THE ENUMERATION OF LATTICE PATHS AND WALKS

by

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ABSTRACT

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A well-known long standing problem in combinatorics and statistical mechanics is to find the generating function for self-avoiding walks (SAW) on a two-dimensional lattice, enumerated by perimeter. A SAW is a sequence of moves on a square lattice which does not visit the same point more than once. It has been considered by more than one hundred researchers in the pass one hundred years, including George Polya, Tony Guttmann, Laszlo Lovasz, Donald Knuth, Richard Stanley, Doron Zeilberger, Mireille Bousquet-Mlou, Thomas Prellberg, Neal Madras, Gordon Slade, Agnes Dittel, E.J. Janse van Rensburg, Harry Kesten, Stuart G. Whittington, Lincoln Chayes, Iwan Jensen, Arthur T. Benjamin, and many others. More than three hundred papers and a few volumes of books were published in this area. A SAW is interesting for simulations because its properties cannot be calculated analytically. Calculating the number of self-avoiding walks is a common computational problem. A recently proposed model called prudent self-avoiding walks (PSAW) was first introduced to the mathematics community in an unpublished manuscript of Pra, who called them exterior walks. A prudent walk is a connected path on square lattice such that, at each step, the extension of that step along its current trajectory will never intersect any previously occupied vertex. A lattice path composed of connected horizontal and vertical line segments, each passing between adjacent lattice points. We will discuss some enumerative problems in self-avoiding walks, lattice paths and walks with several step vectors. Many open problems are posted.

DEDICATION

This manuscript is dedicated to my family, particularly to my understanding and patient wife, Jing, who has put up with these many years of research, and to my son, Jia, and my daughter, May, who are the joy of our lives. I also dedicate this work to my parents, my parents-in-law, and my grandparents, all of whom believed in the pursuit of dreams.

THE ENUMERATION OF LATTICE PATHS AND WALKS

CHAPTER 1

PATTERN AVOIDANCE IN LATTICE PATHS

1.1 Introduction, Notations and Preliminaries

Notations:

East step: E or $\rightarrow$ or $(1,0)$, or x-step

You can see more in the table below:

$(0,1)$	$(1,0)$	$(1,1)$	$(0,-1)$	$(-1,0)$	$(-1,-1)$	$(-1,1)$	$(1,-1)$
$\uparrow$	$\rightarrow$	$\nearrow$	$\downarrow$	$\leftarrow$	$\swarrow$	$\nwarrow$	$\searrow$
N	E	NE	S	W	SW	NW	SE

$\uparrow^{\geq k}$: k or more than k consecutive $\uparrow$ steps

$\uparrow^{=k}$: k consecutive $\uparrow$ steps

avoiding $\uparrow^{\geq k}$: no k or more than k consecutive $\uparrow$ steps

avoiding $\uparrow^{=k}$: no k consecutive $\uparrow$ steps, but can have more than or less than k consecutive $\uparrow$ steps

$\lfloor x \rfloor$: the largest integer not greater than x, $floor(x)$

$\lceil x \rceil$: is the smallest integer not less than x, $ceiling(x)$

$[x^n]f(x)$ denotes the coefficient of x^n in the power series expansion of a function $f(x)$.

$[x^m y^n]f(x,y)$ denotes the coefficient of $x^m y^n$ in the power series expansion of a function $f(x,y)$.

$\binom{n}{r}$, the number of combinations of n things r at a time.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

A lattice path [37] is a path from the lattice point (x_1, y_1) to the lattice point (x_2, y_2), $x_1 \leq x_2$, $y_1 \leq y_2$, we mean a directed path from (x_1, y_1) to (x_2, y_2) which passes through lattice points with movements parallel to the positive direction of either axis. Here, we refer to two types steps, $x-$steps and $y-$steps, where an x (y)$-$step is a directed line segment parallel to the x (y) axis going right (up) joining two neighboring points. For counting purposes we may, without loss of generality, consider lattice paths from the origin to (m, n) and observe that each such path is characterized by having exactly m horizontal steps and n vertical steps. If we denote by $f(m, n)$ the number of paths from $(0, 0)$ to (m, n), elementary reasoning gives the results

$$f(m, n) = \binom{m+n}{n}.$$

Lattice paths are encountered in a natural way in various problems, e.g., ballot problems [36], compositions, random walks, fluctuations, queues, and the tennis ball problem.

The number of lattice paths from the origin to (m, n), $m > n + t$, not touching the line $x = y + t$, where t is a nonzero integer satisfies [34]

$$\binom{m+n}{n} - \binom{m+n}{m-t}.$$

The number of lattice paths of length n starting from the origin that are below the line $y = x + t$, where t is a positive integer satisfies

$$\sum_{i=\max\{\lfloor 1+n/2 \rfloor, t\}}^{n} (\binom{n}{n-i} - \binom{n}{i+t}).$$

When $t = 0$, the paths have to touch the line $x = y$ at the origin, and therefore the number paths from the origin to (m, n) that do not touch the line $x = y$ except at the origin is given by

$$\frac{m-n}{m+n}\binom{m+n}{n}.$$

The number of lattice paths of length n starting from the origin that are below the line $x = y$ except at the origin is given by

$$\sum_{i=\lfloor 1+n/2 \rfloor}^{n} \frac{2i-n}{n}\binom{n}{n-i}.$$

The number of lattice paths from (r, s) to (m, n) that never rise above the line $y = x$ is [1]

$$\binom{n+m-r-s}{m-r} - \binom{n+m-r-s}{m-s+1}.$$

Then the number of lattice paths from $(0, 0)$ to (m, n) that never rise above the line $y = x$ is

$$\binom{n+m}{m} - \binom{n+m}{m+1}.$$

The number of n-step lattice paths starting from $(0, 0)$ that never rise above the line $y = x$ is

$$\sum_{i=\lceil n/2 \rceil}^{n} \frac{n!(2i+1-n)}{(i+1)!(n-i)!}$$
$$= \binom{n}{\lfloor n/2 \rfloor}.$$

The number of paths from $(0,0)$ to (n,n) that never rise above the line $y=x$ is the m-th Catalan number, denoted by C_n, and define $C_0 = 1$.

$$\begin{aligned} C_n &= \frac{1}{n+1}\binom{2n}{n} \\ &= \binom{2n}{n} - \binom{2n}{n+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}^2 \end{aligned}$$

with generating function

$$\frac{1-\sqrt{1-4x}}{2x}.$$

Also,

$$\begin{aligned} C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-1} \\ &= \frac{2(2n+1)}{n+2} C_n. \end{aligned}$$

The follow table is the number of lattice path from $(0,0)$ to (m,n) and weakly above the line $y = x$:

$n = 8$	1	8	35	110	275	572	1001	1430
$n = 7$	1	7	27	75	165	297	429	429
$n = 6$	1	6	20	48	90	132	132	
$n = 5$	1	5	14	28	42	42		
$n = 4$	1	4	9	14	14			
$n = 3$	1	3	5	5				
$n = 2$	1	2	2					
$n = 1$	1	1						
$n = 0$	1							
	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$

The numbers on the diagonal are the Catalan numbers.

