CONSTRUCTION OF COMBINATORIAL DESIGNS WITH PRESCRIBED AUTOMORPHISM GROUPS

by

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ABSTRACT

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In this dissertation, we study some open problems concerning the existence or non-existence of some combinatorial designs. We give the construction or proof of non-existence of some Steiner systems, large sets of designs, and graph designs, with prescribed automorphism groups.

CONSTRUCTION OF COMBINATORIAL DESIGNS WITH PRESCRIBED AUTOMORPHISM GROUPS

TABLES

CHAPTER 1
INTRODUCTION

A combinatorial design is an arrangement of elements of a finite set into subsets that satisfy certain balance properties. Types of combinatorial designs include balanced incomplete block designs, t-designs, pairwise balanced designs, Steiner systems, projective geometries, affine geometries, graph designs, large sets of t-designs, orthogonal Latin squares, and orthogonal arrays. The fundamental question in combinatorial design theory is whether a design of a specified type exists or not. Modern design theory includes many existence and non-existence results. However, many problems remain open concerning the existence of certain types of designs.

Many problems in design theory that are studied today have their roots in the research of Euler, Kirkman, Cayley, Hamilton, Sylvester, Moore, Witt, M. Hall Jr., and others in the last three centuries. Combinatorial design theory arise as an academic subject in its own rights after Fisher's work on the design of biological experiments in the 1920's. Modern design theory has applications in many areas including cryptography, coding theory, algorithm design and analysis, finite geometry, mathematical biology, tournament scheduling, etc., see [5, 27].

In this dissertation, we study some open problems concerning the existence of certain types of combinatorial designs. We give the construction or proof of non-existence of some Steiner systems, large sets of designs, and graph designs, with prescribed automorphism groups.

Let $t \le k \le v$ and λ be positive integers. A t-(v, k, λ) *design* is a pair $(X, \mathcal{B})$, where X is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, where each t-subset of X is contained in precisely λ blocks. A t-design $(X, \mathcal{B})$ is said to be *simple* if $\mathcal{B}$ is a set, i.e. there are no repeated blocks. A basic counting argument shows that the number of blocks in a t-(v, k, λ) design is $b = \lambda\binom{v}{t}/\binom{k}{t}$.

Let $\mathcal{D} = (X, \mathcal{B})$ be a t-(v, k, λ) design, and $x \in X$. It is well known that $\mathcal{D}_x = (X \setminus \{x\}, \mathcal{B}_x)$, where $\mathcal{B}_x = \{B \setminus \{x\} \ : \ x \in B \in \mathcal{B}\}$, is a $(t-1)$-$(v-1, k-1, \lambda)$ design, called the *derived design at* x. The number of blocks in $\mathcal{D}_x$ is $\lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$. Repeating this process, we can construct a second derived design $\mathcal{D}_{x,y}$, a third derived design $\mathcal{D}_{x,y,z}$, and so on. Therefore, for $0 \leq s < t$, the number $\lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$, which is the number of blocks in the s^{th} derived design, has to be an integer. These conditions are called *the necessary conditions* for the existence of a t-(v, k, λ) design.

A t-$(v, k, 1)$ design is also called a *Steiner system*, denoted by $S(t, k, v)$. A Steiner system is necessarily simple. For $t \in \{2, 3\}$, infinite families of Steiner systems are known to exist. For $t \in \{4, 5\}$, only finitely many Steiner systems are known to exists. No Steiner system with $t \geq 6$ has been constructed yet, see [11].

Steiner systems $S(2, 3, v)$, $S(3, 4, v)$, and $S(4, 5, v)$ are called *Steiner triple*, *quadruple* and *quintuple systems*, respectively. The parameter v is called the *order* of the system.

For Steiner triple and quadruple systems, it has been proven that the necessary conditions are also sufficient. An $S(2, 3, v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, and an $S(3, 4, v)$ exists if and only if $v \equiv 2$ or $4 \pmod 6$, see [11].

The necessary conditions for the existence of a Steiner quintuple system is that $v \equiv 3, 5, 11, 15, 17, 21, 23$ or $27 \pmod{30}$. The sufficient conditions are still unknown. The existence of a Steiner quintuple system of order 5 is trivial. The first non-trivial Steiner quintuple system, which is of order 11, was constructed in 1908 by Barrau [3].

In total, only finitely many Steiner quintuple systems are known to exist, namely of orders $5, 11, 23, 35, 47, 71, 83, 107, 131, 167$, and 243, all of which are derived designs of known $S(5, 6, v)$ [11]. The non-existence of a Steiner quintuple system of order 15 was shown in 1972 by Mendelsohn and Hung [24]. Since then, it has been a challenge for many researchers to construct a Steiner quintuple system of order 17 or 21. Recently, Östergård and Pottonen [25] have proven the non-existence of an $S(4, 5, 17)$, leaving 21 as the smallest order for which the existence, or otherwise, of a Steiner quintuple system is unknown. We get the following theorem.

Theorem 1.0.1. *If there exists an $S(4, 5, v)$ for $v < 21$, then $v = 5$ or $v = 11$.*

The problem of the existence of a Steiner quintuple system of order 21 is much harder than the one of order 17. In Chapter 3, we study the possible automorphism groups of such a system, and prove that, if an $S(4, 5, 21)$ exists, the order of its full automorphism group is $1, 2, 3, 4, 5, 6, 7, 10$ or 21.

Let $\binom{X}{k}$ denote the set of all k-subsets of a set X. A *large set* $LS[N](t, k, v)$ is a pair $(X, \mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N})$, where $(X, \mathcal{B}_i)$ is a simple t-(v, k, λ) design for all $\mathcal{B}_i \in \mathbb{B}$, and $\{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\binom{X}{k}$. Arithmetically, for a large set $LS[N](t, k, v)$, we have $N = M_\lambda(t, k, v) = \binom{v-t}{k-t}/\lambda$. Let $D_\lambda(t, k, v)$ denote the maximum number of mutually disjoint simple t-(v, k, λ) designs on a particular set of v points. So, a large set of t-(v, k, λ) designs exists if $D_\lambda(t, k, v) = M_\lambda(t, k, v)$. We denote $M_1(t, k, v)$ and $D_1(t, k, v)$ simply as $M(t, k, v)$ and $D(t, k, v)$.

A projective plane of order n, if such exists, is a 2-$(n^2 + n + 1, n + 1, 1)$ design. If a large set of projective planes of order n exists, it is an $LS[N](2, n + 1, n^2 + n + 1)$, where $N = \binom{n^2+n-1}{n-1}$.

In 1850, Cayley [9] proved by a brief argument that a large set $LS[5](2, 3, 7)$ of Fano planes does not exist. In 1978, Magliveras conjectured that a large set of projective planes of order n will exist, i.e. $D(2, n+1, n^2+n+1) = M(2, n+1, n^2+n+1)$, for all $n \geq 3$, provided that n is the order of a projective plane. In 1983, Chouinard II [10] constructed such large sets for $n = 3$, namely $LS[55](2, 4, 13)$, by prescribing an automorphism of order 11 which acts semiregularly on the set of 55 planes. In Chapter 4, we construct new large sets $LS[55](2, 4, 13)$ by prescribing an automorphism of order 13. We classify all such large sets and determine their full automorphism groups.

The existence, or otherwise, of a large set of projective planes of order n for $n \geq 4$, is still an unsettled problem. For $n = 4$, we have $M(2, 5, 21) = 969$. Kramer and Magliveras have shown in [20] that $D(2, 5, 21) \geq 197$. Later, they constructed over 600 mutually disjoint projective planes of order 4 by probabilistic means. In our effort to construct a large set $LS[969](2, 5, 21)$, we show in Chapter 4 that $D(2, 5, 21) \geq 912$, which improves the lower bound for $D(2, 5, 21)$ obtained by Kramer and Magliveras.

Let K_n denote the complete graph with n vertices. Note that a 2-$(v, k, 1)$ design may also be considered as a decomposition of K_v into mutually disjoint subgraphs each isomorphic to K_k. If we replace K_k by any subgraph of K_v, we get the notion of a graph design.

In general, for a (finite, undirected) graph G, let $V = V(G)$ and $E = E(G)$ denote the set of vertices (or points) and the multiset of edges of G respectively. A graph G is called *simple* if it has no multiple edges or loops. Let $\mathbb{G} = \{G_1, G_2, ..., G_r\}$ be a set of graphs. A $\mathbb{G}$-*decomposition* of a graph K (or a $(K, \mathbb{G})$-*decomposition*) is a set $\mathcal{D} = \{B_1, B_2, ..., B_s\}$ of subgraphs of K, called *blocks*, such that $\{E(B_1), E(B_2), ..., E(B_s)\}$ partition $E(K)$, and for $1 \leq j \leq s$ we have $B_j \cong G_i$ for some i, $1 \leq i \leq r$. A $(K_n, \mathbb{G})$-decomposition is called a $\mathbb{G}$-*design of order* n. When $\mathbb{G} = \{G\}$, we simply denote it as a *G-design*. The *spectrum* for a graph G is the set of positive integers n such that there exists a G-design of order n. The known results on the spectrum of graphs may be found in [2, 7, 8].

There are three obvious necessary conditions for the existence of a G-design.

Theorem 1.0.2. *If a G-design of order n exists, then:*

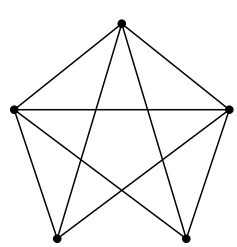
(1) $n = 1$ *or* $n \geq |V(G)|$,

(2) $n(n-1) \equiv 0$ *(mod* $2|E(G)|$*), and*

(3) $n - 1 \equiv 0$ *(mod* d*), where* d *is the g.c.d. of the degrees of the points in* G.

The problem of determining the spectrum of a graph has been considered for several types of graphs, such as complete graphs, trees, cycles, matchings, paths, stars, cubes, graphs of geometric solids, even graphs, theta graphs, unions of graphs, all graphs with up to five vertices, and all graphs with six vertices and up to eight edges. The spectrum problem has been completely solved for cycles, matchings, paths, stars, and graphs with up to four vertices, see [2, 7, 8].

The spectrum problem is almost completely solved for graphs with five vertices. Up to isomorphism, there are 23 simple graphs with 5 vertices and no isolated vertex. The spectrum problem has been completely solved for 20 of these graphs. For the remaining 3 graphs, in total 15 orders are unsolved. Twelve of these unsolved cases are for the graph with 5 vertices and 9 edges, which is shown below. We denote this graph as $K_5 \setminus e$.



The necessary conditions for the existence of a $(K_5 \setminus e)$-design of order n is that $n \equiv 0, 1 \pmod 9$. The first result on the existence of a $(K_5 \setminus e)$-design has been given in 1980. Bermond et al. [4] constructed a $(K_5 \setminus e)$-design of order 19, and showed that $(K_5 \setminus e)$-designs of orders 9, 10, and 18 do not exist. In [17], it was stated that a $(K_5 \setminus e)$-design of order n exists for all $n \equiv 1 \pmod{18}$ with the possible exceptions of $n \in \{37, 55, 73, 109, 397, 415, 469, 487, 505, 541, 613, 685\}$. Although this result is now known to be true, a proof is not given in any of the references cited there. In 2005, Li and Chang [23] eliminated the twelve exceptions stated above. In addition they constructed $(K_5 \setminus e)$-designs of orders 28, 46, and 82. Finally, in 2007, Ge and Ling [14] solved the problem for almost all orders satisfying the necessary conditions including a complete solution for the case $n \equiv 1 \pmod{18}$. Their result is the following:

Theorem 1.0.3. ([14]) *There exists a $(K_5 \setminus e)$-design of order n for all $n \equiv 0, 1 \pmod 9$ except for $n \in \{9, 10, 18\}$, and except possibly for $n \in \{27, 36, 54, 64, 72, 81, 90, 135, 144, 162, 216, 234\}$.*

In Chapter 5, we settle the problem for four of these twelve unsolved orders. We construct $(K_5 \setminus e)$-designs of order 27 by prescribing an automorphism of order 6. We classify all such $(K_5 \setminus e)$-designs and determine their full automorphism groups. Then, we give the construction of $(K_5 \setminus e)$-designs of orders 135, 162 and 216 which follow immediately by the recursive constructions given in [14].

In Chapter 6, we consider the spectrum problem for complete bipartite graphs. Let $K_{s,t}$ denote the complete bipartite graph with one part of size s and one part of size t. The necessary conditions for the existence of a $K_{s,t}$-design of order n are:

(1) $n = 1$ or $n \geq s + t$,

(2) $n(n-1) \equiv 0 \pmod{2st}$, and

(3) $n - 1 \equiv 0 \pmod{gcd(s,t)}$.

In 1966, Rosa [26] proved the following theorem.

Theorem 1.0.4. [26] *For $s, t \geq 1$, there exists a $K_{s,t}$-design of order n for all $n \equiv 1$ (mod $2st$).*

As mentioned in [2], this result completely solves the spectrum problem when s and t are both powers of 2. The spectrum problem for stars, i.e. the graphs $K_{1,k}$ for $k \geq 1$, was completely solved in 1975 by Yamamoto et al. [31], and independently in 1979 by Tarsi [29]. The spectrum problems for $K_{2,3}$ and $K_{3,3}$ were completely solved in 1980 by Bermond et al. [4] and in 1968 by Guy and Beineke [16], respectively.

Some non-existence results for $K_{s,t}$-designs have also been proven by Graham and Pollak [15] and de Caen and Hoffman [12]. These results are given in the following theorem.

Theorem 1.0.5. [12, 15] *For $s, t \geq 1$, there does not exist a $K_{s,t}$-design of order n for $1 < n < 2st$. Moreover, for $s, t \geq 2$, there does not exist a $K_{s,t}$-design of order $2st$.*

In Chapter 6, we prove a recursive construction theorem which provides a way of constructing infinite families of $K_{s,t}$-designs. We consider the complete bipartite graphs with fewer than 18 edges, for which the spectrum problem has not been completely solved yet, namely the graphs $K_{2,5}$, $K_{2,6}$, $K_{3,4}$, $K_{2,7}$ and $K_{3,5}$. Giving necessary direct constructions, we provide an almost complete solution for the spectrum problem for these 5 graphs, leaving 5 orders in total unsolved.

CHAPTER 2

PRELIMINARIES

An action of a group $\mathcal{G}$ on a set X will be denoted by $\mathcal{G}|X$. For $x \in X$ and $\alpha \in \mathcal{G}$, x^α denotes the image of x under α. For $A \subseteq X$, $A^\alpha := \{x^\alpha \mid x \in A\}$. A group action $\mathcal{G}|X$ is called *transitive* if X is a single $\mathcal{G}-$orbit, that is, for all $x, y \in X$, $x^\alpha = y$ for some $\alpha \in \mathcal{G}$, *k-transitive* if the induced action on the ordered k-sets of elements of X is transitive, *doubly transitive* if it is 2-transitive, *semiregular* if only the identity element fixes points, that is, $x^\alpha = x$ implies $\alpha = 1$, and *regular* if it is transitive and semiregular.

If $A \subseteq X$, $\mathcal{G}_{[A]} = \{\alpha \in \mathcal{G} \mid x^\alpha = x \text{ for all } x \in A\}$ denotes the *pointwise stabilizer* of A in $\mathcal{G}$. If $A = \{x\}$, $\mathcal{G}_{[A]}$ is simply denoted as $\mathcal{G}_x$. The set $x^{\mathcal{G}} = \{x^\alpha \;:\; \alpha \in \mathcal{G}\}$ denotes the *orbit of* x *under* $\mathcal{G}$ or the $\mathcal{G}$*-orbit of* x. The number of $\mathcal{G}$-orbits on X is denoted by $o(\mathcal{G})$. If $\mathcal{G}$ has precisely r_i orbits of size u_i for $1 \leq i \leq s$, where $\sum_{i=1}^{s} r_i u_i = |X|$, we say that $\mathcal{G}$ is of type $u_1^{r_1}...u_s^{r_s}$. For $\alpha \in \mathcal{G}$, $fix(\alpha) = \{x \in X \;:\; x^\alpha = x\}$ denotes the set of points fixed by α, and $fix_s(\alpha) = \{S \in \binom{X}{s} \;:\; S^\alpha = S\}$ the set of s-subsets of X fixed by α.

We denote the symmetric group on a set X as $\mathcal{S}_X$, the cyclic group of order n as C_n, the dihedral group of order $2n$ as D_{2n}, and the Klein four-group as V_4. If p is a prime and r divides $p-1$, we denote by C_p^r the unique subgroup of order pr of the affine group $C_p^{p-1} = C_p \rtimes Aut(C_p)$.

If $\mathcal{G}$ is a group and $\alpha \in \mathcal{G}$, $\langle\alpha\rangle$ denotes the subgroup generated by α. Throughout the dissertation, we will use α and $\langle\alpha\rangle$ interchangebly. The subgroup $C_{\mathcal{G}}(\alpha) =$

$\{\beta \in \mathcal{G} \; : \; \alpha\beta = \beta\alpha\}$ denotes the *centralizer* of α in $\mathcal{G}$. For a subgroup $\mathcal{H} \leq \mathcal{G}$, $N_{\mathcal{G}}(\mathcal{H}) = \{\beta \in \mathcal{G} \mid \beta\mathcal{H}\beta^{-1} = \mathcal{H}\}$ denotes the *normalizer* of $\mathcal{H}$ in $\mathcal{G}$.

The statements in the following theorem are basic results in group theory and may be found in [6, 18, 30].

Theorem 2.0.6. *Let $\mathcal{G} \mid X$, $x \in X$ and $\alpha \in \mathcal{G}$. Then,*

(1) $|x^{\mathcal{G}}| = |\mathcal{G}|/|\mathcal{G}_x|$.

(2) $|\mathcal{G}| \cdot o(\mathcal{G}) = \sum_{\alpha \in \mathcal{G}} |fix(\alpha)|$.

(3) $N_{\mathcal{G}}(\alpha)/C_{\mathcal{G}}(\alpha)$ *is isomorphic to a subgroup of* $Aut(\alpha)$.

(4) If $|\mathcal{G}| = pqr$, *where* $p < q < r$ *are primes, then* $\mathcal{G}$ *has a normal Sylow* r*-subgroup.*

(5) If $|\mathcal{G}| = 2h$, *where* h *is odd, then* $\mathcal{G}$ *has a subgroup of order* h.

Two t-(v, k, λ) designs $\mathcal{D} = (X, \mathcal{B})$ and $\mathcal{D}' = (X, \mathcal{B}')$ are called *isomorphic* if there exists $\alpha \in \mathcal{S}_X$ such that $\mathcal{B}^\alpha = \mathcal{B}'$, that is, $B^\alpha \in \mathcal{B}'$ for all $B \in \mathcal{B}$, and we say $\mathcal{D}^\alpha = \mathcal{D}'$. If $\mathcal{D}^\alpha = \mathcal{D}$, then α is called an *automorphism* of $\mathcal{D}$. The group of all automorphisms of $\mathcal{D}$ is called *the full automorphism group* of $\mathcal{D}$, and is denoted by $Aut(\mathcal{D})$. Any subgroup $\mathcal{G}$ of the full automorphism group is called *an automorphism group* of $\mathcal{D}$, and we say that $\mathcal{D}$ is $\mathcal{G}$*-invariant*. If $\mathcal{D} = (X, \mathcal{B})$ is a $\mathcal{G}$-invariant t-(v, k, λ) design, then the orbits of $\mathcal{G}|X$ and $\mathcal{G}|\mathcal{B}$ are called the *point orbits* and *block orbits* of $\mathcal{G}$ respectively, and for $\alpha \in \mathcal{G}$, $fix'(\alpha) = \{B \in \mathcal{B} \; : \; B^\alpha = B\}$ denotes the set of blocks in $\mathcal{D}$ fixed by α.

Two large sets $\mathcal{L} = (X, \mathbb{B})$ and $\mathcal{L}' = (X, \mathbb{B}')$ are said to be *isomorphic* if there exists $\alpha \in \mathcal{S}_X$ such that $\mathbb{B}^\alpha = \mathbb{B}'$, that is, $\mathcal{B}_i^\alpha \in \mathbb{B}'$ for all $\mathcal{B}_i \in \mathbb{B}$, and we write $\mathcal{L}^\alpha = \mathcal{L}'$. If $\mathcal{L}^\alpha = \mathcal{L}$, then α is called an *automorphism* of $\mathcal{L}$. The *full automorphism* *group* of $\mathcal{L}$ is denoted by $Aut(\mathcal{L})$. Any subgroup $\mathcal{G}$ of the full automorphism group is called *an automorphism group* of $\mathcal{L}$, and we say that $\mathcal{L}$ is $\mathcal{G}$-*invariant*. If $\mathcal{L} = (X, \mathbb{B})$ is a $\mathcal{G}$-invariant $LS[N](t, k, v)$, then the orbits of $\mathcal{G}|X$, $\mathcal{G}|\binom{X}{k}$ and $\mathcal{G}|\mathbb{B}$ are called the *point orbits*, *block orbits* and *design orbits* of $\mathcal{G}$ respectively, and for $\alpha \in \mathcal{G}$, $fix'(\alpha) = \{\mathcal{B} \in \mathbb{B} \mid \mathcal{B}^\alpha = \mathcal{B}\}$ denotes the set of designs in $\mathcal{L}$ fixed by α.

Two $(K, \mathbb{G})$-decompositions $\mathcal{D}$ and $\mathcal{D}'$ are called *isomorphic* if there exists $\alpha \in \mathcal{S}_{V(K)}$ such that $\mathcal{D}^\alpha = \mathcal{D}'$. Automorphism groups of graph decompositions are defined in a similar way to those of t-designs. If $\mathcal{D}$ is a $\mathcal{G}$-invariant G-design, for $\alpha \in \mathcal{G}$, $fix'(\alpha) = \{B \in \mathcal{D} \mid B^\alpha = B\}$ denotes the set of blocks in $\mathcal{D}$ fixed by α.

CHAPTER 3

ON THE POSSIBLE AUTOMORPHISM GROUPS OF AN $S(4,5,21)$

3.1 BASIC RESULTS

Let $\mathcal{G} \leq \mathcal{S}_X$, and $\{\mathcal{T}_i \ : \ 1 \leq i \leq r\}$ and $\{\mathcal{K}_j \ : \ 1 \leq j \leq s\}$ be the collection of orbits of $\mathcal{G} \mid \binom{X}{t}$ and $\mathcal{G} \mid \binom{X}{k}$ respectively. The *Kramer-Mesner matrix* is defined as the $r \times s$ matrix $\mathbf{A} = \mathbf{A_{t,k}}(\mathcal{G}) = (a_{i,j})$, where $a_{i,j} = |\{K \in \mathcal{K}_j \ : \ K \supset T\}|$ for a representative $T \in \mathcal{T}_i$. In practice, the identity $a_{i,j} = |\{T \in \mathcal{T}_i \ : \ T \subset K\}| \cdot |\mathcal{K}_j|/|\mathcal{T}_i|$ where K is a representative k-set in $\mathcal{K}_j$, can be used to compute the matrix $\mathbf{A}$. The following theorem was given by Kramer and Mesner in 1976.

Theorem 3.1.1. *[22] There exists a $\mathcal{G}$-invariant t-(v,k,λ) design if and only if there exists an s-dimensional column vector $\mathbf{u}$ with nonnegative integer entries such that $\mathbf{Au} = \lambda\mathbf{j}$ where $\mathbf{j}$ is the r-dimensional column vector of all 1's.*

To reduce computation time in applying Theorem 3.1.1, the following lemma is useful.

Lemma 3.1.1. *Let $\mathcal{D} = (X, \mathcal{B})$ be a t-(v,k,λ) design, and $\mathcal{G} \leq \mathcal{S} = \mathcal{S}_X$. If $\mathcal{D}$ is $\mathcal{G}$-invariant, then $\mathcal{D}^\omega$ is also $\mathcal{G}$-invariant for any $\omega \in N_{\mathcal{S}}(\mathcal{G})$.*

Proof: For any $\omega \in N_{\mathcal{S}}(\mathcal{G})$, we have $\mathcal{D}^{\omega\mathcal{G}\omega^{-1}} = \mathcal{D}^{\mathcal{G}} = \mathcal{D}$. Hence, $\mathcal{D}^{\omega\mathcal{G}} = \mathcal{D}^\omega$. □

3.2 AUTOMORPHISMS OF PRIME ORDER

Throughout this chapter, let $\mathcal{D} = (X, \mathcal{B})$ be a putative $S(4,5,21)$. Note that the number of blocks in $\mathcal{D}$ is $\binom{21}{4}/\binom{5}{4} = 1197$. In this section, we derive some results on possible elements of prime order in $Aut(\mathcal{D})$.

Lemma 3.2.1. *If $\alpha \in Aut(\mathcal{D})$ is of prime order p, then $|fix(\alpha)| \in \{0, 1, 2, 3, 5, 11\}$.*

Proof: Assume that $|fix(\alpha)| \geq 4$. For any distinct $u, w, x, y \in fix(\alpha)$, there exists a unique $z \in X$ such that $B = \{u, w, x, y, z\} \in \mathcal{B}$. Since $\{u, w, x, y\} \in fix_4(\alpha)$, we necessarily have $B \in fix'(\alpha)$ and hence $z \in fix(\alpha)$. Therefore, the set of blocks in $\mathcal{D}$ which are pointwise fixed by α form a Steiner quintuple system of order $|fix(\alpha)|$. Therefore, the result follows by Theorem 1.0.1. □

Lemma 3.2.2. *Let $\alpha \in Aut(\mathcal{D})$ be of prime order p. Then, $|fix(\alpha)| \neq 11$.*

Proof: Assume that $|fix(\alpha)| = 11$. For $0 \leq i \leq 4$, let $\mathcal{T}_i = \{T \in \binom{X}{4} : |T \cap fix(\alpha)| = i\}$ and $t_i = |\mathcal{T}_i| = \binom{11}{i} \cdot \binom{10}{4-i}$. For $0 \leq j \leq 5$, let $\mathcal{B}_j = \{B \in \mathcal{B} : |B \cap fix(\alpha)| = j\}$ and $b_j = |\mathcal{B}_j|$. Counting the number of pairs (T, B), where $T \in \mathcal{T}_i$, $B \in \mathcal{B}$, and $T \subseteq B$, we get $(5 - i) \cdot b_i + (i + 1) \cdot b_{i+1} = t_i$. Therefore we get the following equations:

$$5b_0 + b_1 = 210$$

$$4b_1 + 2b_2 = 1320$$

$$3b_2 + 3b_3 = 2475$$

$$2b_3 + 4b_4 = 1650$$

$$b_4 + 5b_5 = 330$$

By the arguments in the proof of Lemma 3.2.1, we have $b_4 = 0$ and $b_5 = 66$. Therefore, $b_3 = 825$, $b_2 = 0$, $b_1 = 330$, and $b_0 = -24$, contradiction. □

Lemma 3.2.3. *Let $\alpha \in Aut(\mathcal{D})$ be of prime order p. Then, $|fix(\alpha)| \neq 5$.*

Proof: Assume that $|fix(\alpha)| = 5$. Then, $p \mid 16$, and hence $p = 2$. Let $\{a, b\}$ be an α-orbit of size 2, and let $F_{a,b} = \{\{a, b, x, y, z\} \in \mathcal{B} : x, y, z \in fix(\alpha)\}$. For any distinct $x', y' \in fix(\alpha)$, there exists a unique $z' \in X$ such that $B = \{a, b, x', y', z'\} \in \mathcal{B}$. Since $\{a, b, x', y'\} \in fix_4(\alpha)$, we necessarily have $z' \in fix(\alpha)$ and hence $B \in F_{a,b}$. Since there are $\binom{5}{2} = 10$ pairs of fixed points of α, and each block in $F_{a,b}$ contains 3 such pairs, we get $|F_{a,b}| = 10/3$, contradiction. □

Using similar arguments, we can get the following lemma. We omit the proof.

Lemma 3.2.4. *$\mathcal{D}$ cannot have an automorphism of type $1^3 3^6$.*

Theorem 3.2.1. *Let $\alpha \in Aut(\mathcal{D})$ be of prime order p. Then, one of the following holds:*

(1) $|fix(\alpha)| = 0$ *and* $p \in \{3, 7\}$.

(2) $|fix(\alpha)| = 1$ *and* $p \in \{2, 5\}$.

(3) $|fix(\alpha)| = 2$ *and* $p = 19$.

(4) $|fix(\alpha)| = 3$ *and* $p = 2$.

