DERIVATION OF PLANAR DIFFEOMORPHISMS FROM HAMILTONIANS WITH A KICK

by

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ABSTRACT

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In this thesis we will discuss connections between Hamiltonian systems with a periodic kick and planar diffeomorphisms. After a brief overview of Hamiltonian theory we will focus, as an example, on derivations of the Hénon map that can be obtained by considering kicked Hamiltonian systems. We will conclude with examples of Hénon maps of interest.

DEDICATION

This is dedicated to my partner Jeff
for his unquestioning confidence in me,
and to my mother for a lifetime of devotion
and for her unending love.

DERIVATION OF PLANAR DIFFEOMORPHISMS

FROM HAMILTONIANS WITH A KICK

CHAPTER 1

INTRODUCTION

In dynamical systems, the motion of a point on a given manifold, referred to as the phase space of the system, is often determined by an ordinary differential equation, or ODE, most of the time involving parameters. This is equivalent to considering a family of vector fields [4]. Since solving a given ODE or finding the flow associated with the corresponding vector field is, in general, difficult (if not impossible) the problem becomes one of how to describe the dynamics of the system. To overcome this difficulty we often use discrete maps to describe parts of the behavior of the system.

To begin with, let us recall some basic definitions and properties. A flow associated with a vector field X defined on a manifold M is a mapping

$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi_t(x) = \phi(x,t)$$

that satisfies

- $\phi(x,0) \equiv x$, i.e., $\phi_0 \equiv \mathrm{Id}$,
- $\phi(\phi(x,t),s) = \phi(\phi(x,s),t) = \phi(x,t+s)$,
- $\dfrac{d\phi}{dt}(x,0) = X(x)$.

The flow is uniquely defined and we often consider the so-called time-T map, where $T > 0$, i.e., ϕ_T. Up to a time rescaling, we can set $T = 1$. However as we mentioned above, such a solution is, in general, very difficult to find explicitly. Traditionally other discretizations are employed.

An example of a discretization that leads to a reduction in dimension of the phase space is the so-called Poincaré return onto a transversal co-dimensional one section [7]. We no longer have a flow, but a diffeomorphism from the section to itself. Periodic orbits of the Poincaré return map correspond to periodic orbits of the flow, invariant circles correspond to invariant tori of the flow, etc. Studying the dynamics of the Poincaré return map and the dynamics of the original system amounts to the same thing. However, such a reduction is not always possible. For example, a reduction is not possible when the original system admits singularities, thereby preventing the Poincaré return map from being globally defined.

A Hamiltonian system is another example of a dynamic system that can be studied through a reduction in dimension. These reductions are used to describe many fundamental physical and mechanical models. In the physical interpretation, the Hamiltonian gives the total energy of a system in terms of the position and momentum of a particle. An isolated system is conservative and the Hamiltonian H remains constant. Rather than considering a Hamiltonian of this kind in terms of all possible energy levels, it can be reduced to an examination of energy surfaces [4]. For $c \in \mathbb{R}$ and the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the corresponding energy surface with energy c is defined to be the set

$$H^{-1}\{c\} = \{(q, p) \in \mathbb{R} \times \mathbb{R} : H(q, p) = c\}. \tag{1.1}$$

An interesting case of the Hamiltonian is the non-conservative case that involves a "kick," an additional term that involves a periodic impulse function. The periodicity of these Hamiltonians can provide a convenient way of deriving a discrete map that can be used to study the respective dynamics described by the original continuous function.

In 1976, Michel Hénon introduced what is now called the Hénon map, which provided a simplified model of the Poincaré section of the Lorenz model. This map was considered strictly as a mathematical transformation of the plane, in principle corresponding to the dynamics of some physical system. In 1990, J.F. Heagy [5] showed that the Hénon map could be directly derived from a kicked Hamiltonian, thereby directly addressing the dynamics of a Hamiltonian system and identical to the period one return of a periodically kicked harmonic oscillator nonlinerarly coupled to the kicking term. This enables the study of the parameters of the kicked harmonic oscillator, e.g., driving amplitude, driving period and damping constant, in the discrete form provided by the Hénon map.