We can see that the sum of the numbers on the n-th slope -1 line is $\binom{n}{\lceil n/2 \rceil}$:

$$1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, ...$$

It is the number of n-step lattice paths starting from $(0, 0)$ that never rise above the line $y = x$.

The number of ways of putting n like objects into r different cells is [41]

$$\binom{n+r-1}{n} = \binom{n+r-1}{r-1}.$$

It is also the number of nonnegative integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

The above number also equals

$$
\begin{aligned}
&[x^n](1+x+x^2+x^3+...)^r \\
&= [x^n](\frac{1}{1-x})^r \\
&= [x^n](1-x)^{-r} \\
&= [x^n]\sum_{i\geq 0}\binom{-r}{i}(-1)^i x^i \\
&= [x^n]\sum_{i\geq 0}(-1)^i\binom{i+r-1}{i}(-1)^i x^i \\
&= \binom{n+r-1}{n}.
\end{aligned}
$$

The number of ways of putting n like objects into r different cells with no cell is empty is

$$\binom{n-1}{r-1}.$$

It is also the number of positive integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

The above number also equals

$$
\begin{aligned}
&[x^n](x+x^2+x^3+...)^r \\
&=[x^n](\frac{x}{1-x})^r \\
&=[x^n]x^r(1-x)^{-r} \\
&=[x^{n-r}]\sum_{i\geq 0}\binom{-r}{i}(-1)^i x^i \\
&=[x^{n-r}]\sum_{i\geq 0}(-1)^i\binom{i+r-1}{i}(-1)^i x^i \\
&=\binom{n-1}{n-r}=\binom{n-1}{r-1}.
\end{aligned}
$$

If p is a prime, then $\binom{p}{i}$ is divisible by p for $1 \leq i \leq p-1$. [42]

Fibonacci number: F_n is defined as $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n \geq 2$. The generating function is

$$\frac{1}{1-x-x^2}.$$

Definition 1. *A B-Sheffer sequence is a sequence of polynomials* $(p_i)_{i\in\mathbb{N}_0}$ *such that* $\deg p_i = i$*,* $Bp_i = p_{i-1}$*, and* $p_0 \neq 0$*, associated with an operator* B *that can be written as a power series of order* 1 *in the derivative operator,* $\mathcal{D}$*. If a B-Sheffer sequence has the initial value* $p_n(0) = \delta_{0,n}$*, then it is a B-basic sequence. [38]*

1.2 Main Results

Theorem 2. *The number of lattice paths avoiding* $\uparrow^{\geq 2}$*, from* $(0,0)$ *to* (m,n) *is*

$$\binom{m+1}{n}.$$

Proof. The m East steps provide $m+1$ positions (we can say $m+1$ different cells) for n North steps to be inserted with each cell containing at most one element (North step). Then there are $\binom{m+1}{n}$ ways to choose n cells. $\square$

Corollary 3. *The number of lattice paths from* $(0,0)$ *to* $(ns+1, nt-1)$, *avoiding* $\uparrow^{\geq 2}$*is*

$$\binom{ns+2}{nt-1}.$$

Corollary 4. *The number of* n*-step paths with east and north steps and with two consecutive north steps forbidden is equal to [7]*

$$\sum_{i=0}^{\lceil n/2 \rceil} \binom{n+1-i}{i} = F_{n+2}.$$

Theorem 5. *The number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 3}$*is*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{m+1}{n-i} \binom{n-i}{i}.$$

Proof. Without loss of generality, we assume that there are i copies of double North Steps, $n-2i$ copies of single North step in a lattice path from $(0,0)$ to (m,n) and avoiding $\uparrow^{\geq 3}$. It is clear that $0 \leq i \leq \lfloor n/2 \rfloor$. The m East steps provide $m+1$ positions (we can say $m+1$ different cells) for n North steps to be inserted with each cell containing at most one element ($\uparrow$ or $\uparrow^2$). There are $\binom{m+1}{n-i}$ ways to choose n cells for the i copies $\uparrow^2$ and $n-2i$ copies of $\uparrow$. We have $\binom{n-i}{i}$ ways to distribute the i copies $\uparrow^2$. $\square$

Corollary 6. *The number of lattice paths of length* l, *starting from* $(0,0)$, *avoiding* $\uparrow^{\geq 3}$*is*

$$\sum_{n=0}^{\lfloor 2(l+1)/3 \rfloor} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{l-n+1}{n-i} \binom{n-i}{i}.$$

Corollary 7. *The number of lattice paths from* $(0,0)$ *to* $(ns+1, nt-1)$ *avoiding* $\uparrow^{\geq 3}$*is*

$$\sum_{i=0}^{\lfloor (nt-1)/2 \rfloor} \binom{ns+2}{nt-1-i} \binom{nt-1-i}{i}.$$

Theorem 8. *Let* $f(m,n)$ *be the number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq k}$ *and* $\longrightarrow^{\geq k}$*, taking steps from* $\{\uparrow, \longrightarrow\}$*. Then*

$$f(m,n) = f(m-1,n) + f(m,n-1) - f(m-k,n-1) - f(m-1,n-k) + f(m-k,n-k).$$

Theorem 9. *The number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 2}$ *and* $\rightarrow^{\geq 3}$*is*

$$\binom{n+1}{m-n-1} + 2\binom{n}{m-n} + \binom{n-1}{m-n+1}.$$

Corollary 10. *The number , say* $f(n)$*, of lattice paths of length* l*, starting from* $(0,0)$*, avoiding* $\uparrow^{\geq 2}$ *and* $\rightarrow^{\geq 3}$*is*

$$\sum_{m=\lceil (l+1)/2 \rceil}^{\lfloor 2(l+1)/3 \rfloor} \binom{l-m+1}{2m-l-1} + 2\sum_{m=\lceil l/2 \rceil}^{\lfloor 2l/3 \rfloor} \binom{l-m}{2m-l} + \sum_{m=\lceil (l-1)/2 \rceil}^{\lfloor 2(l-1)/3 \rfloor} \binom{l-m-1}{2m-l+1}).$$

$$f(n) = f(n-2) + f(n-3)$$

$$f(1) = 2,\ f(2) = 3,\ f(3) = 4.$$

The generating function is

$$\frac{1+2x+2x^2+x^3}{1-x^2-x^3}.$$

Theorem 11. *The generating function of the number of lattice paths from* $(0,0)$ *to*