Proof: Follows by Lemmas 3.2.1-3.2.4. □

3.3 AUTOMORPHISMS OF ORDER 19

In this section, we prescribe an automorphism of order 19 and prove:

Theorem 3.3.1. *$\mathcal{D}$ cannot have an automorphism of order 19.*

Throughout this section, suppose that $X = \mathbb{Z}_{19} \cup \{\infty_1, \infty_2\}$, $\mathcal{S} = \mathcal{S}_X$, and define $\alpha \in \mathcal{S}$ as $x^\alpha = x + 1$ for $x \in \mathbb{Z}_{19}$, and $(\infty_i)^\alpha = \infty_i$ for $i \in \{1, 2\}$. For any Y, where $\mathbb{Z}_{19} \subseteq Y \subset X$, we denote the restriction of α on Y also as α.

Let $\mathcal{G} = \langle \alpha \rangle$, and assume that $\mathcal{D}$ is $\mathcal{G}$-invariant. The number of orbits of $\mathcal{G} \mid \binom{X}{4}$ and $\mathcal{G} \mid \binom{X}{5}$ are $\binom{21}{4}/19 = 315$ and $\binom{21}{5}/19 = 1071$ respectively. We construct the

315×1071 Kramer-Mesner matrix $\mathbf{A}$ as in Thereom 3.1.1 and search for solutions of $\mathbf{Au} = \mathbf{j}$. Note that any columns of $\mathbf{A}$ that contain entries greater than 1 can be deleted, since the corresponding entry of $\mathbf{u}$ must be zero in any solution of $\mathbf{Au} = \mathbf{j}$. After deleting such columns, $\mathbf{A}$ reduces to a 315×1035 matrix with entries in $\{0, 1\}$, where the column sum is 5. The problem is now equivalent to the exact cover problem, that is, we search for 63 columns of $\mathbf{A}$ which add up to $\mathbf{j}$. An exhaustive search for such a solution does not seem to be feasible with our computer program. We could make it more efficient by implementing Knuth's dancing links algorithm, see [19]. Instead, we develop an alternative approach.

Note that, for $i \in \{1, 2\}$, the derived design $\mathcal{D}_{\infty_i} = (X \backslash \{\infty_i\}, \mathcal{B}_{\infty_i})$ is a $\mathcal{G}$-invariant $S(3, 4, 20)$. Also, their derived design $\mathcal{D}_{\infty_1,\infty_2} = \mathcal{D}_{\infty_2,\infty_1} = (X \setminus \{\infty_1, \infty_2\}, \mathcal{B}_{\infty_1,\infty_2})$ is a $\mathcal{G}$-invariant $S(2, 3, 19)$. For $i \in \{1, 2\}$, let $\mathcal{B}'_{\infty_i} = \{B \in \mathcal{B}_{\infty_i} \ : \ \infty_{3-i} \notin B\}$, and $\mathcal{B}' = \{B \in \mathcal{B} \ : \ B \subset \mathbb{Z}_{19}\}$. Note that $|\mathcal{B}| = |\mathcal{B}_{\infty_1,\infty_2}| + |\mathcal{B}'_{\infty_1}| + |\mathcal{B}'_{\infty_2}| + |\mathcal{B}'|$, where $\mathcal{B}'_{\infty_1}$, $\mathcal{B}'_{\infty_2}$, and $\mathcal{B}'$ are also $\mathcal{G}$-invariant. An $S(2, 3, 19)$, an $S(3, 4, 20)$, and an $S(4, 5, 21)$ have 57, 285, and 1197 blocks, respectively. Therefore, $|\mathcal{B}_{\infty_1,\infty_2}| = 57 = 3 \cdot 19$, $|\mathcal{B}'_{\infty_1}| = |\mathcal{B}'_{\infty_2}| = 228 = 12 \cdot 19$, and $|\mathcal{B}'| = 684 = 36 \cdot 19$.

Proposition 3.3.1. *Let $Y \subset \binom{\mathbb{Z}_{19}}{4}$, where $|Y| = 228$, and suppose that for any $T \in \binom{\mathbb{Z}_{19}}{3}$, T is contained in at most one 4-set $K \in Y$. Let $Y' = \{T \in \binom{\mathbb{Z}_{19}}{3} \ : \ T \not\subset K$ for any $K \in Y\}$. Then, $(\mathbb{Z}_{19}, Y')$ is an $S(2, 3, 19)$, and hence $(\mathbb{Z}_{19} \cup \{\infty\}, Y'')$, where $Y'' = Y \cup \{T \cup \{\infty\} \ : \ T \in Y'\}$, is an $S(3, 4, 20)$.*

Proof: Let $S \in \binom{\mathbb{Z}_{19}}{2}$, and define $Y_S = \{K \in Y \ : \ K \supset S\}$ and $Y'_S = \{T \in Y' \ : \ T \supset S\}$. By definition of Y and Y', any 3-set $T \in \binom{\mathbb{Z}_{19}}{3}$ is contained in exactly one 4-set $K \in Y''$. The number of 3-sets $T \in \binom{\mathbb{Z}_{19}}{3}$ containing S is 17. Each $K \in Y_S$ contains 2 such 3-sets, and each $T \in Y'_S$ contains one such 3-set. Therefore, we get $2|Y_S| + |Y'_S| = 17$, and hence $|Y'_S| \geq 1$. Note that $|Y'| = \binom{19}{3} - 4 \cdot 228 = 57$, which is the number of blocks in an $S(2,3,19)$. Therefore, we necessarily have $|Y'_S| = 1$, and the result follows. □

Proposition 3.3.1 shows that, to construct $\mathcal{B}_{\infty_i}$, it is sufficient to construct $\mathcal{B}'_{\infty_i}$. Therefore, to construct $\mathcal{B}$, it is sufficient to construct $\mathcal{B}'_{\infty_1}$, $\mathcal{B}'_{\infty_2}$, and $\mathcal{B}'$. Since $\mathcal{B}_{\infty_1,\infty_2} = \mathcal{B}_{\infty_2,\infty_1}$, the following condition also has to be satisfied: For any $T \in \binom{\mathbb{Z}_{19}}{3}$, T is contained in a 4-set in $\mathcal{B}'_{\infty_1}$ if and only if T is contained in a 4-set in $\mathcal{B}'_{\infty_2}$. Also, since $\mathcal{D}$ contains each 4-set exactly once, therefore $\mathcal{B}'_{\infty_1}$ and $\mathcal{B}'_{\infty_2}$ have to be disjoint. We get the following lemma. The details of the proof are left to the reader.

Lemma 3.3.1. *There exists a $\mathcal{G}$-invariant $S(4,5,21)$, if and only if there exists a triple $(\Delta_1, \Delta_2, \Delta_3)$ such that:*

(1) For $i \in \{1,2\}$, Δ_i is a union of 12 orbits of $\mathcal{G} \mid \binom{\mathbb{Z}_{19}}{4}$, where any 3-subset of $\mathbb{Z}_{19}$ is contained in at most one 4-set in Δ_i.

(2) Any 3-subset of $\mathbb{Z}_{19}$ is contained in a 4-set in Δ_1 if and only if it is contained in a 4-set in Δ_2.

(3) $\Delta_1 \cap \Delta_2 = \emptyset$, and

(4) Δ_3 is a union of 36 orbits of $\mathcal{G} \mid \binom{\mathbb{Z}_{19}}{5}$, where any 4-subset of $\mathbb{Z}_{19}$ is contained in exactly one set in $\Delta_1 \cup \Delta_2 \cup \Delta_3$.

In Lemma 3.3.1, (4) already implies (3), but we mention (3) seperately since it will be of use later. The number of orbits of $\mathcal{G} \mid \binom{\mathbb{Z}_{19}}{3}$, $\mathcal{G} \mid \binom{\mathbb{Z}_{19}}{4}$ and $\mathcal{G} \mid \binom{\mathbb{Z}_{19}}{5}$ are $\binom{19}{3}/19 = 51$, $\binom{19}{4}/19 = 204$, and $\binom{19}{5}/19 = 612$, respectively. Let $\mathbf{A'} = \mathbf{A_{3,4}}$ and $\mathbf{A''} = \mathbf{A_{4,5}}$ be the 51×204 and 204×612 Kramer-Mesner matrices, respectively. Then, the following lemma is a consequence of Lemma 3.3.1. We omit the proof.

Lemma 3.3.2. *There exists a $\mathcal{G}$-invariant $S(4,5,21)$, if and only if there exists a triple $(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$ such that:*

(1) *For* $i \in \{1, 2\}$*,* $\mathbf{u_i}$ *is a 204-dimensional* $\{0, 1\}$*-vector with exactly 12 one's, such that* $\mathbf{A'u_i}$ *is a 51-dimensional vector with exactly 48 one's and 3 zero's.*

(2) $\mathbf{A'u_1} = \mathbf{A'u_2}$*.*

(3) $\mathbf{u_1} \cdot \mathbf{u_2} = 0$*, that is* $\mathbf{u_1}$ *and* $\mathbf{u_2}$ *are orthogonal.*

(4) $\mathbf{u_3}$ *is a 612-dimensional* $\{0, 1\}$*-vector with exactly 36 one's, such that* $\mathbf{A''u_3} + \mathbf{u_1} + \mathbf{u_2} = \mathbf{j}$*, where* $\mathbf{j}$ *is the 204-dimensional vector of all one's.*

An exhaustive computer search shows that there are exactly 290 vectors $\mathbf{u}$ satisfying Lemma 3.3.2(1). Out of these vectors, there are exactly 576 pairs $\{\mathbf{u_1}, \mathbf{u_2}\}$ (and hence 1152 ordered pairs $(\mathbf{u_1}, \mathbf{u_2})$) satisfying (2) and (3).

To reduce computation time for the search for $\mathbf{u_3}$ satisfying (4), we use Lemma 3.1.1. Define $\beta, \theta \in \mathcal{S}$ as $x^\beta = 2x$, $x^\theta = x$ for $x \in \mathbb{Z}_{19}$, and $(\infty_i)^\beta = \infty_i$, $(\infty_i)^\theta = \infty_{3-i}$ for $i \in \{1, 2\}$. Note that $N_{\mathcal{S}}(\mathcal{G}) = \langle \alpha, \beta, \theta \rangle$. The action of θ just switches the roles of $\mathbf{u_1}$ and $\mathbf{u_2}$. Therefore, it is sufficient to consider only one pair from each β-orbit on the set of pairs $\{\Delta_1, \Delta_2\}$ (see Lemma 3.3.1) corresponding to the 576 pairs $\{\mathbf{u_1}, \mathbf{u_2}\}$. A computer search shows that there are 32 orbits of β, each of size 18, on these 576 pairs. Let $\{\mathbf{u_1^1}, \mathbf{u_2^1}\}, ..., \{\mathbf{u_1^{32}}, \mathbf{u_2^{32}}\}$ be orbit representatives.

For $1 \leq i \leq 32$, we modify $\mathbf{A''}$ and construct $\mathbf{A''_i}$ as follows. We delete any column of $\mathbf{A''}$ that contain entries greater than 1, or that are not orthogonal to $\mathbf{u_1^i}$ or $\mathbf{u_2^i}$, since the corresponding entry of $\mathbf{u_3}$ must be zero in any solution of $\mathbf{A''u_3} + \mathbf{u_1^i} + \mathbf{u_2^i} = \mathbf{j}$. Then, we delete any row of $\mathbf{A''}$ where the corresponding entry in $\mathbf{u_1^i}$ or $\mathbf{u_2^i}$ is 1. We obtain $180 \times s_i$ matrices $\mathbf{A''_i}$, where $292 \leq s_i \leq 297$. What is left is to search for 36 mutually orthogonal columns of $\mathbf{A''_i}$. Compared to the 315×1035 matrix $\mathbf{A}$ in the original approach, where we had to search for 63 mutually orthogonal columns, the matrices $\mathbf{A''_i}$ are much smaller and easier to work with. An exhaustive search for all 32 cases shows that there exists no solution, which proves Theorem 3.3.1.

Theorem 3.3.2. *Let $\alpha \in Aut(\mathcal{D})$ be of prime order p. Then, one of the following holds:*

(1) $p = 2$ *and* $|fix(\alpha)| \in \{1, 3\}$.

(2) $p = 3$ *and* $|fix(\alpha)| = 0$.

(3) $p = 5$ *and* $|fix(\alpha)| = 1$.

(4) $p = 7$ *and* $|fix(\alpha)| = 0$.

Proof: Follows by Theorems 3.2.1 and 3.3.1. □

3.4 AUTOMORPHISM GROUPS OF PRIME POWER ORDER

In this section, we derive some results on possible Sylow subgroups of $Aut(\mathcal{D})$.

Lemma 3.4.1. *Let $\mathcal{P} \leq Aut(\mathcal{D})$, where $|\mathcal{P}| = p^k$, $p \in \{3, 5, 7\}$ and $k \geq 1$. Then, $k = 1$.*

Proof: First let $p = 3$. If $\mathcal{P}$ has a fixed point, then $\mathcal{P}$ has an element of order 3 with fixed points, which contradicts Theorem 3.3.2(2). Therefore, $\mathcal{P}$ has no fixed points. Since $21 \equiv 3 \pmod 9$, $\mathcal{P}$ has at least one orbit of size 3. Let Y be such an orbit and $y \in Y$. Then, $\mathcal{P}_y$ has fixed points and by the above argument, $|\mathcal{P}_y| = 1$. Therefore, by Theorem 2.0.6(1), $|\mathcal{P}| = |\mathcal{P}_y| \cdot |Y| = 3$. The cases $p = 5$ and $p = 7$ follow by similar arguments. □

Lemma 3.4.2. *Let $\mathcal{P} \leq Aut(\mathcal{D})$, where $|\mathcal{P}| = 4$. Then,*

(1) $\mathcal{P}$ *is of type* $1^1 4^5$.

(2) *For any $\omega \in \mathcal{P}$, where $\omega \neq 1$, we have $|fix(\omega)| = 1$.*

Proof: (1) If $\mathcal{P} \cong C_4$, the result follows by Theorem 3.3.2(1). If $\mathcal{P} \cong V_4$, say $\mathcal{P} = \{1, \alpha, \beta, \theta\}$, then by Theorems 2.0.6(2) and 3.3.2(1), we get $(21 + 3 \cdot 1)/4 \leq o(\mathcal{P}) \leq (21 + 3 \cdot 3)/4$, and hence $o(\mathcal{P}) \in \{6, 7\}$. Assume that $o(\mathcal{P}) = 7$. Then, $\mathcal{P}$ is necessarily of type $1^1 2^2 4^4$, where two elements of order 2 have 3 fixed points and one element of order 2 has 1 fixed point. Hence, without loss of generality, we have

$\alpha = (1)(2)(3)(4, 5)(6, 7)(8, 9)(10, 11)(12, 13)(14, 15)(16, 17)(18, 19)(20, 21)$,

$\beta = (1)(2, 3)(4)(5)(6, 8)(7, 9)(10, 12)(11, 13)(14, 16)(15, 17)(18, 20)(19, 21)$, and

$\theta = (1)(2, 3)(4, 5)(6, 9)(7, 8)(10, 13)(11, 12)(14, 17)(15, 16)(18, 21)(19, 20)$.

Let $T = \{2, 3, 6, 7\}$, $T' = \{2, 3, 6, 9\}$, and B and B' be the blocks containing T and T' respectively. Since $T \in fix_4(\alpha)$, $B \in fix'(\alpha)$ and we necessarily have $B = \{1, 2, 3, 6, 7\}$. Similarly, since $T' \in fix_4(\theta)$, we get $B' = \{1, 2, 3, 6, 9\}$, which is a contradiction since the 4-set $\{1, 2, 3, 6\}$ cannot be contained in more than one block. So, $o(\mathcal{P}) = 6$ and hence $\mathcal{P}$ is of type $1^1 4^5$.

(2) Follows by (1) and Theorem 2.0.6(1). □

Lemma 3.4.3. *$\mathcal{D}$ cannot have an automorphism group of order 8.*

Proof: Assume that $\mathcal{P} \leq Aut(\mathcal{D})$, where $|\mathcal{P}| = 8$. Any element α of order 2 in $\mathcal{P}$ is contained in a subgroup of order 4. Hence, by Lemma 3.4.2(2), α is of type $1^1 2^{10}$. Therefore, for any $\omega \in \mathcal{P}$, where $\omega \neq 1$, we have $|fix(\omega)| = 1$. Therefore, we get $o(\mathcal{P}) = (21 + 7 \cdot 1)/8 = 3.5$, contradiction. □

Theorem 3.4.1. *$|Aut(D)|$ divides $2^2 \cdot 3 \cdot 5 \cdot 7$.*

Proof: Follows by Theorem 3.3.2 and Lemmas 3.4.1 and 3.4.3. □

3.5 NORMALIZERS OF AUTOMORPHISMS OF ORDERS 5 AND 7

Lemma 3.5.1. *Let $\alpha \in \mathcal{G} = Aut(\mathcal{D})$ be of order 7. Then,* *(1)* $|C_{\mathcal{G}}(\alpha)|$ *divides 21.*

(2) $|N_{\mathcal{G}}(\alpha)|$ *divides 42.*

(3) If $\beta \in N_{\mathcal{G}}(\alpha)$ *is of order 2, then* β *is of type* $1^3 2^9$.

Proof: (1) An element of type 7^3 cannot be centralized by an element of type $1^3 2^9$, $1^1 2^{10}$, or $1^1 5^4$. Therefore, the result follows by Theorems 3.3.2 and 3.4.1.

(2) Since $Aut(C_7) \cong C_6$, by (1) and Theorem 2.0.6(3), we get $|N_{\mathcal{G}}(\alpha)|$ divides $21 \cdot 6$. Therefore, the result follows by Theorem 3.4.1.

(3) Assume that β is of type $1^1 2^{10}$. Then, without loss of generality, we have $\alpha = (1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14)(15, 16, 17, 18, 19, 20, 21)$, and $\beta = (1)(2, 7)(3, 6)(4, 5)(8, 15)(9, 21)(10, 20)(11, 19)(12, 18)(13, 17)(14, 16)$.

Let $T = \{2, 3, 6, 7\}$ and B be the block containing T. Since $T \in fix_4(\beta)$, we necessarily have $B = \{1, 2, 3, 6, 7\}$, which is a contradiction since the 4-set $\{1, 2, 3, 7\}$ is contained in both B and $B^\alpha = \{1, 2, 3, 4, 7\}$. □

Lemma 3.5.2. *Let* $\alpha \in Aut(\mathcal{D})$ *be of order 5, and* $\Delta = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ *be the set of* α*-orbits of size 5. Then,* $fix'(\alpha) = \mathcal{B} \cap \Delta$*, and* $|fix'(\alpha)| = 2$.

Proof: We have $fix'(\alpha) = \mathcal{B} \cap fix_5(\alpha) = \mathcal{B} \cap \Delta$. Since $\mathcal{D}$ has 1197 blocks, we have $|fix'(\alpha)| \equiv 2 \pmod 5$, and hence the result follows. □

Lemma 3.5.3. *Let* $\alpha \in \mathcal{G} = Aut(\mathcal{D})$ *be of order 5. Then,* $|N_{\mathcal{G}}(\alpha)|$ *divides 10.*

Proof: An element of type $1^1 5^4$ cannot be normalized by an element of type 3^7 or 7^3. Therefore, by Theorems 3.3.2 and 3.4.1, $|N_{\mathcal{G}}(\alpha)|$ divides 20. Assume that $|N_{\mathcal{G}}(\alpha)| = 20$, and define Δ as in Lemma 3.5.2. Then, by Lemma 3.4.2, a Sylow 2-subgroup of $N_{\mathcal{G}}(\alpha)$ acts transitively on Δ. Hence, either $\Delta \subset \mathcal{B}$ or $\Delta \cap \mathcal{B} = \emptyset$, which contradicts Lemma 3.5.2. □

3.6 FURTHER RESULTS ON THE POSSIBLE AUTOMORPHISM GROUP ORDERS

Lemma 3.6.1. *$|Aut(\mathcal{D})|$ is not divisible by 35.*

Proof: Let $\mathcal{G} = Aut(\mathcal{D})$, and assume that 35 divides $|\mathcal{G}|$. Let $\alpha \in \mathcal{G}$ be of order 7, and n_7 denote the number of Sylow 7-subgroups of $\mathcal{G}$. By Lemma 3.5.1(2), we get $5 \mid n_7$. Note that $2^k \cdot 5 \not\equiv 1 \pmod 7$ for any $k \geq 0$, and $2^m \cdot 3 \cdot 5 \not\equiv 1 \pmod 7$ for any $m \in \{1, 2\}$. Therefore, by Theorem 3.4.1, we necessarily have $n_7 = 15$. Hence, by Lemma 3.5.1(2), $|N_{\mathcal{G}}(\alpha)|$ divides 14. Therefore, $|\mathcal{G}| = 3 \cdot 5 \cdot 7$ or $|\mathcal{G}| = 2 \cdot 3 \cdot 5 \cdot 7$. By Theorem 2.0.6(4), $|\mathcal{G}| \neq 3 \cdot 5 \cdot 7$, and hence by Theorem 2.0.6(5), $|\mathcal{G}| \neq 2 \cdot 3 \cdot 5 \cdot 7$. □

Lemma 3.6.2. *Let $\alpha \in Aut(\mathcal{D})$ be of order 7. Then $\langle \alpha \rangle$ is normal in $Aut(\mathcal{D})$ and hence $|Aut(\mathcal{D})|$ divides 42.*

Proof: By Theorem 3.4.1 and Lemma 3.6.1, we get $|Aut(\mathcal{D})|$ divides $2^2 \cdot 3 \cdot 7$. Then, the result follows by Sylow theorems and Lemma 3.5.1(2). □

Lemma 3.6.3. *$\mathcal{D}$ cannot have an automorphism group of order 60.*

Proof: Assume that $\mathcal{G} \leq Aut(\mathcal{D})$ and $|\mathcal{G}| = 60$. By Lemma 3.4.2(2), any element of order 2 in $\mathcal{G}$ is of type $1^1 2^{10}$. Therefore, by Theorem 3.3.2, for any $\omega \in \mathcal{G}$ where $\omega \neq 1$, we have $|fix(\omega)| \leq 1$. Therefore, we get $o(G) \leq (21 + 59 \cdot 1)/60$ and hence $o(G) = 1$, which is a contradiction since 21 does not divide 60. □

Lemma 3.6.4. *Let $\alpha \in Aut(\mathcal{D})$ be of order 5. Then $\langle \alpha \rangle$ is normal in $Aut(\mathcal{D})$ and hence $|Aut(\mathcal{D})|$ divides 10.*

Proof: By Theorem 3.4.1 and Lemma 3.6.1, we get $|Aut(\mathcal{D})|$ divides $2^2 \cdot 3 \cdot 5$. Let n_5 be the number of Sylow 5-subgroups of $Aut(\mathcal{D})$, and assume that $\langle \alpha \rangle$ is not normal in $Aut(\mathcal{D})$. Then, we necessarily have $n_5 = 6$, and hence by Lemma 3.6.3, we get $|Aut(\mathcal{D})| = 30$, which contradicts Theorem 2.0.6(4). Therefore, the result follows by Lemma 3.5.3. □

Lemma 3.6.5. *$\mathcal{D}$ cannot have an automorphism group of order 12.*

Proof: Assume that $\mathcal{G} \leq Aut(\mathcal{D})$ and $|\mathcal{G}| = 12$. By Theorem 3.3.2(2), the orbits of $\mathcal{G} \mid X$ are of size 3, 6, or 12. Since $21 \equiv 3 \pmod 6$, $\mathcal{G}$ has at least one orbit of size 3. Let $Y = \{x, y, z\}$ be such an orbit. Then, $|\mathcal{G}_x| = 12/3 = 4$, and $\{y, z\}$ is a union of $\mathcal{G}_x$-orbits on X, which contradicts Lemma 3.4.2(1). □

Theorem 3.6.1. *$|Aut(\mathcal{D})|$ divides 2^2, $2 \cdot 5$, or $2 \cdot 3 \cdot 7$.*

Proof: Follows by Theorem 3.4.1 and Lemmas 3.6.1, 3.6.2, 3.6.4, and 3.6.5.

3.7 AUTOMORPHISM GROUPS OF ORDERS 14 AND 42

After proving Theorem 3.6.1, it is natural to consider groups of order 42 first. By Lemma 3.5.1, there are only 2 possible automorphism groups of order 42. Using an exhaustive computer search for each group, we prove Theorem 3.7.1. We omit the details of this computation since we also consider groups of order 14 and prove Theorem 3.7.2, which already imples Theorem 3.7.1 since any group of order 42 has a subgroup of order 14.

Theorem 3.7.1. *$\mathcal{D}$ cannot have an automorphism group of order 42.*

Theorem 3.7.2. *$\mathcal{D}$ cannot have an automorphism group of order 14.*

In this section, we give the details for the proof of Theorem 3.7.2. By Lemma 3.5.1(1), the cyclic group C_{14} cannot be an automorphism group of $\mathcal{D}$. So, let $\mathcal{G}$ be isomorphic to the dihedral group D_{14}. By Lemma 3.5.1(3), any element of order 2 in $\mathcal{G}$ is of type $1^3 2^9$. Therefore, $\mathcal{G}$ is of type 7^3, and without loss of generality we can say $X = \mathbb{Z}_{21}$ and $\mathcal{G} = \langle \alpha, \beta \rangle$ where $x^\alpha = x + 3$ and $x^\beta = 13x$ for all $x \in X$.

We have $|fix_4(\alpha)| = |fix_5(\alpha)| = 0$, $|fix_4(\beta)| = \binom{9}{1}\binom{3}{2} + \binom{9}{2} = 63$, and $|fix_5(\beta)| = \binom{9}{1}\binom{3}{3} + \binom{9}{2}\binom{3}{1} = 117$. Hence, there are 63 $\mathcal{G}$-orbits of size 7, and $\left(\binom{21}{4} - 63 \cdot 7\right)/14 = 396$ $\mathcal{G}-$orbits of size 14, and hence in total 459 $\mathcal{G}$-orbits on $\binom{X}{4}$. Similarly, there are 117 $\mathcal{G}$-orbits of size 7, and $\left(\binom{21}{5} - 117 \cdot 7\right)/14 = 1395$ $\mathcal{G}-$orbits of size 14, and hence in total 1512 $\mathcal{G}$-orbits on $\binom{X}{5}$. We construct the 459×1512 Kramer-Mesner matrix. After deleting columns that contain entries greater than 1, we obtain a 459×1080 matrix. An exhaustive search for a solution using Theorem 3.1.1 with this matrix is not feasible with our computer program. We develop a special form of Theorem 3.1.1.

Using the terminology of Theorem 3.1.1, suppose that $r = r_1 + r_2 + ... + r_m$ and $s = s_1 + s_2 + ... + s_n$, and consider the matrix $\mathbf{A}$ as an $m \times n$ block matrix, where the $(p,q)^{th}$ block is an $r_p \times s_q$ matrix, say $\mathbf{B_{p,q}}$. Moreover, suppose that for any (p,q), the column sum in $\mathbf{B_{p,q}}$ is constant. Define an $m \times n$ matrix $\mathbf{B} = (b_{p,q})$, where $b_{p,q}$ is the constant column sum in $\mathbf{B_{p,q}}$. Define an m-dimensional column vector $\mathbf{r} = (r_p)$, where r_p is defined as above. Also define an $n \times n$ block matrix $\mathbf{C}$, where the $(q,q')^{th}$ block is an $s_{q'}$-dimensional row vector with all entries equal to $\delta_{q,q'}$. Here, δ is the Kronecker delta function. Then, the following theorem is a consequence of Theorem 3.1.1. The proof is left to the reader.

Theorem 3.7.3. *There exists a $\mathcal{G}$-invariant t-(v,k,λ) design if and only if the following conditions hold:*

(1) There exists an n-dimensional column vector $\mathbf{v}$ with positive integer entries such that $\mathbf{Bv} = \lambda\mathbf{r}$.

(2) There exists an s-dimensional column vector $\mathbf{u}$ with positive integer entries such that $\mathbf{Cu} = \mathbf{v}$ and $\mathbf{Au} = \lambda\mathbf{j}$.

In applying Theorem 3.7.3, we still need to find solutions of $\mathbf{Au} = \lambda\mathbf{j}$ as in Theorem 3.1.1, that is, we need to search for a set of columns of $\mathbf{A}$ that add up to $\lambda\mathbf{j}$. What Theorem 3.7.3 provides additionally is that if $\mathbf{A}$ can be expressed as a block matrix with constant column sum in each block as described above, then solutions of $\mathbf{Bv} = \lambda\mathbf{r}$ give a hint on the number of columns of $\mathbf{A}$ to be selected from each set of s_q columns for $1 \leq q \leq n$.

