Galerkin method or Ritz method are free from perturbative nature, and are applicable to *any* value of coefficients of variation of stochastic parameters/fields and possess the same convergence property as the conventional Galerkin or Ritz method.

Chapter 4 also proposed the finite element method for stochastic beams based on the variational principles derived for the mean displacement and the covariance function. The new finite element method was directly formulated with respect to the mean displacement or the covariance function of the displacement. To avoid the requirement of high-order polynomials in constructing the shape functions, a modified variational principle for the covariance function of the displacement is presented. A four-node element for the variance function is derived based on the modified variational principle.

It has been understood that the reason for adopting perturbation technique in the FEM lies in the difficulty of the probabilistic coupling of the response and the stochastic input parameters. If we can in some manner resolve the problem of coupling between the response and the input parameters, we then, at least conceptually, obtain the moments of the response in terms of the uncertain parameters. Chapter 5 proposed an element level flexibility-based finite element method for stochastic structures. The unconventional step in this formulation is to divide the element-level finite element equilibrium equation by the element stiffness so that the reciprocal of the stiffness (the flexibility) would appear at the right side of the equation, and thus becomes uncoupled from the unknown displacement. The mean and covariance of the displacement are then obtained in terms of the mean and covariance of the flexibility. The new formulation is exemplied through the bending of beams with stochastic stiffness. The formulation is also derived for plane stress or strain elasticity problems which possess stochastic material properties.

The flexibility-based FEM, as well as variational principle based Galerkin, Ritz or finite element method, requires the mean and covariance of the flexibility. However, it is assumed that the stiffness is the initiative stochastic parameter whose mean and covariance are prescribed. To obtain the mean and covariance of the flexibility, we need the information on one and two dimensional probability density functions of the stiffness; whereas the two dimensional probability density function is rarely known. Chapter 6 addressed this situation by using the simulation technique of the random field. Indeed, the conditional simulation of random field itself has become an alluring research field in recent years because of its application in urban earthquake monitoring systems. Chapter 6 developed the conditional simulation technique to multi-variate Gaussian fields and non-Gaussian fields. Furthermore, by giving the one dimensional probability density function and the covariance function of the stochastic stiffness, a set of samples can be generated by the developed simulation technique and are then used to computer the mean and covariance of the flexibility, which is incorporated with the proposed non-perturbative finite element ,methods for stochastic structures to compute the mean, variance and covariances of the displacement.

This dissertation concentrates on the mean and covariance analysis of displacements of structures. The proposed stochastic finite element methods are all free from the perturbation nature or series expansion. Therefore, the proposed methods are applicable to large values of coefficient of variation of the input material or geometrical parameters. In each chapter, necessary numerical results and discussion were also demonstrated.

Appendix I: Proof of Boundary Conditions in Eqs.(11-13) of Chapter 4

To prove boundary conditions of eqs.(11-13), we just consider cases of free and clamped ends. The appropriate combination of boundary conditions for these two cases will yield other boundary conditions, such as those pertinent to simple supports.

Assume first that the beam is clamped at end x=0. The boundary conditions for the displacement $w(x)$ are

$$w(0)=0\ , \qquad w'(0)=\frac{dw}{dx}\Big|_{x=0}=0 \tag{74}$$

By taking expectation operator and noting that the expectation operator and differential operator are inter-changeable, we get

$$\bar{w}(0)=0\ , \qquad \bar{w}'(0)=\frac{d\bar{w}}{dx}\Big|_{x=0}=0 \tag{75}$$

Then, for arbitrary displacement $w(y)$, we get

$$\begin{aligned}
C(0,y) &= E\{[w(0)-\bar{w}(0)][w(y)-\bar{w}(y)]\}=0\\
C_x'(0,y) &= \frac{\partial}{\partial x}E\{[w(x)-\bar{w}(x)][w(y)-\bar{w}(y)]\}\ \Big|_{x=0}\\
&= E\{[\frac{dw(x)}{dx}-\frac{d\bar{w}(x)}{dx}][w(y)-\bar{w}(y)]\}\ \Big|_{x=0}\\
&= E\{[w'(0)-\bar{w}'(0)][w(y)-\bar{w}(y)]\}=0
\end{aligned} \tag{76}$$