(m,n) *avoiding* $\uparrow^{\geq i}$, $\rightarrow^{\geq j}$ *satisfies*

$$[x^m y^n] \frac{(\sum_{k=1}^{i} x^{k-1})(\sum_{k=1}^{j} y^{k-1})}{1-(\sum_{k=2}^{i} x^{k-1})(\sum_{k=2}^{j} y^{k-1})}.$$

Proof. Let $P(x,y)$ be the generating function of the number of lattice paths from $(0,0)$ to (m,n) avoiding $\uparrow^{\geq i}, \rightarrow^{\geq j}$.

$$P(x,y) = \sum_{s\geq 0}\sum_{t\geq 0} p(m,n) x^m y^n.$$

Case 1. The contribution of paths starting with s East steps and followed by t North steps ($0 \leq s \leq i-1$, $0 \leq t \leq j-1$) is

$$x^s y^t.$$

The total contribution for all possible s and t is

$$\begin{aligned}&\left(1+x+x^2+...+x^{i-1}\right)\left(1+y+y^2+...+y^{j-1}\right)\\ &=(\sum_{k=1}^{i} x^{k-1})(\sum_{k=1}^{j} y^{k-1}).\end{aligned}$$

Case 2. For the walks obtained by concatenating a path, u (for possible u) North steps, and then v ($1 \leq v \leq i-1$) East steps, the contribution is

$$y^u x^v P(x,y).$$

The total contribution for all possible u and v is

$$(y + y^2 + ... + y^{j-1})(x + x^2 + ...x^{i-1})P(x, y)$$
$$= (\sum_{k=2}^{i} x^{k-1})(\sum_{k=2}^{j} y^{k-1})P(x, y).$$

Adding these two contributions give the equation

$$P(x, y) = (\sum_{k=1}^{i} x^{k-1})(\sum_{k=1}^{j} y^{k-1}) + (\sum_{k=2}^{i} x^{k-1})(\sum_{k=2}^{j} y^{k-1})P(x, y).$$

Therefore,

$$P(x, y) = \frac{(\sum_{k=1}^{i} x^{k-1})(\sum_{k=1}^{j} y^{k-1})}{1 - (\sum_{k=2}^{i} x^{k-1})(\sum_{k=2}^{j} y^{k-1})}.$$

□

Theorem 12. *The number, say $f(m, n)$, of lattice paths from $(0, 0)$ to (m, n) avoiding $\uparrow^{\geq 3}$ and $\rightarrow^{\geq 3}$ is*

$$2\sum_{i=m-n}^{\lfloor m/2 \rfloor} \binom{m-i}{i}\binom{m-i}{n-m+i}$$
$$+ \sum_{i=m-n-1}^{\lfloor m/2 \rfloor} \binom{m-i}{i}\binom{m-i-1}{n-m+i+1}$$
$$+ \sum_{i=m-n+1}^{\lfloor m/2 \rfloor} \binom{m-i}{i}\binom{m-i+1}{n-m+i-1}.$$

The generating function of the above numbers is

$$\frac{(1 + x + x^2)(1 + y + y^2)}{1 - xy(1 + x + y + xy)}.$$

Also, for $m \geq n$,

$$f(m,n) = 2\sum_{r=\lceil m/2\rceil}^{n}\binom{r}{m-r}\binom{r}{n-r}$$
$$+\sum_{r=\max\{\lceil (m-2)/2\rceil,\lceil n/2\rceil\}}^{\min\{m-1,n\}}\binom{r+1}{m-r-1}\binom{r}{n-r}$$
$$+\sum_{r=\lceil m/2\rceil}^{n-1}\binom{r}{m-r}\binom{r+1}{n-r-1}.$$

Example:

$$[x^m y^n](\frac{(1+x+x^2)\,(1+y+y^2)}{1-xy(1+x+y+xy)})$$

counts the number of lattice paths from $(0,0)$ to (m,n) avoiding $\uparrow^{\geq 3}$and $\rightarrow^{\geq 3}$.

Proof. Now we prove the second formula of $f(m,n)$. We define a level-stretch (up-stretch), one or some unextendable continues level (north) steps.

Case 1. If there are r level stretches and r up stretches in a walk, then

$$\lceil m/2\rceil \leq r \leq n.$$

The generating function of this case is

$$2\sum_{r=\lceil m/2\rceil}^{n}(x+x^2)^r(y+y^2)^r.$$

$$
\begin{aligned}
&[x^m y^n] 2 \sum_{r=\lceil m/2 \rceil}^{n} (x+x^2)^r (y+y^2)^r \\
&= 2[x^m y^n] \sum_{r=\lceil m/2 \rceil}^{n} (\sum_{i=0}^{r} \binom{r}{i} x^{2i} x^{r-i})(\sum_{j=0}^{r} \binom{r}{j} x^{2j} x^{r-j}) \\
&= 2[x^m y^n] \sum_{r=\lceil m/2 \rceil}^{n} (\sum_{i=0}^{r} \binom{r}{i} x^{r+i})(\sum_{j=0}^{r} \binom{r}{j} x^{r+j}) \\
&= 2 \sum_{r=\lceil m/2 \rceil}^{n} \binom{r}{m-r} \binom{r}{n-r}.
\end{aligned}
$$

Case 2. If there are $r+1$ level stretches and r up stretches in a walk, then

$$
\max\{\lceil (m-2)/2 \rceil, \lceil n/2 \rceil\} \leq r \leq \min\{m-1, n\}.
$$

The generating function for this case is

$$
\sum_{r=\max\{\lceil (m-2)/2 \rceil, \lceil n/2 \rceil\}}^{\min\{m-1,n\}} (x+x^2)^{r+1} (y+y^2)^r.
$$

$$
\begin{aligned}
&[x^m y^n] \sum_{r=\max\{\lceil (m-2)/2 \rceil, \lceil n/2 \rceil\}}^{\min\{m-1,n\}} (x+x^2)^{r+1} (y+y^2)^r \\
&= \sum_{r=\max\{\lceil (m-2)/2 \rceil, \lceil n/2 \rceil\}}^{\min\{m-1,n\}} \binom{r+1}{m-r-1} \binom{r}{n-r}.
\end{aligned}
$$