Let $\mathbf{A}$ be the 459×1080 matrix constructed above. We say that the $\mathcal{G}$-orbits on $\binom{X}{5}$ corresponding to the columns of $\mathbf{A}$ (which are obtained by removing those columns that contain entries greater than 1) are *admissible* orbits. We try to express $\mathbf{A}$ as a block matrix with constant column sum in each block. Define $\theta \in \mathcal{S} = \mathcal{S}_X$ as $x^\theta = x + 7$, and let $\Sigma_1, ..., \Sigma_7$ be the θ-orbits on X. For any $Y \in \binom{X}{4} \cup \binom{X}{5}$, define $n_i = |Y \cap \Sigma_i|$ for $1 \leq i \leq 7$. Let $m_1, ..., m_7$ be a permutation of $n_1, ..., n_7$, where $m_i \geq m_{i+1}$ for $1 \leq i \leq 6$, and suppose that k is the largest index such that $m_k > 0$. Then, if the $\mathcal{G}$-orbit on $\binom{X}{|Y|}$ containing Y is a small orbit (of size 7), we say that Y is of *type* $(m_1, m_2, ..., m_k)S$, and if that orbit is a big orbit (of size 14), we say that Y is of *type* $(m_1, m_2, ..., m_k)B$. Since $\mathcal{G} \in N_{\mathcal{S}}(\theta)$, all sets in a $\mathcal{G}$-orbit are of the same type.

The set of $\mathcal{G}$-orbits on $\binom{X}{4}$ partitions into orbits of types $(3,1)B$, $(2,2)S$, $(2,2)B$, $(2,1,1)S$, $(2,1,1)B$, $(1,1,1,1)S$, and $(1,1,1,1)B$. Using a computer program, we see that these sets of orbits are of sizes 9, 9, 9, 27, 189, 27, and 189, respectively. Similarly, the set of admissible $\mathcal{G}$-orbits on $\binom{X}{5}$ partitions into orbits of types $(3,1,1)S$, $(3,1,1)B$, $(2,2,1)S$, $(2,2,1)B$, $(2,1,1,1)B$, $(1,1,1,1,1)S$, and $(1,1,1,1,1)B$, where these sets of orbits are of sizes 9, 63, 27, 99, 558, 27, and 252 respectively.

We partition these sets of $\mathcal{G}$-orbits further. Let Δ_1, Δ_{14}, Δ_{15}, Γ_1, Γ_2, Γ_{12}, and Γ_{13} be the sets of $\mathcal{G}$-orbits of types $(3,1)B$, $(1,1,1,1)S$, $(1,1,1,1)B$, $(3,1,1)S$, $(3,1,1)B$, $(1,1,1,1,1)S$, and $(1,1,1,1,1)B$, respectively. For any $\mathcal{G}$-orbit of type

$(2,2)S$, $(2,2)B$, $(2,2,1)S$, or $(2,2,1)B$, any set Y in the orbit satisfies $Y \cap \Sigma_i = \{x, x+7\}$, and $Y \cap \Sigma_j = \{y, y+7\}$ for some $1 \leq i, j \leq 7$. If Y is of type $(2,2)S$ or $(2,2,1)S$, then $x \equiv y \pmod 3$. If Y is of type $(2,2)B$ or $(2,2,1)B$, then $x \not\equiv y \pmod 3$, and without loss of generality suppose that $y \equiv x+1 \pmod 3$. Let Δ_2-Δ_4, Δ_5-Δ_7, Γ_3-Γ_5, and Γ_6-Γ_8 be the sets of $\mathcal{G}$-orbits of types $(2,2)S$, $(2,2)B$, $(2,2,1)S$, and $(2,2,1)B$, for $x \equiv 0,1,2 \pmod 3$, respectively. We define $\Delta_8, ..., \Delta_{13}$, and $\Gamma_9, ..., \Gamma_{11}$ as the sets of $\mathcal{G}$-orbits of types $(2,1,1)S$, $(2,1,1)B$, and $(2,1,1,1)B$ in a similar way. One can see that, if the matrix $\mathbf{A}$ is considered as a 15×13 block matrix according to this partitioning, then the column sum is a constant in each block of $\mathbf{A}$. This can also be verified easily by using a computer program. Using the terminology of Theorem 3.7.3, we obtain the 15×13 matrix $\mathbf{B}$ as given below.

$$\begin{pmatrix}
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 2 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 5
\end{pmatrix}$$

Moreover, we get the vectors $\mathbf{r} = (r_p) = (9, 3, 3, 3, 3, 3, 3, 9, 9, 9, 63, 63, 63, 27, 189)^T$ and $(s_q) = (9, 63, 9, 9, 9, 33, 33, 33, 186, 186, 186, 72, 252)$. The equation $\mathbf{Bv} = \mathbf{r}$ has a unique solution, namely $\mathbf{v} = (v_q) = (9, 0, 3, 3, 3, 3, 3, 3, 15, 15, 15, 27, 9)^T$. Therefore, while searching for a solution of $\mathbf{Au} = \mathbf{j}$, we basically need to search for 108 mutually orthogonal columns out of 1080 columns of $\mathbf{A}$, where we select exactly v_q columns from the set of s_q columns corresponding to the orbits in Γ_q, for $1 \leq q \leq 13$. Since $v_1 = s_1 = 9$, all of the first 9 columns are selected, and since $v_2 = 0$, none of the next 63 columns are selected. We order the remaining columns of $\mathbf{A}$ according to the ordering $\Gamma_3, \Gamma_6, \Gamma_8, \Gamma_9, \Gamma_4, \Gamma_7, \Gamma_{10}, \Gamma_5, \Gamma_{11}, \Gamma_{12}, \Gamma_{13}$, and perform an 11-step backtracking search. We know that the second derived designs $\mathcal{D}_{0,7}$, $\mathcal{D}_{7,14}$, and $\mathcal{D}_{0,14}$ have to be Steiner triple systems of order 19. The ordering of the columns guarantee that these 3 Steiner triple systems are formed in the 4^{th}, 7^{th}, and 9^{th} steps respectively. Because of this, the program reaches the 5^{th} step rarely, and even if it reaches the 5^{th} step, then it reaches the 8^{th} step rarely, and so on. This makes the program run much faster. We also make use of Lemma 3.1.1. In the first step, there are 27 solutions, and $N_{\mathcal{S}}(\mathcal{G})$ has 6 orbits on these 27 solutions. We consider only one solution from each orbit. We perform the 11-step exhaustive search which shows that there exists no solution. Therefore, we get Theorem 3.7.2 and hence Theorem 3.7.1.

Theorem 3.7.4. $|Aut(\mathcal{D})| \in \{1, 2, 3, 4, 5, 6, 7, 10, 21\}$.

Proof: Follows by Theorems 3.6.1, 3.7.1 and 3.7.2.

3.8 TYPES OF THE POSSIBLE AUTOMORPHISM GROUPS

The isomorphism classes and orbit sizes of the possible automorphism groups of orders 2, 3, 4, 5, and 7 are given by Theorem 3.3.2 and Lemma 3.4.2. In addition, by Theorem 3.3.2, an automorphism group of order 21 is necessarily of type 21^1. In this section, we specify the types of the possible automorphism groups of orders 6 and 10.

Lemma 3.8.1. *Let $\mathcal{G} = Aut(\mathcal{D})$ be of order 6. Then,*

(1) $\mathcal{G} \cong C_6$ *and* $\mathcal{G}$ *is of type* $3^1 6^3$*, or*

(2) $\mathcal{G} \cong D_6$ *and* $\mathcal{G}$ *is of type* $3^3 6^2$*.*

Proof: (1) follows by Theorem 3.3.2. For (2), let $\mathcal{G} \cong D_6$. Since all elements of order 2 are conjugate in $\mathcal{G}$, we get $o(\mathcal{G}) = (21+3\cdot 1)/6 = 4$ or $o(\mathcal{G}) = (21+3\cdot 3)/6 = 5$, that is $\mathcal{G}$ is of type $3^1 6^3$ or $3^3 6^2$. Assume that $\mathcal{G}$ is of type $3^1 6^3$. Then, any element of order 2 in $\mathcal{G}$ have one fixed point, and without loss of generality, we have $\mathcal{G} = \langle \alpha, \beta \rangle$

where $\alpha = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)$, and $\beta = (1)(2,3)(4,7)(5,9)(6,8)(10,13)(11,15)(12,14)(16,19)(17,21)(18,20)$.

Let $T = \{2,3,4,7\}$, $T' = \{2,3,5,9\}$, and B and B' be the blocks containing T and T', respectively. Since $T, T' \in fix_4(\beta)$, we necessarily have $B = \{1,2,3,4,7\}$ and $B' = \{1,2,3,5,9\}$, which is a contradiction since B' and $B^\alpha = \{1,2,3,5,8\}$ both contain the 4-set $\{1,2,3,5\}$. □

Lemma 3.8.2. *Let $\mathcal{G} = Aut(\mathcal{D})$ be of order 10. Then,*

(1) $\mathcal{G} \cong C_{10}$ and $\mathcal{G}$ is of type $1^1 10^2$, or

(2) $\mathcal{G} \cong D_{10}$ and $\mathcal{G}$ is of type $1^1 5^2 10^1$ or $1^1 10^2$.

Proof: (1) follows by Theorem 3.3.2. For (2), let $\mathcal{G} \cong D_{10}$. By Lemma 3.5.2, the non-trivial orbits of $\mathcal{G}$ are of size 5 or 10. If $\mathcal{G}$ is of type $1^1 5^4$, then the elements of order 2 in $\mathcal{G}$ have 5 fixed points, which contradicts Theorem 3.3.2(1). Then, the result follows. □

We summarize all results in the following theorem.

Theorem 3.8.1. *Suppose that an $S(4,5,21)$, say $\mathcal{D}$, exists, and let $\mathcal{G} = Aut(\mathcal{D})$. Then, exactly one of the following holds:*

(1) $|\mathcal{G}| = 1$,

(2) $\mathcal{G} \cong C_2$ and $\mathcal{G}$ is of type $1^3 2^9$ or $1^1 2^{10}$.

(3) $\mathcal{G} \cong C_3$ and $\mathcal{G}$ is of type 3^7.

(4) $\mathcal{G} \cong C_4$ and $\mathcal{G}$ is of type $1^1 4^5$.

(5) $\mathcal{G} \cong V_4$ and $\mathcal{G}$ is of type $1^1 4^5$.

(6) $\mathcal{G} \cong C_5$ and $\mathcal{G}$ is of type $1^1 5^4$.

(7) *$\mathcal{G} \cong C_6$ and $\mathcal{G}$ is of type $3^1 6^3$.*

(8) *$\mathcal{G} \cong D_6$ and $\mathcal{G}$ is of type $3^3 6^2$.*

(9) *$\mathcal{G} \cong C_7$ and $\mathcal{G}$ is of type 7^3.*

(10) *$\mathcal{G} \cong C_{10}$ and $\mathcal{G}$ is of type $1^1 10^2$.*

(11) *$\mathcal{G} \cong D_{10}$ and $\mathcal{G}$ is of type $1^1 5^2 10^1$ or $1^1 10^2$.*

(12) *$\mathcal{G} \cong C_{21}$ and $\mathcal{G}$ is of type 21^1.*

(13) *$\mathcal{G} \cong C_7^3$ and $\mathcal{G}$ is of type 21^1.*

3.9 CONCLUDING REMARKS

By Theorem 3.8.1, the largest possible automorphism group of an $S(4, 5, 21)$ is of order 21. Interestingly, making an exhaustive search for an $S(4, 5, 21)$ with an automorphism group of order 19 or 14 require less computation time than a search for an $S(4, 5, 21)$ with an automorphism group of order 21. In Section 3.3, we considered a group of order 19. Since such a group has 2 fixed points, we were able to consider the first and second derived designs and reduce the computation time. In Section 3.7, we considered the dihedral group of order 14. Since the elements of order 2 have fixed 4-sets and fixed 5-sets, we get a lot of restrictions on the possible number of orbits of each type in the design. These restrictions allowed us to perform an exhaustive search.

The Kramer-Mesner matrix for a group $\mathcal{G}$ of order 21 is a 285×969 matrix. In $\mathcal{G}$, neither an element of order 7 nor an element of order 3 have fixed points, fixed 4-sets or fixed 5-sets, that is, $\mathcal{G}$ necessarily acts regularly on X and semiregularly on $\binom{X}{4}$

and $\binom{X}{5}$. Therefore, we cannot use the ideas used in Section 3.3. Also it is harder to get restrictions on the structure of the design as in Section 3.7.

Automorphism groups of orders 10 or less are even harder to consider. Further analyses and computation techniques are required for making an exhaustive search for an $S(4,5,21)$, with any one of the groups given in Theorem 3.8.1 as an automorphism group.

CHAPTER 4

ON LARGE SETS OF PROJECTIVE PLANES OF ORDERS 3 AND 4

4.1 BASIC RESULTS

The following well known theorem may be found in [5, 28].

Theorem 4.1.1. *Any automorphism of a projective plane fixes as many points as it fixes blocks.*

The following theorem is a consequence of Theorem 4.1.1.

Theorem 4.1.2. *Let $\mathcal{L}$ be a large set of projective planes of order n, and let $\alpha \in Aut(\mathcal{L})$. Then, $|fix(\alpha)| \cdot |fix'(\alpha)| = |fix_{n+1}(\alpha)|$.*

Proof: A design not fixed by α contains no block fixed by α, and by Theorem 4.1.1 the number of blocks fixed by α in any design fixed by α is precisely $|fix(\alpha)|$. □

Let $\mathcal{G} \leq \mathcal{S}_X$. Let $\{\Delta_1, ..., \Delta_r\}$ be the collection of all orbits of $\mathcal{G}$ acting on the family of all t-(v, k, λ) designs with point set X, and $\{\Gamma_1, ..., \Gamma_s\}$ the collection of all orbits of $\mathcal{G}|\binom{X}{k}$. Define an incidence matrix $\mathbf{M} = (m_{ij})$ by:

$$m_{ij} = |\mathcal{B} \cap \Gamma_j| \cdot |\Delta_i|/|\Gamma_j|$$

where $\mathcal{D} = (X, \mathcal{B})$ is any particular design in orbit Δ_i. Thus, m_{ij} is the number of blocks of design $\mathcal{D}$ belonging to the orbit Γ_j, modified by the normalizing factor $|\Delta_i|/|\Gamma_j|$, or equivalently m_{ij} is the number of designs in Δ_i containing a particular k-set $B \in \Gamma_j$ as a block. The following theorem provides a way of constructing $\mathcal{G}$-invariant large sets.

Theorem 4.1.3. *[21] There exists a $\mathcal{G}$-invariant large set of t-(v,k,λ) designs if and only if there exists an r-dimensional $\{0,1\}$ row vector* $\mathbf{u}$ *such that* $\mathbf{uM} = \mathbf{j}$*, where* $\mathbf{j}$ *is the s-dimensional row vector of all 1's.*

Note that any rows of $\mathbf{M}$ that contain entries greater than 1 can be deleted, since the corresponding entry of $\mathbf{u}$ must be zero in any solution of $\mathbf{uM} = \mathbf{j}$.

The following lemmas will also be used in the construction and classification of $\mathcal{G}$-invariant large sets in later sections.

Lemma 4.1.1. *Suppose that $\mathcal{D} = (X, \mathcal{B})$ is a t-(v,k,λ) design which is unique up to isomorphism. Let $\mathcal{P} \leq \mathcal{G} = Aut(\mathcal{D})$ be a Sylow p-subgroup of $\mathcal{S} = \mathcal{S}_X$, and $n_{\mathcal{P}}$ be the number of $\mathcal{P}$-invariant t-(v,k,λ) designs with point set X. Then, $n_{\mathcal{P}} = |N_{\mathcal{S}}(\mathcal{P})|/|N_{\mathcal{G}}(\mathcal{P})|$.*

Proof: Let n be the number of pairs $(\mathcal{P}', \mathcal{D})$, where $\mathcal{P}'$ is a Sylow p-subgroup of $\mathcal{S}$, and $\mathcal{D}$ is a $\mathcal{P}'$-invariant t-(v,k,λ) design. Since $\mathcal{D}$ is unique up to isomorphism, S acts transitively on the family of all t-(v,k,λ) designs with point set X. Thus, there are $|\mathcal{S}|/|\mathcal{G}|$ distinct t-(v,k,λ) designs on X, and the full automorphism group of each one has $|\mathcal{G}|/|N_{\mathcal{G}}(\mathcal{P})|$ Sylow p-subgroups. Therefore, $n = |\mathcal{S}|/|N_{\mathcal{G}}(\mathcal{P})|$. On the other hand, there are $|\mathcal{S}|/|N_{\mathcal{S}}(\mathcal{P})|$ Sylow p-subgroups of $\mathcal{S}$, and since Sylow subgroups are conjugate, each one is an automorphism group of precisely $n_{\mathcal{P}}$ designs. Therefore, $n = n_{\mathcal{P}} \cdot |\mathcal{S}|/|N_{\mathcal{S}}(\mathcal{P})|$, and the result follows. □

Lemma 4.1.2. *Let $\mathcal{P}$ be a Sylow p-subgroup of $\mathcal{S} = \mathcal{S}_X$, and suppose that $\mathcal{L}_1 = (X, \mathbb{B}_1)$ and $\mathcal{L}_2 = (X, \mathbb{B}_2)$ are $\mathcal{P}$-invariant large sets $LS[N](t,k,v)$. Then, $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic if and only if there exists $\alpha \in N_{\mathcal{S}}(\mathcal{P})$ such that $(\mathcal{L}_1)^{\alpha} = \mathcal{L}_2$.*

Proof: The "if" part is trivial. For the "only if" part, suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic. Then, $(\mathcal{L}_1)^{\beta} = \mathcal{L}_2$ for some $\beta \in \mathcal{S}$. Therefore, $(\mathcal{L}_1)^{\beta \mathcal{P} \beta^{-1}} = \mathcal{L}_1$ and hence $\beta\mathcal{P}\beta^{-1}$ is a Sylow p-subgroup of $Aut(\mathcal{L}_1)$. Hence, there exists $\sigma \in Aut(\mathcal{L}_1)$ such that $\sigma\beta\mathcal{P}\beta^{-1}\sigma^{-1} = \mathcal{P}$. So, $\sigma\beta \in N_{\mathcal{S}}(\mathcal{P})$ and $(\mathcal{L}_1)^{\sigma\beta} = \mathcal{L}_2$. □

4.2 PROJECTIVE PLANES OF ORDER 3

It is well known that, up to isomorphism, there is a unique projective plane of order 3, denoted by $PG(2,3)$, whose full automorphism group is $PGL(3,3)$ of order $5,616$. Throughout this section, let $X = \mathbb{Z}_{13}$, $\mathcal{S} = \mathcal{S}_X$, and define $\alpha, \beta, \gamma, \rho, \sigma, \tau, \mu, \nu \in \mathcal{S}$ as

$\alpha : (0,1,2,3,4,5,6,7,8,9,10,11,12)$, i.e. $x^\alpha = x + 1$,

$\beta : (0)(1,2,4,8,3,6,12,11,9,5,10,7)$, i.e. $x^\beta = 2x$,

$\gamma : (0)(2)(3)(8)(12)(1,9)(4,6)(5,11)(7,10)$,

$\rho : (0)(2)(8)(9)(12)(1,3)(4,11)(5,10)(6,7)$,

$\sigma : (0)(1)(3)(9)(2,4,10)(5,6,8)(7,11,12)$,

$\tau : (0)(1)(3)(9)(2,8,12)(4,5,7)(6,11,10)$,

$\mu : (0)(1)(2)(3)(9)(4,10)(5,11)(6,7)(8,12)$,

$\nu : (0)(1)(2)(8)(12)(3,9)(4,10)(5,6)(7,11)$.

Let $\mathcal{D}_1 = (X, \mathcal{B}_1)$ be the projective plane of order 3 developed by the action of $\langle\alpha\rangle$ on the base block $\{0,1,3,9\}$. Note that $\mathcal{G} = Aut(\mathcal{D}_1) = \langle\alpha, \beta^4, \gamma, \rho, \sigma, \tau, \mu, \nu\rangle \cong PGL(3,3)$. It is well known that $\mathcal{G}$ is doubly transitive on X. Here, we have $\mathcal{G}_0 = \langle\beta^4, \gamma, \rho, \sigma, \tau, \mu, \nu\rangle$, $|\mathcal{G}_0| = 5,616/13 = 432$, $\mathcal{G}_{[0,1]} = \langle\sigma, \tau, \mu, \nu\rangle$, and $|\mathcal{G}_{[0,1]}| = 432/12 = 36$. Moreover, $\mathcal{G}_{[0,1]}$ has two orbits on $X \setminus \{0,1\}$. These orbits are $\{3,9\}$ and $\{2,4,5,6,7,8,10,11,12\}$. We have $\mathcal{G}_{[0,1,3]} = \langle\sigma, \tau, \mu\rangle$, $|\mathcal{G}_{[0,1,3]}| = 36/2 = 18$, $\mathcal{G}_{[0,1,2]} = \langle\mu, \nu\rangle$, and $|\mathcal{G}_{[0,1,2]}| = 36/9 = 4$.

Note that $N_{\mathcal{S}}(\alpha) = \langle\alpha, \beta\rangle$, and $N_{\mathcal{G}}(\alpha) = \langle\alpha, \beta^4\rangle$. Hence, $|N_{\mathcal{S}}(\alpha)| = 13 \cdot 12$ and $|N_{\mathcal{G}}(\alpha)| = 13 \cdot 3$. Therefore, by Lemma 4.1.1, the number of $\langle\alpha\rangle$-invariant projective planes of order 3 with point set X is precisely 4. One of these planes is $\mathcal{D}_1$, and the other three can be constructed by the action of $\langle\alpha\rangle$ on the base blocks $\{0,1,4,6\}$, $\{0,1,5,11\}$, and $\{0,1,8,10\}$. Let us call these planes $\mathcal{D}_2 = (X, \mathcal{B}_2)$, $\mathcal{D}_3 = (X, \mathcal{B}_3)$, and $\mathcal{D}_4 = (X, \mathcal{B}_4)$ respectively.

Let $\mathcal{L} = (X, \mathbb{B})$ be a putative large set $LS[55](2,4,13)$ of projective planes of order 3. In [10], Chouinard II states without proving that any element of prime order in $Aut(\mathcal{L})$ must be either an element of order 5 fixing 3 points, or an element of order 11 or 13. This statement is in fact incorrect. Elements of order 3 are missing. In this chapter we construct large sets of projective planes of order 3 having an automorphism of order 3. We now prove the following statement :

Lemma 4.2.1. *Let $\omega \in Aut(\mathcal{L})$ be of prime order p. Then, exactly one of the following holds:*

(i) $p = 3$, $|fix(\omega)| = 1$ *and* $|fix'(\omega)| = 4$.

(ii) $p = 5$, $|fix(\omega)| = 3$ *and* $|fix'(\omega)| = 0$.

(iii) $p = 11$, $|fix(\omega)| = 2$ *and* $|fix'(\omega)| = 0$.

(iv) $p = 13$, $|fix(\omega)| = 0$ *and* $|fix'(\omega)| = 3$.

Proof: If $|fix'(\omega)| = 0$, then $p \mid 55$. If $p = 5$, then $|fix(\omega)| \equiv 3 \pmod 5$. If $|fix(\omega)| > 3$, then $|fix_5(\omega)| > 0$ and therefore by Theorem 4.1.2, $|fix'(\omega)| > 0$, a contradiction. Therefore, we get (ii). If $p = 11$, we clearly get (iii).

If $|fix'(\omega)| > 0$, then ω is an automorphism of $PG(2,3)$. Note that an element of prime order in $PGL(3,3)$ is of type $1^5 2^4$, $1^4 3^3$, $1^1 3^4$, or 13^1. If ω is of type $1^5 2^4$ then $|fix(\omega)| = 5$ and $|fix_5(\omega)| = 51$, while if ω is of type $1^4 3^3$ then $|fix(\omega)| = 4$ and $|fix_5(\omega)| = 13$. Both cases contradict Theorem 4.1.2. If ω is of type $1^1 3^4$, then $|fix_5(\omega)| = 4$ and by Theorem 4.1.2, $|fix'(\omega)| = 4$, so we get (i). If ω is of type 13^1, then $|fix'(\omega)| \equiv 3 \pmod{55}$. Since there are only 4 projective planes of order 3, with point set X, fixed by a specific permutation of order 13, we get (iv). □

In [10], Chouinard II constructs all C_{11}-invariant $LS[55](2,4,13)$, and shows that, up to isomorphism, there are precisely 15 such large sets. Here, we construct all C_{13}-invariant $LS[55](2,4,13)$ using Theorem 4.1.3, and classify them using Lemma 4.1.2.

Let α, β be as above, and $\mathbb{L}$ be the collection of all $\langle\alpha\rangle$-invariant $LS[55](2,4,13)$. Since a projective plane of order 3 is unique up to isomorphism, there are in total $13!/5,616 = 1,108,800$ distinct projective planes of order 3 on X. Four of these planes are fixed by α, namely $\mathbb{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$ given above. Hence, α has $(1,108,800-4)/13 = 85,292$ orbits of size 13 on the family of all projective planes of order 3 on X. Let $\Pi = \{\Delta_1, ..., \Delta_{85292}\}$ be the collection of these orbits. By Lemma 4.2.1, for any large set $\mathcal{L} = (X, \mathbb{B}) \in \mathbb{L}$, we have $\mathbb{B} = \{\mathcal{B}_i, \mathcal{B}_j, \mathcal{B}_k\} \cup \Delta_m \cup \Delta_n \cup \Delta_q \cup \Delta_r$, where $1 \leq i, j, k \leq 4$, and $1 \leq m, n, q, r \leq 85,292$.

Recall that $N_{\mathcal{S}}(\alpha) = \langle\alpha, \beta\rangle$. Also note that β acts transitively on $\mathbb{D}$, and hence on $\binom{\mathbb{D}}{3}$. Therefore, by Lemma 4.1.2, we can choose $i = 1$, $j = 2$, and $k = 3$, and search for 4 appropriate orbits from Π.

Note that α has $\binom{13}{4}/13 = 55$ orbits of size 13 on $\binom{X}{4}$. Let $\{\Gamma_1, ..., \Gamma_{55}\}$ be the collection of these orbits. $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are themselves 3 of these 55 orbits. We first construct the $85,292 \times 55$ matrix $\mathbf{M}$ as in Theorem 4.1.3. Then, we modify $\mathbf{M}$ by deleting rows of $\mathbf{M}$ that contain entries greater than 0 on the columns labeled by $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$, or entries greater than 1 in any column, and finally deleting the columns labeled by $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. We end up with a $10,314 \times 52$ matrix $\mathbf{M}$ whose entries are 0's or 1's, and the row sum is 13. We look for $\{0,1\}$ vectors $\mathbf{u}$ which are solutions of $\mathbf{uM} = \mathbf{j}$, where $\mathbf{j}$ is the 52-dimensional row vector of all 1's. In other words, we search for 4 mutually orthogonal rows of $\mathbf{M}$ which necessarily add up to $\mathbf{j}$. Using an exhaustive computer search, we obtain exactly 5 solutions. The action of β on these 5 solutions yields in total 20 solutions (including the large sets containing $\mathcal{D}_4$), that is, we get that $|\mathbb{L}| = 20$. Since α fixes each one of these large sets, by Lemma 4.1.2, to classify these large sets it is sufficient to look at the action of β on $\mathbb{L}$. It turns out that β has one orbit of size 12 and two orbits of size 4 on $\mathbb{L}$. Therefore, we get the following theorem.