In this paper we will be concerned only with a 2-dimensional symplectic manifold (M, ω), $M = \mathbb{R}^2$, with coordinates (q, p) where the symplectic form ω is expressed as

$$\omega = dq \wedge dp. \tag{1.2}$$

After a brief review the basics of Hamiltonian theory [2][4], we will focus on the example of the derivation of the Hénon map in the area-preserving and dissipative cases from Hamiltonian equations with a "kick" term that can be used to describe physical systems, followed by a derivation of the so-called generalized Hénon map from a kicked Hamiltonian. We will then conclude with plots of Hénon maps of interest.

CHAPTER 2

HAMILTONIAN THEORY

For the systems we will look at in this paper, it is sufficient to examine a 2-dimensional symplectic manifold $(\mathbb{R}, \omega)$ with coordinates (q, p) where the syplectic form ω is expressed as

$$\omega = dq \wedge dp. \tag{2.1}$$

We have an isomorphism $I_\omega : TM \to T^*M$, from the tangent bundle TM and the cotangent bundle T^*M with the inverse $I_\omega^{-1} : T^*M \to TM$ where I_ω^{-1} is often represented by the skew-symmetric matrix

$$I_\omega^{-1} = \begin{pmatrix} 0 & 1 \\ \text{-}1 & 0 \end{pmatrix}. \tag{2.2}$$

Every differentiable function $H : \mathbb{R}^2 \to \mathbb{R}$ determines a unique vector field X_H, called the Hamiltonian vector field with the Hamiltonian H, [2] by requiring that for every vector field Y on $\mathbb{R}^2$, the identity

$$dH(Y) = I_\omega(X_H, Y) \tag{2.3}$$

must hold [1]. For coordinates (q, p) on $\mathbb{R}^2$ the Hamiltonian vector field with Hamiltonian H is of the form

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} = I_\omega^{-1}(dH). \tag{2.4}$$

A planar system of differential equations that can be written in the form of Hamilton's equations,

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{2.5}$$

where $H(p, q, t)$ is the Hamiltonian function is called a Hamiltonian system.

In the context of classical mechanics the Hamiltonian equations can also be derived from Newton's equations of motion. Newton's equations of motion can be written in terms of q as

$$F = m\ddot{q}. \tag{2.6}$$

For a conservative sytem, we can also express force as the partial derivative of the potential function with respect to q, $F = -\dfrac{\partial V}{\partial q}$, so we have

$$m\ddot{q} = -\frac{\partial V}{\partial q}. \tag{2.7}$$

Defining the Hamiltonian function as the total energy of a particle,

$$H(q, p, t) = T(p) + V(q, t) \tag{2.8}$$

where

$$T = \frac{m\dot{q}^2}{2} = \frac{m\dot{q}\dot{q}}{2} = \frac{p\dot{q}}{2} = \frac{p^2}{2m}. \tag{2.9}$$

Since $H = \dfrac{p^2}{2m} + V$ we get

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \frac{m\dot{q}}{m} = \dot{q} \tag{2.10}$$

and

$$\frac{\partial H}{\partial q} = \frac{\partial V}{\partial q} = -m\ddot{q} = -\frac{\partial}{\partial t}(m\dot{q}) = -\dot{p} \tag{2.11}$$

and we obtain the Hamiltonian equations.

Here we have

$$H(q, p, t) = T(p) + V(q, p, t). \tag{2.12}$$

Recall that $\dot{H} = 0$ and the Hamiltonian represents a conservative, closed system. This means that the dynamics of a two-dimensional Hamiltonian system is understood thanks to the study of the energy surfaces $H^{-1}\{c\}$. Such systems are simple in the sense that no chaotic behavior can be observed. [4] However, if we modify the setting of our study, we can increase the complexity by adding kick terms. When a kick is introduced the results are no longer conservative and have unexpected effects on the behavior of the system and the total energy. A kicked term is included by utilizing the Dirac distribution, or Dirac measure, introduced by physicist Paul Dirac, which has the property

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{2.13}$$

for all continuous functions f. Our goal is to retrieve some classical maps such as that of the (generalized) Hénon map [3][8] through this approach. Before introducing this method we recall some basic facts about the Hénon map.