Case 3. If there are r level stretches and $r+1$ up stretches in a walk, then

$$
\lceil m/2 \rceil \leq r \leq n.
$$

The generating function for this case is

$$\sum_{r=\lceil m/2\rceil}^{n-1} (x+x^2)^r (y+y^2)^{r+1}.$$

$$\begin{aligned}[x^m y^n] \sum_{r=\lceil m/2\rceil}^{n-1} (x+x^2)^r (y+y^2)^{r+1} \\ = \sum_{r=\lceil m/2\rceil}^{n-1} \binom{r}{m-r}\binom{r+1}{n-r-1}.\end{aligned}$$

□

Corollary 13. *The number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*is*

$$2\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}\binom{n-i}{i} + 2\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-j}{j}\binom{n-j+1}{j-1}.$$

Theorem 14. *The generating function for the number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*, is [56]*

$$\begin{aligned}&\frac{(1-t)^2\sqrt{(1+t+t^2)(1-3t+t^2)}-(1-3t+t^2)(1+t^2)}{t^2(1-3t+t^2)} \\ &= 2t+6t^2+14t^3+34t^4+84t^5+208t^6+518t^7+\ldots\ .\end{aligned}$$

A proof in detail of this theorem involving Finite Operator Calculus [35] is in [17].

Proof. Let $f(m,n)$ be the number of paths from $(0,0)$ to (n,n) avoiding $\uparrow^k$ and $\longrightarrow^k$,taking steps from $\{\uparrow, \longrightarrow\}$. Then

$$f(m,n) = f(m-1,n)+f(m,n-1)-f(m-k,n-1)-f(m-1,n-k)+f(m-k,n-k).$$

Using Finite Operator Calculus it can be shown that the basic sequence for this recursion for $k = 3$ has the generating function

$$\sum_{n \geq 0} b_n(x) = 2^{-x} \left(1 + t - \sqrt{(t+1)(4t^2 + t + 1)}\right)^x$$

The initial values determine a Sheffer sequence, whose generating function at $x = 0$ is the generating function in the Theorem.

$$\begin{aligned} f_n(m) &= f_n(m-1) + f_{n-1}(m) - f_{n-3}(m-1) - f_{n-4}(m-3) + f_{n-6}(m-3) \\ \nabla &= B - E^{-1}B^3 - E^{-3}B^4 + E^{-3}B^6 \\ E &= \frac{1}{2} + \frac{1}{2}B - \frac{1}{2}\sqrt{1 + 2B + 5B^2 + 4B^3} \\ \sum_{n \geq 0} b_n(x) &= 2^{-x}\left(1 + t - \sqrt{(t+1)(4t^2 + t + 1)}\right)^x = \phi(t)^x \\ \phi_1(t)^x &= \sum_{n \geq 0} x \frac{b_n(n+x)}{n+x} t^n \end{aligned}$$

$$\begin{aligned} f_n(n+x) &= f_n(n+x-1) + f_{n-1}(n-1+x+1) - \ldots \\ E^{-1}Bf_n(n+x) &= f_{n-1}(n-1+x) \end{aligned}$$

with $f_n(n-1) - f_{n-3}(n-2) = 0$ for all $n \geq 5$.

Let $\bar{B} = E^{-1}B$. Then

$$\begin{aligned} s_n(m) &= s_n(m-1) + s_{n-1}(m+1) - s_{n-3}(m+2) - s_{n-4}(m+1) + s_{n-6}(m+3) \\ \nabla &= E^1\bar{B} - E^2\bar{B}^3 - E^1\bar{B}^4 + E^3\bar{B}^6 \\ E^{-1}\nabla &= \bar{B}\nabla + \left(E^{-1}\bar{B} - \bar{B}^4\right)\left(1 - E^2\bar{B}^2\right) \\ \nabla &= E^1\bar{B}\left(E^{-1} - \bar{B}^3\right)\left(1 + E^1\bar{B}\right) = \tau(B) \end{aligned}$$

$f_n(n-1) - f_{n-3}(n-2) = 0$ for all $n \geq 5$

$g_{n+3}(n) - g_n(n-1)$

$f_4(3) - f_1(2) = 1$; $f_3(2) - f_0(1) = -1$; $f_2(1) - f_{-1}(0) = 0$; $f_1(0) = -1$; $f_0(-1) = 1$

$L\langle f_n(n+x)\rangle = f_n(n-1) - f_{n-3}(n-3+1)$

$= Eval_{-1} - Eval_1 (E^{-1}B)^3$

$\Rightarrow op(L) = E^{-1} - E\bar{B}$

$s_n(x) = f_n(n+x) = \sum_{l=0}^{n} \langle L|f_{n-l}\rangle \, op(L)^{-1} \bar{b}_l(x)$

$= op(L)^{-1}\bar{b}_n(x) - op(L)^{-1}\bar{b}_{n-1}(x) - op(L)^{-1}\bar{b}_{n-3}(x) + op(L)^{-1}\bar{b}_{n-5}(x)$

$s(x,t) = \frac{\left(\frac{1}{2t^3}\left(-t^2+1-t-\sqrt{(t^2-3t+1)(t^2+t+1)}\right)\right)^x \left(1-t-t^3+t^4\right)}{\left(\frac{1}{2t^3}\left(-t^2+1-t-\sqrt{(t^2-3t+1)(t^2+t+1)}\right)\right)^{-1} - t^3\left(\frac{1}{2t^3}\left(-t^2+1-t-\sqrt{(t^2-3t+1)(t^2+t+1)}\right)\right)}$

$(s(0,t) - 1 - t^2)/t^2)$ is the generating function. □

Corollary 15. *The number of lattice paths from* $(0,0)$ *to* $((ns+1),(nt-1))$ *avoiding* $\uparrow^{\geq 3}$ *and* $\rightarrow^{\geq 3}$ *is*

$$2\sum_{i=n(s-t)+2}^{\lfloor (ns+1)/2\rfloor} \binom{ns+1-i}{i}\binom{ns+1-i}{nt-1-m+i}$$
$$+\sum_{i=n(s-t)+1}^{\lfloor (ns+1)/2\rfloor} \binom{ns+1-i}{i}\binom{ns-i}{n(t-s)-1+i}$$
$$\sum_{i=n(s-t)+3}^{\lfloor (ns+1)/2\rfloor} \binom{ns+1-i}{i}\binom{ns+1-i+1}{n(t-s)-3+i}$$

Example 16. *For* $m \leq 7$ *and* $n \leq 8$*, the number, say* $f(m,n)$*, of lattice paths from*

$(0,0)$ to (m,n) avoiding $\uparrow^{\geq 3}$and $\rightarrow^{\geq 3}$is as follows:

