A STUDY OF DIVISORS AND ALGEBRAS ON A DOUBLE COVER OF THE AFFINE PLANE

by

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A Dissertation Submitted to the Faculty of

The Charles E. Schmidt College of Science

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Florida Atlantic University

Boca Raton, FL

August 2012

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This dissertation was prepared under the direction of the candidate's dissertation advisor, Dr. Timothy J. Ford, Mathematical Sciences, and has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Dr. Timothy Ford for the continuous support of my Ph.D study and research, for his motivation, enthusiasm, immense knowledge and inexhaustible patience during the correction phase of this dissertation. The good advice and support of Dr. Ford has been invaluable on both an academic and a personal level, for which I am extremely grateful. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor for my Ph.D study.

I also owe my gratitude to Dr. Lee Klingler who has been always played a key role in encouraging and coordinating my study and from whom I learned a lot.

I would like to thank the rest of my thesis committee: Dr. Fred Richman and Dr. Jorge Viola-Prioli, for their encouragement and insightful comments.

I would like to express my gratitude to Dr. Heinrich Niederhausen who has welcomed me to FAU and to Graduate Directors: Dr. William Kalies and Dr. Rainer Steinwandt, who helped me to remain focused and finish my studies.

I am grateful to Ms. Beth Broer, Ms. Emily Cimillo and Ms. Helen Randall for helping the department to run smoothly and for assisting me in many different ways.

I would also like to thank my colleagues and friends from FAU for their kindness, friendship and support.

Last, but by no means least, I thank to Dr. Milos Tomic, who inspired and encouraged me to get involved in scientific research.

ABSTRACT

Author:	Djordje Bulj
Title:	A Study of Divisors and Algebras on a Double Cover of the Affine Plane
Institution:	Florida Atlantic University
Dissertation Advisor:	Dr. Timothy J. Ford
Degree:	Doctor of Philosophy
Year:	2012

An algebraic surface defined by an equation of the form $z^2 = (x+a_1y)\cdots(x+a_ny)(x-1)$ is studied, from both an algebraic and geometric point of view. It is shown that the surface is rational and contains a singular point which is nonrational. The class group of Weil divisors is computed and the Brauer group of Azumaya algebras is studied. Viewing the surface as a cyclic cover of the affine plane, all of the terms in the cohomology sequence of Chase, Harrison and Roseberg are computed.

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CHAPTER 1

INTRODUCTION

In this thesis we investigate both the algebraic and geometric structure of rings $R = k[x, y][(g(x, y))^{-1}]$, $S = R[\sqrt{g(x, y)}]$ and $T = k[x, y][\sqrt{g(x, y)}]$, where $g(x, y) = (x + a_1 y) \cdots (x + a_n y)(x - 1)$ and n is an even number. Throughout, k is an algebraically closed field with characteristic different from two. Geometrically, the ring T represents a nonsmooth surface X defined by the equation $z^2 = (x + a_1 y) \cdots (x + a_n y)(x - 1)$ and S represents the surface X with lines $x + a_1 y = 0, \ldots, x + a_n y = 0, x - 1 = 0$ removed. Figure 1.1 shows the $n = 4$ surface, over the field $\mathbb{R}$ of real numbers.

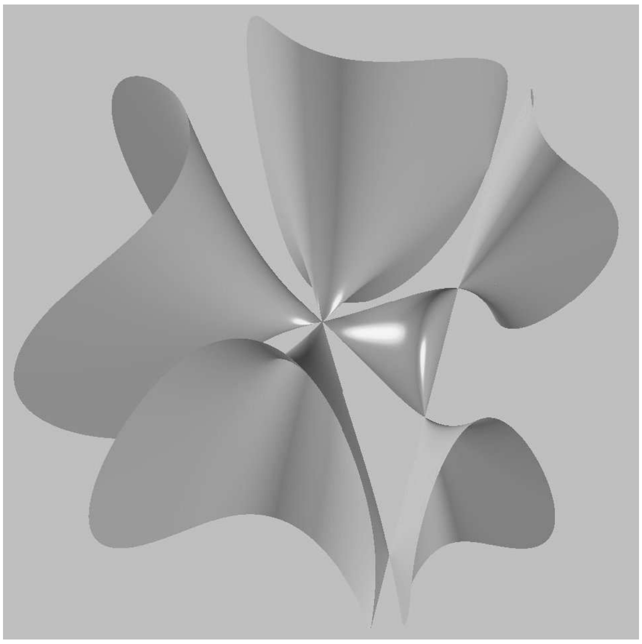

Figure 1.1: The surface X

[1]Figure 1.1 was created by software written by T. J. Ford and used by permission.

The surface X is degenerate at the points $(1, -1/a_1, 0), \ldots, (1, -1/a_n, 0)$ (ordinary double-points) and at $(0, 0, 0)$ (nonrational singular point). In our attempt to improve the surface X we will concentrate on the case of resolving the singularity at the point $(0, 0, 0)$ in section 3.1 and then more thoroughly in section 3.5. For that purpose we will use two methods: blowing-up and normalization. Algebraically, the rings R, S, and T are noetherian, integral domains related by the following commutative diagram

$$\begin{array}{ccc} T = k[x,y][\sqrt{g(x,y)}] & \longrightarrow & S = R[\sqrt{g(x,y)}] \\ \uparrow & & \uparrow \\ k[x,y] & \longrightarrow & R = k[x,y][g(x,y)^{-1}] \end{array}$$

where each arrow represents set inclusion. Moreover, S is a quadratic Galois extension of ring R with Galois group $G = \mathbb{Z}/2$, where $\mathbb{Z}/2$ represents integers modulo 2.

There are three abelian groups associated with every commutative ring: the Picard group, the class group and the Brauer group. The class group of the ring R, denoted by $\mathrm{Cl}(R)$, is defined to be the set of rank-one reflexive $R-$modules (reflexive ideals). The Picard group, $\mathrm{Pic}(R)$, is the subgroup of the class group formed by projective rank-one modules (invertible ideals) and the Brauer group, $\mathrm{B}(R)$, is the set of equivalence classes of Azumaya algebras over R. In section 3.2 we will compute the Picard and class groups of our rings. These results will be obtained by the use of Nagata's Theorem (Proposition 2.11). The Brauer groups of the rings R and S will be computed in section 3.3. This computation involves computing the homology of the graph associated to a divisor on a surface. A second computation is made which involves computing the Betti numbers of the surfaces via a long exact sequence based on restriction and corestriction. It is worth mentioning that this long exact sequence exists only for quadratic extensions. This is one important reason why this study focuses on double planes, rather than covers of higher degree.

Our central result will be presented in section 3.4. Since the Brauer group is a functor, the homomorphism $R \longrightarrow S$ induces a homomorphism of $\mathrm{B}(R)$ into $\mathrm{B}(S)$. The kernel is called the *relative Brauer group* and is denoted $\mathrm{B}(S/R)$. The Chase, Harrison, Rosenberg exact sequence relates $\mathrm{Pic}(R)$ and $\mathrm{B}(S/R)$ to the group cohomology of the Galois group G with coefficients in the groups of units of S and $\mathrm{Pic}(S)$: there are homomorphisms α_i such that

$$1 \longrightarrow \mathrm{H}^1(G, S^*) \xrightarrow{\alpha_1} \mathrm{Pic}(R) \xrightarrow{\alpha_2} \mathrm{Pic}(S)^G \xrightarrow{\alpha_3} \mathrm{H}^2(G, S^*) \xrightarrow{\alpha_4} \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{H}^1(G, \mathrm{Pic}(S)) \quad (1.0.1)$$

is an exact sequence. The existence of such homomorphisms has been proved in [3, Corollary 5.5] and later in [4, Theorem 4.1.1] and [14], with homomorphisms explicitly defined. The original goal of this project was to find a nontrivial example of an extension of rings S/R for which all of the maps and terms in sequence (1.0.1) are computed. The rings R and S which are the subject of this dissertation satisfy these criteria. In section 3.4 we will compute all terms of the Chase, Harrison, Rosenberg exact sequence for S and describe the nontrivial homomorphisms. Section 3.5 is an attempt to understand the divisor class group on the surface $\tilde{Y}$ which is a resolution of the nonrational singularity of X. The motivation for this study is the paper of Lipman [15] in which he constructs some exact sequences involving the class groups of $\mathrm{Cl}(\tilde{Y})$ and $\mathrm{Cl}(X)$. In section 3.5 the groups $\mathrm{Cl}(\tilde{Y})$ and $\mathrm{Pic}(\tilde{Y})$ are computed.

The rings studied in this dissertation are also the subject of the article [2].

CHAPTER 2

PRELIMINARIES

In this section we state definitions and theorems that will be used in the later sections.

Definition 2.1. ***Definition of Progenerator.*** *Let R be a ring and M an R-module. The set*

$$T_R M = \left\{ \sum_{i=1}^{n} f_i(m_i) | n \geq 1, f_i \in \mathrm{Hom}_R(M, R), m_i \in M \right\}$$

is a 2-sided ideal in R. We say that an R-module M is a progenerator *in case M is finitely generated, projective and $T_R M = R$.*

Definition 2.2. ***Definition of faithful module.*** *Let M be an A-module. Let* $\mathrm{Ann}(M)$ *be the set of all $a \in A$ such that $aM = 0$. This ideal of A is called the* annihilator *of M. An A-module is* faithful *if* $\mathrm{Ann}(M) = 0$.

Proposition 2.1. *[4, Corollary 1.10] Let R be a commutative ring and M an R-module. Then M is an R-progenerator if and only if M is finitely generated, projective and faithful.*

Definition 2.3. ***Definition of Enveloping Algebra.*** *For any R-algebra A we can form the* enveloping algebra $A^e = A \otimes A^o$ *of A, where A^o denotes the R-algebra opposite to A. The algebra A has a structure as a left A^e-module induced by*

$$(a \otimes a')b = aba'.$$

There is an induced R-homomorphism $\varphi : A^e \longrightarrow \mathrm{Hom}_R(A, A)$ taking $\alpha \in A^e$ to the element $\varphi(\alpha)$ of $\mathrm{Hom}_R(A, A)$ defined as left multiplication by α. More explicitly, if $\alpha = \sum_i a_i \otimes a'_i$, then $\varphi(\alpha)(a) = \alpha a = \sum_i a_i a a'_i$.

Definition 2.4. *There is a A^e-module homomorphism $\mu : A^e \longrightarrow A$ given by*

$$\mu(\sum_i a_i \otimes a_i') = \sum_i a_i a_i'.$$

Let $J = \mathrm{Ker}(\mu)$. We obtain the exact sequence

$$0 \longrightarrow J \longrightarrow A^e \xrightarrow{\mu} A \longrightarrow 0$$

Proposition 2.2. *[4, Proposition 1.1] The following conditions on an R-algebra A are equivalent:*

1. *A is projective as a left A^e-module under the μ-structure.*

2. *$0 \longrightarrow J \longrightarrow A^e \xrightarrow{\mu} A \longrightarrow 0$ splits as a sequence of left A^e-modules.*

3. *A^e contains an element e such that $\mu(e) = 1$ and $Je = 0$.*

Definition 2.5. ***Definition of Separability.*** *An R-algebra A is called* separable *if it satisfies the equivalent conditions of Proposition 2.2.*

Definition 2.6. ***Definition of Central Algebra.*** *An R-algebra A is called* central *if A is faithful as an R-module and $R \cdot 1$ coincides with the center of A.*

Proposition 2.3. *[4, Theorem 3.4] The following statements about the R-algebra A are equivalent:*

1. *A is central separable over R.*

2. *A is an A^e-progenerator and A is R-central.*

3. *A is an R-progenerator and the map $\varphi : A^e \longrightarrow \mathrm{Hom}_R(A, A)$ of Definition 2.3 is an isomorphism.*

Definition 2.7. ***Definition of an Azumaya algebra.*** *An R-algebra A that satisfies any of the equivalent properties listed in Proposition 2.3 is called an* Azumaya R-algebra*.*

Proposition 2.4. *[20, Proposition 2.5(a)] Let R be a commutative ring and let $\mathrm{Az}(R)$ denote the set of isomorphism classes of Azumaya R-algebras. Let A and B be two Azumaya R-algebras. We say that A and B are Brauer equivalent and write $A \sim B$ in case there exist R-progenerators P and Q such that*

$$A \otimes_R \mathrm{Hom}_R(P, P) \cong B \otimes_R \mathrm{Hom}_R(Q, Q).$$

Let E be any R-progenerator. Then $A = \mathrm{Hom}_R(E, E)$ is Azumaya algebra. Let $\mathrm{Az}^0(R)$ denote the subset of $\mathrm{Az}(R)$ consisting of all isomorphism classes containing a representative of the form $\mathrm{Hom}_R(P, P)$, where P is an R-progenerator module.

Definition 2.8. ***Definition of the Brauer group.*** *Let $\mathrm{B}(R)$ denote the set of equivalence classes of $\mathrm{Az}(R)$ under the Brauer equivalence relation. We call $\mathrm{B}(R)$ the* Brauer group *of R. $\mathrm{B}(R)$ is an abelian group, the binary operation is induced by tensor product and the identity class is the set $\mathrm{Az}^0(R)$.*

Definition 2.9. ***Definition of Normal integral domain.*** *Let R be an integral domain with quotient field K. If R is integrally closed in K, then we say R is a* normal*.*

Definition 2.10. ***Definition of DVR.*** *If F is a field, a* discrete valuation *on F is a valuation $\nu : F^* \longrightarrow \mathbb{Z}$ such that ν is onto. The valuation ring of ν is $R = \{0\} \cup \{x \in F^* | \nu(x) \geq 0\}$. An integral domain A is called a* discrete valuation ring*, or DVR for short, if there exists a discrete valuation on the field of fractions of A such that A is the associated valuation ring.*

Proposition 2.5. *[1, Proposition 9.2] Let R be a noetherian local integral domain with field of fractions K, maximal ideal m and residue field $k = R/m$. If R has Krull dimension one, the following are equivalent.*

1. *R is a DVR.*

2. *R is a PID.*

3. *R is regular.*

4. *R is normal.*

5. *m is a principal ideal.*

6. *There exists an element $\pi \in R$ such that every ideal of R is of the form $R\pi^n$, for some $n \geq 0$.*

Proposition 2.6. *[21, Corollary VI.10.2] Let R be a noetherian normal integral domain with field of fractions K. Let $X_1(R)$ denote the subset of $\operatorname{Spec}(R)$ consisting of all prime ideals P such that $ht(P) = 1$. If $P \in X_1$ then R_P is a one-dimensional noetherian normal local integral domain. By Proposition 2.5 R_P is a DVR of K. Denote by m_P the maximal ideal of R_P and by π_P a generator of m_P. Define $\nu_P : K^* \longrightarrow \mathbb{Z}$ by the following. Any element $x \in K^*$ can be factored uniquely as $x = u\pi_P^{\nu_P(x)}$ for some integer $\nu_P(x)$ and $u \in R_P^*$, where R_P^* designates the group of invertible elements of a ring. Let R be a noetherian normal integral domain with field of fractions K. Let $\alpha \in K^*$.*

1. *$\nu_P(\alpha) = 0$ for all but finitely many $P \in X_1(R)$.*

2. *$\alpha \in R$ if and only if $\nu_P(\alpha) \geq 0$ for all $P \in X_1(R)$.*

3. *$\alpha \in R^*$ if and only if $\nu_P(\alpha) = 0$ for all $P \in X_1(R)$.*

Definition 2.11. ***Definition of Weil divisors and class group.*** *Let R be a noetherian normal integral domain with field of fractions K. The free $\mathbb{Z}-$module on $X_1(R)$,*

$$\operatorname{Div}(R) = \bigoplus_{P \in X_1(R)} \mathbb{Z}P$$

is the called the group of Weil divisors *of R. There is a homomorphism of groups* $\operatorname{Div} : K^* \longrightarrow \operatorname{Div}(R)$ *defined by*

$$\operatorname{Div}(\alpha) = \sum_{P \in X_1(R)} \nu_P(\alpha) P,$$

and the kernel of Div *is equal to the group R^*. The class group of R is the cokernel of*

$$\operatorname{Div} : K^* \longrightarrow \operatorname{Div}(R)$$

and is denoted by $\operatorname{Cl}(R)$. *Equivalently* $\operatorname{Cl}(R)$ *is defined by the following exact sequence:*

$$0 \longrightarrow R^* \longrightarrow K^* \xrightarrow{\operatorname{Div}} \operatorname{Div}(R) \longrightarrow \operatorname{Cl}(R) \longrightarrow 0$$

Proposition 2.7. *[16, (19.A),Theorem 47] Let R be an integral domain. An element $a \neq 0$ of R is said to be irreducible if it is a nonunit of R and if it is not a product of two nonunits of R. The ring R is called a unique factorization domain (UFD) if every nonzero element is a product of a unit and of a finite number of irreducible elements and if such a representation is unique up to order and units. A noetherian domain in which every irreducible element generates a prime ideal is a UFD. Let R be a noetherian integral domain. Then R is a* UFD *if and only if every prime ideal of height one is principal.*

Corollary 2.1. *[12, Proposition 6.2] Let R be a noetherian normal integral domain. Then R is a* UFD *if and only if* $\operatorname{Cl}(R) = (0)$.