Theorem 4.2.1. *Up to isomorphism, there are precisely three C_{13}-invariant large sets $LS[55](2,4,13)$. One large set from each isomorphism is given in Table 4.1, where we give permutations $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathcal{S}$, such that $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ together with the $\langle\alpha\rangle$-orbits of $(\mathcal{D}_1)^{\omega_i}$ for $i \in \{1,2,3,4\}$ form a large set $LS[55](2,4,13)$.*

Table 4.1: Isomorphism classes of C_{13}-invariant $LS[55](2,4,13)$

$\mathcal{L}_1$	$\omega_1 = (0)(1)(2,4,5,12,9,3)(6,11,7,8)(10)$ $\omega_2 = (0)(1)(2,3)(4,9)(5)(6,7,8,12,10,11)$ $\omega_3 = (0)(1)(2,3)(4,8,6)(5,11,10,9)(7,12)$ $\omega_4 = (0)(1)(2,3)(4,7,12,11)(5,10,8,9,6)$
$\mathcal{L}_2$	$\omega_1 = (0)(1)(2,4,6,5,9,3)(7,12,11,10,8)$ $\omega_2 = (0)(1)(2,3)(4,11,10,12,8,7,9,5)(6)$ $\omega_3 = (0)(1)(2,3)(4,10,12,11,9,7,8,5,6)$ $\omega_4 = (0)(1)(2,3)(4,8,6,12)(5)(7)(9,11)(10)$
$\mathcal{L}_3$	$\omega_1 = (0)(1)(2,4,7,9,3)(5,11,6,8,10,12)$ $\omega_2 = (0)(1)(2,3)(4,11,6,7,8,5,10,12,9)$ $\omega_3 = (0)(1)(2,3)(4,8,7,6,12,9,5,10,11)$ $\omega_4 = (0)(1)(2,3)(4,8,11,12,5,7)(6)(9,10)$

Using a computer search, we also determine the full automorphism groups of the large sets given in Theorem 4.2.1, as well as the 15 large sets constructed by Chouinard II in [10].

Theorem 4.2.2. *(1)* $Aut(\mathcal{L}_1) = \langle \alpha \rangle \cong C_{13}$*,*

(2) $Aut(\mathcal{L}_2) = Aut(\mathcal{L}_3) = \langle \alpha, \beta^4 \rangle \cong C_{13}^3$*, and*

(3) $Aut(\mathcal{L}) \cong C_{11}$ *for any* C_{11}*-invariant large set* $\mathcal{L}$*.*

Now let $\mathcal{L}'$ be a putative $LS[55](2, 4, 13)$, which is not isomorphic to the 18 large sets constructed thus far, that is, which does not have an automorphism of order 11 or 13. In the following two lemmas, we give further results on the possible automorphism groups of $\mathcal{L}'$ for future use.

Lemma 4.2.2. *Let* $\mathcal{P} \leq Aut(\mathcal{L}')$ *where* $|\mathcal{P}| = p^k$*,* $p \in \{3, 5\}$*, and* $k \geq 1$*. Then,* $k = 1$*.*

Proof: First let $p = 3$. Orbits of $\mathcal{P}$ on X are of size 1, 3, or 9. If $\mathcal{P}$ has more than one fixed point, then $\mathcal{P}$ has an element of order 3 fixing more than one point, which contradicts Lemma 4.2.1. Therefore, $\mathcal{P}$ has exactly one fixed point. Since $12 \equiv 3 \pmod 9$, $\mathcal{P}$ has at least one orbit of size 3. Let Y be such an orbit and $y \in Y$. Then, $\mathcal{P}_y$ fixes more than one point and by the above argument, $|\mathcal{P}_y| = 1$. Therefore, $|\mathcal{P}| = |\mathcal{P}_y| \cdot |Y| = 3$. The case $p = 5$ follows by similar arguments. □

Lemma 4.2.3. $|Aut(\mathcal{L}')| \in \{1, 3, 5\}$*.*

Proof: By assumption on $\mathcal{L}'$ and Lemmas 4.2.1 and 4.2.2, $|Aut(\mathcal{L}')|$ divides 15. Since an element of type $1^1 3^4$ and an element of type $1^3 5^2$ cannot commute, $|Aut(\mathcal{L}')| \neq 15$. □

4.3 PROJECTIVE PLANES OF ORDER 4

The problem of existence or non-existence of a large set of projective planes of order $n = 4$ is much harder than the case $n = 3$. It is well known that, up to isomorphism, there is a unique projective plane of order 4, denoted by $PG(2,4)$, whose full automorphism group is $P\Gamma L(3,4)$ of order $120,960$. Thus, there are in total $21!/120,960 = 422,378,820,864,000$ distinct projective planes of order 4 on a particular set of 21 points. Our goal is to construct 969 mutually disjoint planes among these 422,378,820,864,000 planes. In this section, we give some results on the possible automorphisms of such a large set, and construct 912 mutually disjoint planes.

Throughout this section, let $\mathcal{L} = (X, \mathbb{B})$ be a putative large set $LS[969](2,5,21)$. It is known that the elements of prime order in $P\Gamma L(3,4)$ acting on the point set of $PG(2,4)$ have types $1^7 2^7$, $1^5 2^8$, $1^6 3^5$, $1^3 3^6$, 3^7, $1^1 5^4$, and 7^3. Using similar arguments with the proofs of Lemmas 4.2.1 and 4.2.2, we can get the following two lemmas. We omit the proofs.

Lemma 4.3.1. *Let $\omega \in Aut(\mathcal{L})$ be of prime order p. Then, exactly one of the following holds:*

(i) $p = 2$, $|fix(\omega)| = 7$, *and* $|fix'(\omega)| = 59$.

(ii) $p = 3$, *and* $|fix(\omega)| = 0$.

(iii) $p = 3$, $|fix(\omega)| = 3$, *and* $|fix'(\omega)| = 6$.

(iv) $p = 5$, $|fix(\omega)| = 1$, *and* $|fix'(\omega)| = 4$.

(v) $p = 7$, *and* $|fix(\omega)| = 0$.

(vi) $p = 17$, $|fix(\omega)| = 4$, *and* $|fix'(\omega)| = 0$.

(vii) $p = 19$, $|fix(\omega)| = 2$, *and* $|fix'(\omega)| = 0$.

Lemma 4.3.2. *Let $\mathcal{P} \leq Aut(\mathcal{L})$ where $|\mathcal{P}| = p^k$, p prime, and $k \geq 1$. Then, either $p = 3$ and $k \leq 3$, or $p \in \{2, 5, 7, 17, 19\}$ and $k = 1$.*

To construct an $LS[969](2,5,21)$ with a prescribed automorphism group, we want to choose a group as large as possible. We try to use the largest possible normalizer of an element of order 19, since 19 is the largest possible prime. Throughout this section, let $X = \mathbb{Z}_{19} \cup \{\infty_1, \infty_2\}$, $\mathcal{S} = \mathcal{S}_X$, and define $\alpha, \beta \in \mathcal{S}$ as

$$x^\alpha = x + 1,\ x^\beta = 4x \text{ for } x \in \mathbb{Z}_{19}, \text{ and}$$
$$(\infty_i)^\alpha = (\infty_i)^\beta = \infty_i \text{ for } i \in \{1, 2\}.$$

Let $\mathcal{G} = \langle \alpha, \beta \rangle$. Note that $\beta \in N_{\mathcal{S}}(\alpha)$, and $\mathcal{G} \cong C_{19}^9$. Also, by Lemma 4.3.1, $\mathcal{G}$ is the largest possible subgroup of $N_{\mathcal{S}}(\alpha)$ which can be an automorphism group of $\mathcal{L}$.

Assume that $\mathcal{L}$ is $\mathcal{G}$-invariant. Let $Y = \{\{x, y, z, \infty_1, \infty_2\} \mid \{x, y, z\} \in \binom{\mathbb{Z}_{19}}{3}\}$, and $\mathcal{O}$ be the collection of all α-orbits on $\binom{X}{5}$. Note that $|Y| = \binom{19}{3} = 969$. We have $|\mathcal{O}| = \binom{21}{5}/19 = 1071$, where Y is the union of $969/19 = 51$ of these 1071 orbits. Since $\beta \in N_{\mathcal{S}}(\alpha)$, β acts on $\mathcal{O}$ and hence on Y. Therefore, $\mathcal{G}$ acts on Y. Each plane in $\mathcal{L}$ has exactly one block containing the points ∞_1 and ∞_2, that is, $|\mathcal{B} \cap Y| = 1$ for all $\mathcal{B} \in \mathbb{B}$. Therefore, the action of $\mathcal{G}$ on $\mathbb{B}$ is determined by its action on Y.

Note that $fix(\beta) = \{0, \infty_1, \infty_2\}$, and β has 2 orbits of size 9 on X, namely $\{1, 4, 16, 7, 9, 17, 11, 6, 5\}$ and $\{2, 8, 13, 14, 18, 15, 3, 12, 10\}$. Thus, $|fix_5(\beta)| = 0$. We also have $fix(\beta^3) = \{0, \infty_1, \infty_2\}$, and β^3 has 6 orbits of size 3 on X, namely $T_1 = \{1, 7, 11\}$, $T_2 = \{4, 9, 6\}$, $T_3 = \{16, 17, 5\}$, $T_4 = \{2, 14, 3\}$, $T_5 = \{8, 18, 12\}$, and $T_6 = \{13, 15, 10\}$. Thus, $fix_5(\beta^3) = \{K_{a,i} \ :\ a \in fix(\beta^3) \text{ and } 1 \leq i \leq 6\}$, where $K_{a,i} = (T_i \cup fix(\beta^3)) \setminus \{a\}$. Therefore, $|fix_5(\beta^3)| = 18$ and $|fix_5(\beta^3) \cap Y| = 6$. Note that these 18 fixed blocks of β^3 are in 18 distinct orbits in $\mathcal{O}$. The action of β on $fix_5(\beta^3)$ has 6 orbits of size 3, namely $\{K_{a,i}, K_{a,i+1}, K_{a,i+2}\}$ for $i \in \{1, 4\}$ and $a \in fix(\beta)$, where 2 of these orbits are contained in Y, namely those for $a = 0$. Therefore, the action of β on $\mathcal{O}$ has 6 orbits of size 3, and hence $(1071 - 6 \cdot 3)/9 = 117$ orbits of size 9, where Y contains 2 of the small orbits and hence $(51 - 2 \cdot 3)/9 = 5$ of the large orbits. Therefore, $\mathcal{G} \mid \binom{X}{5}$ has in total 123 orbits (117 *large orbits* of size $19 \cdot 9 = 171$, and 6 *small orbits* of size $19 \cdot 3 = 57$), where Y is the union of 5 large orbits and 2 small orbits.

Let $\Sigma_1, ..., \Sigma_7$ be the orbits of $\mathcal{G} \mid Y$, and $\Gamma_1, ..., \Gamma_{116}$ be the remaining orbits of $\mathcal{G} \mid \binom{X}{5}$, where Σ_6, Σ_7, and $\Gamma_{113}, ..., \Gamma_{116}$ are the small orbits. Let $B_6 = K_{0,1}$, $B_7 = K_{0,4}$, $K_{\infty_1,1}$, $K_{\infty_1,4}$, $K_{\infty_2,1}$, and $K_{\infty_2,4}$ be orbit representatives for Σ_6, Σ_7, Γ_{113}, Γ_{114}, Γ_{115}, and Γ_{116} respectively. Also, one can verfiy that $B_1 = \{0, 1, 2, \infty_1, \infty_2\}$, $B_2 = \{0, 1, 3, \infty_1, \infty_2\}$, $B_3 = \{0, 2, 3, \infty_1, \infty_2\}$, $B_4 = \{0, 1, 4, \infty_1, \infty_2\}$, and $B_5 = \{0, 3, 4, \infty_1, \infty_2\}$ form a set of orbit representatives for $\Sigma_1, ..., \Sigma_5$. Hence, $\{B_1, ..., B_7\}$ is a full set of orbit representatives for $\mathcal{G} \mid Y$.

Since the action of $\mathcal{G}$ on $\mathbb{B}$ is determined by its action on Y, $\mathcal{G}$ necessarily has 5 large design orbits (of size 171), say $\Phi_1, ..., \Phi_5$, and 2 small design orbits (of size 57), say Φ_6 and Φ_7. Note that, for $1 \leq i \leq 7$, there exists a unique plane $\mathcal{D}_i = (X, \mathcal{B}_i) \in \Phi_i$ such that $B_i \in \mathcal{B}_i$. Define a 116-dimensional row vector $\mathbf{r}(\mathcal{D}_i) = (r^i_j)$, where $r^i_j = |\mathcal{B}_i \cap \Gamma_j| \cdot |(\mathcal{B}_i)^{\mathcal{G}}|/|\Gamma_j|$. One can now see that, by Theorem 4.1.3, the existence of a $\mathcal{G}$-invariant $LS[969](2, 5, 21)$ is equivalent to the existence of 7 planes $\mathcal{D}_1, ..., \mathcal{D}_7$ containing the blocks $B_1, ..., B_7$ respectively, such that $\mathbf{r}(\mathcal{D}_1) + ... + \mathbf{r}(\mathcal{D}_7) = \mathbf{j}$.

We only need to consider the planes $\mathcal{D}_i = (X, \mathcal{B}_i)$ where $\mathbf{r}(\mathcal{D}_i)$ is a $\{0, 1\}$-vector. Therefore, $|\mathcal{B}_i \cap \Gamma_j| \leq |\Gamma_j|/|(\mathcal{B}_i)^{\mathcal{G}}|$ for $1 \leq i \leq 7$ and $1 \leq j \leq 116$.

For $i \in \{1, 2, 3, 4, 5\}$, we have $|(\mathcal{B}_i)^{\mathcal{G}}| = 171$. Hence, $r^i_j = 0$ for $j \in \{113, ..., 116\}$, and $r^i_j = 1$ for exactly 20 values of $j \in \{1, ..., 112\}$.

For $i \in \{6, 7\}$, we have $|(\mathcal{B}_i)^{\mathcal{G}}| = 57$. Hence, $|\mathcal{B}_i \cap \Gamma_j| \in \{0, 1\}$ for $j \in \{113, ..., 116\}$, and $|\mathcal{B}_i \cap \Gamma_j| \in \{0, 3\}$ for $j \in \{1, ..., 112\}$. Since B_i is fixed by β^3, the plane $\mathcal{D}_i$ is necessarily fixed by β^3. By Theorem 4.1.1, $\mathcal{D}_i$ contains exactly 3 fixed blocks of β^3. One of these fixed blocks is B_i. Since $\mathcal{D}_i$ contains exactly one block containing $\{0, \infty_2\}$, that block is necessarily in $\Gamma_{113} \cup \Gamma_{114}$. Similarly, since $\mathcal{D}_i$ contains exactly one block containing $\{0, \infty_1\}$, the third fixed block of β^3 is necessarily in $\Gamma_{115} \cup \Gamma_{116}$. Therefore $r^i_j = 1$ for exactly one $j \in \{113, 114\}$ and one $j \in \{115, 116\}$. The remaining

18 blocks of $\mathcal{D}_i$ are in large orbits and they form 6 orbits of size 3 under β^3. Therefore, $|\mathcal{B}_i \cap \Gamma_j| = 3$, and hence $r^i_j = 1$ for exactly 6 values of $j \in \{1, ..., 112\}$.

For $1 \leq i \leq 7$, using the action of $\mathcal{S}_X$ on a particular projective plane of order 4, we can easily construct the family, say Δ_i, of all projective planes $\mathcal{D}_i$ of order 4 satisfying the conditions given in the above three paragraphs. Let $n_i = |\Delta_i|$. For $1 \leq i \leq 7$, we define an $n_i \times 116$ matrix $\mathbf{M_i}$ whose rows are the row vectors $\mathbf{r}(\mathcal{D}_i)$ of the planes in Δ_i. Note that the row sums are 20 in $\mathbf{M_1}, ..., \mathbf{M_5}$, and 8 in $\mathbf{M_6}$ and $\mathbf{M_7}$. The existence of a $\mathcal{G}$-invariant $LS[969](2, 5, 21)$ is now equivalent to the existence of 7 mutually orthogonal rows (one from each matrix $\mathbf{M_i}$), which necessarily add up to the 116-dimensional vector of all 1's.

Unfortunately, for $i \in \{1, 2, 3, 4, 5\}$, n_i is over 50 billion, and it is hard to store the matrices $\mathbf{M_i}$ and make an exhaustive computer search. Instead, we first construct a random plane $\mathcal{D}_1 \in \Delta_1$. For $2 \leq i \leq 7$, define an $n'_i \times 96$ matrix $\mathbf{M'_i}$ by taking only the rows of $\mathbf{M_i}$ which are orthogonal to $\mathbf{r}(\mathcal{D}_1)$, and then deleting the columns corresponding to the 1's in $\mathbf{r}(\mathcal{D}_1)$. We observe that, for $i \in \{2, 3, 4, 5\}$, n'_i is around 800 million which is still too large. We construct a random plane $\mathcal{D}_2 \in \Delta_2$ such that $\mathbf{r}(\mathcal{D}_1)$ and $\mathbf{r}(\mathcal{D}_2)$ are orthogonal. Then, similar to the above process, for $3 \leq i \leq 7$, define an $n''_i \times 76$ matrix $\mathbf{M''_i}$ by taking only the rows of $\mathbf{M'_i}$ which are orthogonal to $\mathbf{r}(\mathcal{D}_2)$, and then deleting the columns corresponding to the 1's in $\mathbf{r}(\mathcal{D}_2)$. We see that, for $i \in \{3, 4, 5\}$, n''_i is around 4 million. We continue the process by constructing a random plane $\mathcal{D}_3 \in \Delta_3$ such that $\mathbf{r}(\mathcal{D}_3)$ is orthogonal to $\mathbf{r}(\mathcal{D}_1)$ and $\mathbf{r}(\mathcal{D}_2)$, and then constructing an $n'''_i \times 56$ matrix $\mathbf{M'''_i}$ for $i \in \{4, 5, 6, 7\}$ as above. We observe that n'''_4 and n'''_5 are around 2000, while n'''_6 and n'''_7 are around 100. Then, we search for 4 mutually orthogonal rows, one from each matrix $\mathbf{M'''_i}$ for $i \in \{4, 5, 6, 7\}$, and we repeat this process for different choices of $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$. For all choices of $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, we get at least 2 orthogonal rows, one from $\mathbf{M'''_4}$ and one from $\mathbf{M'''_5}$, which yields in total

5 large orbits and hence $5 \cdot 171 = 855$ mutually disjoint planes. For some choices of $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, we get 3 mutually orthogonal rows, which yields in total 5 large orbits and 1 small orbit, and hence $5 \cdot 171 + 57 = 912$ mutually disjoint projective planes of order 4. We get the following theorem, which improves the lower bound for $D(2, 5, 21)$ given by Kramer and Magliveras.

Theorem 4.3.1. $D(2, 5, 21) \geq 912$.

We present one set of 912 mutually disjoint planes in Table 4.2, where we denote ∞_1 and ∞_2 as 19 and 20. We give permutations $\omega_1, ..., \omega_6$, such that the union of the $\mathcal{G}$-orbits on the planes $\mathcal{D}_i = \mathcal{D}^{\omega_i}$ form the set of 912 mutually disjoint planes, where $\mathcal{D}$ is the projective plane of order 4 developed by the permutation $(0, 1, 2, ..., 18, 19, 20)$ on the base block $\{0, 1, 4, 14, 16\}$. The set of 912 mutually disjoint planes given in Table 4.2 is maximal, that is the remaining blocks which form a 2-$(21, 5, 57)$ design contains no 2-$(21, 5, 1)$ design in it.

Table 4.2: 912 mutually disjoint projective planes of order 4

$\omega_1 = (0, 8, 3, 7, 10, 12, 17, 13, 4, 1, 5, 2, 16, 14, 18)(6, 11)(9, 15)(19)(20)$
$\omega_2 = (0, 8, 4, 1, 5, 2, 15, 9, 11, 6, 17, 14, 18)(3, 7, 12, 16, 10, 13)(19)(20)$
$\omega_3 = (0, 7, 12, 17, 10, 6, 8, 5, 2, 15, 16, 14, 18)(1, 4)(3, 11, 9, 13)(19)(20)$
$\omega_4 = (0, 7, 13, 2, 14, 18)(1, 11, 12, 15, 17, 4)(3, 9)(5, 8, 16, 10)(6)(19)(20)$
$\omega_5 = (0, 13, 15, 8, 12, 17, 1, 11, 7, 10, 2, 14, 18)(3, 6, 5, 4)(9)(16)(19)(20)$
$\omega_6 = (0, 9, 10, 15, 18)(1, 5, 3, 13, 17, 8, 14, 11, 16, 4, 2)(6, 12, 7)(19)(20)$

Out of more than 10^{26} choices for $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, we tried 100 choices. We obtained 813 sets of 855 mutually disjoint planes, and 7 sets of 912 mutually disjoint planes. We believe that a C_{19}^9-invariant $LS[969](2, 5, 21)$ does exist. However, an exhaustive search for such a large set does not seem to be feasible. Further analyses could be made to choose $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ in such a way that they are more likely to be extended to a large set.

CHAPTER 5

$(K_5 \setminus E)$-DESIGNS OF ORDERS 27, 135, 162, AND 216

5.1 BASIC RESULTS

For a graph G, we denote an edge between the points x and y as $[x, y]$. An edge $[x, x]$ is also called a *loop*. Let $d(x, G)$ denote the degree of x in the graph G.

We denote by $[x_1, x_2, ..., x_n]$, the graph on the points $\{x_1, x_2, ..., x_n\}$ which consists of the edges $[x_i, x_j]$ for $1 \leq i < j \leq n$ where multiple edges are counted. Also we denote by $[x_1, x_2, ..., x_n \mid y_1, y_2, ..., y_r]$ the graph on the points $\{x_1, x_2, ..., x_n, y_1, y_2, ..., y_r\}$ which consists of the edges $[x_i, x_j]$ for $1 \leq i < j \leq n$ and $[x_k, y_m]$ for $1 \leq k \leq n$ and $1 \leq m \leq r$.

Example 5.1.1. *(1)* $[1, 1, 2]$ *is the graph on the points* $\{1, 2\}$ *which contains the edge* $[1, 1]$ *once and the edge* $[1, 2]$ *twice.*

(2) $[1, 2 \mid 3, 3]$ *is the graph on the points* $\{1, 2, 3\}$ *which contains the edge* $[1, 2]$ *once, and each one of the edges* $[1, 3]$ *and* $[2, 3]$ *twice.*

(3) The graph $[1, 2, 3 \mid 4, 5]$ *is isomorphic to* $K_5 \setminus e$*, where the points* 1*,* 2 *and* 3 *have degree* 4 *and the points* 4 *and* 5 *have degree* 3*.*

Throughout this chapter, let $V = V(K_{27})$, $E = E(K_{27})$, and $\mathcal{D}$ be a putative $(K_5 \setminus e)$-design of order 27. Note that $\mathcal{D}$ has $\binom{27}{2}/9 = 39$ blocks. For $x \in V$, let $a_x = |\{B \in \mathcal{D} \mid d(x, B) = 4\}|$ and $b_x = |\{B \in \mathcal{D} \mid d(x, B) = 3\}|$. Since $d(x, K_{27}) = 26$, we get $4a_x + 3b_x = 26$, which has two non-negative integer solutions.

Either $(a_x, b_x) = (2, 6)$ or $(a_x, b_x) = (5, 2)$. Define $V_1 = \{x \in V \mid (a_x, b_x) = (2, 6)\}$ and $V_2 = \{x \in V \mid (a_x, b_x) = (5, 2)\}$.

Lemma 5.1.1. $|V_1| = 6$ *and* $|V_2| = 21$.

Proof: If we count the total number of occurences of the points as points of degree 3, we get $78 = 2 \cdot 39 = |\{(B, x) \mid B \in \mathcal{D},\ d(x, B) = 3\}| = \sum_{x \in V} b_x \cdot x = 6|V_1| + 2|V_2|$. Since $|V_1| + |V_2| = 27$, the result follows. □

Lemma 5.1.2. *Let* $\alpha \in Aut(\mathcal{D})$. *Then,*

(1) $(V_i)^\alpha = V_i$ *for* $i \in \{1, 2\}$.

(2) *If* $B = [a, b, c \mid d, e] \in fix'(\alpha)$, *then* $\{a, b, c\}^\alpha = \{a, b, c\}$ *and* $\{d, e\}^\alpha = \{d, e\}$.

(3) *If* $x, y \in fix(\alpha)$, $x \neq y$, *and* B *is the block containing* $[x, y]$, *then* $B \in fix'(\alpha)$.

Proof: (1) and (2) are obvious, and (3) follows since B^α contains $[x, y]$. □

Lemma 5.1.3. *Let* $\alpha \in Aut(\mathcal{D})$ *be of odd prime order* p. *Then,* $p = 3$ *and* $fix(\alpha) = fix'(\alpha) = \emptyset$.

Proof: We consider 3 cases on $|fix(\alpha)|$.

Case 1: $|fix(\alpha)| \geq 2$.

Let $x, y \in fix(\alpha)$, $x \neq y$, and B be the block containing the edge $[x, y]$. By Lemma 5.1.2 (3), we get $B^\alpha = B$. Then, since p is odd and at least one of the points x and y have degree 4 in B, by Lemma 5.1.2 (2), B is pointwise fixed. Therefore the set of blocks containing an edge between fixed points form a $(K_5 \setminus e)$-design of order $|fix(\alpha)|$. By Theorem 1.0.3, the only possibility is that $|fix(\alpha)| = 19$, but then $p \mid 27 - 19 = 8$, which is a contradiction since p is odd.

Case 2: $|fix(\alpha)| = 1$.

Let $fix(\alpha) = \{x\}$. If $x \in V_1$, then by Lemma 5.1.2 (1), we have $p \mid 6 - 1 = 5$ and $p \mid 21$, contradiction. If $x \in V_2$, then $p \mid 6$ and $p \mid 21 - 1 = 20$, contradiction.

Case 3: $|fix(\alpha)| = 0$.

We have $p \mid 6$ and $p \mid 21$ and therefore $p = 3$.

Finally, assume that α fixes a block, say $[a, b, c \mid d, e]$. Then, $d, e \in fix(\alpha)$, contradiction. Therefore, $fix'(\alpha) = \emptyset$ as well. □

We now consider 3-groups acting on $\mathcal{D}$.

Lemma 5.1.4. *Let $P \leq Aut(\mathcal{D})$ such that $|P| = 3^k$. Then, $k \leq 1$.*

Proof: By Lemma 5.1.2 (1), P acts on V_1. For all $x \in V_1$ we have $|P_x| = 1$, since otherwise P_x contains an element of order 3 which fixes x (contradiction to Lemma 5.1.3). Therefore P acts semiregularly on V_1, and hence $|P|$ divides 6. □

Theorem 5.1.1. *$|Aut(\mathcal{D})|$ divides $2^k 3$ for some $k \geq 0$.*

Proof: Follows by Lemmas 5.1.3 and 5.1.4.

5.2 AUTOMORPHISMS OF ORDER 3

Our goal is to construct $\mathcal{D}$ using an automorphism group which is as large as possible. In this section we first analyze how $\mathcal{D}$ could be constructed using an automorphism of order 3, say α.

Throughout this chapter, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $U_1 = \{1, 2\}$, $U_2 = U \setminus U_1$, and define V such that $V = \{x_i \mid x \in U,\ i \in \mathbb{Z}_3\}$, $V_j = \{x_i \mid x \in U_j,\ i \in \mathbb{Z}_3\}$ for $j \in \{1, 2\}$. By Lemma 5.1.3, we can define α such that $(x_i)^\alpha = x_{i+1}$ for all $x_i \in V$.

Let K' be the graph, with $V(K') = U$, which contains the edge $[x, x]$ exactly once for all $x \in U$, and the edge $[x, y]$ exactly 3 times for all $x, y \in U$, $x \neq y$. We will denote these 3 edges between the points x and y as $[x, y]_0$, $[x, y]_1$ and $[x, y]_2$.

Let $E' = E(K')$. Define $\phi : (V, E) \to (U, E')$ as $(x_i)^\phi = x$, $[x_i, x_j]^\phi = [x, x]$, and $[x_i, y_k]^\phi = [x, y]_{k-i}$ for $x_i, x_j, y_k \in V$, $x \neq y$.

For a block $B = [x_i, y_j, z_k \mid t_m, u_n] \in \mathcal{D}$, let $C = B^\phi$ be the graph consisting of the images under ϕ of the edges of B. We denote C as $[x, y, z \mid t, u]$, but as long as C is refered as the image under ϕ of the block B, we understand that for example if $x \neq y$ then the edge $[x, y]$ occuring in C is indeed the edge $[x, y]_{j-i}$. When talking about the block C without refering it as the image under ϕ of a block B, we just say that the edge $[x, y]$ occurs in C.

Let $H_1 = [1, 1, 2 \mid 3, 4]$, $H_2 = [1, 2, 3 \mid 1, 2]$, $H_3 = [1, 2, 3 \mid 1, 4]$, $H_4 = [1, 2, 3 \mid 4, 4]$, $H_5 = [1, 2, 3 \mid 4, 5]$, and $\mathbb{H} = \{H_1, H_2, H_3, H_4, H_5\}$.