CHAPTER 3
DERIVATION OF THE HÉNON MAP FROM A HAMILTONIAN WITH A KICK

3.1 THE HÉNON MAP REVISITED

In 1976 Michel Hénon introduced what is now called the Hénon map of the form

$$h(q,p) = (1 + p - aq^2, bq) \tag{3.1}$$

where a and b are real parameters. This map was proposed by Hénon to mimic the dynamics of the Lorenz model. For an initial point of the plane the Hénon map will either produce an area-preserving map that illustrates behavior found in typical low dimensional nonintegrable Hamiltonian systems or when the parameters are near certain values, for instance $a = 1.4$ and $b = 0.3$, a dissipative map which contains a strange attractor known as a Hénon attractor. [5] A strange attractor can be defined in the following way [10]:

A subset A of the phase space is an attractor if

- A is invariant under the map.
- A admits an open basin B such that for $q, p \in B$, $\lim_{n\to\infty} (d(H^n(q,p), A)) = 0$.
- A is not hyperbolic [2][4][8][9].
- A has a positive Lyapunov exponent.

The Lyapunov exponent [4] at any point $\xi \in A$ in the direction of the nonzero vector $v \in \mathbb{R}^2$ for the flow ϕ_t is defined to be

$$\chi(\xi, v) := \limsup_{t \to \infty} \frac{1}{t} \ln \left(\frac{|D\phi_t(\xi)v|}{|v|} \right). \tag{3.2}$$

A positive Lyapunov exponent ensures expansion within the open basin thereby creating a chaotic structure. In the case of the Hénon attractor one can find a Lyapunov exponent arbitrarily close to log(2). The Hénon attractor exhibits a highly complex geometric structure that is suggested by mapping sufficiently large iterations by computer, as in the figure below.

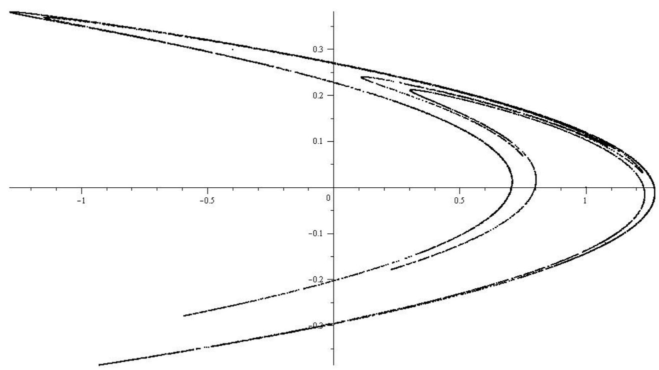

Figure 3.1: Hénon attractor

Although a finite number of iterations produced by computer could be simply interpreted as a periodic orbit of very high period, it was shown by Benedicks and Carleson [3] in 1991 that for $b > 0$ small enough, for values near $a = 2$ the diffeomorphism truly exhibits a strange attractor. For almost all starting points the forward orbit is dense on the unstable manifold [3] and when viewed at finer and finer magnifications, it can be seen that the map exhibits a complicated structure. This result has been generalized for diffeomorphisms near homoclinic tangencies by Mora and Viana [8].

It has later been shown that the structure of that attractor locally resembles that of $[0,1] \times K$, where K is a Cantor set.

Hénon developed the map through the following transformational maps of an initial elongated area of the plane (a) [6]:

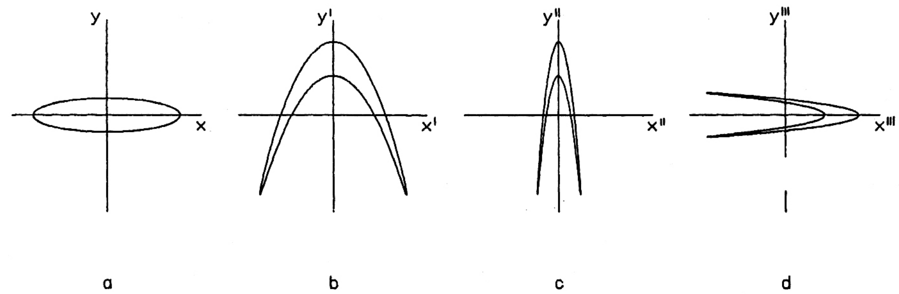

Figure 3.2: Hénon map transformations

1. A vertical stretch (b) of the area is produced by the map

$$T' : x' = x, \quad y' = y + 1 - ax^2, \tag{3.3}$$

   where a is an adjustable parameter.

2. A horizontal contraction (c) of the area is produced by the map

$$T'' : x'' = bx', \quad y'' = y', \tag{3.4}$$

   where b is another adjustable parameter.