Proposition 2.8. *[4, Lemma 1.5.1] For any ring R and any R-module M, set $M^* = \mathrm{Hom}_R(M, R)$. M^* is a right R-module under the operation*

$$(f \cdot r)(m) = f(m)r.$$

Let M be a finitely generated projective faithful module over the commutative ring R. Then the following are equivalent:

1. $\mathrm{Rank}_R(M) = 1$.

2. $\mathrm{Rank}_R(M^*) = 1$.

3. $\mathrm{Hom}_R(M, M) \cong R$.

4. $M^* \otimes_R M \cong R$.

5. *For some R-module N, there is an isomorphism $M \otimes_R N \cong R$.*

Definition 2.12. ***Definition of invertible module.*** *If M is an R-module that satisfies any of the equivalent properties of 2.8, then we say M is* invertible.

Proposition 2.9. *Given a commutative ring R let $\mathrm{Pic}(R)$ be the set of isomorphism classes of finitely generated projective faithful R-modules satisfying any of the five conditions of 2.8. The isomorphism class containing a module M is denoted by $|M|$. Under the binary operation $|P| \cdot |Q| = |P \otimes_R Q|$, $\mathrm{Pic}(R)$ is an abelian group. The identity element is the class $|R|$. The inverse of $|M| \in \mathrm{Pic}(R)$ is $|M^*|$. The group $\mathrm{Pic}(R)$ is called the Picard group of R.*

Proposition 2.10. *[12, Corollary 6.16] If R is a noetherian regular integral domain, then there is a natural isomorphism $\mathrm{Cl}(R) \cong \mathrm{Pic}(R)$.*

Proposition 2.11. ***(Nagata's Theorem)****[11, Theorem 7.1] Let R denote a noetherian normal integral domain with field of fractions K. Let f be a nonzero noninvertible element of R with divisor $\mathrm{Div}(f) = \nu_1 P_1 + ... + \nu_n P_n$. The sequence of abelian groups*

$$1 \longrightarrow R^* \longrightarrow R[f^{-1}]^* \xrightarrow{\mathrm{Div}} \oplus_{i=1}^{n} \mathbb{Z} P_i \longrightarrow \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R[f^{-1}]) \longrightarrow 0$$

is exact.

Definition 2.13. ***Definition of a Galois Extension of Fields.****[18, Definition 3.9] Let F be a field. A finite extension E of F is said to be Galois if F is the fixed field of the group of F-automorphisms of E. This group is then called the Galois group of E over F, and it is denoted $Gal(E/F$.)*

Proposition 2.12. *[18, Theorem 3.10] For an extension E/F, the following statements are equivalent:*

1. *E is the splitting field of a separable polynomial $f \in F[X]$.*
2. *$F = E^G$ for some finite group G of automorphisms of E.*
3. *E is normal and separable, and of finite degree, over F.*
4. *E is Galois over F.*

Definition 2.14. *The ring S is called an extension of the commutative ring R in case S is a commutative faithful R-algebra. If S is an extension of R and H is a set of R-automorphisms of S we let*

$$S^H = \{x \in S | \sigma(x) = x, \text{ for all } \sigma \in H\}.$$

If S is an extension of R and G is the group of all automorphisms of S leaving R elementwise fixed then S is called a normal extension of R in case $S^G = R$. Let G be *a finite group of $R-$automorphisms of the extension S of R. We are going to form two new R-algebras.*

First R***-algebra***, $\Delta(S:G)$*:*

Let $\{u_\sigma|\sigma \in G\}$ be a free basis for an $S-$module. Define multiplication in this module by letting

$$(au_\sigma)(bu_\tau) = a\sigma(b)u_{\sigma\tau}$$

for all $a, b \in S; \sigma, \tau \in G$ and extending by linearity. Define an R-algebra homomorphism by

$$j : \Delta(S:G) \longrightarrow \mathrm{Hom}_R(S, S)$$

by

$$[j(au_\sigma)](x) = a\sigma(x).$$

Second S***-algebra***, $\nabla(S:G)$*:*

Let $\{v_\sigma|\sigma \in G\}$ be a free basis for an S module. Define multiplication in this module by letting

$$(av_\sigma)(bv_\tau) = abv_\sigma\delta_{\sigma\tau}$$

for all $a, b \in S; \sigma, \tau \in G$. Here

$$\delta_{\sigma\tau} = \begin{cases} 0 & \sigma \neq \tau \\ 1 & \sigma = \tau \end{cases}.$$

The map

$$l : S \otimes S \longrightarrow \nabla(S:G)$$

defined by

$$l(a \otimes b) = \sum a\sigma(b)v_\sigma$$

is an S-algebra homomorphism.

Proposition 2.13. *[4, Theorem 3.1.2] Let S be an extension of R and let G be a finite group of automorphisms of S, then the following statements are equivalent:*

1. (a) $S^G = R$

   (b) For each nonzero idempotent $e \in S$ and each pair $\sigma \neq \tau$ in G there is an element $x \in S$ with $\sigma(x)e \neq \tau(x)e$

   (c) S is a separable R-algebra

2. (a) $S^G = R$

   (b) There exist $x_1, ..., x_n; y_1, ..., y_n$ in S with $\sum x_j \sigma(y_j) = \delta_{\sigma,1}$

3. (a) S is a finitely generated projective R-module

   (b) $j : \Delta(S : G) \longrightarrow \mathrm{Hom}_R(S, S)$ is an isomorphism

4. (a) $S^G = R$

   (b) $l : S \otimes S \longrightarrow \nabla(S : G)$ is an isomorphism

5. (a) $S^G = R$

   (b) For each maximal ideal M of S and for each $1 \neq \sigma \in G$ there is an $x \in S$ with $\sigma(x) - x \notin M$

Definition 2.15. ***Definition of a Galois Extension of Rings.*** *Let S be an extension of R and let G be a finite group of automorphisms of S. Then S is called a Galois extension of R with Galois group G in case one of the five equivalent conditions of Proposition 2.13 is satisfied.*

Definition 2.16. *[20, Page 37]* ***Definition of Crossed Product Algebras.*** *Let S be a commutative ring and G a finite group acting as automorphisms of S. Form* *the $S-$module $\Delta = \oplus_{\sigma \in G} S u_\sigma$. For simplicity we set $u_1 = 1$. Let*

$$u_\sigma a = \sigma(a) u_\sigma \tag{2.0.1}$$

$$u_\sigma u_\tau = c(\sigma, \tau) u_{\sigma\tau}, \text{ for some } c(\sigma, \tau) \in S, \tag{2.0.2}$$

$$\sigma(c(\tau, \eta)) c(\sigma, \tau\eta) = c(\sigma, \tau) c(\sigma\tau, \eta), \text{ for all } \sigma, \tau, \eta \in S \tag{2.0.3}$$

$$c(1, \sigma) = c(\sigma, 1) = 1, \text{ for all } \sigma \in G \tag{2.0.4}$$

A function $c : G \times G \longrightarrow S^*$ that satisfies (2.0.3), (2.0.4) is called the factor set. If $R \subset S^G$ then Δ defined above is an R algebra we write as $\Delta(S/R, G, c)$.

Proposition 2.14. *[20, Theorem 7.10(a), Page 49] Let $\Delta = \Delta(S/R, G, c)$ be a crossed product algebra. Suppose G is a cyclic group with generator σ of order n. Then Δ can be alternatively described as*

$$\bigoplus_{0 \le i \le n} S u^i. \tag{2.0.5}$$

where $u^n = b \in S^*$ and $us = \sigma(s) u$ for all $s \in S$. Conversely, any algebra as in (2.0.5) for any $b \in S^*$ is a crossed product with group G.

The algebra defined above is denoted $(S/R, \sigma, b)$ and is called a cyclic crossed product.

Definition 2.17. *[20, Page 50]* ***Definition of Symbol Algebra.*** *Suppose A/F is a cyclic crossed product of degree n, and F contains a primitive n-th root of unity ρ. Then there are $a, b \in F^*$ such that $A = \sum_{0 \le i,j < n} F u^i v^j$ with the relations $v^n = a$, $u^n = b$, $uv = \rho vu$. The algebra described above is called a symbol algebra, and is written $(a, b)_n$.*

Definition 2.18. *[19, Page 265]* ***Definition of Cohomology Group.*** *Let G be a group, A a left G-module, and $\mathbb{Z}$ considered as a trivial G-module. Define*

$$\mathrm{H}^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Definition 2.19. *[19, Page 296] Let G be a finite cyclic group of order k with generator x. Define elements of $\mathbb{Z}G$:*

$$D = x - 1, \ N = 1 + x + x^2 + ... + x^{k-1} \tag{2.0.6}$$

Proposition 2.15. *[19, Theorem 10.35,Page 297] Let G be a finite cyclic group of order k. If A is a G-module, define ${}_N A = \{a \in A | Na = 0\}$. Then*

$$\mathrm{H}^0(G, A) = A^G, \ \mathrm{H}^{2n-1}(G, A) =_N A/DA, \ \mathrm{H}^{2n}(G, A) = A^G/NA.$$

Proposition 2.16. *[4, Theorem 4.1.1]* ***Chase, Harrison, Rosenberg sequence.*** *Let R be any commutative ring and let S be a Galois extension of R with Galois group G. Then there are natural homomorphisms α_i which give the exact sequence*

$$1 \longrightarrow \mathrm{H}^1(G, S^*) \xrightarrow{\alpha_1} \mathrm{Pic}(R) \xrightarrow{\alpha_2} \mathrm{Pic}(S)^G \xrightarrow{\alpha_3}$$
$$\mathrm{H}^2(G, S^*) \xrightarrow{\alpha_4} \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{H}^1(G, \mathrm{Pic}(S)) \longrightarrow \mathrm{H}^3(G, S^*) \tag{2.0.7}$$

Corollary 2.2. *(**Hilbert's Theorem 90**). If $\mathrm{Pic}(R) = \{1\}$ then $\mathrm{H}^1(G, S^*) = \{1\}$.*

Corollary 2.3. *(**Crossed Product Theorem**). If $\mathrm{Pic}(S) = \{1\}$, then $\mathrm{B}(S/R) \cong \mathrm{H}^2(G, S^*)$.*

Definition 2.20. ***Definition of Zariski Topology on $\mathbb{A}^n$.*** *Let k be an algebraically closed field. Let $n \geq 0$. $\mathbb{A}^n = \{(a_1, ..., a_n) | a_i \in k\}$ and $A = k[x_1, ..., x_n]$. If $T \subseteq A$, then $Z(T)$ is defined as $\{P \in \mathbb{A}^n | f(P) = 0, \forall f \in T\}$. A subset $Y \subseteq \mathbb{A}^n$ is an* algebraic set *if there exists $T \subseteq A$ such that $Y = Z(T)$. Define the* Zariski topology *on $\mathbb{A}^n$ by taking the closed sets to be the algebraic sets.*

Definition 2.21. ***Definition of a variety.*** *A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as a union of two proper subsets, $Y = Y_1 \cup Y_2$, each of which is closed in Y. The empty set is not considered to be* *irreducible. An* affine algebraic variety *is an irreducible closed subset of* $\mathbb{A}^n$*. An open subset of an affine variety is called a* quasi-affine variety.

Definition 2.22. *For any* $Y \in \mathbb{A}^n$*, we define the ideal of* Y *in* $\mathbb{A}$ *by* $I(Y) = \{f \in A | f(P) = 0, \forall P \in Y\}$*. If* $Y \subseteq \mathbb{A}^n$ *is an affine algebraic set we define the* affine coordinate ring *of* Y *to be* $A(Y) = A/I(Y)$*. The dimension of* Y*, denoted by* $\dim(Y)$ *is equal to the Krull dimension of its affine coordinate ring* $A(Y)$*. The quotient field of* $A(Y)$ *is called the function field of* Y *and is denoted* $K(Y)$*.*

Definition 2.23. *Let* Y *be a quasi-affine variety,* $Y \subseteq \mathbb{A}^n$*. A function* $f : Y \longrightarrow k$ *is regular at the point* $P \in Y$ *if there exists an open neighborhood* $P \in U \subseteq Y$ *and polynomials* $g, h \in \mathbb{A}$ *such that* h *is nowhere zero on* U *and* $f = g/h$ *on* U*. We say* f *is regular if it is regular at all* $P \in Y$*.*

Definition 2.24. *Let* Y *be a variety. Denote by* $\mathcal{O}(Y)$ *the ring of regular functions on* Y*. If* Y *is an affine variety, the ring* $\mathcal{O}(Y)$ *is equal to* $A(Y)$*.*

Definition 2.25. ***Definition of a rational map.****[12, Definition, Page 24] A rational map on varieties,* $\phi : X \longrightarrow Y$*, is an equivalence class of pairs* $\langle U, \phi_U \rangle$ *where* U *is a nonempty open subset of* X*,* $\phi_U : U \longrightarrow Y$ *is a morphism and where two pairs* $\langle U, \phi_U \rangle$ *and* $\langle V, \phi_V \rangle$ *are equivalent if* $\phi_U = \phi_V$ *on* $U \cap V$*.*

Definition 2.26. ***Definition of a birational map.****[12, Definition, Page 24] A birational map is a rational map* $\phi : X \longrightarrow Y$ *that admits an inverse. That is, there exists a rational map* $\varphi : Y \longrightarrow X$ *such that* $\varphi\phi$ *is equivalent to* 1_X *and* $\phi\varphi$ *is equivalent to* 1_Y*. We say that* X *and* Y *are birationally equivalent, or simply birational. We say* X *is rational if* X *is birationally equivalent to* $\mathbb{P}^n$ *for some* n*.*

Proposition 2.17. *[12, Corollary 4.5] For varieties X and Y the following are equivalent.*

1. *X and Y are birationally equivalent.*

2. *There exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ as varieties.*

3. *$K(X) \cong K(Y)$ as k-algebras.*

Proposition 2.18. *[12, Theorem 1.8A] Let B be an integral domain which is finitely generated as a k-algebra. Denote Krull dimension by* dim *and the height of an ideal by* ht . *Then*

1. $\dim(B)$ *is equal to the transcendence degree of the quotient field $K(B)$ over k.*

2. *For any prime ideal P in B, we have* $\mathrm{ht}(P) + \dim(B/P) = \dim(B)$.