Similarly,

$$C(x,0)=0$$
$$C_y'(x,0)=0 \tag{77}$$

Assume now that the beam is free at end x=0 but subjected to prescribed moment $\bar{M}$ and concentrated force $\bar{Q}$. The boundary conditions for displacement $w(x)$ are

$$EI\frac{d^2w}{dx^2}=\bar{M}\ , \qquad \frac{d}{dx}[EI\frac{d^2w}{dx^2}]=\bar{Q} \tag{78}$$

we re-write the first condition and take expectation to obtain immediately

$$D_0\frac{d^2\bar{w}}{dx^2}=\bar{M} \tag{79}$$

and

$$\frac{d^2[w-\bar{w}]}{dx^2}=\bar{M}[\frac{1}{EI}-\frac{1}{D_0}] \tag{80}$$

Subtracting eq.(4) from eq.(2) and then multiplying by eq.(80), we get

$$\frac{d^2[w(y)-\bar{w}(y)]}{dy^2}\frac{d^2[w-\bar{w}]}{dx^2}=m(y)\bar{M}[\frac{1}{EI(x)}-\frac{1}{D_0(x)}][\frac{1}{EI(y)}-\frac{1}{D_0(y)}] \tag{81}$$

By taking expectation, we obtain

$$D_1(0,y)\frac{\partial^4 C}{\partial x^2 \partial y^2}|_{x=0} = m(y)\bar{M} \tag{82}$$

Similarly

$$D_1(x,0)\frac{\partial^4 C}{\partial x^2 \partial y^2}|_{y=0} = m(x)\bar{M} \tag{83}$$

Re-writing eq.(2) and bearing in mind that $M_0 = \bar{M}$ and $Q_0 = \bar{Q}$ for free ends, we obtain

$$\frac{d^2 w}{dx^2} = \frac{1}{EI(x)}[\bar{Q}x + \bar{M} + \int_0^x\int_0^v q(u)dudv] \tag{84}$$

Taking expectation of eq.(84), multiplying it by D_0 and differentiating the result and then letting x=0, we get

$$\frac{d}{dx}[D_0\frac{d^2\bar{w}}{dx^2}]|_{x=0} = \bar{Q} \tag{85}$$

Taking expectation of eq.(7), multiplying by D_1 and differentiating the result and then letting x=0 yield

$$\frac{\partial}{\partial x}[D_1\frac{\partial^4 C}{\partial x^2 \partial y^2}]|_{x=0} = m(y)\bar{Q} \tag{86}$$

Analogously, the boundary condition at y=0 reads

$$\frac{\partial}{\partial y}[D_1\frac{\partial^4 C}{\partial x^2 \partial y^2}]|_{y=0} = m(x)\bar{Q} \tag{87}$$