Proposition 5.2.1. *Let $\mathcal{D}_1$ be a $(K_5 \setminus e)$-design of order 27. Then, $(\mathcal{D}_1)^\phi$ is a $(K', \mathbb{H})$-decomposition.*

Proof: Let $\mathcal{D}'_1 = (\mathcal{D}_1)^\phi$. Note that $E^\phi = E'$, and B_1 and B_2 are in the same block orbit of α if and only if $(B_1)^\phi = (B_2)^\phi$. Then, the result follows by counting the edges and noting that a block of $\mathcal{D}'_1$ cannot contain a loop more than once and a non-loop edge more than three times. □

By labeling the edges between the points x and y as $[x, y]_0$, $[x, y]_1$ and $[x, y]_2$, we make sure that the converse of Proposition 5.2.1 is also true, i.e. we have the following:

Proposition 5.2.2. *Let $\mathcal{D}'_1 = \{C_i \mid 1 \leq i \leq 13\}$ be a $(K', \mathbb{H})$-decomposition. Define B_i for $1 \leq i \leq 13$ in such a way that $|V(B_i)| = 5$, $(B_i)^\phi = C_i$ and that for $x, y \in U$, $x \neq y$, each of the edges $[x, y]_0$, $[x, y]_1$, and $[x, y]_2$ occurs exactly once in $\mathcal{D}'_1$. Then $\mathcal{D}_1 = (\{B_i \mid 1 \leq i \leq 13\})^{\langle \alpha \rangle}$ is an α-invariant $(K_5 \setminus e)$-design of order 27.*

Proof: Let $[x_j, y_k] \in E$. If $x = y$ and hence $j \neq k$, let $C_i \in \mathcal{D}'_1$ be the block containing the edge $[x, x]$. Then since $|V(B_i)| = 5$, the edge $[x_j, x_k]$ occurs exactly once in $(B_i)^{\langle\alpha\rangle}$. If $x \neq y$, then let $C_{i'} \in \mathcal{D}'_1$ be the block containing the edge $[x, y]_{k-j}$. Then, the edge $[x_j, y_k]$ occurs exactly once in $(B_{i'})^{\langle\alpha\rangle}$. □

In general determining whether a graph decomposition exists or not is an NP-complete problem, see [13]. By prescribing an automorphism of order 3, the problem of finding a $(K_5 \setminus e)$-design of order 27 reduces to the problem of finding a $(K', \mathbb{H})$-decomposition and then inserting subscripts appropriately, but it seems like even the problem of finding a $(K', \mathbb{H})$-decomposition is large enough to attack by a generic algorithm. Instead, we try to construct $(K_5 \setminus e)$-designs of order 27 using a larger automorphism group. By Theorem 5.1.1, in order to understand how a bigger group can act on 27 points and 39 blocks, we study automorphisms of order 2 first.

5.3 AUTOMORPHISMS OF ORDER 2

Let $\beta \in Aut(\mathcal{D})$ be of order 2. If β has r orbits of size 2 in V_1 and s orbits of size 2 in V_2, we say that β is of *type* (r, s). Let each point in $fix(\beta)$ and each point not in $fix(\beta)$ be denoted by a bullet (•) and a circle (∘) respectively. By Lemma 5.1.2 (2) a block in $fix'(\beta)$ is of type F_i, and by Lemma 5.1.2 (3) a block not in $fix'(\beta)$ is of type N_i for some $1 \leq i \leq 4$ (see Table 5.1).

Table 5.1: Types of blocks with respect to an automorphism of order 2

Type	F_1	F_2	F_3	F_4
Block	$[\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$[\bullet, \bullet, \bullet \mid \circ, \circ]$	$[\circ, \circ, \bullet \mid \bullet, \bullet]$	$[\circ, \circ, \bullet \mid \circ, \circ]$
Type	N_1	N_2	N_3	N_4
Block	$[\circ, \circ, \circ \mid \circ, \circ]$	$[\circ, \circ, \circ \mid \bullet, \circ]$	$[\circ, \circ, \circ \mid \bullet, \bullet]$	$[\circ, \circ, \bullet \mid \circ, \circ]$

Note that all blocks in a block orbit of β are of the same type. For $1 \leq i \leq 4$, let f_i and n_i denote the number of orbits with blocks of types F_i and N_i respectively.

Lemma 5.3.1. *Let $x \in (V \setminus fix(\beta))$ and $y = x^\beta$. Let B_1 be the block containing the edge $[x, y]$, and $B_1 \neq B_2 \in \mathcal{D}$ such that $d(x, B_2) = 4$. Then,*

(1) $B_1 \in fix'(\beta)$ *and* $d(x, B_1) = d(y, B_1) = 4$.

(2) $B_2 \notin fix'(\beta)$.

Proof: (1) Since $(B_1)^\beta$ contains the edge $[x, y]$ as well, we have $(B_1)^\beta = B_1$. We also have $d(x, B_1) = d(x^\beta, (B_1)^\beta) = d(y, B_1)$ and therefore $d(x, B_1) = d(y, B_1) = 4$.

(2) Assume that $(B_2)^\beta = B_2$. Then $d(y, B_2) = d(y^\beta, (B_2)^\beta) = d(x, B_2) = 4$, and hence B_2 contains the edge $[x, y]$, contradiction. □

Lemma 5.3.2. *Let β be of type (r, s) and $k = 27 - 2(r + s) = |fix(\beta)|$. Then,*

(1) $f_1 + f_2 + f_3 + f_4 + 2n_1 + 2n_2 + 2n_3 + 2n_4 = 39$

(2) $f_3 + f_4 = r + s$

(3) $9f_1 + 3f_2 + 2f_3 = \binom{k}{2}$

(4) $f_3 + 5f_4 + 18n_1 + 12n_2 + 6n_3 + 10n_4 = \binom{2(r+s)}{2}$

(5) $2f_4 + 9n_1 + 6n_2 + 3n_3 + 5n_4 = (r + s)(r + s - 1)$

(6) $2f_3 + 2f_4 + 6n_1 + 6n_2 + 6n_3 + 4n_4 = 4r + 10s$

(7) $3n_1 + 3n_2 + 3n_3 + 2n_4 = r + 4s$

(8) $3f_1 + 3f_2 + 2n_4 = 117 - 5r - 11s$

(9) $f_2 + f_4 + 2n_1 + n_2 + 2n_4 = 6r + 2s$

(10) $f_1 + f_3 + n_2 + 2n_3 = 39 - 6r - 2s$

Proof: (1) is the number of blocks and (2) follows by Lemma 5.3.1.
(3) and (4) follow by counting the number of edges $[x, y]$ for $x, y \in fix(\beta)$ and the number of edges $[z, t]$ for $z, t \notin fix(\beta)$, and then (5) follows from (2) and (4).
(6) follows by counting the pairs (x, B) for $x \notin fix(\beta)$ and $d(x, B) = 4$, and then (7) and (8) follow from (1), (2) and (6).
(9) follows by counting the pairs (x, B) for $x \notin fix(\beta)$ and $d(x, B) = 3$, and then (10) follows from (1) and (9). □

Lemma 5.3.3. *Let β be of type (r, s). Then, $5 \le r + s \le 11$.*

Proof: First assume that $s = 0$. Then, $1 \le r \le 3$. Let $x \in (V_1 \setminus fix(\beta))$ and $y = x^\beta$. Let B_1 be the block containing the edge $[x, y]$. Since $a_x = 2$ (recall the definitions above Lemma 5.1.1), there exists a block, say $B_2 \ne B_1$, such that $d(x, B_2) = 4$. By Lemma 5.3.1 (2), $B_2 \notin fix'(\beta)$. If two non-fixed points $a, b \in V(B_2)$ belong to the same orbit of β, then $d(a, B_2) = d(b, B_2) = 3$ by Lemma 5.3.1 (1). Therefore, if B_2 is of type N_1 or N_2, then B_2 contains points from at least 4 distinct orbits of size 2, and if B_2 is of type N_3 or N_4, then B_2 contains points from at least 3 distinct orbits of size 2. Therefore $r = 3$ and $n_1 = n_2 = 0$. Then, by Lemma 5.3.2 (7) we get $n_3 = 1$ and $n_4 = 0$, and then by Lemma 5.3.2 (5) we get $2f_4 = 3$, contradiction. Therefore $s > 0$.

Let $z \in (V_2 \setminus fix(\beta))$ and $t = z^\beta$. Let B_3 be the block containing the edge $[z, t]$, and $\Delta = \{B \in \mathcal{D} \mid d(z, B) = 4,\ B \ne B_3\}$. Since $a_z = 5$, we have $|\Delta| = 4$, and by Lemma 5.3.1 (2), we have $\Delta \cap fix'(\beta) = \emptyset$. Therefore $|V(B) \setminus (fix(\beta) \cup \{z\})| \ge 2$ for all $B \in \Delta$ and hence there are at least 8 non-fixed points besides z and t. Therefore, $r + s \ge 5$.

Let $k = |fix(\beta)|$. If $k = 1$ then by Lemma 5.3.2 (2), we get $f_3 + f_4 = 13$, which is impossible since the unique fixed point cannot belong to 13 blocks. Assume that

$k = 3$. Then $f_3 + f_4 = 12$ and by Lemma 5.3.2 (3), we get $f_1 = f_3 = 0$, $f_2 = 1$, and hence $f_4 = 12$. Let B_4 be the block of type F_2, u be a non-fixed point in B_4, and B_5 be the block of type F_4 such that $d(u, B_5) = 4$. Then, B_4 and B_5 contain in total 4 edges of the form $[u, v]$ for $v \in fix(\beta)$, contradiction. Therefore $k \geq 5$ and hence $r + s \leq 11$. $\square$

5.4 AUTOMORPHISM GROUPS OF ORDER 6

We now use the tools derived in the previous sections to construct $(K_5 \setminus e)$-designs of order 27 with an automorphism group of order 6. We first consider the dihedral group D_6.

Theorem 5.4.1. *$D_6 \nleq Aut(\mathcal{D})$.*

Proof: Let α and β be automorphisms of $\mathcal{D}$ of order 3 and 2 respectively and suppose that $\beta\alpha\beta = \alpha^{-1}$. Define α as in Section 5.2, i.e. $(x_i)^\alpha = x_{i+1}$ for all $x_i \in V$. Then without loss, $\beta|_{V_1} = (1_0)(1_1, 1_2)(2_0)(2_1, 2_2)$ or $\beta|_{V_1} = (1_0, 2_0)(1_1, 2_2)(1_2, 2_1)$.

Case 1: If $\beta|_{V_1} = (1_0)(1_1, 1_2)(2_0)(2_1, 2_2)$, let B_1 be the block containing the edge $[1_0, 2_0]$. At least one of the points 1_0 or 2_0 have degree 4 in B_1, without loss say 1_0. Let B_2 be the block containing the edge $[1_1, 1_2]$. By Lemma 5.1.2 (3) $B_1 \in fix'(\beta)$, and by Lemma 5.3.1 (1) $B_2 \in fix'(\beta)$ and $d(1_1, B_2) = d(1_2, B_2) = 4$. Therefore, $(B_2)^\alpha, (B_2)^{\alpha^{-1}} \notin fix'(\beta)$ since $((B_2)^\alpha)^\beta = (B_2)^{\beta\alpha\beta} = (B_2)^{\alpha^{-1}}$. We also have $d(1_0, (B_2)^\alpha) = d(1_2, B_2) = 4$ and $d(1_0, (B_2)^{\alpha^{-1}}) = d(1_1, B_2) = 4$. Therefore, B_1, $(B_2)^\alpha$ and $(B_2)^{\alpha^{-1}}$ are 3 distinct blocks containing the point 1_0 as a point of degree 4, which contradicts the definition of V_1.

Case 2: If $\beta|_{V_1} = (1_0, 2_0)(1_1, 2_2)(1_2, 2_1)$, let B_3 and B_4 be the blocks containing $[1_0, 2_0]$ and $[1_1, 2_2]$ respectively. As in Case 1, $B_i \in fix'(\beta)$ and $(B_i)^{\alpha^j} \notin fix'(\beta)$ for $i \in \{3, 4\}$ and $j \in \{1, 2\}$. By Lemma 5.1.2 and definition of V_1, there is exactly one non-fixed block containing 1_2 as a point of degree 4. Since $d(1_2, (B_3)^{\alpha^{-1}}) = d(1_2, (B_4)^{\alpha}) = 4$, we need to have $(B_3)^{\alpha^{-1}} = (B_4)^{\alpha}$ and hence $(B_3)^{\alpha} = B_4$, contradiction. □

We now consider the cyclic group C_6. We will make a case by case analysis so called isomorph rejection and construct all $(K_5 \setminus e)$-designs of order 27 with an automorphism of order 6, classify them up to isomorphism and determine their full automorphism groups. The results we obtain which are the main results of this chapter are the following theorems.

Theorem 5.4.2. *Suppose that $C_6 \leq Aut(\mathcal{D})$. Then, $\mathcal{D} \cong \mathcal{D}_{i,j,k} = \left(\mathcal{D}^{i,j}_{[1,5]} \cup \mathcal{D}^{j,k}_{[6,9]}\right)^{\langle\omega\rangle}$ for some i, $1 \leq i \leq 6$, $j \in \{0,1\}$ and $k \in \{1,2\}$, where $\mathcal{D}^{i,j}_{[1,5]}$ and $\mathcal{D}^{j,k}_{[6,9]}$ are as in Table 5.2, and ω is defined as $(x_m)^{\omega} = (x^{\delta})_{m-1}$ where $\delta = (1,2)(3,4)(5,6)(7)(8)(9)$.*

Theorem 5.4.3. *The designs given in Theorem 5.4.2 are pairwise non-isomorphic.*

Theorem 5.4.4. *$Aut(\mathcal{D}) \cong C_6$ for any design $\mathcal{D}$ given in Theorem 5.4.2.*

One can check that for each (i,j,k), if we develop the given base blocks in Table 5.2 with the group generated by ω, we obtain a $(K_5 \setminus e)$-design of order 27. In order to classify all C_6-invariant $(K_5 \setminus e)$-designs of order 27 and prove the above theorems, we first derive a few more lemmas.

Let α be defined as $(x_i)^{\alpha} = x_{i+1}$, $\sigma \in Aut(\mathcal{D})$ be of order 6, and $\beta = \sigma^3$. By Lemma 5.1.3 and without loss we can define σ such that $\sigma^2 = \alpha$ and for all $x_i \in V$, if $(x_i)^{\beta} = y_j$ then $i = j$. Note that the point orbits and block orbits of σ are of sizes 6 or 3, and all blocks in a block orbit of σ are of the same type (see Table 5.1).

Recall from Proposition 5.2.1 that $\mathcal{D}^{\phi}$ is a $(K', \mathbb{H})$-decomposition.

Proposition 5.4.1. *Let $\mathcal{D}_1$ be a $(K_5 \setminus e)$-design of order 27, and σ be as above. Define $\lambda \in \mathcal{S}_U$ such that $\phi\lambda = \beta\phi$. Then, λ is an automorphism of order 2 of $\mathcal{D}_1^{\phi}$.*

Table 5.2: C_6-invariant $(K_5 \setminus e)$-designs of order 27

$\mathcal{D}^{1,j}_{[1,5]}$	$\mathcal{D}^{2,j}_{[1,5]}$
$B^{1,j}_1 = [7_0, 8_0, 9_1 \mid 3_j, 4_j]$	$B^{2,j}_1 = [7_0, 8_1, 9_2 \mid 3_j, 4_j]$
$B^{1,j}_2 = [7_1, 7_2, 9_1 \mid 5_0, 6_0]$	$B^{2,j}_2 = [7_2, 9_0, 9_2 \mid 5_0, 6_0]$
$B^{1,j}_3 = [1_j, 2_j, 8_0 \mid 7_1, 7_2]$	$B^{2,j}_3 = [1_j, 2_j, 7_1 \mid 7_2, 8_0]$
$B^{1,j}_4 = [3_j, 4_j, 8_2 \mid 8_1, 9_2]$	$B^{2,j}_4 = [3_j, 4_j, 8_2 \mid 8_0, 9_1]$
$B^{1,j}_5 = [5_0, 6_0, 9_0 \mid 8_1, 9_2]$	$B^{2,j}_5 = [5_0, 6_0, 8_1 \mid 7_1, 9_1]$

$\mathcal{D}^{3,j}_{[1,5]}$	$\mathcal{D}^{4,j}_{[1,5]}$
$B^{3,j}_1 = [8_0, 8_2, 9_1 \mid 3_j, 4_j]$	$B^{4,j}_1 = [8_0, 8_2, 9_1 \mid 3_j, 4_j]$
$B^{3,j}_2 = [7_1, 8_1, 9_1 \mid 5_0, 6_0]$	$B^{4,j}_2 = [7_2, 9_0, 9_2 \mid 5_0, 6_0]$
$B^{3,j}_3 = [1_j, 2_j, 7_1 \mid 7_2, 8_0]$	$B^{4,j}_3 = [1_j, 2_j, 7_1 \mid 7_2, 8_0]$
$B^{3,j}_4 = [3_j, 4_j, 7_0 \mid 8_1, 9_2]$	$B^{4,j}_4 = [3_j, 4_j, 7_0 \mid 8_1, 9_2]$
$B^{3,j}_5 = [5_0, 6_0, 9_0 \mid 7_2, 9_2]$	$B^{4,j}_5 = [5_0, 6_0, 8_1 \mid 7_1, 9_1]$

$\mathcal{D}^{5,j}_{[1,5]}$	$\mathcal{D}^{6,j}_{[1,5]}$
$B^{5,j}_1 = [8_1, 8_2, 9_2 \mid 3_j, 4_j]$	$B^{6,j}_1 = [7_0, 8_1, 9_2 \mid 3_j, 4_j]$
$B^{5,j}_2 = [7_1, 7_2, 9_1 \mid 5_0, 6_0]$	$B^{6,j}_2 = [7_1, 8_1, 9_1 \mid 5_0, 6_0]$
$B^{5,j}_3 = [1_j, 2_j, 8_0 \mid 7_1, 7_2]$	$B^{6,j}_3 = [1_j, 2_j, 7_1 \mid 7_2, 8_0]$
$B^{5,j}_4 = [3_j, 4_j, 7_0 \mid 8_0, 9_1]$	$B^{6,j}_4 = [3_j, 4_j, 8_2 \mid 8_0, 9_1]$
$B^{5,j}_5 = [5_0, 6_0, 9_0 \mid 8_1, 9_2]$	$B^{6,j}_5 = [5_0, 6_0, 9_0 \mid 7_2, 9_2]$

$\mathcal{D}^{j,k}_{[6,9]}$	
$B^{j,k}_6 = [4_0, 5_0, 6_{j+1} \mid 1_0, 4_1]$	$B^{j,k}_8 = [5_0, 5_1, 8_0 \mid 1_2, 2_{1-j}]$
$B^{j,k}_7 = [3_{1-j}, 4_2, 7_0 \mid 1_j, 5_0]$	$B^{j,k}_9 = [1_{j+k}, 3_{1-j}, 9_{j+1} \mid 1_{j-k}, 2_j]$

Proof: Let $\mathcal{D}_1' = \mathcal{D}_1^{\phi}$. First note that λ is well-defined since for $x \in U$, if $(x_0)^{\beta} = y_0$, then $x^{\lambda} = (x_i)^{\phi\lambda} = (x_i)^{\beta\phi} = (y_i)^{\phi} = y$ is independent of the choice of i. The order of λ is clearly 2. To show that λ is an automorphism of $\mathcal{D}_1'$, let $C \in \mathcal{D}_1'$ and $B \in C^{\phi^{-1}}$. Then, $C^{\lambda} = B^{\phi\lambda} = B^{\beta\phi} = (B^{\beta})^{\phi} \in \mathcal{D}_1'$. □

Conversely, we have the following.

Proposition 5.4.2. *Let λ be an automorphism of order 2 of a $(K', \mathbb{H})$-decomposition $\mathcal{D}_1'$, and suppose that λ has precisely m block orbits in $\mathcal{D}_1'$. Let $\{C_i \mid 1 \le i \le m\}$ be a set of representatives for block orbits of λ. Define $\sigma : V \to V$ as $(x_m)^{\sigma} = (x^{\lambda})_{m-1}$, and let $\beta = \sigma^3$. For $1 \le i \le m$, define B_i in such a way that $|V(B_i)| = 5$, $(B_i)^{\phi} = C_i$ (and so $(B_i)^{\beta\phi} = (C_i)^{\lambda}$) and that for $x, y \in U$, $x \ne y$, each of the edges $[x,y]_0$, $[x,y]_1$ and $[x,y]_2$ occurs exactly once in $\mathcal{D}_1'$. Then, $\mathcal{D}_1 = (\{B_i \mid 1 \le i \le m\})^{\langle\sigma\rangle}$ is a $(K_5 \setminus e)$-design of order 27 with σ as an automorphism of order 6.*

Proof: Note that σ satisfies $\sigma^2 = \alpha$ and $\beta\phi = \phi\lambda$. Clearly the order of σ is 6, and the result follows by definition of B_i and Proposition 5.2.2. □

Roughly speaking, in order to construct all $(K_5 \setminus e)$-designs of order 27 with an automorphism of order 6, it is sufficient to construct all $(K', \mathbb{H})$-decompositions with an automorphism λ of order 2 which may possibly be extended to a $(K_5 \setminus e)$-design of order 27 by inserting subscripts to the points appropriately. In order to satisfy the conditions of Proposition 5.4.2, the only things we need to make sure are that the subscripts are inserted in such a way that for $x, y \in U$, $x \ne y$, each of the edges $[x,y]_0$, $[x,y]_1$ and $[x,y]_2$ occur exactly once in $\mathcal{D}_1'$, and the subscripts are inserted the same way to each block in a block orbit of λ.

In what follows, let $\mathcal{D}' = \mathcal{D}^{\phi}$, and σ, α, β and λ be as defined above. If λ has r' orbits of size 2 in U_1 and s' orbits of size 2 in U_2, we say that λ is of type (r', s').

Lemma 5.4.1. *λ is of type $(0,2)$, $(0,3)$, $(1,1)$ or $(1,2)$.*

Proof: Note that if λ is of type (r', s'), then β is of type $(3r', 3s')$. Then the result follows by Lemma 5.3.3. □

As for β, we denote a fixed point of λ by a bullet ($\bullet$), and a non-fixed point of λ by a circle ($\circ$). Moreover let a star (*) denote an arbitrary point. Define types of blocks of $\mathcal{D}'$ as in Table 5.1. Note that for any $B \in \mathcal{D}$, B^{ϕ} is of the same type as B. For $1 \leq i \leq 4$, let f'_i and n'_i denote the number of λ-orbits of blocks of type F_i and N_i in $\mathcal{D}'$ respectively. Then, $f_i = 3f'_i$, $n_i = 3n'_i$, and we get the following corollary of Lemma 5.3.2.

Lemma 5.4.2. *Let λ be of type (r', s') and $k' = 9 - 2(r' + s') = |fix(\lambda)|$. Then,*

(1) $f'_1 + f'_2 + f'_3 + f'_4 + 2n'_1 + 2n'_2 + 2n'_3 + 2n'_4 = 13$

(2) $f'_3 + f'_4 = r' + s'$

(3) $9f'_1 + 3f'_2 + 2f'_3 = \frac{k'(3k'-1)}{2}$

(4) $2f'_4 + 9n'_1 + 6n'_2 + 3n'_3 + 5n'_4 = (r' + s')(3r' + 3s' - 1)$

(5) $3n'_1 + 3n'_2 + 3n'_3 + 2n'_4 = r' + 4s'$

(6) $3f'_1 + 3f'_2 + 2n'_4 = 39 - 5r' - 11s'$

(7) $f'_2 + f'_4 + 2n'_1 + n'_2 + 2n'_4 = 6r' + 2s'$

(8) $f'_1 + f'_3 + n'_2 + 2n'_3 = 13 - 6r' - 2s'$

For $x_i \in V$, $B \in \mathcal{D}$ and $C = B^{\phi}$, if $d(x_i, B) = 4$, we say that x occurs at $left$ in C, and if $d(x_i, B) = 3$, we say that x occurs at *right* in C. For $x \in U$, let a'_x and b'_x denote the number of occurences of x at left and right in $\mathcal{D}'$ respectively. Also, for $1 \leq i \leq 4$, let $f_{i,x,1}$ and $n_{i,x,1}$ denote the number of occurences of x at left in a block of type F_i and N_i respectively. Similarly, let $f_{i,x,2}$ and $n_{i,x,2}$ denote the number of occurences of x at right in a block of type F_i and N_i respectively.

Lemma 5.4.3. *Let λ be of type (r', s'), and $k' = 9 - 2(r' + s') = |fix(\lambda)|$. Then,*

(1) *If $x \notin fix(\lambda)$, then* $f_{3,x,1} + f_{4,x,1} = 1$.

(2) *If $x \in U_1$ then $(a'_x, b'_x) = (2, 6)$, and if $x \in U_2$ then $(a'_x, b'_x) = (5, 2)$.*

(3) *If $x \in fix(\lambda)$, then* $f_{1,x,1} + f_{2,x,1} + f_{3,x,1} + f_{4,x,1} + n_{4,x,1} = a'_x$.

(4) *If $x \in fix(\lambda)$, then* $f_{1,x,2} + f_{3,x,2} + n_{2,x,2} + n_{3,x,2} = b'_x$.

(5) *If $x \notin fix(\lambda)$, then* $n_{1,x,1} + n_{2,x,1} + n_{3,x,1} + n_{4,x,1} = a'_x - 1$.

(6) *If $x \notin fix(\lambda)$, then* $f_{2,x,2} + f_{4,x,2} + n_{1,x,2} + n_{2,x,2} + n_{4,x,2} = b'_x$.

(7) *If $x \in fix(\lambda)$, then* $4f_{1,x,1} + 3f_{1,x,2} + 2f_{2,x,1} + 2f_{3,x,1} + f_{3,x,2} = 3k' - 1$.

Proof: (1) follows by Lemma 5.3.1 and (2) follows by the definitions of V_1 and V_2. Then, (3)-(6) follow by counting the occurences of x at left and right, and finally (7) follows by counting the occurences of x in the edges of the form $[x, y]$ for $y \in fix(\lambda)$. □

Proposition 5.4.3. *Suppose that $1^\lambda = 2$. Let $\mathcal{D}'_{1,2}$ be the set of blocks covering the edges $[x, y]$ for $x, y \in \{1, 2\}$. Then,*

(1) $\mathcal{D}'_{1,2} = \{[1, 2, \bullet \mid 1, 2]\}$, *or*

(2) $\mathcal{D}'_{1,2} = \{[1, 2, \bullet \mid *, *], [1, \circ, * \mid 1, 2], [2, \circ, * \mid 1, 2]\}$.

Proof: Let C_1 be the block containing the points 1 and 2 at left (see Lemma 5.4.3 (1)). By Lemma 5.4.3 (5), each of the points 1 and 2 occur exactly once at left in a nonfixed block, say in $C_2 = [1, \circ, * \mid *, *]$ and $(C_2)^\lambda = [2, \circ, * \mid *, *]$. The only blocks which can cover the edges $[x, y]$ for $x, y \in \{1, 2\}$ are C_1, C_2 and $(C_2)^\lambda$. If $C_1 = [1, 2, \bullet \mid 1, 2]$, all such edges are covered and we get $\mathcal{D}'_{1,2} = \{C_1\}$, otherwise $C_1 = [1, 2, \bullet \mid z, t]$ where $z, t \notin \{1, 2\}$ and hence we need to have $C_2 = [1, \circ, * \mid 1, 2]$. □

Proposition 5.4.4. *Let x, y and z be 3 distinct points in U. Then $\mathcal{D}'$ cannot contain two blocks C_1 and C_2, where $[x, y \mid z, z]$ is a subgraph of C_1, and $[x, y, z]$ is a subgraph of C_2.*

Proof: Without loss of generality, suppose that one of the blocks, say $B_2 \in (C_2)^{\phi^{-1}}$, contains $[x_0, y_0, z_0]$ as a subgraph (otherwise relabel $y's$ and $z's$ by shifting their indices). Let B_1 be the block in $(C_1)^{\phi^{-1}}$ containing x_0, i.e. containing $[x_0, y_i \mid z_j, z_k]$ as a subgraph. Then since the edges $[x, y]_0$ and $[x, z]_0$ occur in C_2, we have $i \in \{1, 2\}$ and $\{j, k\} = \{1, 2\}$, but then $[y, z]_0$ occurs in both C_1 and C_2, contradiction. □

Let $\mathbb{K}_n$ denote the set of all graphs of the form $[a_1, a_2, ..., a_n]$. (Note that $K_n \in \mathbb{K}_n$.)