Proposition 2.19. *[12, Lemma 1.4.2] Let Y be a hypersurface in $\mathbb{A}^n$, given by the equation $f(x_1, ...x_n) = 0$. Then $\mathbb{A}^n - Y$ is isomorphic to the hypersurface H in $\mathbb{A}^{n+1}$ given by $x_{n+1}f - 1 = 0$. In particular, $\mathbb{A}^n - Y$ is affine, and its affine coordinate ring is $k[x_1, \ldots, x_n][f^{-1}]$.*

Definition 2.27. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety, and let*

$$f_1, \ldots, f_t \in k[x_1, \ldots, x_n]$$

be a set of generators for the ideal of Y. We say Y is nonsingular at the point $P \in \mathbb{A}^n$ if the rank of the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(P) \right)$$

is $n - r$, where $r = \dim(Y)$.

CHAPTER 3
DIVISORS AND ALGEBRAS ON A DOUBLE PLANE

Throughout, k is assumed to be an algebraically closed field with characteristic different from 2. Let $A = k[x, y]$ with quotient field $K = k(x, y)$. Let $f(x, y) = f_1 f_2 \cdots f_n$ be a square-free, homogeneous polynomial of even degree n in A, where $f_1, f_2, \ldots, f_n$ are linear polynomials, all of the form $f_i = x + a_i y$ with $a_i \neq 0$ for all i. Also, let $T = \frac{A[z]}{(z^2 - f(x,y)(x-1))}$, $R = k[x, y]\left[\frac{1}{f(x,y)(x-1)}\right]$, and $S = \frac{R[z]}{(z^2 - f(x,y)(x-1))}$. If L denotes quotient field of T it is easy to see that $L = K(z)$. The rings defined so far make up the commutative diagram

$$\begin{array}{ccccc} T = A[\sqrt{f(x,y)(x-1)}] & \longrightarrow & S = R[\sqrt{f(x,y)(x-1)}] & \longrightarrow & L \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & R = A[(f(x,y)(x-1))^{-1}] & \longrightarrow & K \end{array}$$

where each arrow represents set inclusion.

Proposition 3.1. *T and S are normal noetherian domains.*

Proof. It follows from Hilbert's Basis Theorem and [1, Proposition 6.6] that T is noetherian. Since k is a field it is well known that $k[x, y, z]$ is a UFD. The polynomial $z^2 - f(x, y)(x-1)$ is irreducible in $k[x, y, z]$ by Eisenstein's Irreducibility Criterion for a UFD. Therefore $(z^2 - f(x, y)(x-1))$ is a prime ideal in $k[x, y, z]$ so $T = \frac{A[z]}{(z^2 - f(x,y)(x-1))}$ is an integral domain. Since $f(x, y)(x - 1)$ is a square-free, T is integrally closed by [12, Exercise II 6.4]. This proves that T is normal. Since S is a localization of T, S is a normal noetherian domain.

Proposition 3.2. *The ring S is a Galois extension of R with Galois group $G = \{1, \sigma\}$, where $\sigma(z) = -z$.*

Proof. We will check all three conditions from Proposition 2.13(1) for S.

(a) It is true that L is a quadratic Galois extension of K with Galois group $\{1, \Sigma\}$, where $\Sigma(z) = -z$ (Proposition 2.12(1.)). Since every isomorphism between integral domains extends to a unique isomorphism of the corresponding quotient fields ([13, Corollary 4.6, Pg. 145]) it is clear that $G = \{1, \sigma\}$, where $\sigma(z) = -z$.

(b) Since the characteristic of k is different from 2, it is true that for any nonzero idempotent $e \in S$, $ze \neq -ze$.

(c) That S is a separable $R-$algebra follows from [17, Example, Pg.23]. Moreover, we can compute the separability idempotent e from Proposition 2.2(3.) and it is equal to $\frac{1}{2}(1 \otimes 1 + z \otimes \frac{1}{z})$.

□

3.1 GEOMETRY ON AFFINE SURFACE X.

We retain the notation for A, R, f, f_i from the beginning of Chapter 3. The affine plane over k is defined to be $\mathbb{A}^2 = \{(a, b) | a, b \in k\}$. From a geometrical point of view the ring A is the coordinate ring of $\mathbb{A}^2$. Let F be the zero set of $f(x, y)(x-1)$ in $\mathbb{A}^2$, i.e. $F = Z(f(x, y)(x-1))$. Let

$$F_1 = Z(f_1), ..., F_n = Z(f_n), F_{n+1} = Z(x-1) \tag{3.1.1}$$

It is easy to check that when $i \neq j$ and $i, j \in \{1, ..., n\}$

$$F_i \cap F_j = (0, 0). \tag{3.1.2}$$

Let's compute $F_i \cap F_{n+1}$ for an arbitrary $i < n$. It is clear that $x = 1$ for any point in the intersection. Now solving $1 + a_i y = 0$, gives $y = -1/a_i$. Therefore

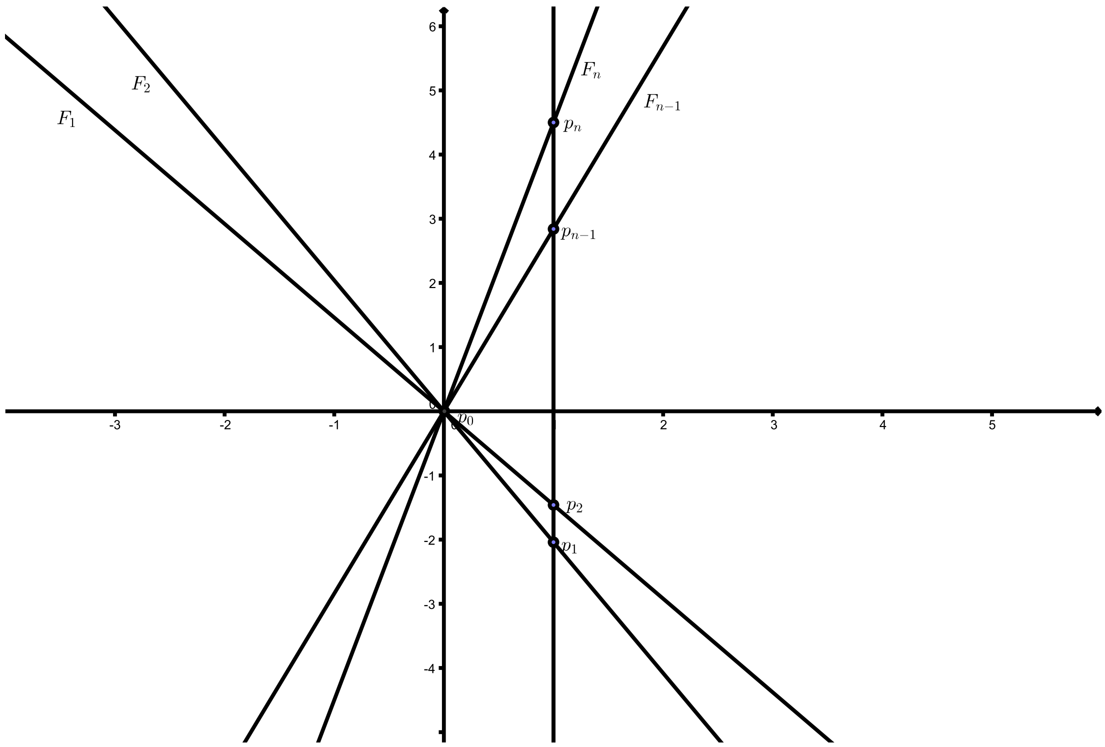

Figure 3.1: The divisor F.

$F_i \cap F_{n+1} = (1, -1/a_i)$. It follows that

$$F_1 \cap F_{n+1} = (1, -1/a_1) \, , ..., F_n \cap F_{n+1} = (1, -1/a_n) \tag{3.1.3}$$

If we denote the points of intersection by $p_0, ..., p_n$ (Fig. 3.1) then we have

$$p_0 = (0, 0), p_1 = (1, -1/a_1), ..., p_n = (1, -1/a_n) \tag{3.1.4}$$

Lemma 3.1. *The singular locus of F is equal to $\{p_i | i = 0, ..., n\}$.*

Proof. The Jacobian matrix for $F = Z(f(x, y)(x - 1))$ is

$$J = \begin{pmatrix} f_x(x,y)(x-1) + f(x,y) \\ f_y(x,y)(x-1) \end{pmatrix}$$

From Proposition 2.27 it is enough to find all points in F for which $J = 0$. Thus we need to solve the system of equations

$$\begin{cases} f_x(x,y)(x-1)+f(x,y)=0 \\ f_y(x,y)(x-1)=0 \end{cases} \tag{3.1.5}$$

(a) if $x=1$, then $f(1,y)=0$. The solution set for the equation $f(1,y)=0$ is equal to $\{(1,-1/a_1),...,(1,-1/a_n)\}$.

(b) if $x \neq 1$, then $f_y(x,y)=0$. We will show that $(0,0)$ is the only point in F that satisfies this equation. It is easy to check that $f_y(0,0)=0$. Is it the only point in F that satisfies $f_y(x,y)=0$? Assume not. If $(a,b)\in F$ and $a\neq 1$, then $f_i(a,b)=0$ for some i. Without loss of generality we can assume $f_1(a,b)=0$. Then from $f_y(a,b)=0$ follows

$$(f_1(a,b))_y f_2(a,b)...f_n(a,b)=0 \tag{3.1.6}$$

Thus $(f_1(a,b))_y=0$ and therefore $a_1=0$ which is a contradiction.

□

Denote by X the surface in $\mathbb{A}^3=\{(a,b,c)|a,b,c\in k\}$ defined by the equation $z^2=f(x,y)(x-1)$. From a geometrical point of view, T is the affine coordinate ring of X. Let

$$P_0=(0,0,0), P_1=(1,-1/a_1,0),\ldots,P_n=(1,-1/a_n,0) \tag{3.1.7}$$

be points on X.

Proposition 3.3. *(a) The singular locus of X is $\{P_i|i=0,...,n\}$.*

(b) X is normal.

(c) X is rational.

Proof. (a) The Jacobian matrix for $z^2 - f(x,y)(x-1)$ is the 3-by-1 matrix

$$J = \begin{pmatrix} f_x(x,y)(x-1) + f(x,y) \\ f_y(x,y)(x-1) \\ 2z \end{pmatrix}$$

Clearly $J = 0$ when

$$\begin{cases} f_x(x,y)(x-1) + f(x,y) = 0 \\ f_y(x,y)(x-1) = 0 \\ z = 0 \end{cases} \tag{3.1.8}$$

From (3.1.8) and (3.1.5) it follows that singular locus of X consists of the set of points lying over the singular locus of F. Combining the result from Lemma 3.1 and the fact that $z = 0$ we get all of the singular points of X.

(b) Since X is an affine variety, X is normal if and only $A(X)$ is a normal ring [12, II, Ex. 3.17]. From $T = A(X)$ and T is normal (Proposition 3.1) it follows that X is normal.

(c) In this proof we will use two methods for resolving a singularity, namely blowing-up and normalization ([12, Pg 27]). We will omit details here (need them only to define the homomorphism λ (3.1.17) and cover it more thoroughly in section 3.5. After setting blowing-up equations

$$xv = yu, xw = zu, yw = zv \tag{3.1.9}$$

and dehomogenizing ($u = 1$) we get

$$v = \frac{y}{x} \tag{3.1.10}$$

$$w = \frac{z}{x} \tag{3.1.11}$$

$$x^2w^2 = x^n f(1, \frac{y}{x})(x-1). \tag{3.1.12}$$

Dividing the last equation by x^n we have

$$\frac{w^2}{x^{n-2}} = f(1, \frac{y}{x})(x-1). \tag{3.1.13}$$

Set

$$t = \frac{w}{x^{\frac{n-2}{2}}} \tag{3.1.14}$$

Then from (3.1.11) and (3.1.14)

$$t = \frac{z}{x^{\frac{n}{2}}}. \tag{3.1.15}$$

Also (3.1.10), (3.1.13) and (3.1.14) give

$$t^2 = f(1, v)(x-1). \tag{3.1.16}$$

Define a map

$$\lambda : \frac{k[x, y, z]}{(z^2 - f(x, y)(x-1))}\left[\frac{1}{xf(x, y)}\right] \longrightarrow k[v, t]\left[\frac{1}{f(1, v)(f(1, v) + t^2)}\right] \tag{3.1.17}$$

by letting

$$x \longmapsto 1 + \frac{t^2}{f(1, v)} \tag{3.1.18}$$

$$y \longmapsto v + \frac{vt^2}{f(1, v)} \tag{3.1.19}$$

$$z \longmapsto t\left(1 + \frac{t^2}{f(1, v)}\right)^{\frac{n}{2}}. \tag{3.1.20}$$

That λ is well defined follows from

$$z^2 - f(x, y)(x-1) \longmapsto$$
$$t^2\left(1 + \frac{t^2}{f(1, v)}\right)^n - \left(1 + \frac{t^2}{f(1, v)}\right)^n f(1, v)(1 + \frac{t^2}{f(1, v)} - 1) \tag{3.1.21}$$

and

$$t^2\left(1 + \frac{t^2}{f(1, v)}\right)^n - \left(1 + \frac{t^2}{f(1, v)}\right)^n f(1, v)(1 + \frac{t^2}{f(1, v)} - 1) = 0. \tag{3.1.22}$$

That λ is onto follows from:

$$\frac{y}{x} \longmapsto v \tag{3.1.23}$$

$$\frac{z}{x^{\frac{n}{2}}} \longmapsto t \tag{3.1.24}$$

$$\frac{x^n}{f(x,y)} \longmapsto \frac{1}{f(1,v)} \tag{3.1.25}$$

$$\frac{x^{n-1}}{f(x,y)} \longmapsto \frac{1}{f(1,v)+t^2}. \tag{3.1.26}$$

Finally, let's prove that λ is one-to-one. Notice first that since $k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)}\right]$ is an integral domain, then $\mathrm{Ker}(\lambda)$ is a prime ideal in the ring $\frac{k[x,y,z]}{(z^2-f(x,y)(x-1))}\left[\frac{1}{xf(x,y)}\right]$. Notice also that both rings have dimension 2. This follows from Proposition 2.18(1.) and the fact that their corresponding quotient fields have transcendence degree two over k. Since λ is onto then $\dim\left(\frac{k[x,y,z]}{(z^2-f(x,y)(x-1))}\left[\frac{1}{xf(x,y)}\right]/\,\mathrm{Ker}(\lambda)\right) = 2$. From Proposition 2.18(2.) it follows that $\mathrm{Ker}(\lambda) = 0$. Therefore

$$\lambda : \frac{k[x,y,z]}{(z^2-f(x,y)(x-1))}\left[\frac{1}{xf(x,y)}\right] \longrightarrow k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)}\right]$$

is an isomorphism. By Proposition 2.19, $k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)}\right]$ is the coordinate ring for $\mathbb{A}^2 - Z(f(1,v)(f(1,v)+t^2))$. It follows directly from Definitions 2.25 and 2.26 that any variety is birationally equivalent to an open subset of itself. That makes $\mathbb{A}^2 - Z(f(1,v)(f(1,v)+t^2))$ birationally equivalent to $\mathbb{A}^2$, which is rational (as open set in $\mathbb{P}^2$). The isomorphism λ implies that $X - Z(xf(x,y))$ is rational and therefore X is rational.