$n=8$				1	15	87	300	720
$n=7$				4	30	114	285	518
$n=6$			1	10	43	113	208	285
$n=5$			3	16	45	84	113	114
$n=4$		1	6	18	34	45	43	30
$n=3$		2	7	14	18	16	10	4
$n=2$	1	3	6	7	6	3	1	
$n=1$	1	2	3	2	1			
$n=0$		1	1					
	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$

$$f(m,n) = f(m-1,n)+f(m,n-1)-f(m-3,n-1)-f(m-1,n-3)+f(m-3,n-3).$$

$$\begin{aligned} f(6,6) &= f(5,6)+f(6,51)-f(3,5)-f(5,3)+f(3,3) \\ &= 113+113-16-16+14 \\ &= 208. \end{aligned}$$

Theorem 17. *The number of lattice path from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 4}, \rightarrow^{\geq 4}$*is*

$$2\sum_{i=0}^{\lfloor n/3\rfloor}\sum_{j=0}^{\lfloor (n-3i)/2\rfloor}\binom{n-2i-j}{i}\binom{n-3i-j}{j}$$

$$\sum_{s=0}^{\min\{\lfloor n/3\rfloor,\lfloor\frac{2i+j}{2}\rfloor\}}\binom{n-2i-j}{s}\binom{n-2i-j-s}{2i+j-2s}$$

$$+2\sum_{i=0}^{\lfloor n/3\rfloor}\sum_{j=0}^{\lfloor (n-3i)/2\rfloor}\binom{n-2i-j}{i}\binom{n-3i-j}{j}$$

$$\sum_{s=0}^{\min\{\lfloor n/3\rfloor,\lfloor\frac{2i+j+1}{2}\rfloor\}}\binom{n-2i-j-1}{s}\binom{n-2i-j-s-1}{2i+j+1-2s}.$$

The above number equals

$$[x^m y^n]\left(\frac{(1+x+x^2+x^3)(1+y+y^2+y^3)}{1-xy(1+y+y^2)(1+x^2+x)}\right).$$

Theorem 18. *Given two positive integers* n *and* r *and let* $p(n,r)$ *be the number of vectors* $(x_1, x_2, ..., x_r)$*, such that*

$$n = \sum_{i=1}^{r} x_i$$

x_i *is positive integer for* $i = 1, 2, ..., r$*. Then*

$$p(n,r) = \binom{n-1}{r-1}.$$

Proof.

$$\begin{aligned}
&[x^n]\,(x + x^2 + x^3 + ...)^r \\
&= [x^n]\,x^r(1-x)^{-r} \\
&= [x^n]\,x^r \sum_{i \geq 0} \binom{i+r-1}{i} x^i \\
&= \binom{n-1}{r-1}.
\end{aligned}$$

Remember $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$. □

Theorem 19. *Given three positive integers n, r and k and let $p(n, r, k)$ be the number of vectors $(x_1, x_2, ..., x_r)$, such that*

$$n = \sum_{i=1}^{r} x_i$$

x_i is positive integer, and $1 \leq x_i \leq k$ for $i = 1, 2, ..., r$. Then

$$p(n, r, k) = \sum_{h=0}^{\min\{r, \lfloor \frac{n-1}{k} \rfloor\}} (-1)^h \binom{n-kh-1}{r-1} \binom{r}{h}.$$

Example:

$p(3, 3, 1) = p(3, 3, 2) = 1$: $(1, 1, 1)$

$p(3, 2, 2) = 2$: $(1, 2)$ and $(2, 1)$

$p(4, 3, 2) = 3$: $(1, 1, 2), (1, 2, 1)$ and $(2, 1, 1)$

$p(n, r, k) - p(n, r, n-1) \geq 0$ for $0 < n \leq r(k+1)/2$.

Proof.

$$
\begin{aligned}
&[x^n]\,(x + x^2 + x^3 + ... + x^k)^r \\
&= \left[x^{n-r}\right](1 + x + x^2 + ... + x^{k-1})^r \\
&= \left[x^{n-r}\right]\frac{(1-x^k)^r}{(1-x)^r} \\
&= \left[x^{n-r}\right](1-x^k)^r(1-x)^{-r} \\
&= \left[x^{n-r}\right](\sum_{h\geq 0}(-1)^h\binom{r}{h}x^{kh})(\sum_{i\geq 0}\binom{i+r-1}{i}x^i) \\
&= \sum_{h=0}^{\min\{r,\lfloor\frac{n-1}{k}\rfloor\}}(-1)^h\binom{n-kh-1}{r-1}\binom{r}{h}.
\end{aligned}
$$

□

Theorem 20. *Given four positive integers n, r, k, and t, and let $p(n,r,k,t)$ be the number of vectors $(x_1, x_2, ..., x_r)$, such that*

$$n = \sum_{i=1}^{r} x_i$$

x_i is positive integer, and $t \leq x_i \leq k$ for $i = 1, 2, ..., r$. Then

$$
\begin{aligned}
p(n,r,k,t) &= p(n-(t-1)r, r, k-(t-1)) \\
&= \sum_{h=0}^{\min\{r,\lfloor\frac{n-rt+r-1}{k-t+1}\rfloor\}}(-1)^h\binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h}.
\end{aligned}
$$

Example:

$p(3,3,1,1) = 1$: $(1,1,1)$

$p(3,2,2,1) = 2$: $(1,2)$ and $(2,1)$

$p(4,3,2,1) = 3$: $(1,1,2)$, $(1,2,1)$ and $(2,1,1)$

Proof.

$$
\begin{aligned}
&[x^n]\,(x^t + x^{t+1} + x^{t+2} + ... + x^k)^r \\
&= \left[x^{n-tr}\right](1 + x + x^2 + ... + x^{k-t})^r \\
&= \left[x^{n-tr}\right]\frac{(1 - x^{k-r+1})^r}{(1-x)^r} \\
&= \left[x^{n-tr}\right](1 - x^{k-r+1})^r(1-x)^{-r} \\
&= \left[x^{n-tr}\right](\sum_{h\geq 0}(-1)^h\binom{r}{h}x^{(k-r+1)h})(\sum_{i\geq 0}\binom{i+r-1}{i}x^i) \\
&= \sum_{h=0}^{\min\{r,\lfloor\frac{n-rt+r-1}{k-t+1}\rfloor\}}(-1)^h\binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h}.
\end{aligned}
$$