Proposition 5.4.5. *Let J be the graph with $V(J) = \{x, y, z, t\}$, which contains the edge $[a, a]$ once for all $a \in V(J)$, the edges $[x, y]$ and $[z, t]$ twice, and the edge $[a, b]$ three times for all $a \in \{x, y\}$ and $b \in \{z, t\}$. Let $\Sigma = \{B_2, B_2^*, B_3, B_3^*, B_4, B_4^*\}$ be a $(\mathbb{K}_2 \cup \mathbb{K}_3 \cup \mathbb{K}_4)$-decomposition of J, where the permutation $\tau = (x, y)(z, t)$ is an automorphism of Σ, $B_k \in \mathbb{K}_k$ and $B_k^* = (B_k)^\tau$ for $k \in \{2, 3, 4\}$. Moreover suppose that the points x and y occur 4 times, and the points z and t occur 5 times in Σ. Then, Σ is isomorphic to the decomposition given in Table 5.3.*

Table 5.3: $(\mathbb{K}_2 \cup \mathbb{K}_3 \cup \mathbb{K}_4)$-decomposition of J

$B_2 = [z, z]$	$B_3 = [x, y, z]$	$B_4 = [x, x, z, t]$
$B_2^* = [t, t]$	$B_3^* = [x, y, t]$	$B_4^* = [y, y, z, t]$

Proof: A point $a \in V(J)$ cannot occur more than twice in $\{B_i, B_i^*\}$, since otherwise the edges $[c, d]$ for $c, d \in \{a, a^\lambda\}$ occur at least 6 times in total. Therefore, each point occurs twice in $\{B_4, B_4^*\}$, and the point z occurs at least once in $\{B_2, B_2^*\}$. Counting the edges $[a, b]$ for $a, b \in \{x, y\}$, we see that x occurs twice in $\{B_3, B_3^*\}$, and hence z occurs twice in $\{B_2, B_2^*\}$ and once in $\{B_3, B_3^*\}$. Therefore, without loss, Σ is as in Table 5.4, where the last row indicates the missing points in each column.

Table 5.4: ($\mathbb{K}_2 \cup \mathbb{K}_3 \cup \mathbb{K}_4$)-decomposition of J (partial)

$B_2 = [*, z]$	$B_3 = [*, *, z]$	$B_4 = [x, *, *, z]$
$B_2^* = [*, t]$	$B_3^* = [*, *, t]$	$B_4^* = [y, *, *, t]$
zt	$xxyy$	$xyzt$

We have $B_4 = [x, a, b, z]$, where $a \in \{x, y\}$ and $b \in \{z, t\}$. Also $(a, b) \neq (x, z)$, since otherwise $[x, z]$ occurs 4 times in B_4. In all other cases the edges $[x, z]$ and $[y, z]$ occur twice in $\{B_4, B_4^*\}$, and hence $B_3 = [x, y, z]$. Therefore, the edge $[x, x]$ occurs in B_4, and hence $B_4 = [x, x, z, t]$. Finally, the edge $[z, z]$ occurs in B_2 and we get the result. □

Proof of Theorem 5.4.2: We consider the four cases for the type of λ (see Lemma 5.4.1).

Case 1: λ is of type $(0, 2)$.

By Lemma 5.4.2 (5), $n'_4 \equiv 1 \pmod 3$, and hence by Lemma 5.4.2 (4), $n'_4 = 1$, $n'_1 = n'_2 = 0$, and $f'_4 = n'_3 = 1$, which contradicts Lemma 5.4.2 (5).

Case 2: λ is of type $(0, 3)$, say $\lambda = (1)(2)(3,4)(5,6)(7,8)(9)$.

By Lemma 5.4.2 (3), $f'_1 \leq 1$.

Case 2.1: $f'_1 = 1$.

Then $f'_2 = 1$ and $f'_3 = 0$. Let $fix(\lambda) = \{x, y, z\}$, and C_1 and C_2 be the blocks of types F_1 and F_2 respectively. Necessarily $C_1 \cong H_2$, say $C_1 = [x, y, z \mid x, y]$. Then the remaining edges on $\{x, y, z\}$ occur in C_2, which is impossible.

Case 2.2: $f'_1 = 0$.

By Lemma 5.4.2 (6), $f'_2 \leq 2$, and so by Lemma 5.4.2 (3), $f'_3 \geq 3$. Therefore, by Lemma 5.4.2 (2), $f'_3 = 3$ and $f'_4 = 0$. So, by Lemma 5.4.2 (3), $f'_2 = 2$ and by

Lemma 5.4.2 (6), $n'_4 = 0$. Since $f'_1 = f'_4 = n'_4 = 0$, by Lemma 5.4.3 (3), we have $f_{2,9,1} + f_{3,9,1} = 5$ which contradicts Lemma 5.4.3 (7).

Case 3: λ is of type $(1,1)$, say $\lambda = (1,2)(3,4)(5)(6)(7)(8)(9)$.

By Lemma 5.4.2 (5), $n'_4 = 1$. Then, by Lemma 5.4.2 (4), $n'_1 = n'_2 = 0$, and $f'_4 = n'_3 = 1$. By Lemma 5.4.2 (2), $f'_3 = 1$, by Lemma 5.4.2 (8), $f'_1 = 2$, and finally by Lemma 5.4.2 (1), $f'_2 = 5$. Therefore, $\mathcal{D}'$ is as in Table 5.5.

Table 5.5: Case 3

Types F_1, F_3 and F_4	Type F_2	Types N_3 and N_4
$C_1 = [\bullet,\bullet,\bullet \mid \bullet,\bullet]$	$C_3 = [\bullet,\bullet,\bullet \mid \circ,\circ]$	$C_{10} = [\circ,\circ,\circ \mid \bullet,\bullet]$
$C_2 = [\bullet,\bullet,\bullet \mid \bullet,\bullet]$	$C_4 = [\bullet,\bullet,\bullet \mid \circ,\circ]$	$(C_{10})^\lambda = [\circ,\circ,\circ \mid \bullet,\bullet]$
	$C_5 = [\bullet,\bullet,\bullet \mid \circ,\circ]$	$C_{11} = [\circ,\circ,\bullet \mid \circ,\circ]$
$C_8 = [\circ,\circ,\bullet \mid \bullet,\bullet]$	$C_6 = [\bullet,\bullet,\bullet \mid \circ,\circ]$	$(C_{11})^\lambda = [\circ,\circ,\bullet \mid \circ,\circ]$
$C_9 = [\circ,\circ,\bullet \mid \circ,\circ]$	$C_7 = [\bullet,\bullet,\bullet \mid \circ,\circ]$	

By Lemma 5.4.3 (5), $n_{3,1,1} + n_{4,1,1} = 1$. If $n_{3,1,1} = 0$, then the blocks C_{10} and $(C_{10})^\lambda$ contain in total 6 edges of the form $[x,y]$ where $x,y \in \{3,4\}$, but there are only 5 such edges. Therefore, $n_{3,1,1} = 1$, say $C_{10} = [1,\circ,\circ \mid \bullet,\bullet]$. By Lemma 5.4.3 (1), one of the edges $[3,4]$ occurs in C_8 or C_9 and the remaining 4 edges of the form $[x,y]$ for $x,y \in \{3,4\}$ occur in the last column. Therefore, $n_{4,3,2} = 0$ and $n_{4,1,2} = 2$. If the edge $[3,3]$ occurs in a block of type N_3, say in C_{10}, then we have $C_{10} = [1,3,3 \mid \bullet,\bullet]$ and $C_{11} = [3,4,\bullet \mid \circ,\circ]$. But then the edge $[1,3]$ occurs 4 times in $\{C_{10}, C_{11}, (C_{11})^\lambda\}$, contradiction. Therefore, the edge $[3,3]$ occurs in a block of type N_4, say in C_{11}, and hence $C_{10} = [1,3,4 \mid \bullet,\bullet]$ and $C_{11} = [3,3,\bullet \mid 1,2]$. Let z be a fixed point occuring in C_{10}. Note that z cannot occur twice in C_{10} and z cannot occur in C_{11}, since otherwise the edge $[3,z]$ occurs more than 3 times. Therefore, without loss, we have $C_{10} = [1,3,4 \mid 5,6]$ and $C_{11} = [3,3,7 \mid 1,2]$. Also by Proposition 5.4.3, we have

$C_9 = [1, 2, \bullet \mid 1, 2]$ and hence $C_8 = [3, 4, \bullet \mid \bullet, \bullet]$. Finally by Lemma 5.4.3 (6), $\mathcal{D}'$ is as in Table 5.6.

Table 5.6: Case 3 (continued)

Type F_1, F_3 and F_4	Type F_2	Types N_3 and N_4
$C_1 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_3 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_{10} = [1, 3, 4 \mid 5, 6]$
$C_2 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_4 = [\bullet, \bullet, \bullet \mid 1, 2]$	$(C_{10})^\lambda = [2, 3, 4 \mid 5, 6]$
	$C_5 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_{11} = [3, 3, 7 \mid 1, 2]$
$C_8 = [3, 4, \bullet \mid \bullet, \bullet]$	$C_6 = [\bullet, \bullet, \bullet \mid 3, 4]$	$(C_{11})^\lambda = [4, 4, 7 \mid 1, 2]$
$C_9 = [1, 2, \bullet \mid 1, 2]$	$C_7 = [\bullet, \bullet, \bullet \mid 3, 4]$	

Counting the occurences of the edge $[1, 7]$, we see that C_9 cannot contain the point 7. Therefore, C_9 contains 5,6,8 or 9.

Case 3.1: C_9 contains 5 or 6, without loss say 5.

Then the blocks $C_1, ..., C_9$ are as in Table 5.7, where the numbers in the last row indicate the points missing in the corresponding column. The values in the first and third columns also indicate which points occur at left and which ones occur at right. We get these values in the second and third columns by counting the edges $[x, y]$ for $x \in \{1, 3\}$ and $y \in \{5, 6, 7, 8, 9\}$. Then the values in the first column, and then the points which occur at right in the third column, follow by counting the number of occurences of each fixed point in $\mathcal{D}'$.

The edge $[7, 7]$ needs to occur in the first column. Therefore the block, say C_1, which contains 7 at left also has to contain 7 at right once, and therefore C_2 contains 7 at right once. Then we need to have $C_1 = [7, 5, 6 \mid 7, \bullet]$ and $C_2 = [5, 5, 6 \mid 7, \bullet]$, which contradicts Proposition 5.4.4 since the graphs $[5, 6 \mid 7, 7]$ and $[5, 6, 7]$ are subgraphs of C_1 and C_2 respectively.

Case 3.2: C_9 contains 8 or 9, without loss say 8.

Table 5.7: Case 3.1 (The blocks $C_1, ..., C_9$)

Type F_1	Blocks with 1 and 2	Blocks with 3 and 4
$C_1 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_3 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_6 = [\bullet, \bullet, \bullet \mid 3, 4]$
$C_2 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_4 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_7 = [\bullet, \bullet, \bullet \mid 3, 4]$
	$C_5 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_8 = [3, 4, \bullet \mid \bullet, \bullet]$
	$C_9 = [1, 2, 5 \mid 1, 2]$	
[555667 | 7789]	667888999	[5678899 | 89]

Then the blocks $C_1, ..., C_9$ are as in Table 5.8. The edge $[7, 7]$ needs to occur in the first column, say in C_1. So, $C_1 = \{7, 7, \bullet \mid \bullet, \bullet\}$ or $C_1 = \{7, \bullet, \bullet \mid 7, \bullet\}$.

Table 5.8: Case 3.2 (The blocks $C_1, ..., C_9$)

Type F_1	Blocks with 1 and 2	Blocks with 3 and 4
$C_1 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_3 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_6 = [\bullet, \bullet, \bullet \mid 3, 4]$
$C_2 = [\bullet, \bullet, \bullet \mid \bullet, \bullet]$	$C_4 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_7 = [\bullet, \bullet, \bullet \mid 3, 4]$
	$C_5 = [\bullet, \bullet, \bullet \mid 1, 2]$	$C_8 = [3, 4, \bullet \mid \bullet, \bullet]$
	$C_9 = [1, 2, 8 \mid 1, 2]$	
5566777889	556678999	567888999

Case 3.2.1: $C_1 = \{7, 7, \bullet \mid \bullet, \bullet\}$.

All occurences of 5 and 6 in the first column have to be at left. So, without loss, say $C_1 = \{7, 7, 5 \mid \bullet, \bullet\}$ and $C_2 = \{5, 6, 6 \mid 7, \bullet\}$. The point 8 cannot occur twice in C_1, and so $C_1 = \{7, 7, 5 \mid 8, 9\}$, $C_2 = \{5, 6, 6 \mid 7, 8\}$, and hence $C_8 = [3, 4, \bullet \mid 7, 9]$. Moreover, the edge $[5, 5]$ has to occur in the second column, say in C_3. So, the blocks $C_1, ..., C_9$ are as in Table 5.9.

Counting the edges $[5, x]$ for $x \in \{6, 7, 8, 9\}$, we get $C_3 = [5, 5, 9 \mid 1, 2]$ and say $C_6 = [5, 6, 8 \mid 3, 4]$. Also, since the edge $[6, 6]$ occurs in C_2, we need to have $C_4 = [6, \bullet, \bullet \mid 1, 2]$

Table 5.9: Case 3.2.1 (The blocks $C_1, ..., C_9$)

Type F_1	Blocks with 1 and 2	Blocks with 3 and 4
$C_1 = \{7,7,5 \mid 8,9\}$	$C_3 = [5,5,\bullet \mid 1,2]$	$C_6 = [\bullet,\bullet,\bullet \mid 3,4]$
$C_2 = \{5,6,6 \mid 7,8\}$	$C_4 = [\bullet,\bullet,\bullet \mid 1,2]$	$C_7 = [\bullet,\bullet,\bullet \mid 3,4]$
	$C_5 = [\bullet,\bullet,\bullet \mid 1,2]$	$C_8 = [3,4,\bullet \mid 7,9]$
	$C_9 = [1,2,8 \mid 1,2]$	
	6678999	5688899

and $C_5 = [6,\bullet,\bullet \mid 1,2]$. But then, the edge $[6,8]$ occurs 4 times, contradiction.

Case 3.2.2: $C_1 = \{7,\bullet,\bullet \mid 7,\bullet\}$.

The points 5 and 6 occur twice at left in the first column. Assume that $C_1 = \{7,5,6 \mid 7,\bullet\}$. Then, $C_2 = [5,6,\bullet \mid \bullet,\bullet]$, which contradicts Proposition 5.4.4 since the graph $[5,6 \mid 7,7]$ is a subgraph of C_1, and C_2 contains the point 7 (and hence has the subgraph $[5,6,7]$). Therefore, C_1 contains only one of the points 5 and 6, without loss, say 5. Then, $C_1 = \{7,5,\bullet \mid 7,\bullet\}$ and $C_2 = [5,6,6 \mid 7,\bullet]$. Therefore the edge $[5,8]$ occurs twice and the edge $[5,9]$ occurs once in the first column. The edge $[5,5]$ needs to occur in the second column and counting the edges $[5,x]$ for $x \in \{6,7,8,9\}$, we get say $C_3 = [5,5,9 \mid 1,2]$ and $C_6 = [5,6,8 \mid 3,4]$. Also, since the edge $[6,6]$ occurs in C_2, the blocks $C_1, ..., C_9$ are as in Table 5.10.

Counting the edges $[6,x]$ for $x \in \{7,8,9\}$, we see that the missing edges containing 6 are one $[6,7]$, two $[6,8]$'s and three $[6,9]$'s. Since one $[6,7]$, one $[6,8]$ and two $[6,9]$'s occur in the second column, therefore the remaining one $[6,8]$ and one $[6,9]$ need to occur in C_2, which is impossible.

Case 4: λ is of type $(1,2)$, say $\lambda = (1,2)(3,4)(5,6)(7)(8)(9)$.

By Lemma 5.4.2 (3), $f_1' \leq 1$, and as in Case 2.1, $f_1' = 1$ is not possible. Therefore, by Lemma 5.4.2 (3) and 5.4.2 (2), $(f_1', f_2', f_3', f_4') = (0,4,0,3)$ or $(f_1', f_2', f_3', f_4') =$

Table 5.10: Case 3.2.2 (The blocks $C_1, ..., C_9$)

Type F_1	Blocks with 1 and 2	Blocks with 3 and 4
$C_1 = \{7, 5, \bullet \mid 7, \bullet\}$	$C_3 = [5, 5, 9 \mid 1, 2]$	$C_6 = [5, 6, 8 \mid 3, 4]$
$C_2 = \{5, 6, 6 \mid 7, \bullet\}$	$C_4 = [6, \bullet, \bullet \mid 1, 2]$	$C_7 = [\bullet, \bullet, \bullet \mid 3, 4]$
	$C_5 = [6, \bullet, \bullet \mid 1, 2]$	$C_8 = [3, 4, \bullet \mid \bullet, \bullet]$
	$C_9 = [1, 2, 8 \mid 1, 2]$	
889	7899	788999

$(0, 2, 3, 0)$.

Case 4.1: $(f_1', f_2', f_3', f_4') = (0, 4, 0, 3)$.

By Lemma 5.4.2 (6), $n_4' = 0$, and then by Lemma 5.4.2 (1) and 5.4.2 (7), we have $(n_1', n_2', n_3', n_4') = (1, 1, 1, 0)$ or $(n_1', n_2', n_3', n_4') = (0, 3, 0, 0)$. Therefore, $\mathcal{D}'$ is as in Table 5.11, where among the points denoted by a star in the third column, 6 of them are fixed points and 6 of them are non-fixed points.

Table 5.11: Case 4.1

Type F_2	Type F_4	Types N_1, N_2 and N_3
$C_1 = [\bullet, \bullet, \bullet \mid \circ, \circ]$	$C_5 = [\circ, \circ, \bullet \mid \circ, \circ]$	$C_8 = [\circ, \circ, \circ \mid *, *]$
$C_2 = [\bullet, \bullet, \bullet \mid \circ, \circ]$	$C_6 = [\circ, \circ, \bullet \mid \circ, \circ]$	$(C_8)^\lambda = [\circ, \circ, \circ \mid *, *]$
$C_3 = [\bullet, \bullet, \bullet \mid \circ, \circ]$	$C_7 = [\circ, \circ, \bullet \mid \circ, \circ]$	$C_9 = [\circ, \circ, \circ \mid *, *]$
$C_4 = [\bullet, \bullet, \bullet \mid \circ, \circ]$		$(C_9)^\lambda = [\circ, \circ, \circ \mid *, *]$
		$C_{10} = [\circ, \circ, \circ \mid *, *]$
		$(C_{10})^\lambda = [\circ, \circ, \circ \mid *, *]$

Since the edges $[x, x]$ for $x \in fix(\lambda)$ have to occur in the first column, we need to have, say $C_1 = [7, 7, \bullet \mid \circ, \circ]$, $C_2 = [8, 8, \bullet \mid \circ, \circ]$, $C_3 = [9, 9, \bullet \mid \circ, \circ]$, and therefore $C_4 = [7, 8, 9 \mid \circ, \circ]$. Without loss, say the third fixed point in C_1 is 8. Then, C_2 and C_3

contain 9 and 7 respectively. Also, by Lemma 5.4.3 (1), we have say $C_5 = [1, 2, \bullet \mid \circ, \circ]$, $C_6 = [3, 4, \bullet \mid \circ, \circ]$ and $C_7 = [5, 6, \bullet \mid \circ, \circ]$. Then, by Lemma 5.4.3 (3), each of the points 7, 8 and 9 occur exactly once in the second column, without loss, say in C_5, C_6, and C_7 respectively. Moreover, by Lemma 5.4.3 (5), the point 1 needs to occur exactly once at left in the third column, say in C_8. Therefore, $\mathcal{D}'$ is as in Table 5.12.

Table 5.12: Case 4.1 (continued)

Type F_2	Type F_4	Types N_1, N_2 and N_3
$C_1 = [7, 7, 8 \mid \circ, \circ]$	$C_5 = [1, 2, 7 \mid \circ, \circ]$	$C_8 = [1, \circ, \circ \mid *, *]$
$C_2 = [8, 8, 9 \mid \circ, \circ]$	$C_6 = [3, 4, 8 \mid \circ, \circ]$	$(C_8)^\lambda = [2, \circ, \circ \mid *, *]$
$C_3 = [9, 9, 7 \mid \circ, \circ]$	$C_7 = [5, 6, 9 \mid \circ, \circ]$	$C_9 = [\circ, \circ, \circ \mid *, *]$
$C_4 = [7, 8, 9 \mid \circ, \circ]$		$(C_9)^\lambda = [\circ, \circ, \circ \mid *, *]$
		$C_{10} = [\circ, \circ, \circ \mid *, *]$
		$(C_{10})^\lambda = [\circ, \circ, \circ \mid *, *]$

By Proposition 5.4.3, there are 2 cases for the blocks C_5 and C_8.

Case 4.1.1: C_5 does not contain 1 or 2 at right and $C_8 = [1, \circ, \circ \mid 1, 2]$.

The edges $[1, x]$ for $x \in \{7, 8, 9\}$ occur 9 times in total, and one of these edges is contained in C_5. The remaining 8 edges need to occur in the first two columns (except for the block C_5). Therefore we get $3f_{2,1,2} + f_{4,1,2} = 8$, and hence $f_{2,1,2} = f_{4,1,2} = 2$. But then, each of the edges $[1, x]$ for $x \in \{7, 8, 9\}$ occur exactly once in the second column and needs to occur exactly 2 times in the first column, which is a contradiction since there exists no pair of blocks in the first column covering each $x \in \{7, 8, 9\}$ twice.

Case 4.1.2: $C_5 = [1, 2, 7 \mid 1, 2]$ and C_8 does not contain 1 or 2 at right.

The missing points occuring at left in the third column are 3, 4, 5 and 6 (each occuring 4 times). Let a, b, and d be the total number of occurences of the points 1 and 2 at right in $\{C_1, C_2, C_3, C_4\}$, $\{C_6, C_7\}$, and $\{C_9, (C_9)^\lambda, C_{10}, (C_{10})^\lambda\}$ respectively,

a', b', c', and d' be the total number of occurences of the points 3, 4, 5 and 6 at right in $\{C_1, C_2, C_3, C_4\}$, $\{C_6, C_7\}$, $\{C_8, (C_8)^\lambda\}$ and $\{C_9, (C_9)^\lambda, C_{10}, (C_{10})^\lambda\}$ respectively, and c'' and d'' be the total number of occurences of the points 7, 8 and 9 at right in $\{C_8, (C_8)^\lambda\}$ and $\{C_9, (C_9)^\lambda, C_{10}, (C_{10})^\lambda\}$ respectively. Clearly, $a + a' = 8$, $b + b' = 4$, $c' + c'' = 4$, and $d + d' + d'' = 8$. Also, since a point in U_1 and U_2 occur at right 6 times and 2 times respectively, we have $a + b + d = 10$, $a' + b' + c' + d' = 8$, and $c'' + d'' = 6$.

There are $3\binom{4}{2} + 4 = 22$ edges of the form $[x, y]$ where $x, y \in \{3, 4, 5, 6\}$, and 16 of these edges are contained in blocks $\{C_6, C_7\} \cup \{C_i, (C_i)^\lambda \mid 8 \leq i \leq 10\}$, in such a way that both x and y occur at left. Therefore, the remaining 6 edges occur in such a way that one of x and y occurs at left and the other one occurs at right. Therefore, we get $2b' + 2c' + 3d' = 6$. Note that since $\{3, 4\}$ and $\{5, 6\}$ are orbits of λ, therefore b', c', and d' are all even. Therefore, $b' = c' = 0$, $d' = 2$, and we get $(a, a', b, b', c', c'', d, d', d'') = (2, 6, 4, 0, 0, 4, 4, 2, 2)$. Let $\{x, y, z, t\} = \{3, 4, 5, 6\}$ such that $\{x, y\} = \{3, 4\}$ or $\{x, y\} = \{5, 6\}$. Then, $\mathcal{D}'$ is as in Table 5.13.

Table 5.13: Case 4.1.2

Type F_2	Type F_4	Types N_1, N_2 and N_3
$C_1 = [7, 7, 8 \mid \circ, \circ]$	$C_5 = [1, 2, 7 \mid 1, 2]$	$C_8 = [1, \circ, \circ \mid \bullet, \bullet]$
$C_2 = [8, 8, 9 \mid \circ, \circ]$	$C_6 = [3, 4, 8 \mid 1, 2]$	$(C_8)^\lambda = [2, \circ, \circ \mid \bullet, \bullet]$
$C_3 = [9, 9, 7 \mid \circ, \circ]$	$C_7 = [5, 6, 9 \mid 1, 2]$	$C_9 = [\circ, \circ, \circ \mid *, \circ]$
$C_4 = [7, 8, 9 \mid \circ, \circ]$		$(C_9)^\lambda = [\circ, \circ, \circ \mid *, \circ]$
		$C_{10} = [\circ, \circ, \circ \mid *, z]$
		$(C_{10})^\lambda = [\circ, \circ, \circ \mid *, t]$
$123456xy$		$[3333444455556666 \mid 1122778899]$

Note that the subgraphs of the blocks in the third column of Table 5.13 on the points $\{3, 4, 5, 6\}$ satisfy the conditions of Proposition 5.4.5. Therefore, according to whether the point marked by a $*$ in C_9 is fixed or not, there are 2 cases for these blocks which are given in Table 5.14. The missing points in each case are 1122778899.

Table 5.14: Cases 4.1.2.1 and 4.1.2.2 (Blocks of types N_1, N_2 and N_3)

Case 4.1.2.1	**Case 4.1.2.2**
$C_8 = [1, z, z \mid \bullet, \bullet]$	$C_8 = [1, z, z \mid \bullet, \bullet]$
$(C_8)^\lambda = [2, t, t \mid \bullet, \bullet]$	$(C_8)^\lambda = [2, t, t \mid \bullet, \bullet]$
$C_9 = [x, y, z \mid \bullet, \circ]$	$C_9 = [x, y, z \mid \circ, \circ]$
$(C_9)^\lambda = [x, y, t \mid \bullet, \circ]$	$(C_9)^\lambda = [x, y, t \mid \circ, \circ]$
$C_{10} = [x, x, t \mid \circ, z]$	$C_{10} = [x, x, t \mid \bullet, z]$
$(C_{10})^\lambda = [y, y, z \mid \circ, t]$	$(C_{10})^\lambda = [y, y, z \mid \bullet, t]$

Since the edge $[1, z]$ occurs once in $\{C_6, C_7\}$ and twice in C_8, the edge $[1, z]$ cannot occur in the blocks in Table 5.14. In the first case, the point 1 needs to occur in $(C_9)^\lambda$ and C_{10}, but then the edge $[1, x]$ occurs 4 times, contradiction. In the second case, the point 1 needs to occur twice in $(C_9)^\lambda$, which contradicts Proposition 5.4.4, since the graphs $[x, y \mid 1, 1]$ and $[x, y, 1]$ are subgraphs of $(C_9)^\lambda$ and C_6 (or C_7) respectively.

Case 4.2: $(f_1', f_2', f_3', f_4') = (0, 2, 3, 0)$.

By Lemma 5.4.2 (6), $n_4' = 3$, and then by Lemma 5.4.2 (1) and 5.4.2 (7), we have $(n_1', n_2', n_3', n_4') = (1, 0, 0, 3)$. Therefore, $\mathcal{D}'$ is as in Table 5.15.

We first consider the blocks of types N_1 and N_4. By Proposition 5.4.3, the block of type N_1 or N_4 containing the point 1 at left contains the points 1 and 2 at right. For $i \in \{1, 4\}$, $(j, k) \in \{(0, 0), (0, 1), (0, 2), (1, 2)\}$, let $m_{i,j,k}$ be the number of blocks C_t for $6 \leq t \leq 9$ such that C_t is of type N_i and the point 1 occurs j times at left and k times at right in $\{C_t, (C_t)^\lambda\}$. Also let m' be the number of occurences of the point 1 at right in a block of type F_2. Clearly, we have

(1) $m_{1,0,0} + m_{1,0,1} + m_{1,0,2} + m_{1,1,2} = 1,$

Table 5.15: Case 4.2

Types F_2 and F_3	Type N_1	Type N_4
$C_1 = [\bullet, \bullet, \bullet \mid \circ, \circ]$	$C_6 = [\circ, \circ, \circ \mid \circ, \circ]$	$C_7 = [\circ, \circ, \bullet \mid \circ, \circ]$
$C_2 = [\bullet, \bullet, \bullet \mid \circ, \circ]$	$(C_6)^\lambda = [\circ, \circ, \circ \mid \circ, \circ]$	$(C_7)^\lambda = [\circ, \circ, \bullet \mid \circ, \circ]$
$C_3 = [\circ, \circ, \bullet \mid \bullet, \bullet]$		$C_8 = [\circ, \circ, \bullet \mid \circ, \circ]$
$C_4 = [\circ, \circ, \bullet \mid \bullet, \bullet]$		$(C_8)^\lambda = [\circ, \circ, \bullet \mid \circ, \circ]$
$C_5 = [\circ, \circ, \bullet \mid \bullet, \bullet]$		$C_9 = [\circ, \circ, \bullet \mid \circ, \circ]$
		$(C_9)^\lambda = [\circ, \circ, \bullet \mid \circ, \circ]$

(2) $m_{4,0,0} + m_{4,0,1} + m_{4,0,2} + m_{4,1,2} = 3$, and

(3) $m_{1,1,2} + m_{4,1,2} = 1$. (see Proposition 5.4.3)

Counting the occurences of the point 1 at right, we get

(4) $m_{1,0,1} + m_{4,0,1} + 2m_{1,0,2} + 2m_{4,0,2} + 2m_{1,1,2} + 2m_{4,1,2} + m' = 6$

and hence by (3) and (4), we get

(5) $m_{1,0,1} + m_{4,0,1} + 2m_{1,0,2} + 2m_{4,0,2} + m' = 4$.