□

Because the lines $f_i = 0$ and $x = 1$ intersect with normal crossings, it is well known that the singular points $P_1, \ldots, P_n$ are rational A_1 double points [5]. The singular point P_0 with affine coordinates $(0,0,0)$ will receive our attention in section 3.5.

3.2 ON THE CLASS GROUPS OF T AND S.

We will keep the same notation A, R, S, T, f, f_i as in the beginning of Chapter 3. In this section we will study invariants of X known as *divisor classes*. To be more precise, we will compute the groups of Weil divisors for the rings T and S.

Proposition 3.4. *If Q is a UFD and q is an irreducible element in Q, then the group of units in the ring $Q[q^{-1}]$ is isomorphic to $Q^* \times \langle q \rangle$.*

Proof. Let α be a unit in $Q[q^{-1}]$. Then

$$\alpha = \frac{r_1}{q^{k_1}} \tag{3.2.1}$$

and there exists $\beta \in Q[q^{-1}]$ such that

$$\alpha\beta = 1. \tag{3.2.2}$$

If

$$\beta = \frac{r_2}{q^{k_2}} \tag{3.2.3}$$

then from (3.2.1), (3.2.2) and (3.2.3) we have

$$\frac{r_1}{q^{k_1}} \frac{r_2}{q^{k_2}} = 1 \tag{3.2.4}$$

Since Q is a UFD and $r_1 r_2 = q^{k_1+k_2}$, it follows that $r_1 = uq^k$, $u \in Q^*$. Then $\alpha = uq^{k-k_1}$. This proves that $Q[q^{-1}]^* \subset Q^* \times \langle q \rangle$. The other inclusion is trivial. □

Corollary 3.1. *Let k be a field and $A_n = k[x_1, ..., x_n]$. Let $q \in A_n$ be a square-free polynomial such that $q = q_1 \cdots q_m$. The group of units in $R_n = A_n[q^{-1}]$ is equal to $k^* \times \langle q_1 \rangle \times \langle q_2 \rangle \times \cdots \times \langle q_m \rangle$.*

Proof. Follows from Proposition 3.4 by induction. □

Proposition 3.5. *Let R be the ring defined in the opening paragraph of Chapter 3.*

(a) $\mathrm{Cl}(R) = (1)$

(b) $\mathrm{Pic}(R) = (1)$

(c) $R^* \cong k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x - 1 \rangle$.

Proof. (a) Follows from Corollary 2.1.

(b) Follows from Proposition 2.10

(c) Follows from Corollary 3.1. □

Lemma 3.2. *(a)* $\mathrm{Cl}\left(T\left[\frac{1}{xf(x,y)}\right]\right) = 0$.

(b) $T\left[\frac{1}{xf(x,y)}\right]^* = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x \rangle$.

Proof. (a) From Proposition 3.3*(b)* we have that $T[\frac{1}{xf(x,y)}]$ is isomorphic to the UFD $k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)}\right]$. That $\mathrm{Cl}\left(T\left[\frac{1}{xf(x,y)}\right]\right) = 0$ follows from Proposition 2.1.

(b) From Corollary 3.1 we have that

$$\left(k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)}\right]\right)^* = k^* \times \langle f_1(1,v) \rangle \times ... \times \langle f_n(1,v) \rangle \times \left\langle f(1,v) + t^2 \right\rangle.$$

Since $\lambda\left(\frac{f_i}{x}\right) = f_i(1,v)$ and $\lambda\left(\frac{f(x,y)}{x^{n-1}}\right) = t^2 + f(1,v)$, then $\left(T\left[\frac{1}{xf(x,y)}\right]\right)^* = k^* \times \left\langle \frac{f_1}{x} \right\rangle \times ... \times \left\langle \frac{f_n}{x} \right\rangle \times \left\langle \frac{f(x,y)}{x^{n-1}} \right\rangle$. We will show that this group is equal to the group $k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x \rangle$. Call the first group G_1 and the second group G_2. We will prove both inclusions, i.e. $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$.

Proof for $G_1 \subseteq G_2$. It is easy to see that $\frac{f_1}{x}, ..., \frac{f_n}{x}, \frac{f(x,y)}{x^{n-1}}$ are all elements in G_2.

Proof for $G_2 \subseteq G_1$.

$$x = \left(\frac{f_1}{x}\right)^{-1} \cdots \left(\frac{f_n}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}},$$

$$f_1 = \left(\frac{f_2}{x}\right)^{-1} \cdots \left(\frac{f_n}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}},$$

$$\vdots$$

$$f_n = \left(\frac{f_1}{x}\right)^{-1} \cdots \left(\frac{f_{n-1}}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}}.$$

□

Proposition 3.6. *a)* $\mathrm{Cl}(T) \cong (\mathbb{Z}/2)^n \oplus \mathbb{Z}$

b) $T^* = k^*$.

Proof. a) We will use Nagata's Theorem (Proposition 2.11) to compute $\mathrm{Cl}(T)$. From Lemma 3.2 we have $T\left[\frac{1}{xf(x,y)}\right]^* = k^* \times \langle f_1\rangle \times ... \times \langle f_n\rangle \times \langle x\rangle$. What is $\mathrm{Div}(f_1)$? Let $f_1 \in P$, where P is a height-one prime ideal in T. Since $z^2 = f(x,y)(x-1)$ in T, then $z^2 = 0$ in T/P and since P is prime ideal, $z \in P$. Let $P_1 = (f_1, z)$. It is easy to check that $T/P_1 \cong k[t]$. Therefore T/P_1 is an integral domain and P is a prime ideal in T. That $\mathrm{ht}(P_1) = 1$ follows from $\mathrm{ht}(P_1) + \dim(T/P_1) = \dim(T)$ (Proposition 2.18(b)). From [12, Proposition 1.13] we have that $\dim(T) = 2$, and from $T/P_1 \cong k[t]$ it follows that $\dim(T/P_1) = 1$. Look at T_{P_1}. It is a one-dimensional, noetherian, normal, local integral domain. By [1, Theorem 9.2], T_{P_1} is a DVR. From $z^2 = f_1 f_2 ... f_n(x-1)$ it follows that $f_1 = z^2 f_2^{-1} ... f_n^{-1}(x-1)^{-1}$ therefore z generates the maximal ideal in T_{P_1}, $v_{P_1}(f_1) = 2$, and

$$\mathrm{Div}(f_1) = 2P_1. \tag{3.2.5}$$

Analogously, we show that

$$\mathrm{Div}(f_2) = 2P_2, \;\; P_2 = (f_2, z)$$

$$\vdots$$

$$\mathrm{Div}(f_n) = 2P_n, \;\; P_n = (f_n, z). \tag{3.2.6}$$

What is $\mathrm{Div}(x)$? Let $x \in Q$, where Q is a height-one prime ideal in T. Write $f(x,y)$ as $y^n + xg(x,y)$. In T

$$z^2 = (y^n + xg(x,y))(x-1) \tag{3.2.7}$$

Then in T/Q we have $z^2 + y^n = 0$. Since Q is a height-one prime ideal in T, either $z + iy^{\frac{n}{2}} \in Q$, or $z - iy^{\frac{n}{2}} \in Q$. Let $Q_1 = (x, z + iy^{\frac{n}{2}})$ and $Q_2 = (x, z - iy^{\frac{n}{2}})$. In T_{Q_1}, $z + iy^{\frac{n}{2}} = xf(x, y)(z - iy^{\frac{n}{2}})^{-1}$ which means that x generates the maximal ideal and $v_{Q_1}(x) = 1$. Analogously, $v_{Q_2}(x) = 1$. Thus

$$\mathrm{Div}(x) = Q_1 + Q_2. \tag{3.2.8}$$

Nagata's sequence for T is

$$1 \longrightarrow T^* \longrightarrow T\left[\frac{1}{xf(x, y)}\right]^* \xrightarrow{\mathrm{Div}} \oplus_{i=1}^{n} \mathbb{Z}P_i \oplus \mathbb{Z}Q_1 \oplus \mathbb{Z}Q_2 \longrightarrow \mathrm{Cl}(T) \longrightarrow 0. \tag{3.2.9}$$

It follows from (3.2.5), (3.2.6), (3.2.8) and Nagata's sequence (3.2.9) that the group $\mathrm{Cl}(T)$ is isomorphic to $(\mathbb{Z}/2)^n \oplus \mathbb{Z}$, and is generated by $P_1, ..., P_n, Q_1$.

b) From part a) the homomorphism Div in Nagata's sequence (3.2.9) can be represented by the $(n + 1) \times (n + 2)$ matrix of rank $n + 1$.

$$\mathrm{Div} = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}$$

Therefore $T^* = k^*$. ☐

Lemma 3.3. *a)* $\mathrm{Cl}\left(S\left[\frac{1}{x}\right]\right) = 0$.

b) $S\left[\frac{1}{x}\right]^* = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x \rangle \times \langle z \rangle$.

Proof. a) We will prove that $S\left[\frac{1}{x}\right] \cong k[v, t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)t}\right]$. We will use the map defined in Proposition 3.3(c) to show $S\left[\frac{1}{x}\right] \cong k[v, t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)t}\right]$. Let

$$\chi : S\left[\frac{1}{x}\right] \longrightarrow k[v, t]\left[\frac{1}{f(1, v)(f(1, v) + t^2)t}\right] \tag{3.2.10}$$

be defined by

$$x \longmapsto 1 + \frac{t^2}{f(1,v)} \tag{3.2.11}$$

$$y \longmapsto v + \frac{vt^2}{f(1,v)} \tag{3.2.12}$$

$$z \longmapsto t\left(1 + \frac{t^2}{f(1,v)}\right)^{\frac{n}{2}}. \tag{3.2.13}$$

Combining results from Proposition 3.3(b) and the fact that $\frac{zx^{\frac{n}{2}}}{(x-1)f(x,y)} \longmapsto \frac{1}{t}$ we have that χ is onto. That χ is well defined and one-to-one follows from the proof of Proposition 3.3(b). Now since $k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)t}\right]$ is a UFD, $S\left[\frac{1}{x}\right]$ is a UFD. $\mathrm{Cl}\left(S\left[\frac{1}{x}\right]\right) = 0$ follows from Corollary 2.1.

b) From Corollary 3.1 we have $k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)t}\right]^* = k^* \times \langle f_1(1,v)\rangle \times ... \times \langle f_n(1,v)\rangle \times \langle f(1,v)+t^2\rangle \times \langle t\rangle$. Combining results from the proof of Lemma 3.2 (b) and $\chi(\frac{z}{x^{\frac{n}{2}}}) = t$ we have that this group is isomorphic to the group $k^* \times \left\langle \frac{f_1}{x}\right\rangle \times ... \times \left\langle \frac{f_n}{x}\right\rangle \times \left\langle \frac{f(x,y)}{x^{n-1}}\right\rangle \times \left\langle \frac{z}{x^{\frac{n}{2}}}\right\rangle$. We will use results from the proof of Lemma 3.2 (b) again to show that this group is equal to $k^* \times \langle f_1\rangle \times ... \times \langle f_n\rangle \times \langle x\rangle \times \langle z\rangle$. Name the first group G_1 and the second group G_2. The inclusion $G_1 \subseteq G_2$ is trivial. Now we prove that $G_2 \subseteq G_1$. From the proof of Lemma 3.2 (b) we have

$$x = \left(\frac{f_1}{x}\right)^{-1} \cdots \left(\frac{f_n}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}},$$

$$f_1 = \left(\frac{f_2}{x}\right)^{-1} \cdots \left(\frac{f_n}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}},$$

$$\vdots$$

$$f_n = \left(\frac{f_1}{x}\right)^{-1} \cdots \left(\frac{f_{n-1}}{x}\right)^{-1} \frac{f(x,y)}{x^{n-1}}$$

□

Proposition 3.7. *(a)* $\mathrm{Cl}(S) \cong \mathbb{Z}$

(b) $\mathrm{Pic}(S) \cong \mathbb{Z}$

(c) $S^* = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle z \rangle$.

Proof. (a) To compute $\mathrm{Cl}(S)$ we will apply Lemma 3.3 and Nagata's Theorem (Proposition 2.11). We look at the ring S and its element x, which is not a unit in S. First compute im(Div). By using the same proof as in Proposition 3.6 (a), we get $\mathrm{Div}(x) = Q_1 + Q_2$, where $Q_1 = (x, z + iy^{\frac{n}{2}})$ and $Q_2 = (x, z - iy^{\frac{n}{2}})$. (equation 3.2.8 in Proposition 3.6). Nagata's sequence for S becomes

$$1 \longrightarrow S^* \longrightarrow S\left[\frac{1}{x}\right]^* \xrightarrow{\mathrm{Div}} \bigoplus_{i=1}^{2} \mathbb{Z}Q_i \longrightarrow \mathrm{Cl}(S) \longrightarrow 0. \tag{3.2.14}$$

From $\mathrm{Div}(x) = Q_1 + Q_2$, and (3.2.14), it is clear that $\mathrm{Cl}(S) \cong \mathbb{Z}$.

(b) Follows from Proposition 2.10 and part(a).

(c) The homomorphism Div from (3.2.14) can be represented by the $(n+2) \times 2$ matrix of rank 1.

$$\mathrm{Div} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Thus, $S^* = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle z \rangle$.

□

3.3 ON THE BRAUER GROUPS OF *R* AND *S*.

Throughout this section the notation R, S, f, f_i means the same as it did in the beginning of Chapter 3. In this section we will study Azumaya algebra classes on X

(Definition 2.8). Our main concern will be computing the groups $\mathrm{B}(R)$ and $\mathrm{B}(S)$. Let $Y_0, ..., Y_m$ be distinct hyperplanes in $\mathbb{P}^n$. Let $Y = Y_0 \cup ... \cup Y_m$. Let ωY denote the singular set of Y, $\omega Y = \{Y_i \cap Y_j | i \neq j\}$. Write $\omega Y = p_1 \cup ... \cup p_s$, where the p_j are the irreducible components of ωY. If $\tilde{Y} = Y_0 \coprod ... \coprod Y_m$ (disjoint union), then there is an obvious projection $\pi : \tilde{Y} \to Y$. Let $\omega \tilde{Y}$ be the inverse image $\pi^{-1}(\omega Y)$. Write $\tilde{Y} = q_1 \cup ... \cup q_e$, where the q_i are the irreducible components of $\tilde{Y}$. Define a graph Γ associated with Y. The vertices of Γ are the hyperplanes $Y_0, ..., Y_m$ and the varieties $p_1, ..., p_s$. The edges of Γ are the varieties $q_1, ..., q_e$. The edge q connects Y_i and p_j if and only if $p_j = \pi(q)$ and q is a subvariety of Y_i. The graph Γ is bipartite and oriented. We orient Γ by taking the positive end of an edge q the Y_i and the negative end p_j. There are $v = 1 + m + s$ vertices and e edges.