□

Theorem 21. *The number lattice paths from* $(0,0)$ *to* (m,n) *avoiding patterns* $\rightarrow^{\geq k+1}$, $\uparrow^{\geq k+1}$, $\rightarrow^{\leq t-1}$, *and* $\uparrow^{\leq t-1}$, *is [48]*

$$
\begin{aligned}
&2\sum_{r=\max\{\lceil m/k\rceil,\lceil n/k\rceil\}}^{\min\{\lfloor m/t\rfloor,\lfloor n/t\rfloor\}}((\sum_{h=0}^{\lfloor\frac{m-rt+r-1}{k-t+1}\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})\times \\
&(\sum_{h=0}^{\lfloor\frac{n-rt+r-1}{k-t+1}\rfloor}(-1)^h\binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})) \\
&+\sum_{r=\max\{\lceil m/k\rceil,\lceil n/k-1\rceil\}}^{\min\{\lfloor m/t\rfloor,\lfloor n/t-1\rfloor\}}((\sum_{h=0}^{\lfloor\frac{m-rt+r-1}{k-t+1}\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-1}{r-1}\binom{r}{h}) \\
&\times(\sum_{h=0}^{\lfloor\frac{n-rt+r-t}{k-t+1}\rfloor}(-1)^h\binom{n-rt+r-hk+ht-h-t}{r}\binom{r+1}{h})) \\
&+\sum_{r=\max\{\lceil m/k-1\rceil,\lceil n/k\rceil\}}^{\min\{\lfloor m/t-1\rfloor,\lfloor n/t\rfloor\}}((\sum_{h=0}^{\lfloor\frac{m-rt+r-t}{k-t+1}\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-t}{r}\binom{r+1}{h}) \\
&\times(\sum_{h=0}^{\lfloor\frac{n-rt+r-1}{k-t+1}\rfloor}(-1)^h\binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})).
\end{aligned}
$$

Theorem 22. *The number lattice paths from* $(0,0)$ *to* (m,n)*, avoiding patterns* $\rightarrow^{\geq k+1}$, $\rightarrow^{\leq t-1}$,$\uparrow^{\geq u+1}$*, and* $\uparrow^{\leq v-1}$*, is*

$$
\begin{aligned}
&2\sum_{r=\max\{\lceil m/k\rceil,\lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor,\lfloor n/v\rfloor\}}((\sum_{h=0}^{\left\lfloor\frac{m-rt+r-1}{k-t+1}\right\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})\\
&\times(\sum_{h=0}^{\left\lfloor\frac{n-rv+r-1}{u-v+1}\right\rfloor}(-1)^h\binom{n-rv+r-hu+hv-h-1}{r-1}\binom{r}{h}))\\
&+\sum_{r=\max\{\lceil m/k\rceil,\lceil n/u-1\rceil\}}^{\min\{\lfloor m/t\rfloor,\lfloor n/v-1\rfloor\}}((\sum_{h=0}^{\left\lfloor\frac{m-rt+r-1}{k-t+1}\right\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})\times\\
&(\sum_{h=0}^{\left\lfloor\frac{n-rv+r-t}{u-t+1}\right\rfloor}(-1)^h\binom{n-rv+r-hu+hv-h-v}{r}\binom{r+1}{h}))\\
&+\sum_{r=\max\{\lceil m/k-1\rceil,\lceil n/u\rceil\}}^{\min\{\lfloor m/t-1\rfloor,\lfloor n/v\rfloor\}}((\sum_{h=0}^{\left\lfloor\frac{m-rt+r-t}{k-t+1}\right\rfloor}(-1)^h\binom{m-rt+r-hk+ht-h-t}{r}\binom{r+1}{h})\\
&\times(\sum_{h=0}^{\left\lfloor\frac{n-rt+r-1}{k-t+1}\right\rfloor}(-1)^h\binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})).
\end{aligned}
$$

Proof. Case 1. If there are r level stretches and r up stretches in a walk, then

$$\max\{\lceil m/k\rceil,\lceil n/u\rceil\}\leq r\leq\min\{\lfloor m/t\rfloor,\lfloor n/v\rfloor\}.$$

The generating function of this case is

$$2(x^t+x^{t+1}+...+x^k)^r(y^v+y^{v+1}+...+y^u)^r.$$

$$[x^m y^n]2(x^t + x^{t+1} + ... + x^k)^r (y^v + y^{v+1} + ... + y^u)^r$$

$$= 2[x^{m-tr} y^{n-vr}]\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} (1 + x + ... + x^{k-t})^r\Big)$$

$$\times \Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} (1 + y + ... + y^{u-v})^r\Big)$$

$$= 2[x^{m-tr} y^{n-vr}]\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \frac{(1 - x^{k-t+1})^r}{(1-x)^r}\Big)\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \frac{(1 - x^{u-v+1})^r}{(1-x)^r}\Big)$$

$$= 2[x^{m-tr} y^{n-vr}]\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \binom{r}{h}(-1)^h x^{(k-t+1)h} \binom{-r}{a} (-1)^a x^a\Big)$$

$$\times \Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \binom{r}{h}(-1)^h y^{(u-v+1)h} \binom{-r}{a} (-1)^a y^a\Big)$$

$$= 2[x^{m-tr} y^{n-vr}]\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \binom{r}{h}(-1)^h x^{(k-t+1)h} (-1)^a \binom{r+a-1}{a} (-1)^a x^a\Big)$$

$$\times \Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} \binom{r}{h}(-1)^h y^{(u-v+1)h} (-1)^a \binom{r+a-1}{a} (-1)^a y^a\Big)$$

$$= 2[x^{m-tr} y^{n-vr}]\Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} (-1)^h \binom{r}{h} \binom{r+a-1}{a} x^{(k-t+1)h+a}\Big)$$

$$\times \Big(\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v\rfloor\}} (-1)^h \binom{r}{h} \binom{r+a-1}{a} y^{(u-v+1)h+a}$$

$$= 2\Big(\sum_{h=0}^{\left\lfloor \frac{m-rt+r-1}{k-t+1} \right\rfloor} (-1)^h \binom{m - rt + r - hk + ht - h - 1}{r-1} \binom{r}{h}\Big)$$

$$\times \Big(\sum_{h=0}^{\left\lfloor \frac{n-rv+r-1}{u-v+1} \right\rfloor} (-1)^h \binom{n - rv + r - hu + hv - h - 1}{r-1} \binom{r}{h}\Big)\Big).$$

Case 2. If there are $r+1$ level stretches and r up stretches in a walk, then

$$\sum_{r=\max\{\lceil m/k\rceil, \lceil n/u-1\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v-1\rfloor\}} \leq r \leq \min\{\lfloor m/t\rfloor, \lfloor n/v-1\rfloor\}.$$