Appendix II: Proof of Functional π_2 in Eq.(15) of Chapter 4

The variation of the first integral in the functional π_2 of eq.(15) reads

$$
\begin{aligned}
\delta I &= \int_0^L \int_0^L D_1 \frac{\partial^4 C}{\partial x^2 d\partial^2} \frac{\partial^4 \delta C}{\partial x^2 \partial y^2} dxdy \\
&= \int_0^L D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2} \frac{\partial^3 \delta C}{\partial x \partial y^2} \Big|_{x=0}^{x=L} dy - \int_0^L \frac{\partial}{\partial x}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial^2 \delta C}{\partial x \partial y} \Big|_{y=0}^{y=L} dx \\
&+ \int_0^L \frac{\partial^2}{\partial x \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial y} \Big|_{x=0}^{x=L} dy - \int_0^L \int_0^L \frac{\partial^3}{\partial x^2 \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial y} dxdy \\
&= \int_0^L \int_0^L \frac{\partial^4}{\partial x^2 \partial y^2}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \delta C dxdy \\
&+ \{ \int_0^L [\frac{\partial^2}{\partial y^2}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial x} - \frac{\partial^3}{\partial x \partial y^2}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \delta C] dy \} \Big|_{x=0}^{x=L} \\
&+ \{ \int_0^L [\frac{\partial^2}{\partial x^2}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial y} - \frac{\partial^3}{\partial x^2 \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \delta C] dx \} \Big|_{y=0}^{y=L} \\
&+ [D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2} \frac{\partial^2 \delta C}{\partial x \partial y} - \frac{\partial}{\partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial x} - \frac{\partial}{\partial x}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \frac{\partial \delta C}{\partial y} + \frac{\partial^2}{\partial x \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) \delta C] \Big|_{y=0}^{y=L} \Big|_{x=0}^{x=L}
\end{aligned}
\tag{88}
$$

The variation of the functional π_2 is then

$$
\begin{aligned}
\delta\pi_2 = & \int_0^L\int_0^L [\frac{\partial^4}{\partial x^2\partial y^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - q(x)q(y)]\delta C dxdy \\
& + \{\int_0^L \{[\frac{\partial^2}{\partial y^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}q(y)]\frac{\partial\delta C}{\partial x} - [\frac{\partial^3}{\partial x\partial y^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{Q}q(y)]\delta C\}dy\}\Big|_{x=0}^{x=L} \\
& + \{\int_0^L \{[\frac{\partial^2}{\partial x^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}q(x)]\frac{\partial\delta C}{\partial y} - [\frac{\partial^3}{\partial x^2\partial y}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{Q}q(x)]\delta C\}dx\}\Big|_{y=0}^{y=L} \\
& + \{(D_1\frac{\partial^4 C}{\partial x^2\partial y^2} - \bar{M}\bar{M})\frac{\partial^2\delta C}{\partial x\partial y} - [\frac{\partial}{\partial y}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}\bar{Q}]\frac{\partial\delta C}{\partial x} \\
& - [\frac{\partial}{\partial x}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}\bar{Q}]\frac{\partial\delta C}{\partial y} + [\frac{\partial^2}{\partial x\partial y}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{Q}\bar{Q}]\delta C\}\Big|_{y=0}^{y=L}\Big|_{x=0}^{x=L}
\end{aligned} \tag{89}
$$

The stationarity condition of π_2 requires:

(i) the governing equation

$$\frac{\partial^4}{\partial x^2\partial y^2}(D_1\frac{\partial^4}{\partial x^2\partial y^2}) = q(x)q(y), \qquad \forall x,y \tag{90}$$

(ii) boundary conditions at sides

$$[\frac{\partial^2}{\partial y^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}q(y)]\delta(\frac{\partial C}{\partial x}) = 0\ , \qquad \forall\ y\ \ at\ \ x=0,L \tag{91}$$

$$[\frac{\partial^3}{\partial x\partial y^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{Q}q(y)]\delta C = 0\ , \qquad \forall\ y\ \ at\ \ x=0,L \tag{92}$$

$$[\frac{\partial^2}{\partial x^2}(D_1\frac{\partial^4 C}{\partial x^2\partial y^2}) - \bar{M}q(x)]\delta(\frac{\partial C}{\partial y}) = 0\ , \qquad \forall\ x\ \ at\ \ y=0,L \tag{93}$$

$$[\frac{\partial^3}{\partial x^2 \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) - \bar{Q}q(x)]\delta C = 0 \ , \qquad \forall\ x \quad at \quad y=0,L \tag{94}$$