Proposition 3.8. *[7, Theorem 4] Let k be an algebraically closed field of characteristic p. Let $f_1, ..., f_m$ be linear polynomials in $k[x_1, ..., x_n]$ and $R = k[x_1, ..., x_n][f_1^{-1}, ..., f_m^{-1}]$. Let Y_0 be the hyperplane at infinity and $Y_1, ..., Y_m$ the complete hyperplanes in $\mathbb{P}^n$ defined by $f_1, ..., f_m$. Assume that the Y_i are distinct. Let $Y = Y_0 \cup ... \cup Y_m$ and Γ the graph of Y. Then modulo $p-$groups $\mathrm{B}(R) \cong \mathrm{H}_1(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^r$, where $r = e - v + 1$. The cup product map*

$$\beta : \mathrm{H}^1(R, z/\nu) \otimes \mathrm{H}^1(R, z/\nu) \longrightarrow_\nu \mathrm{B}(R) \tag{3.3.1}$$

is surjective for all ν relatively prime to p and $ker\beta$ is generated by

$$\{f_1 \otimes f_j | Y_i \cap Y_0 = Y_j \cap Y_0\} \cup \left\{(f_i \otimes f_j)(f_j \otimes f_t)(f_i \otimes f_t)^{-1} | Y_i \cap Y_j = Y_j \cap Y_t\right\}.$$

In Proposition 3.8 above, the homomorphism β takes a generator $f_i \otimes f_j$ to the class of the cyclic algebra $(f_i, f_j)_v$. Symbol algebras $(f_i, f_j)_v$ are explained in Definition 2.17.

In our setup (3.1.1) $Y_1 = F_1, ..., Y_n = F_n$ are affine curves embedded in projective

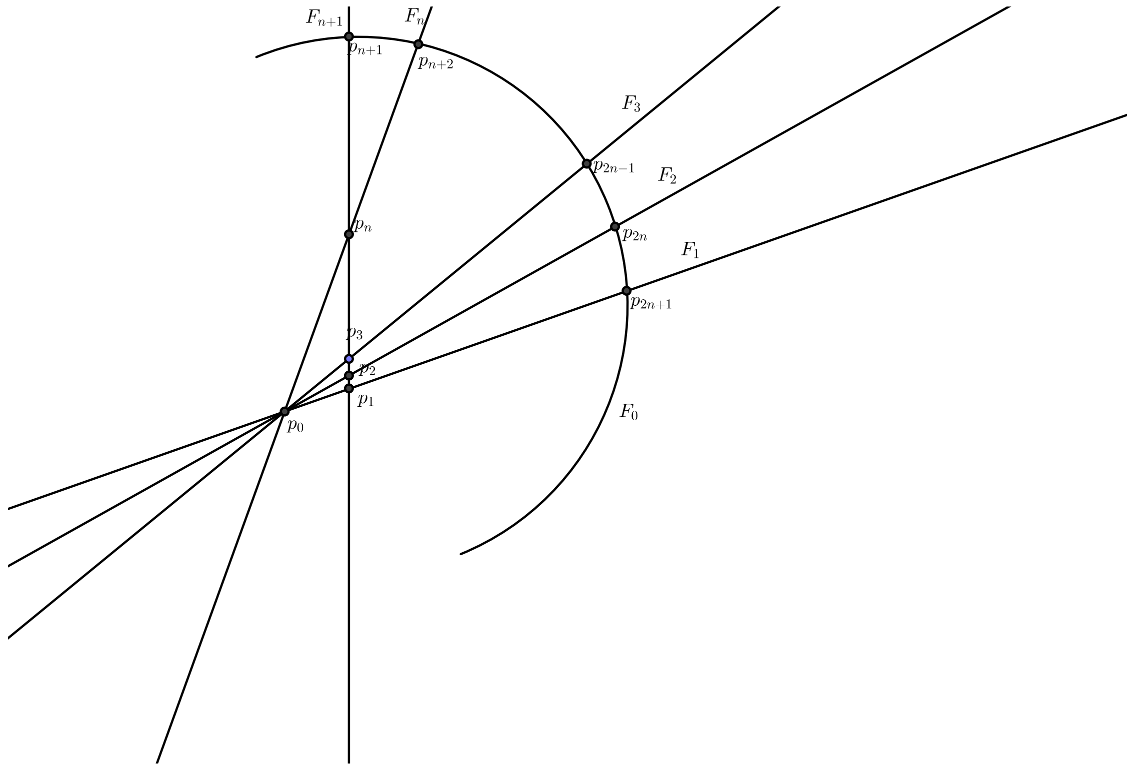

Figure 3.2: The lines in the plane.

space $\mathbb{P}^2$ in the usual way. Denote the line at infinity by F_0. Let $p_0, p_1, ..., p_n$ be as defined in (3.1.4). Let $p_{n+1} = F_0 \cap F_1, ..., p_{2n+1} = F_0 \cap F_{n+1}$. Figure 3.2 shows the divisors, Figure 3.3 shows the associated graph Γ.

Proposition 3.9. *a)* $\mathrm{B}(R) \cong (\mathbb{Q}/\mathbb{Z})^{2n-1}$. *A basis for $v-$torsion consists of the Brauer classes of symbol algebras $(f_1, f_2)_v, ..., (f_1, f_n)_v, (x-1, f_1)_v, ..., (x-1, f_n)_v$ over R.*
b) ${}_2\,\mathrm{B}(R) \cong (\mathbb{Z}/2)^{2n-1}$ *with basis $(f_1, f_2)_2, ..., (f_1, f_n)_2, (x-1, f_1)_2, ..., (x-1, f_n)_2$ over R.*

Proof. a) We will use Proposition 3.8 to prove this statement. In our setup, the vertices of the graph Γ are $p_0, ..., p_{2n+1}, F_0, ..., F_n, F_{n+1}$ (defined above). Thus $v = 3n+4$. Counting the number of edges we get $e = 5n+2$. (Figures 3.2 and 3.3) Therefore $e - v + 1 = 2n - 1$ and $\mathrm{B}(R) \cong (\mathbb{Q}/\mathbb{Z})^{2n-1}$. That it is generated by the

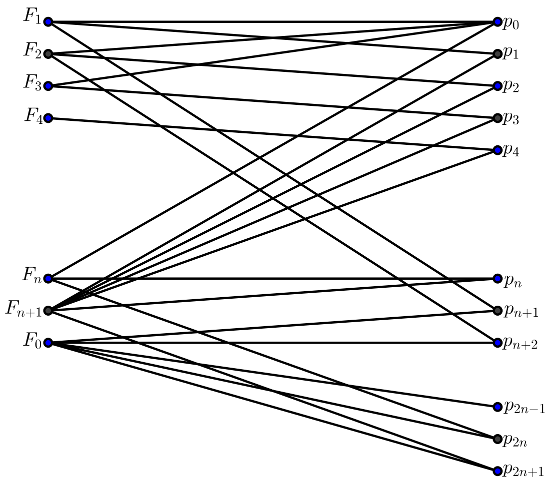

Figure 3.3: The graph of the divisor.

Brauer classes of the symbol algebras $(f_1, f_2)_v, ..., (f_1, f_n)_v, (x-1, f_1)_v, ..., (x-1, f_n)_v$ follows again from Proposition 3.8.

b) Combining Proposition 3.8 (when $\nu = 2$) and the result obtained in (a) we get ${}_2\,\mathrm{B}(R) \cong (\mathbb{Z}/2)^{2n-1}$. We also have a generating set for ${}_2\,\mathrm{B}(R)$. They are the cyclic algebras $(f_1, f_2)_2, ..., (f_n, x-1)_2$.

□

In our attempt to compute $\mathrm{B}(S)$ we used two different approaches. As they both worked well we found it hard to commit to only one, so we decided to present both of them.

Proposition 3.10. $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{3n-3}$.

Approach 1: We will need the three propositions stated below to complete the proof.

Proposition 3.11. *[8, Lemma 0.1] There is an exact sequence*

$$0 \longrightarrow \mathrm{B}(X) \longrightarrow \mathrm{B}(X - Y) \longrightarrow \mathrm{H}^3_Y(X, \mu) \longrightarrow \mathrm{H}^3(X, \mu) \longrightarrow \mathrm{H}^3(X - Y, \mu) \longrightarrow \ . \ . \ .$$

Proposition 3.12. *[8, Corollary 1.3] Suppose X is a complete, smooth surface and Y is a closed curve on X. Then $\mathrm{H}^3_Y(X,\mu)$ is isomorphic to $\mathrm{H}^1(\widetilde{Y},\mathbb{Q}/\mathbb{Z}) \oplus \mathrm{H}_1(\Gamma,\mu(-1))$.*

Proposition 3.13. *[8, Corollary 1.4] If the valence of each vertex $Z_1,...,Z_s$ in Γ is 1 then there is an exact sequence*

$$0 \longrightarrow B(X) \longrightarrow B(X-Y) \longrightarrow \mathrm{H}^1(\widetilde{Y},\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^3(X,\mu) \longrightarrow \mathrm{H}^3(X-Y,\mu) \quad (3.3.2)$$

Proof of Proposition 3.10. We will embed $S\left[\frac{1}{x}\right]$ in $\mathbb{P}^2$ first. From the proof of Lemma 3.3 we have an isomorphism $S\left[\frac{1}{x}\right] \cong k[v,t]\left[\frac{1}{f(1,v)(f(1,v)+t^2)t}\right]$. After homogenizing the polynomial $f(1,v)(f(1,v)+t^2)t$ we get $f(z,v)(f(z,v)+z^{n-2}t^2)t$. The exact sequence from Proposition 3.11 then becomes

$$0 \longrightarrow B(\mathbb{P}^2) \longrightarrow B(\mathbb{P}^2-Y) \longrightarrow \mathrm{H}^3_Y(\mathbb{P}^2,\mu) \longrightarrow \mathrm{H}^3(\mathbb{P}^2,\mu) \longrightarrow \mathrm{H}^3(\mathbb{P}^2-Y,\mu) \longrightarrow \ . \ . \ . \quad (3.3.3)$$

where $Y = Z\left(f(z,v)(f(z,v)+z^{n-2}t^2)t\right)$. Observe that $B(\mathbb{P}^2)=1$ [9, Section 1] and $\mathrm{H}^3(\mathbb{P}^2,\mu)=1$, [17, Example VI.5.6] therefore

$$B\left(S\left[\frac{1}{x}\right]\right) \cong \mathrm{H}^3_Y(\mathbb{P}^2,\mu) \quad (3.3.4)$$

From Proposition 3.12, $\mathrm{H}^3_Y(\mathbb{P}^2,\mu) \cong \mathrm{H}_1(\Gamma,\mu(-1)) \oplus \mathrm{H}^1(\widetilde{Y},\mathbb{Q}/\mathbb{Z}))$. The number of vertices in Γ (Figure 3.4) is $2n+5$ and the number of edges $4n+5$. We have $\mathrm{H}_1(\Gamma,\mu(-1)) \cong (\mathbb{Q}/\mathbb{Z})^{(4n+5)-(2n+5)+1}$, which gives

$$\mathrm{H}_1(\Gamma,\mu(-1)) \cong (\mathbb{Q}/\mathbb{Z})^{2n+1} \quad (3.3.5)$$

Solving the Riemann-Hurwitz Formula $2(g-1) = 2(-2)+n$ [12, Corollary 4.2.4] for the genus g of the hyperelliptic curve $f(1,v)+z^{n-2}t^2$, we have

$$g = \frac{n-2}{2} \quad (3.3.6)$$

Then $\mathrm{H}^1(\widetilde{Y}, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{2g}$, (see [17, p.127]), i.e.

$$\mathrm{H}^1(\widetilde{Y}, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{n-2} \tag{3.3.7}$$

Combining results from (3.3.4), (3.3.5) and (3.3.7) we have

$$B(S\left[\frac{1}{x}\right]) \cong (\mathbb{Q}/\mathbb{Z})^{3n-1} \tag{3.3.8}$$

Next we embed $S\left[\frac{1}{x}\right]$ in S. Since we are in the affine plane we can use the sequence from Proposition 3.13 which becomes

$$0 \longrightarrow \mathrm{B}(S) \longrightarrow B(S\left[\frac{1}{x}\right]) \longrightarrow \mathrm{H}^1(S/(x), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^3(S, \mu) \longrightarrow \mathrm{H}^3(S\left[\frac{1}{x}\right], \mu) \tag{3.3.9}$$

From $S/(x) = \frac{k[z,y]}{(z^2+\beta y^n)}\left[\frac{1}{y}\right]$ and $z^2 + \beta y^n = (z - by^{\frac{n}{2}})(z + by^{\frac{n}{2}})$ we have

$$S/(x) \cong \frac{k[z,y]}{(z + by^{\frac{n}{2}})}\left[\frac{1}{y}\right] \oplus \frac{k[z,y]}{(z - by^{\frac{n}{2}})}\left[\frac{1}{y}\right] \tag{3.3.10}$$

Since $\frac{k[z,y]}{(z+by^{\frac{n}{2}})}\left[\frac{1}{y}\right] \cong k[y, y^{-1}]$ and $\frac{k[z,y]}{(z-by^{\frac{n}{2}})}\left[\frac{1}{y}\right] \cong k[y, y^{-1}]$

$$S/(x) \cong k[y, y^{-1}] \oplus k[y, y^{-1}] \tag{3.3.11}$$

The ring $k[y, y^{-1}]$ is the coordinate ring of an algebraic torus. We will use the Kummer sequence

$$0 \to k[y, y^{-1}]^*/(k[y, y^{-1}]^*)^n \to \mathrm{H}^1(k[y, y^{-1}], \mathbb{Z}/n) \to {}_n\operatorname{Pic}(k[y, y^{-1}]) \to 0$$

to compute $\mathrm{H}^1(k[y, y^{-1}], \mathbb{Q}/\mathbb{Z})$. The group of units in $k[y, y^{-1}]$ is $k^* \times \langle y \rangle$ (Proposition 3.4). The units modulo nth powers, $k[y, y^{-1}]^*/(k[y, y^{-1}]^*)^n$, is cyclic of order n. Since $k[y, y^{-1}]$ is factorial, $\operatorname{Pic}(k[y, y^{-1}]) = 0$. Therefore

$$\mathrm{H}^1(k[y, y^{-1}], \mathbb{Z}/n) \cong \mathbb{Z}/n \tag{3.3.12}$$

From (3.3.12) follows that the first Betti number (rank of the free module $\mathrm{H}^1(T, \mathbb{Z}/n)$) is 1. Thus $H^1(k[y, y^{-1}], \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})$, and

$$\mathrm{H}^1(S/(x), \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^2 \tag{3.3.13}$$

From (3.3.8) and (3.3.13) and the exactness of (3.3.9) it follows that $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{3n-3}$.