The generating function of this case is

$$(x^t + x^{t+1} + ... + x^k)^{r+1}(y^v + y^{v+1} + ... + y^u)^r.$$

$$\begin{aligned}
&[x^m y^n](x^t + x^{t+1} + ... + x^k)^{r+1}(y^v + y^{v+1} + ... + y^u)^r \\
&= \sum_{r=\max\{\lceil m/k\rceil, \lceil n/u-1\rceil\}}^{\min\{\lfloor m/t\rfloor, \lfloor n/v-1\rfloor\}} ((\sum_{h=0}^{\left\lfloor \frac{m-rt+r-1}{k-t+1} \right\rfloor} (-1)^h \binom{m-rt+r-hk+ht-h-1}{r-1}\binom{r}{h}) \\
&\times (\sum_{h=0}^{\left\lfloor \frac{n-rv+r-1}{u-v+1} \right\rfloor} (-1)^h \binom{n-rv+r-hu+hv-h-1}{r-1}\binom{r}{h})).
\end{aligned}$$

Case 3. There are r level stretches and $r+1$ up stretches in a walk, then we have

$$\begin{aligned}
&\sum_{r=\max\{\lceil m/k-1\rceil, \lceil n/k\rceil\}}^{\min\{\lfloor m/t-1\rfloor, \lfloor n/t\rfloor\}} ((\sum_{h=0}^{\left\lfloor \frac{m-rt+r-t}{k-t+1} \right\rfloor} (-1)^h \binom{m-rt+r-hk+ht-h-t}{r}\binom{r+1}{h}) \\
&\times (\sum_{h=0}^{\left\lfloor \frac{n-rt+r-1}{k-t+1} \right\rfloor} (-1)^h \binom{n-rt+r-hk+ht-h-1}{r-1}\binom{r}{h})).
\end{aligned}$$

□

1.3 Open Problems

Problem 23. *How to find the number of lattice paths from $(0,0)$ to (m,n) avoiding $\uparrow^{\geq i}, \rightarrow^{\geq j}$, and weakly above the diagonal $y = x$. And how to find a good generating function for this problem?*

Problem 24. *How to find the number, say $f(m,n)$, of lattice path from $(0,0)$ to (m,n) avoiding $\uparrow^{\geq 3}$, $\longrightarrow^{\geq 3}$, and weakly above $y = x$? (This problem is a special case of the above problem: $i = j = 3$)*

You can see some numbers of this problem:

$n = 14$							1	35	364
$n = 13$							7	104	693
$n = 12$						1	27	226	1023
$n = 11$						6	70	370	1156
$n = 10$					1	20	130	454	956
$n = 9$					5	44	176	402	522
$n = 8$				1	14	67	168	231	164
$n = 7$				4	25	70	102	77	
$n = 6$			1	9	29	45	36		
$n = 5$			3	12	20	17			
$n = 4$		1	5	9	8				
$n = 3$		2	4	4					
$n = 2$	1	2	2						
$n = 1$	1	1							
$n = 0$									
	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$

For $m + 2 \leq n$,

$$\begin{aligned} f(m,n) = {} & f(m-1,n) + f(m-1,n-1) + f(m-1,n-2) \\ & - f(m-3,n) - f(m-3,n-1) - f(m-3,n-2). \end{aligned}$$

For $m + 1 = n$,

$$f(n,n+1) = f(n-1,n+1) + f(n-1,n) - f(n-3,n+1) - f(n-3,n).$$

For $m = n$,

$$f(n, n) = f(n-1, n) - f(n-3, n).$$

Example:

$$\begin{aligned} f(9,12) &= f(8,12) + f(8,11) + f(8,10) \\ &\quad - f(6,12) - f(6,11) - f(6,10) \\ &= 1023 + 1156 + 956 - 27 - 70 - 130 \\ &= 2098 \end{aligned}$$

$$\begin{aligned} f(6,7) &= f(5,7) + f(5,6) - f(3,7) - f(3,6) \\ &= 70 + 45 - 4 - 9 = 103 \end{aligned}$$

$$\begin{aligned} f(5,5) &= f(4,5) - f(2,5) \\ &= 20 - 3 = 17 \end{aligned}$$

(1) We can say the numbers on the first line with slope 2 are:

$$1, 1, 1, 1, 1, 1, 1, 1, 1, ...$$

(2) The numbers on the second line with slope 2 are:

$$1, 2, 3, 4, 5, 6, 7, ..$$

(3) The numbers on the third line with slope 2 are:

$$0, 2, 5, 9, 14, 20, 27, 35, 44, 54, ...$$

$$\binom{n+1}{2} - 1, \text{ for } n \geq 2$$

(4) The numbers on the fourth line with slope 2 are:

$$0, 1, 4, 12, 25, 44, 70, 104, 147, 200...$$

$$f(n) = \frac{1}{6}n(n+5)(n-2), \text{ for } n \geq 3$$

$f(n+3)$ is the number in the sequence [58, A000297] which counts other things:

Some kind of partitions by P. Erdos, R. K. Guy and J. W. Moon [12].

Rooted trees by J. Riordan [41].

(5) The numbers on the fifth line with slope 2 are:

$$0, 0, 2, 9, 29, 67, 130, 226, 364, 554, 807, ...$$

$$\frac{1}{24}(n-2)(n^3 + 8n^2 - 9n - 48), \text{ for } n \geq 4$$

(6) The numbers on the sixth line with slope 2 are:

$$0, 0, 0, 4, 20, 70, 176, 370, 693, ...$$

$$\frac{1}{120}(n-2)\left(n^4 + 12n^3 - 21n^2 - 232n + 360\right), \text{ for } n \geq 5$$

(7) The numbers on the seventh line with slope 2 are:

$$0, 0, 0, 0, 8, 45, 168, 454, 1023, 2045, 3751, ...$$

$$\frac{1}{720}n\left(n^5+15n^4-65n^3-675n^2+3304n-3300\right), \text{ for } n \geq 6.$$

CHAPTER 2

WALKS WITH SEVERAL STEP VECTORS

We will discuss some new enumerative problems including some pattern avoidance problems in walks with several step vectors. And some open problems are posted.

The number of n-step walks with steps $(0,1)$, $(1,0)$ and $(-1,-1)$ is

$$\frac{(3n)!}{(n!)^3}.$$

2.1 Main Results

Theorem 25. *The number of $3n$-step walks from $(0,0)$ to $(0,0)$, taking steps from $\{E, N, SW\}$, and staying above the line $y = x$ (i.e., any point (x, y) along the path satisfies $y \geq x$) is given by*

$$\frac{(3n)!}{(n!)^2(n+1)!}.$$

Example: $n = 1$, three walks: $NE(SW)$, $(SW)NE$, $N(SW)E$.