(iii) and boundary conditions at corners

$$(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2} - \bar{M}\bar{M})\delta(\frac{\partial^2 C}{\partial x \partial y}) = 0 \ , \qquad at \quad x=0,L \ ; \quad y=0,L \tag{95}$$

$$[\frac{\partial}{\partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) - \bar{M}\bar{Q}]\delta(\frac{\partial C}{\partial x}) = 0 \ , \qquad at \quad x=0,L \ ; \quad y=0,L \tag{96}$$

$$[\frac{\partial}{\partial x}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) - \bar{M}\bar{Q}]\delta(\frac{\partial C}{\partial y}) = 0 \ , \qquad at \quad x=0,L \ ; \quad y=0,L \tag{97}$$

$$[\frac{\partial^2}{\partial x \partial y}(D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}) - \bar{Q}\bar{Q}]\delta C = 0 \ , \qquad at \quad x=0,L \ ; \quad y=0,L \tag{98}$$

The boundary conditions both at sides and corners can be combined to be

$$\begin{aligned} &\frac{\partial C}{\partial x} = 0 \quad or \quad D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2} = \bar{M} m(y) \ , \quad for \quad x = 0,\ L \\ &C = 0 \quad or \quad \frac{\partial}{\partial x}[D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}] = \bar{Q} m(y) \ , \quad for \quad x = 0,\ L \end{aligned} \tag{99}$$

$$\begin{aligned} &\frac{\partial C}{\partial y} = 0 \quad or \quad D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2} = \bar{M} m(x) \ , \quad for \quad y = 0,\ L \\ &C = 0 \quad or \quad \frac{\partial}{\partial y}[D_1 \frac{\partial^4 C}{\partial x^2 \partial y^2}] = \bar{Q} m(x) \ , \quad for \quad y = 0,\ L \end{aligned} \tag{100}$$

It can be further proved that the functional behaves the minimum property for the exact solutions. Assume that C_0 is the exact solution of equation (10) pertinent to boundary conditions in eqs.(12,13). Denoting $C=C_0+C_1$ and substituting it into eq.(15), we have

$$\pi_2(C)=\pi_2(C_0+C_1)=\int_0^L\int_0^L\{\frac{1}{2}D_1(x,y)[\frac{\partial^4(C_0+C_1)}{\partial x^2\partial y^2}]^2-q(x)q(y)(C_0+C_1)\}d$$

$$-\{\int_0^L[\bar{M}\frac{\partial(C_0+C_1)}{\partial x}-\bar{Q}(C_0+C_1)]q(y)dy\}\Big|_{x=0}^{x=L}-\{\int_0^L[\bar{M}\frac{\partial(C_0+C_1)}{\partial y}-\bar{Q}(C_0+C_1)$$

$$-[\bar{M}\bar{M}\frac{\partial^2(C_0+C_1)}{\partial x\partial y}-\bar{M}\bar{Q}\frac{\partial(C_0+C_1)}{\partial x}-\bar{M}\bar{Q}\frac{\partial(C_0+C_1)}{\partial y}+\bar{Q}\bar{Q}(C_0+C_1)]\Big|_{y=}^{y=}$$

$$=\pi_2(C_0)+\pi_2(C_1)+\int_0^L\int_0^L D_1(x,y)\frac{\partial^4C_0}{\partial x^2\partial y^2}\frac{\partial^4C_1}{\partial x^2\partial y^2}dxdy$$

(101)

Comparing the integral term in the above equation and the integral δI in eq.(88), we can easily obtain, by replacing δC and C in eq.(88) with C_1 and C_0, respectively,

$$J=\int_0^L\int_0^L D_1\frac{\partial^4C_0}{\partial x^2\partial^2 y}\frac{\partial^4C_1}{\partial x^2\partial y^2}dxdy$$

$$=\int_0^L\int_0^L\frac{\partial^4}{\partial x^2\partial y^2}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})C_1dxdy$$

$$+\{\int_0^L[\frac{\partial^2}{\partial y^2}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})\frac{\partial C_1}{\partial x}-\frac{\partial^3}{\partial x\partial y^2}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})C_1]dy\}\Big|_{x=0}^{x=L}$$

$$+\{\int_0^L[\frac{\partial^2}{\partial x^2}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})\frac{\partial C_1}{\partial y}-\frac{\partial^3}{\partial x^2\partial y}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})C_1]dx\}\Big|_{y=0}^{y=L}$$