Counting the edges $[1, x]$ for $x \in fix(\lambda)$, we get

(6) $m_{4,0,1} + 2m_{4,0,2} + 3m_{4,1,2} + 3m' = 6$,

and hence by (5), (6), and (1) we get

(7) $3m_{4,1,2} + 2m' = m_{1,0,1} + 2m_{1,0,2} + 2 \leq 4$.

By (3), $m_{1,1,2} \leq 1$. Assume that $m_{1,1,2} = 1$. Then we have, say $C_6 = [1, \circ, \circ \mid 1, 2]$. By (1), we get $m_{1,0,0} = m_{1,0,1} = m_{1,0,2} = 0$, and then by (7), we get $m' = 1$. So, say $C_1 = [\bullet, \bullet, \bullet \mid 1, 2]$ and $C_2 = [\bullet, \bullet, \bullet \mid 3, 4]$. Then, counting the edges $[3, x]$ for $x \in fix(\lambda)$, we get $n_{4,3,1} + n_{4,3,2} = 3$, and then counting the occurences of the point 3, we get $n_{1,3,1} = 2$. Therefore $C_6 = [1, 3, 4 \mid 1, 2]$, which now contradicts Proposition 5.4.4 since C_6 and $(C_6)^\lambda$ contain the subgraphs $[3, 4 \mid 1, 1]$ and $[3, 4, 1]$ respectively. Therefore $m_{1,1,2} = 0$ and hence $m_{4,1,2} = 1$.

Then, by (7), we get $m' = 0$, $m_{1,0,2} = 0$ and $m_{1,0,1} = 1$, and then by (1) we get $m_{1,0,0} = 0$. Finally, by (2) and (6), we get $m_{4,0,0} = 0$, and $m_{4,0,1} = m_{4,0,2} = 1$.

Moreover, for $y \in \{3,4,5,6\}$, assume that $f_{2,y,2} = 2$. Then, counting edges $[y,x]$ for $x \in fix(\lambda)$, we get $n_{4,y,1} = n_{4,y,2} = 0$. But then counting the occurences of y at left, we get $n_{1,y,1} = 4$, contradiction. Therefore, $f_{2,y,2} = 1$ for all $y \in \{3,4,5,6\}$.

Also, for $x \in fix(\lambda)$, by Lemma 5.4.3 (4), we get $f_{3,x,2} = 2$, and then by Lemma 5.4.3 (7), $f_{2,x,1} + f_{3,x,1} = 3$. Finally, by Lemma 5.4.3 (3), $n_{4,x,1} = 2$. Therefore, without loss, $\mathcal{D}'$ is as in Table 5.16, where $\{a,b\} = \{1,2\}$.

Table 5.16: Case 4.2 (continued)

Type F_2 and F_3	Type N_1	Type N_4
$C_1 = [\bullet,\bullet,\bullet \mid 3,4]$	$C_6 = [\circ,\circ,\circ \mid 1,\circ]$	$C_7 = [\circ,\circ,7 \mid 1,\circ]$
$C_2 = [\bullet,\bullet,\bullet \mid 5,6]$	$(C_6)^\lambda = [\circ,\circ,\circ \mid 2,\circ]$	$(C_7)^\lambda = [\circ,\circ,7 \mid 2,\circ]$
$C_3 = [1,2,\bullet \mid \bullet,\bullet]$		$C_8 = [\circ,\circ,8 \mid 1,a]$
$C_4 = [3,4,\bullet \mid \bullet,\bullet]$		$(C_8)^\lambda = [\circ,\circ,8 \mid 2,b]$
$C_5 = [5,6,\bullet \mid \bullet,\bullet]$		$C_9 = [1,3,9 \mid 1,2]$
		$(C_9)^\lambda = [2,4,9 \mid 1,2]$

Note that the subgraphs of the blocks of types N_1 and N_4 on the points $\{3,4,5,6\}$ satisfy the conditions of Proposition 5.4.5 (with $\{x,y\} = \{3,4\}$ and $\{z,t\} = \{5,6\}$). Therefore, without loss, say $C_8 = [5,5,8 \mid 1,a]$ and hence $a = 2$.

By Proposition 5.4.5, the missing points in C_6 are either 3356 or 4456, and the missing points in C_7 are either 345 or 346. Counting the edges $[1,3]$, we see that the missing points in C_6 are 4456. Without loss, the point 5 occurs at right either in C_6 or C_7. In the first case, we get $C_6 = [4,4,6 \mid 1,5]$ and then counting the edges $[1,c]$ for $c \in \{3,4,5,6\}$, we get $C_7 = [3,5,7 \mid 1,4]$. In the second case, we get $C_7 = [3,4,7 \mid 1,5]$ and then counting the edges $[1,c]$ for $c \in \{3,4,5,6\}$, we get

$C_6 = [4,5,6 \mid 1,4]$. Therefore $\{C_i \mid 6 \leq i \leq 9\} = \mathcal{D}'^{k}_{[6,7]} \cup \mathcal{D}'_{[8,9]}$ for some $k \in \{1,2\}$ (see Table 5.17).

Table 5.17: Case 4.2 (Two cases for the blocks $C_6, ..., C_9$)

$\mathcal{D}'^{1}_{[6,7]}$	$\mathcal{D}'^{2}_{[6,7]}$	$\mathcal{D}'_{[8,9]}$
$C_6 = [4,4,6 \mid 1,5]$	$C_6 = [4,5,6 \mid 1,4]$	$C_8 = [5,5,8 \mid 1,2]$
$C_7 = [3,5,7 \mid 1,4]$	$C_7 = [3,4,7 \mid 1,5]$	$C_9 = [1,3,9 \mid 1,2]$

Now we consider the blocks of types F_2 and F_3. For $1 \leq i \leq 5$, let C_i^* be the subgraph of C_i on the fixed points of λ, and $\mathcal{D}^* = \{C_1^*, C_2^*, C_3^*, C_4^*, C_5^*\}$. We will show that $\mathcal{D}^* \cong \mathcal{D}_j^*$ for some $1 \leq j \leq 6$ (see Table 5.18).

Table 5.18: Case 4.2 (Six cases for the blocks $C_1^*, ..., C_5^*$)

$\mathcal{D}_1^*$	$\mathcal{D}_2^*$	$\mathcal{D}_3^*$	$\mathcal{D}_4^*$	$\mathcal{D}_5^*$	$\mathcal{D}_6^*$
$[7,7,8]$	$[7,7,8]$	$[7,7,9]$	$[7,7,9]$	$[7,7,8]$	$[7,8,9]$
$[7,8,9]$	$[7,8,9]$	$[7,8,8]$	$[7,8,8]$	$[8,9,9]$	$[7,8,9]$
$[8 \mid 8,9]$	$[8 \mid 8,9]$	$[8 \mid 7,9]$	$[8 \mid 7,9]$	$[7 \mid 8,9]$	$[7 \mid 7,8]$
$[9 \mid 7,7]$	$[9 \mid 7,8]$	$[9 \mid 7,8]$	$[9 \mid 8,8]$	$[8 \mid 8,9]$	$[8 \mid 8,9]$
$[9 \mid 8,9]$	$[9 \mid 7,9]$	$[9 \mid 8,9]$	$[9 \mid 7,9]$	$[9 \mid 7,7]$	$[9 \mid 7,9]$

Let $x \in fix(\lambda)$. Recall that $f_{3,x,2} = 2$ and $f_{2,x,1} + f_{3,x,1} = 3$. (See the paragraph above Table 5.16.) If x occurs 3 times at left in $\{C_3, C_4, C_5\}$, then the edge $[x,x]$ occurs twice, contradiction. Therefore, either $\{C_1^*, C_2^*\}$ contains one point say 7 three times, one point say 8 twice, and hence the point 9 once, or $\{C_1^*, C_2^*\}$ contains all three points twice.

If $\{C_1^*, C_2^*\}$ contains 777889, then $\{C_1^*, C_2^*\}$ is either as in $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ or as in $\mathcal{D}_3^*$ and $\mathcal{D}_4^*$. In both cases, 8 occurs once and 9 occurs twice at left in $\{C_3^*, C_4^*, C_5^*\}$. Then, if we count the edges $[8,8]$, $[9,9]$, $[7,8]$, $[8,9]$ and $[7,9]$, we get $\mathcal{D}_1^*$, $\mathcal{D}_2^*$, $\mathcal{D}_3^*$ and $\mathcal{D}_4^*$.

If $\{C_1^*, C_2^*\}$ contains 778899, then $\{C_1^*, C_2^*\}$ is either as in $\mathcal{D}_5^*$ or as in $\mathcal{D}_6^*$. In both cases, each point occurs once at left in $\{C_3^*, C_4^*, C_5^*\}$, and the block containing the point 8 at left is either $[8 \mid 8, 7]$ or $[8 \mid 8, 9]$, and without loss, we can say it is $[8 \mid 8, 9]$. Then, counting the remaining edges, we get $\mathcal{D}_5^*$ and $\mathcal{D}_6^*$. Therefore, $\mathcal{D}^* \cong \mathcal{D}_j^*$ for some $1 \leq j \leq 6$ as claimed. Note that $\mathcal{D}_i^* \ncong \mathcal{D}_j^*$ for $i \neq j$.

Now if we count the edges $[a, x]$ for $a \in \{1, 3, 5\}$ and $x \in fix(\lambda)$, we see that the fixed points in $\{C_3\}$, $\{C_1, C_4\}$ and $\{C_2, C_5\}$ are 778, 788899 and 778999 respectively. Then, looking for such configurations for $\mathcal{D}^*$, one can see from Table 5.18 that, if we let θ and π be the permutations $(7, 8, 9)$ and $(7, 9)$ respectively, then $\mathcal{D}^* \in \{(\mathcal{D}_1^*)^{\pi\theta}, (\mathcal{D}_1^*)^{\pi}, (\mathcal{D}_2^*)^{\theta^2}, (\mathcal{D}_2^*)^{\theta}, (\mathcal{D}_3^*)^{\pi}, (\mathcal{D}_5^*)^{\theta^2}, (\mathcal{D}_5^*)^{\pi\theta}, \mathcal{D}_6^*\}$, where for the case $\mathcal{D}^* = (\mathcal{D}_3^*)^{\pi}$, there are 2 choices for the blocks $\{C_4, C_5\}$. Note that the cases $\mathcal{D}^* = (\mathcal{D}_6^*)^{\theta}$ and $\mathcal{D}^* = (\mathcal{D}_6^*)^{\theta^2}$ are included since $(\mathcal{D}_6^*)^{\theta} = \mathcal{D}_6^*$. These 9 cases for the blocks $\{C_i \mid 1 \leq i \leq 5\}$ are given in Table 5.19, where 3 of these cases are not possible for the following reasons: If $\mathcal{D}^* = (\mathcal{D}_1^*)^{\pi}$ or $\mathcal{D}^* = (\mathcal{D}_5^*)^{\theta^2}$, then the blocks $\{C_2, C_5\}$ contradict Proposition 5.4.4, and if $\mathcal{D}^* = (\mathcal{D}_3^*)^{\pi}$ and $C_4 = [3, 4, 8 \mid 7, 9]$, then the blocks $\{C_1, C_4\}$ contradict Proposition 5.4.4. Therefore, $\mathcal{D}' = \mathcal{D}'^{j}_{[1,5]} \cup \mathcal{D}'^{k}_{[6,7]} \cup \mathcal{D}'_{[8,9]}$ for some $j \in \{1, 2, 3, 4, 5, 6\}$ and $k \in \{1, 2\}$ (see Tables 5.17 and 5.19).

Now for each of these 12 cases, we try to extend $\mathcal{D}'$ to $\mathcal{D}$ by inserting subscripts. For $1 \leq i \leq 9$, let $B_i \in \mathcal{D}$ be such that $(B_i)^{\phi} = C_i$ and that the conditions of Proposition 5.4.2 are satisfied (see the remarks below Proposition 5.4.2). During the process of inserting subscripts to the points, for $x \in U$ and $C_i \in \mathcal{D}'$, we are free to choose the subscripts for all occurences of x in B_i without any loss, if

(1) The point x occurs twice at left or once in C_i and no subscript is inserted to a point in C_i yet (since we can choose B_i as the block in $(C_i)^{\phi^{-1}}$ where the subscript(s) of x in B_i are as required), or

Table 5.19: Case 4.2 (Nine cases for the blocks $C_1, ..., C_5$ (Only six cases are possible))

If $\mathcal{D}^* = (\mathcal{D}_1^*)^{\pi\theta}$	If $\mathcal{D}^* = (\mathcal{D}_1^*)^{\pi}$	If $\mathcal{D}^* = (\mathcal{D}_2^*)^{\theta^2}$
$\mathcal{D}'^1_{[1,5]}$	(*not possible*)	$\mathcal{D}'^2_{[1,5]}$
$C_1 = [7,8,9 \mid 3,4]$	$C_1 = [8,9,9 \mid 3,4]$	$C_1 = [7,8,9 \mid 3,4]$
$C_2 = [7,7,9 \mid 5,6]$	$C_2 = [7,8,9 \mid 5,6]$	$C_2 = [7,9,9 \mid 5,6]$
$C_3 = [1,2,8 \mid 7,7]$	$C_3 = [1,2,7 \mid 7,8]$	$C_3 = [1,2,7 \mid 7,8]$
$C_4 = [3,4,8 \mid 8,9]$	$C_4 = [3,4,8 \mid 7,8]$	$C_4 = [3,4,8 \mid 8,9]$
$C_5 = [5,6,9 \mid 8,9]$	$C_5 = [5,6,7 \mid 9,9]$	$C_5 = [5,6,8 \mid 7,9]$
If $\mathcal{D}^* = (\mathcal{D}_2^*)^{\theta}$	If $\mathcal{D}^* = (\mathcal{D}_3^*)^{\pi}$ (1)	If $\mathcal{D}^* = (\mathcal{D}_3^*)^{\pi}$ (2)
$\mathcal{D}'^3_{[1,5]}$	(*not possible*)	$\mathcal{D}'^4_{[1,5]}$
$C_1 = [8,8,9 \mid 3,4]$	$C_1 = [8,8,9 \mid 3,4]$	$C_1 = [8,8,9 \mid 3,4]$
$C_2 = [7,8,9 \mid 5,6]$	$C_2 = [7,9,9 \mid 5,6]$	$C_2 = [7,9,9 \mid 5,6]$
$C_3 = [1,2,7 \mid 7,8]$	$C_3 = [1,2,7 \mid 7,8]$	$C_3 = [1,2,7 \mid 7,8]$
$C_4 = [3,4,7 \mid 8,9]$	$C_4 = [3,4,8 \mid 7,9]$	$C_4 = [3,4,7 \mid 8,9]$
$C_5 = [5,6,9 \mid 7,9]$	$C_5 = [5,6,7 \mid 8,9]$	$C_5 = [5,6,8 \mid 7,9]$
If $\mathcal{D}^* = (\mathcal{D}_5^*)^{\theta^2}$	If $\mathcal{D}^* = (\mathcal{D}_5^*)^{\pi\theta}$	If $\mathcal{D}^* = \mathcal{D}_6^*$
(*not possible*)	$\mathcal{D}'^5_{[1,5]}$	$\mathcal{D}'^6_{[1,5]}$
$C_1 = [7,8,8 \mid 3,4]$	$C_1 = [8,8,9 \mid 3,4]$	$C_1 = [7,8,9 \mid 3,4]$
$C_2 = [7,9,9 \mid 5,6]$	$C_2 = [7,7,9 \mid 5,6]$	$C_2 = [7,8,9 \mid 5,6]$
$C_3 = [1,2,7 \mid 7,8]$	$C_3 = [1,2,8 \mid 7,7]$	$C_3 = [1,2,7 \mid 7,8]$
$C_4 = [3,4,8 \mid 9,9]$	$C_4 = [3,4,7 \mid 8,9]$	$C_4 = [3,4,8 \mid 8,9]$
$C_5 = [5,6,9 \mid 7,8]$	$C_5 = [5,6,9 \mid 8,9]$	$C_5 = [5,6,9 \mid 7,9]$

(2) The point x occurs twice at left or once in C_i and no subscript is inserted to x or $y = x^\lambda$ in any block of $\mathcal{D}'$ yet (since we can relabel x_t's and y_t's by shifting their indices), or

(3) The point x occurs once at left and once at right in C_i and no subscript is inserted to any point in any block of $\mathcal{D}'$ yet (since we can choose the subscript of x occuring at left, say as t, and then if necessary relabel the points by switching z_{t+1} and z_{t-1} for all $z \in U$ and replacing σ by σ^{-1}).

We now consider the 2 cases for $\{C_6, C_7\}$.

Case 4.2.1: $\{C_6, C_7\} = \mathcal{D}'^1_{[6,7]}$ (see Table 5.17).

Say $B_6 = [4_0, 4_1, 6_0 \mid 1_0, 5_i]$ for some $i \in \mathbb{Z}_3$. Then the edges $[5,6]_{-i}$ and $[5,6]_i$ occur in C_6 and $(C_6)^\lambda$ respectively, and hence $i \neq 0$. Say $B_7 = [3_*, 5_0, 7_0 \mid 1_*, 4_*]$. Since the edges $[3,5]_0$, $[3,5]_2$, $[4,5]_i$ and $[4,5]_{i-1}$ occur in $\{C_6, (C_6)^\lambda\}$, we get $B_7 = [3_2, 5_0, 7_0 \mid 1_j, 4_{2-i}]$ for some $j \in \mathbb{Z}_3$. Say $B_8 = [5_0, 5_1, 8_0 \mid 1_*, 2_*]$. Since the edges $[1,5]_{-j}$ and $[2,5]_0$ occur in C_7 and $(C_6)^\lambda$, we get $B_8 = [5_0, 5_1, 8_0 \mid 1_{j-1}, 2_2]$. Then the edges $[1,8]_{1-j}$ and $[1,8]_1$ occur in C_8 and $(C_8)^\lambda$ respectively, and hence $j \neq 0$. Say $B_9 = [1_*, 3_*, 9_0 \mid 1_*, 2_0]$. Since the edges $[2,3]_0$ and $[2,3]_1$ occur in $(C_6)^\lambda$, we get $B_9 = [1_k, 3_2, 9_0 \mid 1_m, 2_0]$ for some $k, m \in \mathbb{Z}_3$. Since the edge $[1,3]_{2-j}$ occurs in C_7, we get $k \neq j$ and $m \neq j$. Also, since the edge $[1,9]_0$ occurs in $(C_9)^\lambda$, we get $k \neq 0$ and $m \neq 0$. Since $j \neq 0$, we get $k = m$, contradiction since a point cannot occur more than once in a block.

Case 4.2.2: $\{C_6, C_7\} = \mathcal{D}'^2_{[6,7]}$ (see Table 5.17).

Say $B_6 = [4_0, 5_0, 6_i \mid 1_0, 4_1]$ for some $i \in \mathbb{Z}_3$. Then the edges $[5,6]_i$ and $[5,6]_{-i}$ occur in C_6 and $(C_6)^\lambda$ respectively, and hence $i \neq 0$. Say $B_7 = [3_*, 4_*, 7_0 \mid 1_*, 5_0]$. Since the edges $[4,5]_0$, $[4,5]_2$, $[3,5]_i$ and $[3,5]_{i-1}$ occur in $\{C_6, (C_6)^\lambda\}$, we get $B_7 = [3_{2-i}, 4_2, 7_0 \mid 1_j, 5_0]$ for some $j \in \mathbb{Z}_3$. Say $B_8 = [5_0, 5_1, 8_0 \mid 1_*, 2_*]$. Since the edges

$[1,5]_0$ and $[2,5]_i$ occur in C_6 and $(C_6)^\lambda$, we get $B_8 = [5_0, 5_1, 8_0 \mid 1_2, 2_{2-i}]$. Say $B_9 = [1_p, 3_{2-i}, 9_i \mid 1_q, 2_r]$ for some $p, q, r \in \mathbb{Z}_3$. Since the edge $[1,3]_{2-i-j}$ occurs in C_7, we have $p \neq j$ and $q \neq j$. Also, since the edge $[1,9]_{i-r}$ occurs in $(C_9)^\lambda$, we get $p \neq r$ and $q \neq r$. Since $p \neq q$, we need to have $r = j$. Since the edges $[1,4]_0$, $[1,4]_{2-j}$ and $[1,4]_{2-i-j}$ occur in $\{C_6, C_7, (C_9)^\lambda\}$, we get $j \neq 2$ and $2 - i - j = j - 2$, i.e. $i = j + 1$, and hence $B_9 = [1_{j+k}, 3_{1-j}, 9_{j+1} \mid 1_{j-k}, 2_j]$ for some $k \in \{1, 2\}$. Therefore, we get $\mathcal{D}^{j,k}_{[6,9]}$ in Theorem 5.4.2.

We now consider the 6 cases for $\mathcal{D}'_{[1,5]} = \{C_t \mid 1 \leq t \leq 5\}$ (see Table 5.19), and complete the construction. Counting the edges of the form $[x,y]_m$ for $x \notin fix(\lambda)$ and $y \in fix(\lambda)$ in the blocks $\{C_t, (C_t)^\lambda \mid 6 \leq t \leq 9\}$, we see that the edges $[x,y]_m$ for $x \in \{1,2\}$ and $y_m \in \{7_{1-j}, 7_{2-j}, 8_{-j}\}$, for $x \in \{3,4\}$ and $y_m \in \{7_{-j}, 8_0, 8_1, 8_2, 9_{1-j}, 9_{2-j}\}$, and for $x \in \{5,6\}$ and $y_m \in \{7_1, 7_2, 8_1, 9_0, 9_1, 9_2\}$ need to occur in $\mathcal{D}'_{[1,5]}$.

Say $B_1 = [*,*,* \mid 3_j, 4_j]$, $B_2 = [*,*,* \mid 5_0, 6_0]$, $B_3 = [1_j, 2_j, * \mid *,*]$, $B_4 = [3_j, 4_j, * \mid *,*]$, and $B_5 = [5_0, 6_0, * \mid *,*]$. Then, $\{B_3\}$, $\{B_1, B_4\}$ and $\{B_2, B_5\}$ contain the points $\{7_1, 7_2, 8_0\}$, $\{7_0, 8_0, 8_1, 8_2, 9_1, 9_2\}$, and $\{7_1, 7_2, 8_1, 9_0, 9_1, 9_2\}$ respectively. For $1 \leq i \leq 6$, define $\mathcal{D}^{i,j}_{[1,5]}$ such that $\left(\mathcal{D}^{i,j}_{[1,5]}\right)^\phi = \mathcal{D}'^{i}_{[1,5]}$ (see Table 5.19). Then, we get Table 5.20 where $m, n, p, u, v \in \{1,2\}$ and $q, r \in \{0,1,2\}$.

Then, looking at the edges $[x,y]_t$ for each pair $x, y \in fix(\lambda)$, we get the constraints given on Table 5.21. For example, for the case $\mathcal{D}_{[1,5]} = \mathcal{D}^{2,j}_{[1,5]}$, the edges $[7,8]_t$ occur for $t \in \{q, -m, 1+p\}$, and hence $-m \neq 1 + p$, i.e. $m + p \neq 2$, and $q - m + 1 + p = 0$, i.e. $q = m - p - 1$. The edges $[7,9]_t$ occur for $t \in \{n, r-1, r+1\}$, and hence $r = n$. Finally, the edges $[8,9]_t$ occur for $t \in \{n-q, -n-q-u, r+1\}$, and hence $n - q \neq -n - q - u$, i.e. $n \neq u$, and $(n-q) + (-n-q-u) + (r+1) = 0$, i.e. $q + r = u - 1$. We get the constraints for the other cases using similar arguments.

One can now see from the constraints in Table 5.21 that the following holds:

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{1,j}_{[1,5]}$, then $(n, q, r, u, v) = (1, 0, 1, 2, 2)$.

Table 5.20: Case 4.2.2 (Six cases for the blocks $B_1, ..., B_5$)

$\mathcal{D}^{1,j}_{[1,5]}$	$\mathcal{D}^{2,j}_{[1,5]}$
$B^{1,j}_1 = [7_0, 8_q, 9_n \mid 3_j, 4_j]$	$B^{2,j}_1 = [7_0, 8_q, 9_n \mid 3_j, 4_j]$
$B^{1,j}_2 = [7_1, 7_2, 9_r \mid 5_0, 6_0]$	$B^{2,j}_2 = [7_p, 9_r, 9_{r+1} \mid 5_0, 6_0]$
$B^{1,j}_3 = [1_j, 2_j, 8_0 \mid 7_1, 7_2]$	$B^{2,j}_3 = [1_j, 2_j, 7_m \mid 7_{-m}, 8_0]$
$B^{1,j}_4 = [3_j, 4_j, 8_{q+u} \mid 8_{q-u}, 9_{-n}]$	$B^{2,j}_4 = [3_j, 4_j, 8_{q+u} \mid 8_{q-u}, 9_{-n}]$
$B^{1,j}_5 = [5_0, 6_0, 9_{r+v} \mid 8_1, 9_{r-v}]$	$B^{2,j}_5 = [5_0, 6_0, 8_1 \mid 7_{-p}, 9_{r-1}]$

$\mathcal{D}^{3,j}_{[1,5]}$	$\mathcal{D}^{4,j}_{[1,5]}$
$B^{3,j}_1 = [8_q, 8_{q+1}, 9_n \mid 3_j, 4_j]$	$B^{4,j}_1 = [8_q, 8_{q+1}, 9_n \mid 3_j, 4_j]$
$B^{3,j}_2 = [7_p, 8_1, 9_r \mid 5_0, 6_0]$	$B^{4,j}_2 = [7_p, 9_r, 9_{r+1} \mid 5_0, 6_0]$
$B^{3,j}_3 = [1_j, 2_j, 7_m \mid 7_{-m}, 8_0]$	$B^{4,j}_3 = [1_j, 2_j, 7_m \mid 7_{-m}, 8_0]$
$B^{3,j}_4 = [3_j, 4_j, 7_0 \mid 8_{q-1}, 9_{-n}]$	$B^{4,j}_4 = [3_j, 4_j, 7_0 \mid 8_{q-1}, 9_{-n}]$
$B^{3,j}_5 = [5_0, 6_0, 9_{r+v} \mid 7_{-p}, 9_{r-v}]$	$B^{4,j}_5 = [5_0, 6_0, 8_1 \mid 7_{-p}, 9_{r-1}]$

$\mathcal{D}^{5,j}_{[1,5]}$	$\mathcal{D}^{6,j}_{[1,5]}$
$B^{5,j}_1 = [8_q, 8_{q+1}, 9_n \mid 3_j, 4_j]$	$B^{6,j}_1 = [7_0, 8_q, 9_n \mid 3_j, 4_j]$
$B^{5,j}_2 = [7_1, 7_2, 9_r \mid 5_0, 6_0]$	$B^{6,j}_2 = [7_p, 8_1, 9_r \mid 5_0, 6_0]$
$B^{5,j}_3 = [1_j, 2_j, 8_0 \mid 7_1, 7_2]$	$B^{6,j}_3 = [1_j, 2_j, 7_m \mid 7_{-m}, 8_0]$
$B^{5,j}_4 = [3_j, 4_j, 7_0 \mid 8_{q-1}, 9_{-n}]$	$B^{6,j}_4 = [3_j, 4_j, 8_{q+u} \mid 8_{q-u}, 9_{-n}]$
$B^{5,j}_5 = [5_0, 6_0, 9_{r+v} \mid 8_1, 9_{r-v}]$	$B^{6,j}_5 = [5_0, 6_0, 9_{r+v} \mid 7_{-p}, 9_{r-v}]$

Table 5.21: Constraints for m, n, p, q, r, u, and v

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{1,j}_{[1,5]}$ then	If $\mathcal{D}_{[1,5]} = \mathcal{D}^{2,j}_{[1,5]}$ then	If $\mathcal{D}_{[1,5]} = \mathcal{D}^{3,j}_{[1,5]}$ then
$q = 0$	$m + p \neq 2$	$p \neq m + 1$
$r = n$	$q = m - p - 1$	$q = m + p$
$n \neq u$	$r = n + p + 1$	$p \neq v$
$q + r = u - v + 1$	$n \neq u$	$r = v - n$
	$q + r = u - 1$	$q + r = n - 1$
If $\mathcal{D}_{[1,5]} = \mathcal{D}^{4,j}_{[1,5]}$ then	**If $\mathcal{D}_{[1,5]} = \mathcal{D}^{5,j}_{[1,5]}$ then**	**If $\mathcal{D}_{[1,5]} = \mathcal{D}^{6,j}_{[1,5]}$ then**
$m + p \neq 2$	$q = 1$	$p \neq m + 1$
$q = m - p$	$r = -n$	$q = m + p - 1$
$r = p - n + 1$	$q + r = n - v - 1$	$p \neq v$
$q + r = n$		$r = n + v$
		$n \neq u$
		$q + r = u + 1$

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{2,j}_{[1,5]}$, then $(m, n, p, q, r, u) = (1, 2, 2, 1, 2, 1)$.