Approach 2: We present another proof of Proposition 3.10.

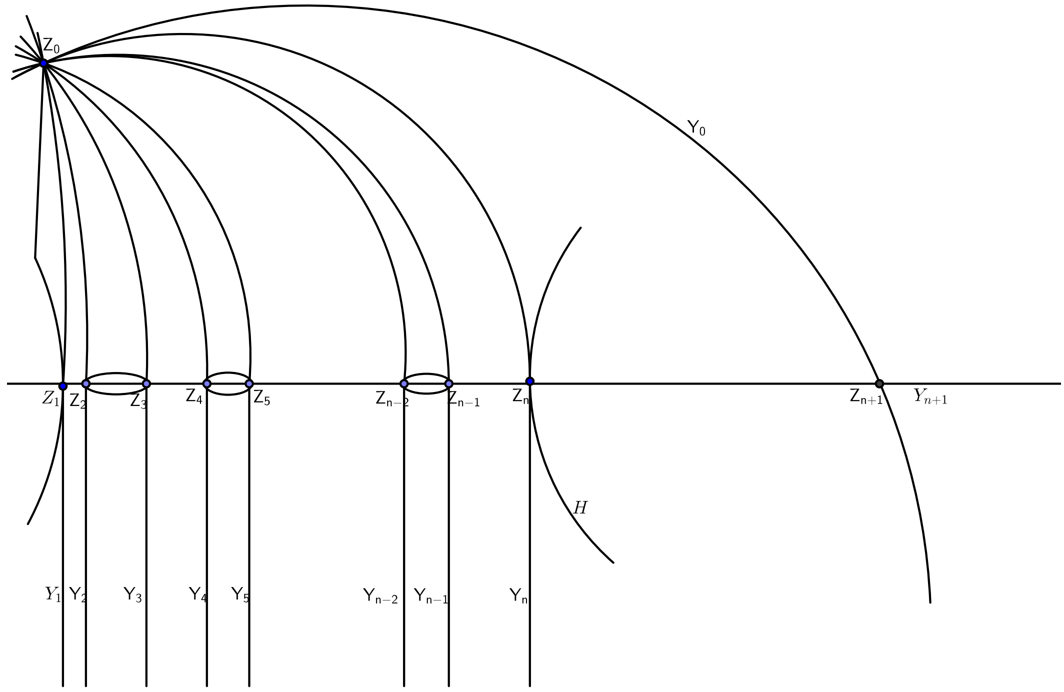

Figure 3.4: The curves that appear in the proof of Prop. 3.10

Proof of Proposition 3.10. We will use the exact sequence of vector spaces over $(\mathbb{Z}/2)$ from [10, Proposition 2.1,(9)]. That is

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{H}^1(R, \mathbb{Z}/2) \longrightarrow \mathrm{H}^1(S, \mathbb{Z}/2) \longrightarrow$$
$$\mathrm{H}^1(R, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(R, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(S, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(R, \mathbb{Z}/2) \longrightarrow 1 \tag{3.3.14}$$

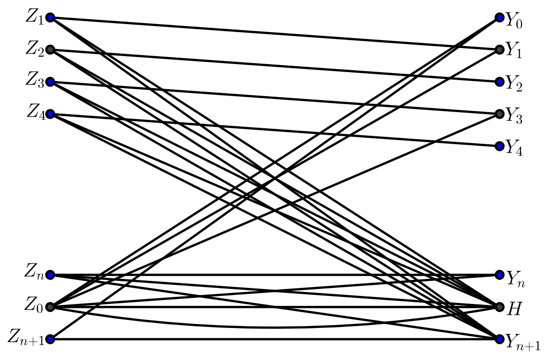

Figure 3.5: The graph that appears in the proof of Prop. 3.10

This is the so-called restriction-corestriction sequence. From (4) in [10] we have the exact Kummer sequence

$$1 \longrightarrow R^*/(R^*)^2 \longrightarrow \mathrm{H}^1(R, \mathbb{Z}/2) \longrightarrow {}_2\operatorname{Pic}(R) \longrightarrow 1 \tag{3.3.15}$$

Since the group of units of R is equal to $k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x-1 \rangle$ (Proposition 3.5) and since R is factorial ($\operatorname{Pic}(R) = (1)$) we have

$$\mathrm{H}^1(R, \mathbb{Z}/2) \cong \left(\frac{\mathbb{Z}}{2}\right)^{n+1}. \tag{3.3.16}$$

From (5) in [10] we have exact Kummer sequence

$$1 \longrightarrow {}_2\operatorname{Pic}(R) \otimes \mathbb{Z}/2 \longrightarrow \mathrm{H}^2(R, \mathbb{Z}/2) \longrightarrow {}_2\operatorname{B}(R) \longrightarrow 1 \tag{3.3.17}$$

From Proposition 3.9 we have ${}_2\operatorname{B}(R) \cong (\mathbb{Z}/2)^{2n-1}$. Since $\operatorname{Pic}(R) = (1)$, it follows that

$$\mathrm{H}^2(R, \mathbb{Z}/2) \cong \left(\frac{\mathbb{Z}}{2}\right)^{2n-1}. \tag{3.3.18}$$

Similarily, from $1 \longrightarrow S^*/(S^*)^2 \longrightarrow \mathrm{H}^1(S, \mathbb{Z}/2) \longrightarrow \operatorname{Pic}(S) \longrightarrow 1$ and $\operatorname{Pic}(S) = \mathbb{Z}$ it follows that

$$\mathrm{H}^1(S, \mathbb{Z}/2) \cong \left(\frac{\mathbb{Z}}{2}\right)^{n+1}. \tag{3.3.19}$$

If we apply the rank-nullity theorem (the alternating sum of the dimensions in an exact sequence of finite-dimensional vector spaces is zero) in (3.3.14) we can compute the rank of $\mathrm{H}^2(S,\mathbb{Z}/2)$. After solving the equation $1+(n+1)+(2n-1)+(2n-1) = (n+1)+(n+1)+x$ for x, where x is equal to the $\mathbb{Z}/2$-dimension of $\mathrm{H}^2(S,\mathbb{Z}/2)$, we get that $x = 3n-2$. Therefore

$$\mathrm{H}^2(S,\mathbb{Z}/2) \cong \left(\frac{\mathbb{Z}}{2}\right)^{3n-2}. \tag{3.3.20}$$

Equation (5) in [10] applied on S is

$$1 \longrightarrow \mathrm{Pic}(S)\otimes\mathbb{Z}/2 \longrightarrow \mathrm{H}^2(S,\mathbb{Z}/2) \longrightarrow {}_2\,\mathrm{B}(S) \longrightarrow 1 \tag{3.3.21}$$

Because of rank-nullity theorem we have

$${}_2\,\mathrm{B}(S) \cong \left(\frac{\mathbb{Z}}{2}\right)^{3n-3}. \tag{3.3.22}$$

□

3.4 TERMS OF THE CHASE, HARRISON, ROSENBERG SEQUENCE.

In this section we will compute all terms of the Chase, Harrison, Rosenberg sequence

$$1 \longrightarrow \mathrm{H}^1(G,S^*) \xrightarrow{\alpha_1} \mathrm{Pic}(R) \xrightarrow{\alpha_2} \mathrm{Pic}(S)^G \xrightarrow{\alpha_3}$$
$$\mathrm{H}^2(G,S^*) \xrightarrow{\alpha_4} \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{H}^1(G,\mathrm{Pic}(S)) \longrightarrow \mathrm{H}^3(G,S^*) \tag{3.4.1}$$

where S is Galois extension of R (introduced in the beginning of Chapter 3) with Galois group G (Proposition 3.2).

Proposition 3.14. *(a)* $\mathrm{Pic}(R) = (1)$

(b) $\mathrm{H}^1(G,S^*) = (1)$

(c) $\mathrm{H}^3(G,S^*) = (1)$

(d) $\mathrm{H}^1(G,\mathrm{Pic}(S)) = \mathrm{Pic}(S)\otimes\mathbb{Z}/2$

(e) $\mathrm{Pic}(S)^G = (1)$

Proof. (a) This follows from Proposition 2.1 and Proposition 2.10 and the fact that R is a UFD.

(b) Follows from Corollary 2.2 and part (a) of this Proposition.

(c) Follows from Proposition 2.15 and part (b).

(d) Follows from [9, Lemma 2.5(b)]

(e) We have that $\mathrm{Pic}(S) \cong \mathbb{Z}$ is generated by $Q_1 = (x, z + iy^{\frac{n}{2}})$. From $\sigma(Q_1) = Q_2$ follows $\mathrm{Pic}(S)^G = (1)$.

$\square$

Before computing the two remaining terms, $\mathrm{H}^2(G, S^*)$ and $\mathrm{B}(S/R)$, we will take a look at the construction of $\alpha_4 : \mathrm{H}^2(G, S^*) \longrightarrow \mathrm{B}(S/R)$ which is described in [14]. It assigns to a unit $u \in (S^*)^G$ the cyclic crossed product $\Delta(S, G, u)$.

Proposition 3.15. *(a)* $\mathrm{H}^2(G, S^*) \cong (\mathbb{Z}/2)^n$
(b) $\mathrm{im}(\alpha_4)$ *has basis* $(f(x,y)(x-1), f_1)_2, ..., (f(x,y)(x-1), f_n)_2$

Proof. (a) The first part of the statement follows from Proposition 2.15. Because G is cyclic of order two, $\mathrm{H}^2(G, S^*) = (S^*)^G / NS^*$. From $(S^*)^G = R^*$ and Proposition 3.5 follows $(S^*)^G = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x - 1 \rangle$. From $G = (1, \sigma)$ and Definition 2.19 we have

$$Nf_1 = f_1^2, \ . \ . \ . \ , \ Nf_n = f_n^2, \ N(x-1) = (x-1)^2.$$

Then

$$\mathrm{H}^2(G, S^*) = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x - 1 \rangle / k^* \times \left\langle f_1^2 \right\rangle \times ... \times \left\langle f_n^2 \right\rangle \times \left\langle z^2 \right\rangle . \quad (3.4.2)$$

Since $z^2 = f_1 ... f_n (x - 1)$, (3.4.2) becomes

$$\mathrm{H}^2(G, S^*) = k^* \times \langle f_1 \rangle \times ... \times \langle f_n \rangle \times \langle x - 1 \rangle / k^* \times \left\langle f_1^2 \right\rangle \times ... \times \left\langle f_n^2 \right\rangle \times \langle f_1 ... f_n (x - 1) \rangle . \quad (3.4.3)$$

This is clearly isomorphic to $(\mathbb{Z}/2)^n$.

(b) From Proposition 3.8, $\mathrm{im}(\alpha_4)$ is generated by $(f(x,y)(x-1), f_1)_2, ..., (f(x,y)(x-1), f_n)_2, (f(x,y)(x-1), x-1)_2$. The proof will have 2 parts. First we will prove that $(f(x,y)(x-1), f_1)_2, ..., (f(x,y)(x-1), f_n)_2$ are independent and after that we will show that $(f(x,y)(x-1), x-1)_2$ is generated by them. From Proposition 3.9(b) a basis for ${}_2\,\mathrm{B}(R)$ is $(f_1, f_2)_2, ..., (f_1, f_n)_2, (x-1, f_1)_2, ..., (x-1, f_n)_2$. Observe first that

$$(f(x,y)(x-1), f_1)_2 \sim (f_1, f_1)_2(f_2, f_1)_2 \cdots (f_n, f_1)_2(x-1, f_1)_2 \sim \\ \sim (f_1, f_2)_2(f_1, f_3)_2...(f_1, f_n)_2(f_1, x-1)_2 \quad (3.4.4)$$

$$(f(x,y)(x-1), f_2)_2 \sim (f_1, f_2)_2(f_2, f_2)_2 \cdots (f_n, f_2)_2(x-1, f_2)_2 \sim \\ \sim (f_1, f_2)_2(f_1, f_3)_2...(f_1, f_n)_2(x-1, f_2)_2 \quad (3.4.5)$$

$$\vdots$$

$$(f(x,y)(x-1), f_n)_2 \sim (f_1, f_n)_2(f_2, f_n)_2 \cdots (f_n, f_n)_2(x-1, f_n)_2 \sim \\ \sim (f_1, f_2)_2(f_1, f_3)_2...(f_1, f_n)_2(x-1, f_n)_2 \quad (3.4.6)$$

Look at the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The matrix has rank n so the elements $(f(x,y)(x-1), f_1)_2, ..., (f(x,y)(x-1), f_n)_2$ are independent. $(f(x,y)(x-1), x-1)_2$ is clearly generated by them. □

The map α_5 in (3.4.1) is onto. Since the $\mathbb{Z}/2$ rank of the map is equal to n and α_4 is one-to-one there is one element from $\mathrm{B}(S/R)$ that maps into $\mathrm{H}^1(G, \mathrm{Pic}(S))$. Our next task is to find that element. More precisely we will describe Δ (an Azumaya R-algebra that is split by S) such that $\alpha_5(\Delta) \in \mathrm{H}^1(G, \mathrm{Pic}(S))$. The reference for this is [14]. Let I be a rank-one prime ideal in S. Then I is a projective $S-$module representing a class in $\mathrm{Pic}(S)$. By Proposition 3.14 (e), $I\sigma(I) = Sg$ is a principal ideal in S. By ${}_1I_\sigma$ we denote the $S-$bimodule where multiplication by S from the left is $s \cdot x = x\sigma(x)$, and from the right is $x \cdot s = x\sigma(s)$. There are four $S-$bimodule homomorphisms

$$ {}_1I_\sigma \otimes_S {}_1I_\sigma \xrightarrow{f_{\sigma,\sigma}} S \tag{3.4.7}$$

$$ {}_1I_\sigma \otimes_S S \xrightarrow{f_{\sigma,1}} {}_1I_\sigma \tag{3.4.8}$$

$$ S \otimes_S {}_1I_\sigma \xrightarrow{f_{1,\sigma}} {}_1I_\sigma \tag{3.4.9}$$

$$ S \otimes_S S \xrightarrow{f_{1,1}} S \tag{3.4.10}$$

defined by the assignments

$$ f_{\sigma,\sigma}(a \otimes b) = a\sigma(b)g^{-1} \tag{3.4.11}$$

$$ f_{\sigma,1}(a \otimes b) = a\sigma(b) \tag{3.4.12}$$

$$ f_{1,\sigma}(a \otimes b) = ab \tag{3.4.13}$$

$$ f_{1,1}(a \otimes b) = ab \tag{3.4.14}$$

Let $\Delta_1 = S$ and $\Delta_\sigma =_1 I_\sigma$. Then $\Delta(I) = \Delta_1 \oplus \Delta_\sigma$ is a $S-$bimodule. Turn $\Delta(I)$ into an $R-$algebra by the following rule. On homogeneous elements $a \in \Delta_\alpha$ and $b \in \Delta_\beta$, the product is defined by $ab = f_{\alpha,\beta}(a \otimes b)$. For typical elements $(a, b), (c, d), (e, f)$ in $\Delta(I)$, this translates into the product rule

$$ (a, b)(c, d) = (ac + b\sigma(d)g^{-1}, b\sigma(c) + ad) \tag{3.4.15}$$

In the terminology of [14], $\Delta(I)$ is a generalized crossed product and $\{f_{\alpha,\beta}\}$ is a factor set. It follows from [14] that $\Delta(I)$ is an Azumaya $R-$algebra which is split by S. The generalized crossed product $\Delta(\{f_{\alpha,\beta}\}, S,_1 I_\sigma, \sigma)$ is an Azumaya R-algebra. From Proposition 3.7 we have that $\mathrm{Pic}(S)$ is generated by the rank-one prime ideals $Q_1 = (x, z+iy^{\frac{n}{2}})$ and $\sigma Q_1 = (x, z-iy^{\frac{n}{2}})$. We call the latter ideal Q_2. First notice that Q_1 and Q_2 are comaximal. It is because $z \in Q_1+Q_2$ and therefore $\frac{z^2}{f(x,y)}+x \in Q_1+Q_2$. From $\frac{z^2}{f(x,y)} + x = -1$ in S we get $Q_1 + Q_2 = S$. Then

$$Q_1Q_2 = Q_1 \cap Q_2 = Sx \tag{3.4.16}$$

Lemma 3.4. *Q_1 is a free $R-$module of rank 2 generated by x and $z + iy^{\frac{n}{2}}$.*