$$+[D_1\frac{\partial^4C_0}{\partial x^2\partial y^2}\frac{\partial^2C_1}{\partial x\partial y}-\frac{\partial}{\partial y}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})\frac{\partial C_1}{\partial x}-\frac{\partial}{\partial x}(D_1\frac{\partial^4C_1}{\partial x^2\partial y^2})\frac{\partial C_1}{\partial y}+\frac{\partial^2}{\partial x\partial y}(D_1\frac{\partial^4C_0}{\partial x^2\partial y^2})C_1]\Big|_{y=0}^{y=L}\Big|_{x=0}^{x=L}$$

(102)

Bearing in mind that C_0 satisfies the equation (10) and boundary conditions in eqs(12,13), therefore

$$J = [\bar{M}\bar{M}\frac{\partial^2 C_1}{\partial x \partial y} - \bar{M}\bar{Q}\frac{\partial C_1}{\partial x} - \bar{M}\bar{Q}\frac{\partial C_1}{\partial y} + \bar{Q}\bar{Q}C_1]\Big|_{x=0}^{x=L}\Big|_{y=0}^{y=L} + \int_0^L [\bar{M}\frac{\partial C_1}{\partial x} - \bar{Q}C_1]q(y)dy\Big|_{x=0}^{x=L}$$

$$+ \int_0^L [\bar{M}\frac{\partial C_1}{\partial y} - \bar{Q}C_1]q(x)dx\Big|_{y=0}^{y=L} + \int_0^L\int_0^L \frac{\partial^4}{\partial x^2 \partial y^2}[D_1(x,y)\frac{\partial^4 C_0}{\partial x^2 \partial y^2}]C_1 dxdy \tag{103}$$

$$= \int_0^L\int_0^L \frac{1}{2}D_1(x,y)(\frac{\partial^4 C_1}{\partial x^2 \partial y^2})^2 dxdy - \pi_2(C_1)$$

Substituting it back into eq.(101), we obtain

$$\pi_2(C_0 + C_1) = \pi_2(C_0) + \int_0^L\int_0^L \frac{1}{2}D_1(x,y)(\frac{\partial^4 C_1}{\partial x^2 \partial y^2})^2 dxdy \geq \pi_2(C_0) \tag{104}$$

Q.E.D.

Appendix III: Proof of Variational Principle in Eq.(16) of Chapter 4

The variation of the first integral in the functional π_3 of eq.(16) reads

$$\delta L = \int_0^L \int_0^L \frac{\partial^2 C}{\partial x d\partial} \frac{\partial^2 \delta C}{\partial x \partial y} dxdy = \int_0^L \frac{\partial^2}{\partial x \partial y} \frac{\partial \delta C}{\partial y} \Big|_{x=0}^{x=L} dy - \int_0^L \int_0^L \frac{\partial^3 C}{\partial x^2 \partial y} \frac{\partial \delta C}{\partial y} dxdy$$

$$= \int_0^L \frac{\partial^2 C}{\partial x \partial y} \frac{\partial \delta C}{\partial y} \Big|_{x=0}^{x=L} dy - \int_0^L \frac{\partial^3 C}{\partial x^2 \partial y}) \delta C \Big|_{y=0}^{y=L} dx + \int_0^L \int_0^L \frac{\partial^4 C}{\partial x^2 \partial y^2} \delta C dxdy$$

$$= \frac{\partial^2 C}{\partial x \partial y} \delta C \Big|_{x=0, y=0}^{x=L, y=L} - \int_0^L \frac{\partial^3 C}{\partial x \partial y^2} \delta C \Big|_{x=0}^{x=L} dy - \int_0^L \frac{\partial^3 C}{\partial x^2 \partial y} \delta C \Big|_{y=0}^{y=L} dx + \int_0^L \int_0^L \frac{\partial^4 C}{\partial x^2 \partial y^2} \delta C dxdy$$