3. A reflection over the line $y = x$ (d) by the map

$$T''' : x''' = y'', \quad y''' = x''. \tag{3.5}$$

The map T defined as $T = T'''T''T'$, writing (x_n, y_n) for (x, y) and (x_{n+1}, y_{n+1}) for (x''', y''') gives the iterated map

$$T : x_{n+1} = y_n + 1 - ax_n^2, \quad y_{n+1} = bx_n. \tag{3.6}$$

The Jacobian of the Hénon map is constant:

$$\frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} = -b \tag{3.7}$$

and the inverse mapping is

$$T_{-1} : x_i = b^{-1}y_{n+1}, \quad y_n = x_{n+1} - 1 + ab^{-2}y_{n+1}^2, \tag{3.8}$$

and so we have a diffeomorphic map. The "standard form" of the Hénon map is

$$X_{n+1} = 1 + Y_n - aX_n^2, \quad Y_{n+1} = -bX_n \tag{3.9}$$

where a is the nonlinearity parameter and b is the Jacobian determinant of the map. [5]

3.2 DERIVATION OF THE HENON MAP

The area-preserving map can be derived from the following Hamiltonian:

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + F(q)\sum_{n\in\mathbb{Z}}\delta(t - \frac{\pi}{2}n) \tag{3.10}$$

where $F(q)$ is a differentiable function such that $\dfrac{dF}{dq} = f(q)$. From this we obtain the Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p} = p \text{ and } \dot{p} = -\frac{\partial H}{\partial q} = -q - f(q)\sum_{n\in\mathbb{Z}}\delta(t - \frac{\pi}{2}n). \tag{3.11}$$

In order to derive a map from H, we evaluate H from the n to $n+1$ kick by first integrating over the small interval over the n-th kick and then evaluating the evolution of H over the rest of the period. We let (q_n, p_n) be a point just before the n kick, (q_n^*, p_n^*) be a point just after the n kick, and (q_{n+1}, p_{n+1}) be a point just before the $n+1$ kick. Integrating the Hamiltonian equations $dq/dt = p$ on the interval $(\frac{\pi}{2}n - \epsilon, \frac{\pi}{2}n + \epsilon)$ we get

$$\int_{q_n}^{q_n^*} dq = \int_{\frac{\pi}{2}n-\epsilon}^{\frac{\pi}{2}n+\epsilon} pdt \tag{3.12}$$

which gives us

$$q_n^* - q_n = \int_{\frac{\pi}{2}n-\epsilon}^{\frac{\pi}{2}n+\epsilon} p dt \underset{\epsilon\to 0}{=} 0. \tag{3.13}$$

Since p is a bounded function of time, the integral on the right side goes to zero as ϵ goes to zero and we get

$$q_n^* = q_n. \tag{3.14}$$

Then we integrate the equation

$$dp/dt = -q - f(q)\sum_{n\in\mathbb{Z}} \delta(t - \frac{\pi}{2}n). \tag{3.15}$$

By taking the limit as epsilon tends to zero, the integral of $f(q)\sum_{n\in\mathbb{Z}} \delta(t - \frac{\pi}{2}n)$ tends to $f(q_n)$ since the Delta distribution charges only points, however the contribution from the continuous part is zero as in the case of the p term in the integral (3.13). We then get

$$\int_{p_n}^{p_n^*} dp = \int_{\frac{\pi}{2}n-\epsilon}^{\frac{\pi}{2}n+\epsilon} -q - f(q)\sum_{n\in\mathbb{Z}} \delta(t - \frac{\pi}{2}n)dt \tag{3.16}$$

which gives

$$p_n^* - p_n = \int_{\frac{\pi}{2}n-\epsilon}^{\frac{\pi}{2}n+\epsilon} -q - f(q)\sum_{n\in\mathbb{Z}} \delta(t - \frac{\pi}{2}n)dt \underset{\epsilon\to 0}{=} -f(q_n) \tag{3.17}$$

therefore we have

$$p_n^* = p_n - f(q_n). \tag{3.18}$$

For the interval $(\frac{\pi}{2}n + \epsilon, \frac{\pi}{2}(n+1) - \epsilon)$, since $\dot{q} = p$ and $\dot{p} = -q$ are linear, we obtain the solution

$$q(t) = C_1 \cos(t - \frac{\pi}{2}n) + C_2 \sin(t - \frac{\pi}{2}n) \tag{3.19}$$

by combining the two Hamiltonian equations into one second-order equation

$$\ddot{q} + q = 0 \tag{3.20}$$

and using the characteristic equation to find the roots. The solution for $p(t)$ is then obtained directly from $q(t)$:

$$p(t) = \frac{dq}{dt} = -C_1 \sin(t - \frac{\pi}{2}n) + C_2 \cos(t - \frac{\pi}{2}n). \tag{3.21}$$