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{3,j}_{[1,5]}$, then $(m, n, p, q, r, v) = (1, 1, 1, 2, 1, 2)$.

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{4,j}_{[1,5]}$, then $(m, n, p, q, r) = (1, 1, 2, 2, 2)$.

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{5,j}_{[1,5]}$, then $(n, q, r, v) = (2, 1, 1, 2)$.

If $\mathcal{D}_{[1,5]} = \mathcal{D}^{6,j}_{[1,5]}$, then $(m, n, p, q, r, u, v) = (1, 2, 1, 1, 1, 1, 2)$.

Therefore, we get Theorem 5.4.2. □

We now show that the designs given in Theorem 5.4.2 are pairwise non-isomorphic and then determine the full automorphism group of these designs.

Lemma 5.4.4. *For $i, i' \in \{1, 2, 3, 4, 5, 6\}$, $j, j' \in \{0, 1\}$ and $k, k' \in \{1, 2\}$, suppose that ψ is an isomorphism from $\mathcal{D}_{i,j,k}$ to $\mathcal{D}_{i',j',k'}$ such that $(1_0)^\psi = 1_0$. Then, $\psi = 1$ and $(i, j, k) = (i', j', k')$.*

Proof: The blocks in $\mathcal{D}_{i,j,k}$ containing the point 1_0 are given in Table 5.22.

Table 5.22: The set of blocks containing the point 1_0

$A_1^{i,j} = \left(B_3^{i,j}\right)^{\omega^{-2j}}$	$=$	$[1_0, 2_0, * \mid *, *]$
$A_2^{j,k} = \left(B_9^{j,k}\right)^{\omega^{-2j-2k}}$	$=$	$[1_0, 3_{1+j-k}, 9_{1-k} \mid 1_k, 2_k]$
$A_3^{j} = B_6^{j}$	$=$	$[4_0, 5_0, 6_{j+1} \mid 1_0, 4_1]$
$A_4^{j} = \left(B_7^{j}\right)^{\omega^{-2j}}$	$=$	$[3_{1+j}, 4_{2-j}, 7_{-j} \mid 1_0, 5_{-j}]$
$A_5^{j} = \left(B_8^{j}\right)^{\omega^{2}}$	$=$	$[5_1, 5_2, 8_1 \mid 1_0, 2_{2-j}]$
$A_6^{j} = \left(B_8^{j}\right)^{\omega^{2j+1}}$	$=$	$[6_{j-1}, 6_j, 8_{j-1} \mid 2_{j+1}, 1_0]$
$A_7^{j,k} = \left(B_9^{j,k}\right)^{\omega^{2k-2j}}$	$=$	$[1_{-k}, 3_{1+k+j}, 9_{k+1} \mid 1_0, 2_k]$
$A_8^{j,k} = \left(B_9^{j,k}\right)^{\omega^{3-2j}}$	$=$	$[2_k, 4_{1+j}, 9_1 \mid 2_{-k}, 1_0]$

Since $(V_1)^\psi = V_1$ and $d(1_0, B) = d(1_0, B^\psi)$ for all $B \in \mathcal{D}_{i,j,k}$, therefore ψ maps $A_1^{i,j} \to A_1^{i',j'}$ and hence $(2_0)^\psi = 2_0$. Also, $\{A_7^{j,k}, A_8^{j,k}\} \to \{A_7^{j',k'}, A_8^{j',k'}\}$ and hence $\{1_{-k}, 2_k\} \to \{1_{-k'}, 2_{k'}\}$ and $\{2_k, 2_{-k}\} \to \{2_{k'}, 2_{-k'}\}$. Therefore, we get $(x_k)^\psi = x_{k'}$ and $(x_{-k})^\psi = x_{-k'}$ for all $x \in \{1, 2\}$, and hence $A_8^{j,k} \to A_8^{j',k'}$. Therefore, $\{4_{1+j}, 9_1\} \to \{4_{1+j'}, 9_1\}$. Similarly, looking at the blocks containing 2_0, we get $\{3_{1+j}, 9_1\} \to \{3_{1+j'}, 9_1\}$, and hence $(3_{1+j}, 4_{1+j}, 9_1) \to (3_{1+j'}, 4_{1+j'}, 9_1)$. Then if we look at the blocks containing the points 1_k, 2_k, 1_{-k} and 2_{-k}, we see that $(9_{1+k}, 9_{1-k}) \to (9_{1+k'}, 9_{1-k'})$ and $(x_{1+j+k}, x_{1+j-k}) \to (x_{1+j'+k'}, x_{1+j'-k'})$ for all $x \in \{3, 4\}$. We also have that $A_2^{j,k} \to A_2^{j',k'}$. Therefore, $(4_m)^\psi = 4_m$ for $m \in \mathbb{Z}_3$, and hence $j' = j$ and $k' = k$. Therefore, $(x_m)^\psi = x_m$ for all $x \in \{1, 2, 3, 4, 9\}$ and $m \in \mathbb{Z}_3$. Now, looking at the blocks containing 3_m for $m \in \mathbb{Z}_3$, one can see that we also have $(x_m)^\psi = x_m$ for all $x \in \{5, 6, 7\}$ and $m \in \mathbb{Z}_3$. Finally, since $\left(A_5^j, A_6^j\right) \to \left(A_5^j, A_6^j\right)$, we see that $(8_m)^\psi = 8_m$ for all $m \in \mathbb{Z}_3$. Therefore, $\psi = 1$ and hence $i' = i$. □

Proof of Theorem 5.4.3: Suppose that there is an isomorphism ψ from $\mathcal{D}_{i,j,k}$ to $\mathcal{D}_{i',j',k'}$. Since $(V_1)^\psi = V_1$ and the automorphism ω given in Theorem 5.4.2 is transitive on V_1, we can choose ψ such that $(1_0)^\psi = 1_0$. Then, the result follows by Lemma 5.4.4. □

Proof of Theorem 5.4.4: Let $\mathcal{G} = Aut(\mathcal{D}_{i,j,k})$. We have $\langle\omega\rangle \leq \mathcal{G}$ and hence $(1_0)^{\mathcal{G}} = V_1$. By Lemma 5.4.4, $|\mathcal{G}_{1_0}| = 1$ and so $|\mathcal{G}| = |\mathcal{G}_{1_0}| \cdot |(1_0)^{\mathcal{G}}| = 6$. □

5.5 $(K_5 \setminus E)$-DESIGNS OF ORDERS 135, 162 AND 216

Finally, we give the construction of $(K_5 \setminus e)$-designs of orders 135, 162 and 216. It was already known by G. Ge and A. C. H. Ling that the existence of these designs are implied by the existence of a $(K_5 \setminus e)$-design of order 27. The recursive constructions given in this section are their work [14]. We are just filling in the holes.

A G-decomposition of the complete multipartite graph, which has exactly u_i parts of size g_i for $1 \leq i \leq s$, is called a G-group divisible design of type $g_1^{u_1} g_2^{u_2} ... g_s^{u_s}$, and denoted as G-GDD. A transversal design $TD(k, n)$ is a K_k-GDD of type n^k.

Lemma 5.5.1. *[14] There exists a $(K_5 \setminus e)$-GDD of type 9^k for all $k \geq 5$.*

Theorem 5.5.1. *[1] There exists a $TD(q + 1, q)$ for any prime power q.*

Lemma 5.5.2. *[5] Suppose that there exists a $TD(4, m)$. Then, if there exists a $(K_5 \setminus e)$-GDD of type T, then there exists a $(K_5 \setminus e)$-GDD of type mT.*

Corollary 5.5.1. *There exists a $(K_5 \setminus e)$-GDD of type 27^k for all $k \geq 5$.*

Proof: By Theorem 5.5.1, there exists a $TD(4, 3)$. Then, the result follows by Lemmas 5.5.1 and 5.5.2. □

Theorem 5.5.2. *There exist $(K_5 \setminus e)$-designs of orders 135, 162 and 216.*

Proof: Starting with a $(K_5 \setminus e)$-GDD of type 27^k for $k \in \{5, 6, 8\}$ and filling in the holes with a $(K_5 \setminus e)$-design of order 27, we get the result. □

CHAPTER 6

SOME $K_{S,T}$-DESIGNS

6.1 MAIN RECURSIVE CONSTRUCTIONS

We denote a complete bipartite graph with parts $\{a_1, ..., a_s\}$ and $\{b_1, ..., b_t\}$ as $\{a_1, ..., a_s \ : \ b_1, ..., b_t\}$.

The following two propositions are standard constructions which can be found in [5]. Even so, we give the proofs since they are simple.

Proposition 6.1.1. *Let G be a graph. Suppose that there exist G-designs of orders u and v, and a G-decomposition of $K_{u-1,v-1}$. Then, there exists a G-design of order $u+v-1$.*

Proof: Let A and B be disjoint sets of size $u-1$ and $v-1$ respectively, and let $x \notin A \cup B$. We obtain a G-design of order $u+v-1$ on the vertex set $A \cup B \cup \{x\}$ as follows. By assumption the complete graph with vertex set $A \cup \{x\}$, the complete graph with vertex set $B \cup \{x\}$, and the complete bipartite graph with bipartition (A, B) all have G-decompositions, and the result follows. □

Proposition 6.1.2. *There exists a $K_{s,t}$-decomposition of $K_{ps,qt}$ for all positive integers s, t, p and q.*

Proof: Let $A = \{a_1, ..., a_{ps}\}$ and $B = \{b_1, ..., b_{qt}\}$ be the parts of $K_{ps,qt}$. The blocks $\{a_{is+1}, ..., a_{is+s} \ : \ b_{jt+1}, ..., b_{jt+t}\}$ for $0 \le i < p$, $0 \le j < q$ form the desired decomposition. □

Corollary 6.1.1. *Suppose that there exist $K_{s,t}$-designs of orders u and v for some $u \equiv 1 \pmod s$ and $v \equiv 1 \pmod t$. Then, there exists a $K_{s,t}$-design of order $u+v-1$.*

Proof: If $u = 1$ or $v = 1$, the result is trivial. Otherwise, it follows by Propositions 6.1.1 and 6.1.2. □

The following lemma is a basic result from number theory and we omit the proof.

Lemma 6.1.1. *Let s, t, and m be positive integers such that $gcd(s,t)|m$ and $m \geq st$. Then, there exist nonnegative integers p and q such that $m = ps + qt$.*

Proposition 6.1.3. *Let s, t, m and k be positive integers such that $gcd(s,t)|m$ and $m \geq st$. Then, there exists a $K_{s,t}$-decomposition of $K_{m,kst}$.*

Proof: Let $K_{m,kst}$ have bipartition (A, B) with $|A| = m$ and $|B| = kst$. By Lemma 6.1.1, there exist nonnegative integers p and q such that $m = ps + qt$. Say $A = A_1 \cup A_2$ where $|A_1| = ps$ and $|A_2| = qt$. If $p = 0$ or $q = 0$, the result follows immediately by Proposition 6.1.2. Otherwise, by Proposition 6.1.2, the complete bipartite graphs with bipartitions (A_1, B) and (A_2, B) both have $K_{s,t}$-decompositions, and the result follows. □

The following theorem is the main result of this chapter.

Theorem 6.1.1. *Let $s, t \geq 1$. If there exists a $K_{s,t}$-design of order N, then there exists a $K_{s,t}$-design of order n for all $n \equiv N \pmod{2st}$ and $n \geq N$.*

Proof: Suppose that there exists a $K_{s,t}$-design of order N. If $N = 1$, then the claim is true by Theorem 1.0.4. Let $N > 1$, and let k be a positive integer. By Theorem 1.0.5, we have $N \geq 2st > st$. Also, by the third necessary condition, we have $gcd(s,t)|(N-1)$. Therefore, by Proposition 6.1.3, there exists a $K_{s,t}$-decomposition of $K_{N-1,2kst}$. Also, by Theorem 1.0.4, there exists a $K_{s,t}$-design of order $2kst + 1$. Therefore, the result follows by Proposition 6.1.1. □

6.2 SOME APPLICATIONS OF THEOREM 6.1.1

Theorem 6.1.1 is a powerful tool for constructing infinite families of $K_{s,t}$-designs. The spectrum problem is already completely solved for the case $s = 1$. For $s, t \geq 2$, to provide a complete solution for the spectrum problem, it is suffient to construct a $K_{s,t}$-design of order n for all values of n that satisfy the necessary conditions and also satisfy $2st + 1 < n \leq 4st$ (see also Theorems 1.0.4 and 1.0.5). We give direct constructions of these designs for the graphs $K_{2,5}$, $K_{2,6}$, $K_{3,4}$, $K_{2,7}$ and $K_{3,5}$, except for when $n \in \{3st, 4st\}$. For these exceptional cases, we construct a $K_{s,t}$-design for the next smallest order in the congruence classes, namely a $K_{s,t}$-design of order $n + 2st$, and hence leave five isolated orders unresolved. The necessary conditions for the 5 graphs are listed in Table 6.1.

Table 6.1: Necessary conditions for the existence of $K_{s,t}$-designs

Graph	Necessary Conditions
$K_{2,5}$	$n \equiv 0, 1, 5, 16 \pmod{20}$
$K_{2,6}$	$n \equiv 1, 9 \pmod{24}$
$K_{3,4}$	$n \equiv 0, 1, 9, 16 \pmod{24}$
$K_{2,7}$	$n \equiv 0, 1, 8, 21 \pmod{28}$
$K_{3,5}$	$n \equiv 0, 1, 6, 10 \pmod{15}$

For the direct constructions, we use cyclic groups which act semiregularly on blocks and either act semiregularly on points or fix one point and act semiregularly on the remaining points.

For $p, q \geq 2$, let $S_{p,q} = \{x_i \ : \ x \in \{1, 2, ..., p\}, \ i \in \mathbb{Z}_q\}$. Define $\rho_{p,q} : S_{p,q} \to S_{p,q}$ and $\rho^{\infty}_{p,q} : S_{p,q} \cup \{\infty\} \to S_{p,q} \cup \{\infty\}$ as $\rho_{p,q}(x_i) = \rho^{\infty}_{p,q}(x_i) = x_{i+1}$ and $\rho^{\infty}_{p,q}(\infty) = \infty$.

Theorem 6.2.1. *There exists a $K_{2,5}$-design of order n for all $n \in \{25, 36, 60\}$.*

Proof: For $n = 25$, take $V = S_{5,5}$ and develop the following 6 blocks with $\rho_{5,5}$:

$\{1_0, 2_0 \;:\; 1_1, 1_2, 2_1, 2_2, 3_0\}$ $\{1_0, 3_0 \;:\; 3_1, 3_2, 4_0, 4_3, 4_4\}$ $\{1_0, 4_0 \;:\; 2_0, 3_3, 3_4, 4_1, 4_2\}$

$\{1_0, 2_0 \;:\; 5_0, 5_1, 5_2, 5_3, 5_4\}$ $\{3_0, 4_0 \;:\; 2_1, 2_2, 2_3, 2_4, 5_0\}$ $\{5_0, 5_1 \;:\; 3_2, 3_4, 4_2, 4_4, 5_2\}$.

For $n = 36$, take $V = S_{4,9}$ and develop the following 7 blocks with $\rho_{4,9}$:

$\{1_4, 2_0 \;:\; 1_1, 1_2, 1_3, 3_2, 3_3\}$ $\{1_0, 2_1 \;:\; 2_2, 2_3, 2_4, 2_5, 3_0\}$ $\{1_0, 3_1 \;:\; 1_4, 3_2, 3_3, 3_4, 3_5\}$

$\{2_0, 2_1 \;:\; 1_0, 3_5, 3_7, 4_6, 4_8\}$ $\{1_0, 2_0 \;:\; 3_1, 4_1, 4_2, 4_3, 4_4\}$ $\{1_0, 3_0 \;:\; 4_0, 4_5, 4_6, 4_7, 4_8\}$

$\{3_0, 4_0 \;:\; 2_0, 4_1, 4_2, 4_3, 4_4\}$.

For $n = 60$, apply Corollary 6.1.1 with $u = 25$ and $v = 36$. □

Theorem 6.2.2. *There exists a $K_{2,6}$-design of order 33.*

Proof: Take $V = S_{3,11}$ and develop the following 4 blocks with $\rho_{3,11}$:

$\{1_0, 1_1 \;:\; 1_2, 1_5, 2_4, 2_6, 3_3, 3_9\}$ $\{1_2, 2_1 \;:\; 1_5, 2_4, 2_0, 3_1, 3_8, 3_9\}$

$\{1_1, 2_7 \;:\; 2_0, 2_2, 2_9, 3_2, 3_5, 3_6\}$ $\{2_0, 3_0 \;:\; 1_0, 3_1, 3_2, 3_3, 3_4, 3_5\}$. □

Theorem 6.2.3. *There exists a $K_{3,4}$-design of order n for all $n \in \{33, 40, 72\}$.*

Proof: For $n = 33$, take $V = S_{3,11}$ and develop the following 4 blocks with $\rho_{3,11}$:

$\{1_0, 1_1, 2_0 \;:\; 1_3, 3_0, 3_2, 3_4\}$ $\{1_0, 2_0, 2_4 \;:\; 1_1, 1_4, 1_6, 2_1\}$

$\{1_0, 2_0, 3_5 \;:\; 2_2, 2_4, 2_6, 3_6\}$ $\{1_0, 2_0, 3_0 \;:\; 3_5, 3_7, 3_8, 3_9\}$.

For $n = 40$, take $V = S_{3,13} \cup \{\infty\}$ and develop the following 5 blocks with $\rho^{\infty}_{3,13}$:

$\{1_0, 1_1, 2_0 \;:\; 1_2, 1_4, 2_1, 2_3\}$ $\{1_0, 2_0, 2_1 \;:\; 1_6, 2_5, 3_0, 3_2\}$ $\{1_0, 1_5, 2_3 \;:\; 3_1, 3_8, 3_{10}, 3_{12}\}$

$\{2_0, 3_0, 3_2 \;:\; 1_9, 3_3, 3_4, 3_8\}$ $\{1_0, 2_4, 3_3 \;:\; 1_5, 2_6, 2_{10}, \infty\}$.

For $n = 72$, apply Corollary 6.1.1 with $u = 40$ and $v = 33$. □

Theorem 6.2.4. *There exists a $K_{2,7}$-design of order n for all $n \in \{36, 49, 84\}$.*

Proof: For $n = 36$, take $V = S_{4,9}$ and develop the following 5 blocks with $\rho_{4,9}$:

$\{1_0, 2_0 \;:\; 1_1, 2_1, 2_2, 3_0, 3_1, 3_2, 4_0\}$ $\{1_0, 3_1 \;:\; 1_2, 1_5, 1_6, 2_5, 2_6, 2_7, 4_1\}$

$\{2_0, 3_3 \ : \ 1_0, 1_5, 1_6, 2_4, 2_6, 3_7, 4_8\}$ $\{1_0, 2_1 \ : \ 4_2, 4_3, 4_4, 4_5, 4_6, 4_7, 4_8\}$

$\{3_0, 4_0 \ : \ 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4\}$.

For $n = 49$, take $V = S_{7,7}$ and develop the following 12 blocks with $\rho_{7,7}$:

$\{1_0, 3_0 \ : \ 1_1, 2_1, 2_2, 2_3, 2_4, 2_5, 2_6\}$ $\{1_0, 4_6 \ : \ 1_2, 2_0, 3_1, 3_2, 3_3, 3_4, 3_5\}$

$\{1_3, 5_0 \ : \ 1_0, 4_1, 4_2, 4_3, 4_4, 4_5, 4_6\}$ $\{1_0, 6_0 \ : \ 3_0, 5_1, 5_2, 5_3, 5_4, 5_5, 5_6\}$

$\{2_0, 4_6 \ : \ 2_1, 2_2, 2_3, 3_0, 4_0, 4_1, 4_2\}$ $\{3_0, 6_0 \ : \ 3_1, 3_2, 3_3, 4_0, 6_1, 6_2, 6_3\}$

$\{2_0, 3_0 \ : \ 5_0, 5_1, 5_2, 5_3, 5_4, 5_5, 5_6\}$ $\{1_0, 2_0 \ : \ 6_0, 6_1, 6_2, 6_3, 6_4, 6_5, 6_6\}$

$\{1_0, 2_0 \ : \ 7_0, 7_1, 7_2, 7_3, 7_4, 7_5, 7_6\}$ $\{3_0, 4_0 \ : \ 7_0, 7_1, 7_2, 7_3, 7_4, 7_5, 7_6\}$

$\{4_0, 7_0 \ : \ 5_0, 6_1, 6_2, 6_3, 6_4, 6_5, 6_6\}$ $\{5_0, 7_0 \ : \ 5_1, 5_2, 5_3, 6_0, 7_1, 7_2, 7_3\}$.

For $n = 84$, apply Corollary 6.1.1 with $u = 49$ and $v = 36$. □

Theorem 6.2.5. *There exists a $K_{3,5}$-design of order $n \in \{36, 40, 46, 51, 55, 75, 90\}$.*

Proof: For $n = 36$, let $V = S_{5,7} \cup \{\infty\}$. Develop the following 6 blocks with $\rho^{\infty}_{5,7}$:

$\{\infty, 3_1, 4_1 \ : \ 1_0, 2_0, 3_0, 4_0, 5_0\}$ $\{1_0, 2_0, 4_0 \ : \ 1_1, 1_4, 3_3, 4_2, 4_4\}$

$\{2_0, 3_1, 3_3 \ : \ 3_6, 4_3, 4_6, 5_5, 5_6\}$ $\{1_0, 2_6, 3_0 \ : \ 1_5, 2_0, 2_2, 5_0, 5_1\}$

$\{1_0, 1_1, 2_3 \ : \ 2_5, 3_0, 3_5, 5_3, 5_6\}$ $\{4_0, 4_2, 5_6 \ : \ 1_2, 2_2, 5_0, 5_3, 5_4\}$.

For $n = 40$, take $V = S_{3,13} \cup \{\infty\}$ and develop the following 4 blocks with $\rho^{\infty}_{3,13}$:

$\{1_0, 1_1, 3_0 \ : \ 1_2, 1_4, 1_6, 2_0, 2_2\}$ $\{1_0, 1_1, 1_2 \ : \ 2_5, 2_8, 2_{11}, 3_1, 3_4\}$

$\{2_5, 2_7, 3_5 \ : \ 2_0, 2_3, 2_8, 3_1, 3_6\}$ $\{1_0, 2_2, 3_3 \ : \ 3_5, 3_6, 3_8, 3_{10}, \infty\}$.

For $n = 46$, take $V = S_{2,23}$ and develop the following 3 blocks with $\rho_{2,23}$:

$\{1_0, 1_1, 2_0 \ : \ 1_3, 1_5, 1_7, 1_9, 1_{11}\}$ $\{1_1, 2_0, 2_1 \ : \ 1_2, 2_2, 2_4, 2_6, 2_8\}$

$\{2_0, 2_2, 2_4 \ : \ 1_0, 1_8, 1_{14}, 1_{17}, 2_{13}\}$.

For $n = 51$, take $V = S_{3,17}$ and develop the following 5 blocks with $\rho_{3,17}$:

$\{1_0, 1_4, 3_0 \ : \ 1_5, 1_6, 1_7, 1_8, 2_0\}$ $\{1_0, 1_5, 3_1 \ : \ 2_7, 2_8, 2_9, 2_{10}, 2_{11}\}$

$\{1_0, 1_5, 2_9 \ : \ 3_4, 3_5, 3_6, 3_7, 3_8\}$ $\{1_7, 2_0, 2_1 \ : \ 2_2, 2_4, 2_6, 2_8, 3_4\}$

$\{2_2, 3_0, 3_2 \ : \ 1_4, 3_3, 3_4, 3_7, 3_8\}$.

For $n = 55$, take $V = S_{5,11}$ and develop the following 9 blocks with $\rho_{5,11}$:

$\{1_1, 2_6, 2_7 \ : \ 1_0, 2_4, 2_0, 3_9, 4_6\}$ $\{1_0, 3_8, 3_9 \ : \ 2_0, 2_2, 2_4, 2_8, 4_2\}$

$\{1_0, 3_0, 4_4 \ : \ 3_1, 3_2, 3_3, 3_4, 3_5\}$ $\{1_1, 1_3, 1_5 \ : \ 3_1, 4_4, 5_5, 5_6, 5_0\}$

$\{1_0, 1_7, 5_0 \ : \ 1_2, 1_4, 2_5, 3_6, 4_0\}$ $\{1_0, 2_0, 3_0 \ : \ 2_1, 4_6, 4_7, 4_8, 4_9\}$

$\{2_0, 4_0, 5_0 \ : \ 4_1, 4_2, 4_3, 4_4, 4_5\}$ $\{2_0, 2_5, 3_6 \ : \ 5_1, 5_2, 5_3, 5_4, 5_5\}$

$\{3_1, 4_0, 5_0 \ : \ 5_1, 5_2, 5_3, 5_4, 5_5\}$.

For $n \in \{75, 90\}$, apply Corollary 6.1.1 with $u = 40$ and $v \in \{36, 51\}$. □

Finally, applying Theorem 6.1.1 to the designs given in Theorems 6.2.1-6.2.5, we obtain an almost complete solution for the spectrum problem for the complete bipartite graphs with fewer than 18 edges, leaving 5 orders in total unsolved. We summarize all known results on the spectrum of $K_{s,t}$, for $s, t \geq 2$, in Table 6.2.

Table 6.2: The Known Results on The Spectrum of $K_{s,t}$ for $s, t \geq 2$

st	Graph	Spectrum	Possible Exceptions	References
2^k	$K_{2^a,2^{k-a}}$	$n \equiv 1 \pmod{2^{k+1}}^*$	$\emptyset$	Thm 1.0.4
6	$K_{2,3}$	$n \equiv 0, 1, 4, 9 \pmod{12}^{**}$	$\emptyset$	[4]
9	$K_{3,3}$	$n \equiv 1 \pmod 9^{**}$	$\emptyset$	[16]
10	$K_{2,5}$	$n \equiv 0, 1, 5, 16 \pmod{20}^{**}$	$n = 40$	Thms 1.0.4, 6.1.1, 6.2.1
12	$K_{2,6}$	$n \equiv 1, 9 \pmod{24}^{**}$	$\emptyset$	Thms 1.0.4, 6.1.1, 6.2.2
	$K_{3,4}$	$n \equiv 0, 1, 9, 16 \pmod{24}^{**}$	$n = 48$	Thms 1.0.4, 6.1.1, 6.2.3
14	$K_{2,7}$	$n \equiv 0, 1, 8, 21 \pmod{28}^{**}$	$n = 56$	Thms 1.0.4, 6.1.1, 6.2.4
15	$K_{3,5}$	$n \equiv 0, 1, 6, 10 \pmod{15}^{**}$	$n \in \{45, 60\}$	Thms 1.0.4, 6.1.1, 6.2.5

* For all $k \geq 2$ and $1 \leq a \leq k/2$

** There does not exist a $K_{s,t}$-design of order n for $1 < n \leq 2st$ (Thm 1.0.5)

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