(105)

The variation of the functional π_3 is then

$$\delta \pi_3 = \int_0^L \int_0^L [\frac{\partial^4 C}{\partial x^2 \partial y^2} - f_1(x,y) m(x) m(y) \delta C] dxdy - \int_0^L \frac{\partial^3 C}{\partial x^2 \partial y} \delta C \Big|_{y=0}^{y=L} dx$$

$$+ \int_0^L m(x) H_1(x) \delta C \Big|^{y=L} dx - \int_0^L \frac{\partial^3 C}{\partial x \partial y^2} \delta C \Big|_{x=0}^{x=L} dy + \int_0^L m(y) H_2(y) \delta C \Big|^{x=L} dy \qquad \textbf{(106)}$$

$$+ \frac{\partial^2 C}{\partial x \partial y} \delta C \Big|_{x=0, y=0}^{x=L, y=L} - G \delta C \Big|^{x=L, y=L}$$

The stationarity condition of π_3 leads to set of the governing equation and the boundary conditions.

(i) The governing equation reads

$$\frac{\partial^4}{\partial x^2 \partial y^2} = f_1(x,y)m(x)m(y) \ , \qquad \forall x,y \tag{107}$$

(ii) the boundary conditions at sides read

$$\begin{aligned}
&[\frac{\partial^3 C}{dxdy^2} - m(y)H_2(y)]\delta C = 0 \ , \qquad \forall \ y \quad at \quad x=L \\
&\frac{\partial^3 C}{dxdy^2}\delta C = 0 \ , \qquad \forall \ y \quad at \quad x=0 \\
&[\frac{\partial^3 C}{dx^2 dy} - m(x)H_1(x)]\delta C = 0 \ , \qquad \forall \ x \quad at \quad y=L \\
&\frac{\partial^3 C}{dx^2 dy}\delta C = 0 \ , \qquad \forall \ x \quad at \quad y=0
\end{aligned} \tag{108}$$

(iii) and boundary conditions at corners

$$\begin{aligned}
&[\frac{\partial^2 C}{\partial x \partial y} - G]\delta C = 0 \ , \qquad at \quad x=L \ ; \ y=L \\
&\frac{\partial^2 C}{\partial x \partial y}\delta C = 0 \ , \qquad at \quad x=0 \ ; \ y=0,L \quad or \quad x=L \ ; \ y=0
\end{aligned} \tag{109}$$

The boundary conditions both at sides and corners can be combined to be

$$C=0 \ , \qquad at \quad x=0,L \ ; \quad or \quad y=0,L \tag{110}$$

$$C=\frac{\partial C}{\partial x}=0 \quad at \quad x=0 \ ; \qquad C=\frac{\partial C}{\partial y}=0 \quad at \quad y=0 \tag{111}$$

The boundary conditions in eq.(110) is associated with simply-supported beams, whereas the boundary conditions in eq.(111) refer as the left-side clamped beams.

Appendix IV: Exact Solutions for Cantilever Beam in Example 1 of Chapter 4

For the cantilever beam in example 1, the governing equations for the mean value $\bar{w}$ and covariance function $C(x,y)$ are, respectively,

$$D_0 \frac{d^4\bar{w}}{dx^4} = q_0 \tag{112}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2}\left[D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2}\right] = q_o^2 \tag{113}$$

where D_0 and $D_1(x,y)$ are given by eq.(51) and eq.(52), respectively. Boundary conditions in eqs.(11-13) are simplified to

$$\begin{aligned} &\frac{d^2\bar{w}}{dx^2} = 0 \quad \frac{d^3\bar{w}}{dx^3} = 0 \qquad at \quad x=0, \\ &\bar{w} = 0, \quad \frac{d\bar{w}}{dx} = 0 \qquad at \quad x=L \end{aligned} \tag{114}$$

and

$$\frac{\partial^2 C}{\partial x^2}=0, \quad \frac{\partial^3 C}{\partial x^3}=0 \qquad at \quad x=0,$$
$$C=0, \quad \frac{\partial C}{\partial x}=0 \qquad at \quad x=L \tag{115}$$

$$\frac{\partial^2 C}{\partial y^2}=0, \quad \frac{\partial^3 C}{\partial y^3}=0 \qquad at \quad y=0,$$
$$C=0, \quad \frac{\partial C}{\partial y}=0 \qquad at \quad y=L \tag{116}$$