To determine the integration constants $C_1,\ C_2$ we define initial conditions as

$$q(\frac{\pi}{2}n + \epsilon) \equiv q_n^* \tag{3.22a}$$

$$p(\frac{\pi}{2}n + \epsilon) \equiv p_n^* \tag{3.22b}$$

and direct calculation yields

$$C_1 = q_n \tag{3.23a}$$

$$C_2 = p_n - f(q_n). \tag{3.23b}$$

Then denoting

$$q_{n+1} \equiv q(\frac{\pi}{2}(n+1) - \epsilon) \tag{3.24a}$$

$$p_{n+1} \equiv p(\frac{\pi}{2}(n+1) - \epsilon) \tag{3.24b}$$

we obtain

$$\begin{aligned} q_{n+1} = q(\frac{\pi}{2}n(n+1) - \epsilon) = q_n \cos(\frac{\pi}{2}(n+1) - \epsilon - \frac{\pi}{2}n) + \\ (p_n - f(q_n)) \sin(\frac{\pi}{2}(n+1) - \epsilon - \frac{\pi}{2}n) \\ \underset{\epsilon \to 0}{=} q_n \cos(\frac{\pi}{2}) + (p_n - f(q_n)) \sin(\frac{\pi}{2}) = 0 + p_n - f(q_n) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} p_{n+1} = p(\frac{\pi}{2}(n+1) - \epsilon) = -q_n \sin(\frac{\pi}{2}(n+1) - \epsilon - \frac{\pi}{2}n) + \\ (p_n - f(q_n)) \cos(\frac{\pi}{2}(n+1) - \epsilon - \frac{\pi}{2}n) \\ \underset{\epsilon \to 0}{=} -q_n \sin(\frac{\pi}{2}) + (p_n - f(q_n)) \cos(\frac{\pi}{2}) = -q_n + 0. \end{aligned} \tag{3.26}$$

so we get the map

$$q_{n+1} = p_n - f(q_n) \tag{3.27a}$$

$$p_{n+1} = -q_n. \tag{3.27b}$$

If we choose $f(q) = aq^2 - 1$ where a is the non-linearity parameter and the Jacobian $b = 1$ we have

$$q_{n+1} = 1 + p_n - aq_n^2 \tag{3.28a}$$

$$p_{n+1} = -bq_n \tag{3.28b}$$

which gives us equivalence to the standard form of the Hénon map.

In order to treat the Hénon map in the dissipative case, we add a damping term γp to the equations of motion:

$$\dot{q} = p \text{ and } \dot{p} = -q - \gamma p - f(q) \sum_{n \in \mathbb{Z}} \delta(t - nT). \tag{3.29}$$

This can be derived from the following Hamiltonian (see Heagy for more details):

$$H = e^{\gamma t} \left(e^{-2\gamma t} \frac{1}{2} p^2 + \frac{1}{2} q^2 + F(q) \sum_{n \in \mathbb{Z}} \delta(t - nT) \right). \tag{3.30}$$

From this Hamiltonian we get the equations

$$\dot{q} = e^{-\gamma t} p \text{ and } \dot{p} = e^{\gamma t} \left(-q - f(q) \sum_{n \in \mathbb{Z}} \delta(t - nT) \right). \tag{3.31}$$

Making the substitution $\tilde{p} = e^{-\gamma t} p$, we have $\dot{\tilde{p}} = e^{-\gamma t} \dot{p} - \gamma e^{-\gamma t} p = e^{-\gamma t} \dot{p} - \gamma \tilde{p}$ and we get

$$\dot{q} = \tilde{p} \tag{3.32a}$$

$$\dot{\tilde{p}} = -q - \gamma \tilde{p} - f(q) \sum_{n \in \mathbb{Z}} \delta(t - nT). \tag{3.32b}$$