Proof. Let a, b, c, d be arbitrary elements of R. Then $a + bz, c + dz$ are arbitrary elements of S. A typical element of Q_1 is

$$\begin{aligned}
(a + bz)x + (c + dz)(z + iy^{\frac{n}{2}}) &= ax + bzx + c(z + iy^{\frac{n}{2}}) + dz^2 + dizy^{\frac{n}{2}} \\
&= ax + bzx + bixy^{\frac{n}{2}} - bixy^{\frac{n}{2}} + c(z + iy^{\frac{n}{2}}) \\
&\quad + dz^2 + dizy^{\frac{n}{2}} - dy^{\frac{n}{2}} + dy^{\frac{n}{2}} \\
&= x(a + by^{\frac{n}{2}} + (z + iy^{\frac{n}{2}})(bx + c + idy^{\frac{n}{2}}) \\
&\quad + d(f(x, y)x - f(x, y) + y^n) \\
&= x(a + by^{\frac{n}{2}} + (z + iy^{\frac{n}{2}})(bx + c + idy^{\frac{n}{2}}) \\
&\quad + d(f(x, y)x - (y^n + xg(x, y) - y^n)) \\
&= x(a + by^{\frac{n}{2}} + (z + iy^{\frac{n}{2}})(bx + c + idy^{\frac{n}{2}}) \\
&\quad + dx(f(x, y) - g(x, y)) \\
&= (df(x, y) - dg(x, y) + a + iby^{\frac{n}{2}})x \\
&\quad + (bx + c + idy^{\frac{n}{2}})(z + iy^{\frac{n}{2}})
\end{aligned} \tag{3.4.17}$$

□

Now we tie our example into this. We construct the generalized crossed product algebra $\Delta(Q_1)$ which is split by S. Using the isomorphism $Q_1Q_2 = Sx$, we have the generalized crossed product algebra $\Delta(\{f_{\alpha,\beta}\}, S, J_\sigma, \sigma)$, where J_σ is ${}_\sigma S_1 \otimes_S Q_1$. To shorten the notation we call this algebra $\Delta(Q_1)$. Then $\Delta(Q_1)$ is an Azumaya R-algebra. As an R-module, $\Delta(Q_1)$ is free of rank four. As an R-algebra, $\Delta(Q_1)$ is generated by three elements $u = (z, 0), v = (0, x), w = (0, z + iy^{\frac{n}{2}})$. Applying (3.4.15) we get

$$u^2 = (z, 0)(z, 0) = (z^2, 0)$$

$$v^2 = (0, x)(0, x) = (x, 0)$$

$$w^2 = (0, z + iy^{\frac{n}{2}})(0, z + iy^{\frac{n}{2}}) = (-\frac{z^2 + y^n}{x}, 0)$$

$$uv = (0, zx)$$

$$vu = (0, -zx)$$

$$uw = (0, z(z + iy^{\frac{n}{2}})$$

$$wu = (0, -z(z + iy^{\frac{n}{2}})$$

$$vw = (-z + iy^{\frac{n}{2}}, 0)$$

$$wv = (z + iy^{\frac{n}{2}}, 0)$$

From the multiplication table for $\Delta(Q_1)$ (Table 3.1) it is clear that over the field K, this algebra becomes the symbol $(f, x)_2$.

Corollary 3.2. $\mathrm{B}(S/R) \cong (\mathbb{Z}/2)^{n+1}$ *with basis* $(f(x, y)(x - 1), f_1)_2, ..., (f(x, y)(x - 1), f_n)_2, (x, f(x, y))_2$.

3.5 CLASS GROUP, PICARD GROUP FOR THE RESOLUTION $\widetilde{Y}$

The goal of this section is to see how the divisor classes on our example fit into the exact sequences constructed in the paper of Lipman [15]. As a consequence of the

Table 3.1: Multiplication table for $\Delta(Q_1)$

	$(1,0)$	$(z,0)$	$(0,x)$	$(0,z+iy^{\frac{n}{2}})$
$(1,0)$	$(1,0)$	$(z,0)$	$(0,x)$	$(0,z+iy^{\frac{n}{2}})$
$(z,0)$	$(z,0)$	$(z^2,0)$	$(0,zx)$	$(0,z(z+iy^{\frac{n}{2}}))$
$(0,x)$	$(0,x)$	$(0,-zx)$	$(x,0)$	$(-z+iy^{\frac{n}{2}},0)$
$(0,z+iy^{\frac{n}{2}})$	$(0,z+iy^{\frac{n}{2}})$	$(0,-z(z+iy^{\frac{n}{2}}))$	$(z+iy^{\frac{n}{2}},0)$	$(-(z^2+y^n)/x,0)$

computations, we see that the singularity is nonrational. Let P_0 be the point with affine coordinates $(0,0,0)$, defined in (3.1.7).

Proposition 3.16. *The singularity P_0 on X is resolved by one blowing-up of the closed point followed by a normalization. The reduced exceptional fiber E is nonsingular and nonrational.*

Proof. Following [12, pp.28–29], the blowing-up of X at $P_0 = (0,0,0)$ is a subvariety of the product space $\mathbb{A}^3 \times \mathbb{P}^2$. Introduce projective coordinates u, v, w. The strict transform of X in $\mathbb{A}^3 \times \mathbb{P}^2$ is denoted $Y \to X$. Then Y is a subvariety of the algebraic set defined by the equations $z^2 = f_1 \cdots f_n(x-1)$, $xv = yu$, $xw = zu$, $yw = zv$. The affine open subsets of Y where $u \neq 0$, $v \neq 0$, $w \neq 0$, are written Y_u, Y_v, Y_w respectively. We investigate Y by considering each of these sets in turn. For the following steps refer to the proof of Proposition 3.3(b). On the open set Y_u, we set $u = 1$ and the blowing-up equations reduce to

$$y = xv \tag{3.5.1}$$

$$z = xw \tag{3.5.2}$$

Eliminating y and z in the defining equations we see from (3.1.11)

$$x^2w^2 = x^n f(1, \frac{y}{x})(x-1). \tag{3.5.3}$$

The equation for the reduced exceptional curve is $x = 0$. Dividing by x^2 in the latter equation, we see that the affine surface Y_u is defined in (3.1.13) by the single equation

$$w^2 = x^{n-2} f(1, \frac{y}{x})(x-1) \tag{3.5.4}$$

Notice that Y_u is not normal. The singular locus contains the exceptional line defined by $x = 0$, $w = 0$. The affine coordinate ring of this surface is not integrally closed. If we adjoin $t = \frac{w}{x^{\frac{n-2}{2}}}$, then from (3.1.13)

$$\frac{w^2}{x^{n-2}} = f(1, \frac{y}{x})(x-1)$$

or from (3.1.15)

$$t^2 = f(1, v)(x-1) \tag{3.5.5}$$

Denote by $\tilde{Y}_u$ the surface defined by (3.5.5). Let R_u be the affine coordinate ring for $\tilde{Y}_u$ i.e.

$$R_u = \frac{k[x, v, t]}{(t^2 - f(1, v)(x-1))} \tag{3.5.6}$$

We have the finite morphism $\tilde{Y}_u \to Y_u$. Let E_u denote the reduced exceptional curve on $\tilde{Y}_u$. From (3.5.5) we see that E_u is the nonsingular affine curve defined by

$$t^2 = -f(1, v) \tag{3.5.7}$$

Next, consider the open set Y_v. Set $v = 1$ and blowing-up equations reduce to $x = yu$ and $z = yw$. The counterpart of (3.5.3) on this open set is

$$y^2 w^n = y^n f(\frac{y}{x}, 1)(yu - 1). \tag{3.5.8}$$

Dividing by y^n we have

$$\frac{w^2}{y^{n-2}} = f(\frac{y}{x}, 1)(yu - 1) \tag{3.5.9}$$

If we set $s = \frac{w}{y^{\frac{n}{2}}}$ and normalize as above we obtain

$$s^2 = f(u, 1)(yu - 1) \tag{3.5.10}$$

which defines the normalization $\tilde{Y}_v \to Y_v$. Denote by R_v the affine ring of $\tilde{Y}_v$ i.e.

$$R_v = \frac{k[y,u,s]}{(s^2 - f(u,1)(yu-1))} \tag{3.5.11}$$

The reduced exceptional curve on $\tilde{Y}_v$ is denoted E_v. Then E_v is defined by

$$s^2 = -f(u,1) \tag{3.5.12}$$

hence is nonsingular. As in [12, Exercise II.3.8] the normalization of Y, denoted by $\tilde{Y}$, is obtained by gluing together three morphisms $\tilde{Y}_u \to Y_u$, $\tilde{Y}_v \to Y_v$, and $\tilde{Y}_w \to Y_w$. The composition of $\tilde{Y} \longrightarrow Y$ with $Y \longrightarrow X$ gives a resolution of the singularity P_0, which is written $\pi : \tilde{Y} \longrightarrow X$. Let E be the reduced exceptional fiber of π. We have seen that E has an open cover consisting of the two nonsingular affine curves E_u and E_v. From Riemann-Hurwitz formula we have that genus of E is equal to $\frac{n-2}{2}$ which is at least one. It follows from equations (3.5.7) and (3.5.12) that E is nonrational. □

Proposition 3.17. *(a)* $\mathrm{Cl}(R_u) \cong \left(\frac{\mathbb{Z}}{2}\right)^n$

(b) $\mathrm{Cl}(R_v) \cong \left(\frac{\mathbb{Z}}{2}\right)^n \oplus \mathbb{Z}$

Proof. (a) Look at the homomorphism

$$\chi : \frac{k[x,v,t]}{(t^2 - f(1,v)(x-1))}\left[\frac{1}{f(1,v)}\right] \to k[v,t]\left[\frac{1}{f(1,v)}\right] \tag{3.5.13}$$

defined by

$$x \longmapsto 1 + \frac{t^2}{f(1,v)} \tag{3.5.14}$$

$$v \longmapsto v \tag{3.5.15}$$

$$t \longmapsto t \tag{3.5.16}$$

That χ is well defined follows from

$$(t^2 - f(1,v)(x-1)) \longmapsto t^2 - f(1,v)(1 + \frac{t^2}{f(1,v)} - 1) \tag{3.5.17}$$

and

$$t^2 - f(1,v)(1 + \frac{t^2}{f(1,v)} - 1) = 0 \tag{3.5.18}$$

χ is trivially onto. Using the same dimension argument we used in Proposition 3.3 we can prove that χ is one-to-one. Therefore, it is an isomorphism. We will use Nagata's sequence (Proposition 2.11) to compute the class group of R_u. We will compute the image of the group of units of the ring $\frac{k[x,v,t]}{(t^2-f(1,v)(x-1))}\left[\frac{1}{f(1,v)}\right]$ under the map Div. From (3.5.13) the group of units for this ring is equal to $\langle f_1 \rangle \times ... \times \langle f_n \rangle$. If $f_1(1,v) \in P_{u1}$, where P_{u1} is height-one prime ideal in $\frac{k[x,v,t]}{(t^2-f(1,v)(x-1))}\left[\frac{1}{f(1,v)}\right]$, then $P_1 = (f_1, t)$. After localizing at P_1 we have that t is a local parameter, hence $\mathrm{Div}(f_1) = 2P_{u1}$. We use the same argument to compute $\mathrm{Div}(f_2), \ldots, \mathrm{Div}(f_n)$, and get

$$\mathrm{Div}(f_2) = 2P_{u2},\ P_{u2} = (f_2, t)$$

$$\vdots$$

$$\mathrm{Div}(f_n) = 2P_{un},\ P_{un} = (f_n, t). \tag{3.5.19}$$

From Nagata's sequence it follows that $\mathrm{Cl}(R_u) \cong \left(\frac{\mathbb{Z}}{2}\right)^n$, and is generated by $P_{u1}, ..., P_{un}$.

(b) Look at homomorphism

$$\eta : \frac{k[y,u,s]}{(s^2 - f(u,1)(yu-1))}\left[\frac{1}{f(u,1)u}\right] \to k[u,s]\left[\frac{1}{f(u,1)u}\right] \tag{3.5.20}$$

defined by

$$y \longmapsto \frac{1}{u}(1 + \frac{s^2}{f(u,1)}) \tag{3.5.21}$$

$$u \longmapsto u \tag{3.5.22}$$

$$s \longmapsto s \tag{3.5.23}$$

To prove that η is well defined we need to compute the image of $s^2 - f(u,1)(yu-1)$.

$$(s^2 - f(u,1)(yu-1)) \longmapsto s^2 - f(u,1)(\frac{1}{u}(1 + \frac{s^2}{f(u,1)})u - 1) \tag{3.5.24}$$

and

$$s^2 - f(u,1)(\frac{1}{u}(1+\frac{s^2}{f(u,1)})u - 1) = 0 \tag{3.5.25}$$

We use proof similar to the proof in (a) to show that η is an isomorphism. For computing divisors for f_i we use the same argument we have used in the proof of (a).

$$\mathrm{Div}(f_1) = 2P_{v1},\ P_{v1} = (f_1, s)$$

$$\vdots$$

$$\mathrm{Div}(f_n) = 2P_{vn},\ P_{vn} = (f_n, s). \tag{3.5.26}$$

To compute the divisor for u we look at all prime ideals in $\frac{k[y,u,s]}{(s^2-f(u,1)(yu-1))}\left[\frac{1}{f(u,1)u}\right]$ containing u. Let $u \in Q_v$. Then $s^2 + a$ has to be in Q_v, where $a = a_1 \cdots a_n$. Since $s^2 + a$ factors into $s - b$ and $s + b$, we get two prime ideals $Q_{v1} = (u, s - b)$ and $Q_{v2} = (u, s + b)$. Look at the ring $\frac{k[y,u,s]}{(s^2-f(u,1)(yu-1))}\left[\frac{1}{f(u,1)u}\right]$ localized at Q_{v1}. We will prove that u is a local parameter for this ring. Start with $s^2 = f(u,1)(yu - 1)$. This is equivalent to

$$s^2 + a = u(f(u,1)y + g(u,1) \tag{3.5.27}$$

From this

$$s - b = u(f(u,1)y + g(u,1))(s+b)^{-1} \tag{3.5.28}$$

Therefore

$$\mathrm{Div}(u) = Q_{v1} \oplus Q_{v2} \tag{3.5.29}$$

□

Proposition 3.18. *(a)* $\mathrm{Pic}(R_u) = 0$.