The solution for the mean displacement $\bar{w}(x)$ is straightforward and is obtained by integrating eq.(112) four times and satisfying the boundary conditions eq.(114). The result reads

$$\bar{w}(x)=\frac{q_0}{24D_0}(x^4-4L^3x+3L^4) \tag{117}$$

It is seen that the mean displacement $\bar{w}$ coincides with that of a beam which has a deterministic stiffness D_0. To obtain the solution of covariance function $C(x,y)$ from eq.(113) and boundary conditions eqs.(115,116), we first integrate eq.(113) with respect to x twice and with respect to y twice and reach

$$\frac{\partial^4 C(x,y)}{\partial^2 x \partial y^2}=\frac{q_0^2}{4\bar{D}_{1u}}x^2y^2(1-\frac{|x-y|}{L}) \tag{118}$$

under the free boundary conditions at x=0 and y=0. The solution of eq.(118) is comprised of a complementary solution $\psi(x,y)$ and a particular solution $\phi(x,y)$. The complementary solution

$\psi(x,y)$ can be written as follows

$$\psi(x,y)=f_1(x)+yf_2(x)+g_1(y)+xg_2(y) \tag{119}$$

where $f_1(x)$,$f_2(x)$,$g_1(y)$ and $g_2(y)$ are four arbitrary functions. The particular solution of eq.(118) reads

$$\phi(x,y)=\frac{q_0^2y^4}{20160L\bar{D}_{1u}}(35Lx^4-21x^5+21yx^4-15xy^4+7y^5)\ , \qquad x\geq y \tag{120}$$

The particular solution for $x\leq y$ can be obtained from eq.(120) by formal replacement x by y and y by x, due to symmetry in x and y. The boundary conditions at fixed end $x=L$ and $y=L$ require

$$\begin{aligned}
f_1(L)+yf_2(L)+g_1(y)+Lg_2(y)+\phi(L,y)&=0\\
f_1'(L)+yf_2'(L)+g_2(y)+\phi_1(L,y)&=0\\
f_1(x)+Lf_2(x)+g_1(L)+xg_2(L)+\phi(L,x)&=0\\
f_2(x)+g_1'(L)+xg_2'(L)+\phi_1(L,x)&=0
\end{aligned} \tag{121}$$

Solving out functions $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(x)$ from above conditions, we get

$$\begin{aligned}
f_2(x)&=-\phi_1(L,x)-g_1'(L)-xg_2'(L)\\
f_1(x)&=-\phi(L,x)+\phi_1(L,x)-g_1(L)-xg_2(L)+Lg_1'(L)+xLg_2(L)\\
g_2(y)&=\phi_1(L,L)-\phi_1(L,y)+(L-y)\phi_{12}(L,L)+g_2(L)-(L-y)g_2'(L)\\
g_1(y)&=\phi(L,L)-\phi(L,y)-(L-y)\phi_1(L,L)+g_1(L)+Lg_2(L)-Lg_2(y)\\
&\quad-(L-y)g_1'(L)-L(L-y)g_2'(L)
\end{aligned} \tag{122}$$

By substituting eq.(122) back into eq.(119) and combining complementary and particular solution, we obtain the covariance function $C(x,y)$

$$C(x,y) = \frac{q^2}{20160L\bar{D}_{1u}}[259L^9-336L^8(x+y)+440L^7xy+21L^5(x^4+y^4)$$
$$+63L^4(x^5+y^5)-35L^4xy(x^3+y^3)-84L^3xy(x^4+y^4)$$
$$+35Lx^4y^4+21x^4y^4(x-y)-7x^9]\ , \qquad x\geq y \tag{123}$$