Eliminating the ˜ notation and integrating over the n-th kick as in the undamped case, we get

$$q_n^* = q_n \tag{3.33a}$$

$$p_n^* = p_n - f(q). \tag{3.33b}$$

For the solution between kicks, we again combine the Hamiltonian equations into one second-order linear equation

$$\ddot{q} + \gamma\dot{q} + q = 0. \tag{3.34}$$

From the characteristic equation we get

$$r^2 + \gamma r + 1 = 0 \Rightarrow r = \frac{-\gamma}{2} \pm \sqrt{1 - \frac{\gamma^2}{4}}i \tag{3.35}$$

For $0 < \gamma < 2$ by the same method as in the area-preserving case we obtain the map

$$q_{n+1} = e^{-\frac{\gamma T}{2}} \left(q_n \cos(\omega T) + \frac{1}{\omega}(p_n - f(q_n) + \frac{\gamma}{2} q_n) \sin(\omega T) \right) \tag{3.36a}$$

$$p_{n+1} = e^{-\frac{\gamma T}{2}} \left[-\omega q_n \sin(\omega T) + (p_n - f(q_n) + \frac{\gamma}{2} q_n) \cos(\omega T) \right] - \frac{\gamma}{2} q_{n+1} \tag{3.36b}$$

where $\omega = \sqrt{1 - \frac{1}{4}\gamma^2}$.

We choose the parameter T to be such that $\omega T = \dfrac{\pi}{2}$ and we get

$$q_{n+1} = \frac{e^{-\frac{\gamma T}{2}}}{\omega} \left(p_n - f(q_n) + \frac{\gamma}{2} q_n \right) \tag{3.37a}$$

$$p_{n+1} = -\omega e^{-\frac{\gamma T}{2}} q_n - \frac{\gamma}{2} q_{n+1}. \tag{3.37b}$$

Making the substitutions

$$\tilde{q}_n = \omega e^{-\frac{\gamma T}{2}} q_n \tag{3.38a}$$

$$\tilde{p}_n = p_n + \frac{\gamma}{2} q_n \tag{3.38b}$$

we then have

$$\tilde{q}_{n+1} = \omega e^{\frac{\gamma T}{2}} \cdot \frac{e^{-\frac{\gamma T}{2}}}{\omega} \left(p_n - f(q_n) + \frac{\gamma}{2} q_n \right) = p_n - f(q_n) + \frac{\gamma}{2} q_n \tag{3.39}$$

and then since $\tilde{p}_n = p_n + \frac{\gamma}{2}q_n$ implies that $p_n = \tilde{p}_n - \frac{\gamma}{2}q_n$, we get

$$\tilde{q}_{n+1} = \tilde{p}_n - f(q_n). \tag{3.40}$$

We also have

$$\tilde{p}_{n+1} = p_{n+1} + \frac{\gamma}{2}q_{n+1}. \tag{3.41}$$

Since $\tilde{q}_n = \omega e^{\frac{\gamma T}{2}} q_n$ implies that $q_n = \frac{1}{\omega} e^{-\frac{\gamma T}{2}} \tilde{q}_n$, we have

$$\tilde{p}_{n+1} = -\omega e^{-\frac{\gamma T}{2}} \left(\frac{1}{\omega} e^{-\frac{\gamma T}{2}} \tilde{q}_n \right) = -e^{-\gamma T} \tilde{q}_n. \tag{3.42}$$

Choosing $f(q_n) = aq^2 - 1$ and omitting the ˜ notation and we get

$$q_{n+1} = p_n + 1 - aq_n^2 \tag{3.43a}$$

$$p_{n+1} = -e^{-\gamma T} q_n \tag{3.43b}$$

where a is a non-linearity parameter and $b = e^{-\gamma T}$ is the Jacobian, and we have equivalence to the standard form of the Hénon map.

3.3 THE GENERALIZED HÉNON MAP

The generalized Hénon map is of the form

$$(q, p) \mapsto (c_0 + c_1 q + c_2 q^2 + \cdots + c_{k-1} q^{k-1} - q^{k+1} + bp, -q), \tag{3.44}$$

for constants $c_k,\ k \in \mathbb{Z}$. This map can be derived in the same way as the area-preserving Hénon map by choosing

$$f(q) = q^{k+1} - (c_{k-1} q^{k-1} + c_{k-2} q^{k-2} + \cdots + c_1 q + c_0) \tag{3.45}$$

in place of $f(q) = aq^2 - 1$ in the map (3.21).