(b) $\mathrm{Pic}(R_v) \cong \mathbb{Z}$.

Proof. (a) There is an exact sequence [6]

$$0 \longrightarrow \mathrm{Pic}(R_u) \longrightarrow \mathrm{Cl}(R_u) \longrightarrow \oplus\, \mathrm{Cl}(R_u)_{m_{P_i'}} \tag{3.5.30}$$

Since $P_{ui} \subseteq m_{P'_i}$, all generators for $\mathrm{Cl}(R_u)$ are mapped into nonzero elements in $\oplus\,\mathrm{Cl}(R_u)_{m_{P'_i}}$, therefore $(\mathrm{Pic}\,R_u) = 0$.

(b) The counterpart of (3.5.30) for R_v is

$$0 \longrightarrow \mathrm{Pic}(R_v) \longrightarrow \mathrm{Cl}(R_v) \longrightarrow \oplus\,\mathrm{Cl}(R_v)_{m_{P''_i}} \tag{3.5.31}$$

It is true that $P_{vi} \subseteq m_{P''_i}$. Since u is not an element of $m_{P''_i}$, for all i we have that Q_{v1} is an element of $\mathrm{Pic}(R_v)$ and $\mathrm{Pic}(R_v) \cong \mathbb{Z}$.

□

Lemma 3.5. $R_u\left[\frac{1}{v}\right] = R_v\left[\frac{1}{u}\right]$.

Proof. Consider the homomorphism

$$\frac{k[x,v,t]}{(t^2 - f(1,v)(x-1))}\left[\frac{1}{v}\right] \xrightarrow{\theta} \frac{k[y,u,s]}{(s^2 - f(u,1)(yu-1))}\left[\frac{1}{u}\right] \tag{3.5.32}$$

defined by

$$x \longmapsto yu \tag{3.5.33}$$

$$v \longmapsto \frac{1}{u} \tag{3.5.34}$$

$$t \longmapsto \frac{s}{u^{\frac{n}{2}}} \tag{3.5.35}$$

It is well defined because $t^2 - f(1,v)(x-1)$ maps to $\frac{s^2}{u^n} - \frac{1}{u^n}f(u,1)(yu-1)$. It is easy to see that θ is onto. Since θ is onto and $R_u\left[\frac{1}{v}\right]$ and $R_v\left[\frac{1}{u}\right]$ are integral domains with same dimension, θ is one-to-one. Therefore $R_u\left[\frac{1}{v}\right] \cong R_v\left[\frac{1}{u}\right]$. Now, from (3.5.2) follows that $w = \frac{z}{x}$. Combining this with $t = \frac{w}{x^{\frac{n}{2}}}$ we get $t = \frac{z}{x^{\frac{n+2}{2}}}$. This proves that ring $R_u\left[\frac{1}{v}\right]$ sits inside L, the quotient field of T. Similarly, from $u = \frac{x}{y}$ (3.5.1) and $s = \frac{z}{xy^{\frac{n}{2}}}$, it follows that $R_v\left[\frac{1}{u}\right]$ is in L too. The isomorphism θ fixes the field L element-wise, therefore the rings must be equal. □

Let $\tilde{Y}_u$ be the surface defined by (3.5.5), whose coordinate ring is R_u, and let R_v be the affine ring for $\tilde{Y}_v$.

Proposition 3.19. *Let $\tilde{Y}_{uv} = \tilde{Y}_u \cap \tilde{Y}_v$. Then $A(\tilde{Y}_{uv}) = R_u\left[\frac{1}{v}\right]$.*

Proof. From [12, Theorem 1.2] ring $A(\tilde{Y}_{uv})$ contains both R_u and R_v. Therefore, both u and v are in $A(\tilde{Y}_{uv})$. From Lemma 3.5 we have that $u = \frac{1}{v}$, so $R_u\left[\frac{1}{v}\right]$ is subring of $A(\tilde{Y}_{uv})$. From [12, Theorem 3.2] $A(\tilde{Y}_{uv})$ is the smallest ring containing R_u and R_v, so $A(\tilde{Y}_{uv}) = R_u\left[\frac{1}{v}\right]$. □

Proposition 3.20. *The singular locus for $\tilde{Y}_u$ is $\left\{P'_1 = (1, -\frac{1}{a_1}, 0), \ldots,\ P'_n = (1, -\frac{1}{a_n}, 0)\right\}$ and for $\tilde{Y}_v$ it is the set $\left\{P''_1 = (-\frac{1}{a_1}, -a_1, 0), \ldots,\ P''_n = (-\frac{1}{a_n}, -a_n, 0)\right\}$.*

Proof. The Jacobian matrix for $t^2 - f(1, v)(x - 1)$ is

$$J = \begin{pmatrix} f(1, v) \\ f_v(1, v)(x - 1) \\ 2t \end{pmatrix}$$

Then $J = 0$, when $t = 0$ and $(1 + a_1 v) \cdots (1 + a_n v) = 0$. After solving this equation we have that the set of singular points on $\tilde{Y}_u$ is equal to $(1, -\frac{1}{a_1}, 0), \ldots, (1, -\frac{1}{a_n}, 0)$. From the Lemma 3.5 we have that $u = \frac{1}{v}$ and $y = \frac{x}{u}$. Therefore, points corresponding to $(1, -\frac{1}{a_1}, 0), \ldots, (1, -\frac{1}{a_n}, 0)$ in the ring $R_v\left[\frac{1}{u}\right]$ are $\left\{(-\frac{1}{a_1}, -a_1, 0), \ldots, (-\frac{1}{a_n}, -a_n, 0)\right\}$. Since the singular locus on $\tilde{Y}_u$ is same as the singular locus on $\tilde{Y}_{uv}$ and the singular locus on $\tilde{Y}_v$ is equal to the singular locus on $\tilde{Y}_{uv}$ we have that set of singular points on $\tilde{Y}_v$ is equal to $\left\{(-\frac{1}{a_1}, -a_1, 0), \ldots, (-\frac{1}{a_n}, -a_n, 0)\right\}$. □

Since the two sets of singular points from Proposition 3.20 are both equal to the set of rational double points on X we will use their original notation $P_1, \ldots, P_n$ defined in (3.1.7) for future reference.

Proposition 3.21. *(a)* $\mathrm{Cl}(R_u\left[\frac{1}{v}\right]) \cong \left(\frac{\mathbb{Z}}{2}\right)^n$.

(b) $\mathrm{Pic}(R_u\left[\frac{1}{v}\right]) = 0$.

(c) $R_u\left[\frac{1}{v}\right]^* = k^* \times (v)$.

Proof. (a) We will find divisors for v first. Let Q_u be the prime ideal in R_u generated by v. Then it is easy to check that $t^2-(x-1)=0$ in $\frac{R_u}{(v)}$. Therefore $Q=(v,t^2-(x-1))$ is the only prime ideal in $R-u$ containing v. Take a look at the ring $\frac{R_u}{(v)}$. Its equation is $\frac{k[x,v,t]}{(t^2-f(1,v)(x-1))(v)}$ which is isomorphic to $\frac{k[x,t]}{(t^2-(x-1))}$. Since $t^2-(x-1)$ is irreducible, $\frac{k[x,t]}{(t^2-(x-1))}$ is an integral domain, so is $\frac{R_u}{(v)}$. The ideal (v) is a prime and thus is equal to Q. So $Q=(v)$ is a principal ideal and

$$\mathrm{Div}(v)=Q. \tag{3.5.36}$$

Consider Nagata's sequence for R_u (Propostion 2.11)

$$1 \longrightarrow R_u^* \longrightarrow R_u\left[\frac{1}{v}\right]^* \xrightarrow{\mathrm{Div}} \mathbb{Z}Q \longrightarrow \mathrm{Cl}(R_u) \longrightarrow \mathrm{Cl}\left(R_u\left[\frac{1}{v}\right]\right) \longrightarrow 0 \tag{3.5.37}$$

From (3.5.36) we have that $\mathrm{Cl}\left(R_u\left[\frac{1}{v}\right]\right) \cong \mathrm{Cl}(R_u)$, and therefore $\mathrm{Cl}(R_u\left[\frac{1}{v}\right]) \cong \left(\frac{\mathbb{Z}}{2}\right)^n$.

From Proposition 3.18 we have that $\mathrm{Pic}(R_v)$ is generated by the prime ideal $Q_{v1}=(u,s-b)$. After inverting u in $R_v\left[\frac{1}{u}\right]$, this ideal vanishes so $\mathrm{Pic}(R_v\left[\frac{1}{u}\right])=0$.

(b) Consider the homomorphism

$$\frac{k[x,v,t]}{(t^2-f(1,v)(x-1))}\left[\frac{1}{f(1,v)v}\right] \xrightarrow{\rho} k[v,t]\left[\frac{1}{f(1,v)v}\right] \tag{3.5.38}$$

defined by

$$x \longmapsto \frac{f(1,v)+t^2}{f(1,v)} \tag{3.5.39}$$

$$v \longmapsto v \tag{3.5.40}$$

$$t \longmapsto t \tag{3.5.41}$$

It is well defined since $t^2-f(1,v)(x-1) \longmapsto 0$ and it is trivially onto. Both rings are integral domains of Krull dimension two, hence ρ is one-to-one. The ring on the right hand side is a factorial ring and from Corollary 3.1 we have $k[v,t]\left[\frac{1}{f(1,v)v}\right]^* = k^* \times \langle v \rangle \times$

$\langle f_1 \rangle \times \ . \ . \ . \ \times \langle f_n \rangle$. We use ρ to compute the group of units $\frac{k[x,v,t]}{(t^2 - f(1,v)(x-1))}\left[\frac{1}{f(1,v)v}\right]^*$.

$$R_u \left[\frac{1}{vf(1,v)}\right]^* = k^* \times \langle v \rangle \times \langle f_1 \rangle \times \cdots \times \langle f_n \rangle . \tag{3.5.42}$$

Consider Nagata's sequence for $R_u \left[\frac{1}{v}\right]$ (Propostion 2.11)

$$1 \to R_u \left[\frac{1}{v}\right]^* \longrightarrow R_u \left[\frac{1}{vf(1,v)}\right]^* \xrightarrow{\text{Div}} \oplus P_{ui} \oplus \mathbb{Z}Q \longrightarrow \\ \mathrm{Cl}\left(R_u \left[\frac{1}{v}\right]\right) \longrightarrow \mathrm{Cl}\left(R_u \left[\frac{1}{vf(1,v)}\right]\right) \longrightarrow 1 \tag{3.5.43}$$

The homomorphism Div is defined by the $n+1 \times n+1$ matrix with rank n

$$\mathrm{Div} = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & \\ 0 & 2 & \cdots & 0 & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2 & 0 & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \end{pmatrix}$$

Therefore $R_u \left[\frac{1}{v}\right]^* = k^* \times (v)$.

□

Lemma 3.6. *(a)* $R_u^* = k^*$.

(b) $R_v^* = k^*$.

Proof. (a) Follows from (3.5.36), (3.5.37) and Proposition 3.21.

(b) From (3.5.29) we have that $\mathrm{Div}(u) = Q_{v1} \oplus Q_{v2}$ in R_v. Consider Nagata's sequence for R_v

$$1 \longrightarrow R_v^* \longrightarrow R_v \left[\frac{1}{u}\right]^* \xrightarrow{\text{Div}} \mathbb{Z}Q_{v1} \oplus \mathbb{Z}Q_{v2} \longrightarrow \\ \mathrm{Cl}(R_v) \longrightarrow \mathrm{Cl}\left(R_v \left[\frac{1}{u}\right]\right) \longrightarrow 1 \tag{3.5.44}$$

The homomorphism Div from (3.5.44) can be represented by a 2×2 matrix.

$$\text{Div} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is clear now that $R_v^* = k^*$.

□

From Proposition 3.20, the singular locus for Y is equal to $P_1, \ldots, P_n$. Denote by $\tilde{R}$ the ring of regular functions on $\tilde{Y}$ (Definition 2.24) i.e. $\tilde{R} = \mathcal{O}(\tilde{Y})$.

Proposition 3.22. *(a)* $\text{Pic}(\tilde{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$

(b) $\text{Cl}(\tilde{R}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$

Proof. (a) Consider the Meyer-Vietoris sequence [17, Exercise III.2.24]

$$(\tilde{Y})^* \longrightarrow R_u^* \oplus R_v^* \longrightarrow (R_u\left[\frac{1}{v}\right])^* \longrightarrow$$
$$\text{Pic}(\tilde{R}) \longrightarrow \text{Pic}(R_u) \oplus \text{Pic}(R_v) \longrightarrow \text{Pic}(R_u\left[\frac{1}{v}\right]) \longrightarrow 1 \quad (3.5.45)$$

Applying results obtained in this section, the above exact sequence becomes

$$(\tilde{Y})^* \longrightarrow (1) \longrightarrow \mathbb{Z} \longrightarrow \text{Pic}(\tilde{R}) \longrightarrow \mathbb{Z} \longrightarrow (1) \longrightarrow (1) \quad (3.5.46)$$

Since it splits, we have that $\text{Pic}(\tilde{Y}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(b) Let $\tilde{Y}' = \tilde{Y} - \{P_1, \ldots, P_n\}$ and $\tilde{R}' = \mathcal{O}(\tilde{Y}')$. Since $\tilde{Y}'$ is nonsingular we have that $\text{Pic}(\tilde{R}') = \text{Cl}(\tilde{R}')$ (Proposition 2.10). It is also true that $(\tilde{Y}')^* = \tilde{Y}^*$. If we tie this up with the fact that the group of divisors doesn't change when a finite number of points are removed from the variety, the Meyer-Vietoris sequence for $\tilde{Y}$ can be also written as

$$(\tilde{Y})^* \longrightarrow R_u^* \oplus R_v^* \longrightarrow (R_{uv})^* \longrightarrow$$
$$\text{Cl}(\tilde{R}) \longrightarrow \text{Cl}(R_u) \oplus \text{Cl}(R_v) \longrightarrow \text{Cl}(R_u\left[\frac{1}{v}\right]) \longrightarrow (1) \quad (3.5.47)$$

With results obtained earlier in this section, the exact sequence above is equivalent to the sequence

$$(\tilde{Y})^* \longrightarrow (1) \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(\tilde{R}) \longrightarrow (\mathbb{Z}/2)^n \oplus (\mathbb{Z}/2)^n \oplus \mathbb{Z} \longrightarrow (\mathbb{Z}/2)^n \longrightarrow (1) \quad (3.5.48)$$

This exact sequence above splits and $\mathrm{Cl}(\tilde{R}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$.

□

CHAPTER 4

FUTURE RESEARCH

Open questions for future research: $\mathrm{Pic}(T) = ?$ $\mathrm{B}(T) = ?$

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