Appendix V: Exact Solution for Simply-Supported Beam in Example 2 of Chapter 4

For the simply-supported beam described in example 2, the governing equations for the mean and the covariance function are identically given by eq.(112) and eq.(113), respectively, where D_0 and $D_1(x,y)$ are given, respectively, by eq.(60) and eq.(61). The attendant boundary conditions are

$$\bar{w}=0\ ,\quad \frac{d^2\bar{w}}{dx^2}=0 \qquad at\quad x=0,\ L \tag{124}$$

and

$$C=0,\quad \frac{\partial^2 C}{\partial x^2}=0 \qquad at\quad x=0,\ L \tag{125}$$

$$C=0,\quad \frac{\partial^2 C}{\partial y^2}=0 \qquad at\quad y=0,\ L \tag{126}$$

The mean displacement is obtained by direct integration. It reads

$$\bar{w}(x)=\frac{qx}{24\bar{D}_0}(L^3-2Lx^2+x^3) \tag{127}$$

As previously mentioned, the mean displacement is identical to the displacement of a beam with deterministic stiffness D_0. Integrating eq.(113) twice yields

$$\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2}=\frac{q^2}{4\bar{D}_{1t}}(Lx-x^2)(Ly-y^2)e^{-|x-y|} \tag{128}$$

with boundary conditions

The complementary solution of eq.(128) is the same as that in eq.(119). The particular solution

$$C=0, \quad at \quad x=0,\, L \quad or \quad y=0,\, L \tag{129}$$

reads

$$\begin{aligned}\phi(u,v)=\frac{q^2}{4\bar{D}_{1t}}[&(8L+3u)(4L-v)L^6-\frac{2}{3}(L+u)L^4v^3+\frac{1}{6}(8L+3u)L^3v^4\\&-\frac{1}{10}(7L+2u)L^2v^5+\frac{1}{15}(2L+u)Lv^6-\frac{1}{21}Lv^7\\&+L^4(4L^2+3Lu+u^2)(8L^2-5Lv+v^2)e^{-\frac{(u-v)}{L}}\\&-L^5(8L+3v)(4L^2+3Lu+u^2)e^{-\frac{u}{L}}\\&-L^5(8L+3u)(4L^2+3Lv+v^2)e^{-\frac{v}{L}}]\end{aligned} \tag{130}$$

for $x \geq y$. The boundary conditions require

$$\begin{aligned}f_1(0)+yf_2(0)+g_1(y)+\phi(y,0)&=0\\f_1(L)+yf_2(L)+g_1(L)+Lg_2(y)+\phi(L,y)&=0\\f_1(x)+g_1(0)+xg_2(0)+\phi(x,0)&=0\\f_1(x)+Lf_2(x)+g_1(L)+xg_2(L)+\phi(L,x)&=0\end{aligned} \tag{131}$$

By solving for $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(x)$ and noting that $\phi(x,0)=\phi(y,0)=0$, the solution of covariance function becomes

$$C(x,y)=\phi(x,y)+\frac{xy}{L^2}\phi(L,L)-\frac{1}{L}[x\phi(L,y)+y\phi(L,x)]\ , \quad for\ x \geq y \tag{132}$$

To sum up, new stochastic Galerkin and Ritz methods, which are based on either the

$$C(x,y) = \phi(y,x) + \frac{xy}{L^2}\phi(L,L) - \frac{1}{L}[x\phi(L,y) + y\phi(L,x)] \ , \quad for \ x \le y \tag{133}$$

variational principles or the governing equations, can be applied to obtain the approximate mean and variance functions of bending beams for any value of coefficient of variation of stochastic bending stiffness. The variational principle based stochastic finite element method also are applicable to any value of coefficient of variation of bending stiffness and possesses the same convergence property as the deterministic finite element method.

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IMAGE EVALUATION TEST TARGET (QA-3)

150mm

6"

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