The Jacobian of the standard form of the Hénon map is

$$\frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} = b \tag{3.46}$$

so the map is invertible. The generalized Hénon map can be defined as follows. Let

$$g(q) = c_0 + c_1 q + c_2 q^2 + \cdots + c_{k-1} q^{k-1} - q^{k+1}, \tag{3.47}$$

a polynomial of degree $k+1$ and we write the map as

$$h(q,p) = (g(q) + bp, -q). \tag{3.48}$$

The inverse can then be written as

$$h^{-1}(q,p) = \left(-p, \frac{q - g(-p)}{b}\right). \tag{3.49}$$

With the parameters γ and T chosen to correspond to the parameters $a = 2.1$ and $b = 0.3$ in the standard form, the Henon map forms the strange attractor shown at 100,000 iterations in figure (3.1).

From the generalized Hénon map we can form the map below [9] (figure 3.3) for $k = 2$.

$$(q,p) \mapsto (a + bq + q^3 + cp, -cq) \tag{3.50}$$

where $a = 0.3,\ b = -2.5,\ c = 0.5$.

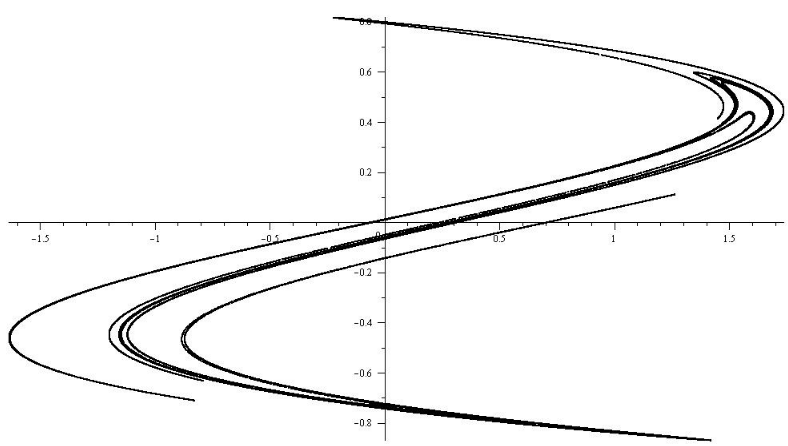

Figure 3.3: Hénon map, $k = 2$

Also from the generalized Hénon map we can form the map below (figure 3.4) for $k = 3$:

$$(q, p) \mapsto (a + bq + cq^2 + dq^4 + ep, -fq) \tag{3.51}$$

where $a = -0.965,\ b = 0.25,\ c = -0.925,\ d = 0.305,\ e = 0.32,\ f = 0.58$. The complexity at high magnification can be seen in the inset.

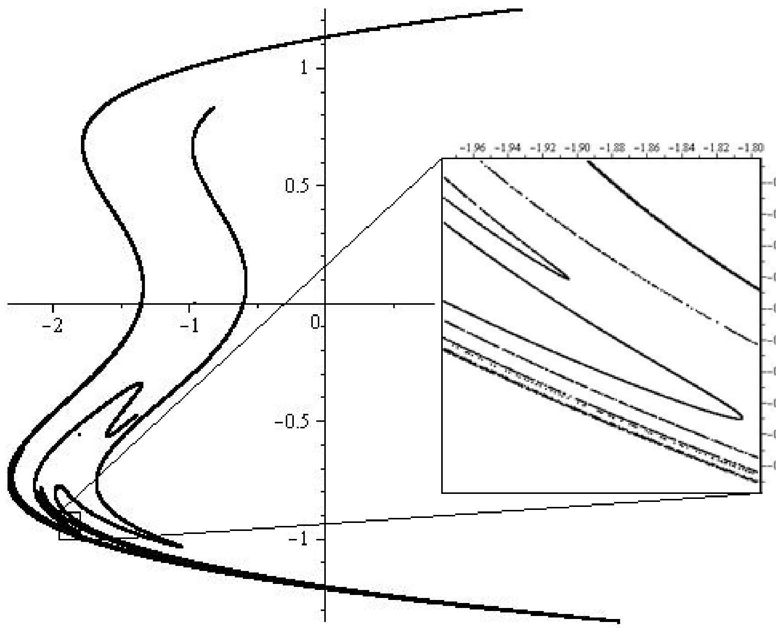

Figure 3.4: Hénon map, $k = 3$

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