This is the sequence [58, A007004]:

$$1, 3, 30, 420, 6930, 126126, ...$$

Proof. It is clear that such a $3n$-step walk contains n copies of north, east and southwest steps, respectively. It is also true that the total number of north and east steps is greater or equal to the number of southwest steps at any lattice point on a walk.

Now we arrange n north and east steps (total is n) with n southwest steps to get a $2n$-step walk according to: the total number of the chosen steps is greater or equal to the number of southwest steps at any lattice point on a walk, which gives C_n (We do not consider the difference of north steps and east steps at this moment). Next, we have $2n+1$ positions to insert the remaining n steps of the north steps and east steps into the $2n$-step walk, giving $\binom{3n}{n}$ ways. Now combine them:

$$C_n \binom{3n}{n} = \frac{(3n)!}{(n!)^2(n+1)!}.$$

□

This theorem also could be proved by using *André's Reflection Method*:

$$\binom{3n}{n, n, n} - \binom{3n}{n+1, n-1, n} = \frac{(3n)!}{(n!)^2(n+1)!}.$$

Theorem 26. *The number of $3n$-step walks from $(0,0)$ to $(0,0)$, taking steps from $\{W, S, NE\}$, and staying within the first quadrant (i.e., any point (x, y) along the walk satisfies $x,\ y \geq 0$) is given by [5], [6], [28]*

$$\frac{4^n (3n)!}{(n+1)!(2n+1)!}.$$

Example: $n = 1$, two walks: $(NE)SW$, $(NE)WS$.

This is the sequence [58, A006335]:

$$1, 2, 16, 192, 2816, 46592, 835584, ...$$

Theorem 27. *The number of walks from $(0,0)$ to (m, n) ($m \geq n$) taking steps from*

$\{E, N, NE\}$ *is*

$$\sum_{k=0}^{n} \binom{m+n-2k}{n-k} \binom{m+n-k}{k}.$$

Proof. Without loss of generality, we assume that there are k Northwest steps, $m-k$ East steps and $n-k$ North steps in a walk from $(0,0)$ to (m,n). It is clear that $0 \leq k \leq n$.

Firstly, we only consider the number of arrangements of $m-k$ East steps and $n-k$ North steps, which give us $\binom{m+n-2k}{n-k}$ ways.

Secondly, $m-k$ East steps and $n-k$ North steps provide $m+n-2k+1$ positions (we can say $m+n-2k+1$ different cells) for k Northwest steps to be inserted, which give $\binom{m+n-k}{k}$.

Therefore, we get the number:

$$\sum_{k=0}^{n} \binom{m+n-2k}{n-k} \binom{m+n-k}{k}.$$

□

We obtain sequence [58, A001850] for $m = n$:

$$1, 3, 13, 63, 321, 1683, 8989, 48639, ...$$

Sequence [58, A002002] for $m = n + 1$:

$$1, 5, 25, 129, 681, 3653, 19825, ...$$

The number of walks from $(0,0)$ to $(n, n-1)$ ($m \geq n$) take steps from $\{E, N, NE\}$.

Sequence [58, A026002] for $m = n + 2$:

$$1, 7, 41, 231, 1289, 7183, 40081, ...$$

Sequence [58, A190666] for $m = n + 3$:

$$9, 61, 377, 2241, 13073, 75517, 433905, ...$$

Example 28. *There are* 13 *walks in the above theorem for* $m = n = 2$*:* 6 *walks with* 2 *East steps and* 2 *North steps,* 1 *walk with two Northeast steps,* 6 *walks with* 1 *Northeast step,* 1 *East step and* 1 *North step:*

$$(NE)NE,\ (NE)EN,\ NE(NE),\ EN(NE),\ E(NE)N,\ N(NE)E.$$

Corollary 29. *The number of walks from* $(0, 0)$ *to* (n, n) *taking steps from* $\{E, N, NE\}$ *is*

$$\sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!(k!)^2}.$$

Now we consider the number of walks from $(0, 0)$ to (n, m), taking steps from $\{\uparrow, \longrightarrow, \nearrow\}$, weakly above $y = x$, avoiding $\uparrow^{\geq 2}, \longrightarrow^{\geq 2}, \nearrow^{\geq 2}$. You can see some numbers for this problem:

m							237
5			1	13	45	47	
4			4	18	20		
3		1	7	9			
2		3	4				
1	1	2					
0	$*$						
	0	1	2	3	4	5	n

$(Y_{n,m}, C_{n,m}, X_{n,m})$: the number of paths from $(0, 0)$ to (n, m), with a last step $\uparrow$, $\nearrow$, $\longrightarrow$ respectively (avoiding $\rightarrow^{\geq 2}, \uparrow^{\geq 2}, \nearrow^{\geq 2}$)

m							
5			$(1,0,0)$	$(9,3,1)$	$(20,13,12)$	$(0,14,33)$	
4			$(3,1,0)$	$(9,5,4)$	$(0,6,14)$		
3		$(1,0,0)$	$(4,2,1)$	$(0,3,6)$			
2		$(2,1,0)$	$(0,1,3)$				
1	$(1,0,0)$	$(0,1,1)$					
0	$*$						
	0	1	2	3	4	5	n

Then

$$Y_{n,m} = C_{n,m-1} + X_{n,m-1}$$

$$C_{n,m} = Y_{n-1,m-1} + X_{n-1,m-1}$$

$$X_{n,m} = Y_{n-1,m} + C_{n-1,m}$$

$$Y_{n,m} = 0 \text{ for } n \geq m$$

Also,

$$C_{n,m} = C_{n-1,m-2} + 2C_{n-2,m-2} + C_{n-2,m-1} + C_{n-1,m-1} \text{ for } m > n$$

$$C_{n,n} = C_{n-1,n-2} + C_{n-2,n-1} + C_{n-1,n-1}$$

$$X_{n,m} = X_{n-1,m-2} + 2X_{n-2,m-2} + X_{n-2,m-1} + X_{n-1,m-1} \text{ for } m > n$$

$$X_{n,n} = X_{n-1,n-1} + 2X_{n-2,n-2} + X_{n-2,n-1} + X_{n-3,n-3}$$

$$Y_{n,m} = Y_{n-1,m-1} + Y_{n-1,m-2} + 2Y_{n-2,m-2} + Y_{n-2,m-1} \text{ for } m > n$$

$$Y_{n,n} = 0$$

$Y_{n,m}$:

m							
9					1	22	151
8					5	49	179
7				1	15	67	112
6				4	25	47	0
5			1	9	20	0	
4			3	9	0		
3		1	4	0			
2		2	0				
1	1	0					
0	*						
	0	1	2	3	4	5	6 n

,

Rewrite:

m							
6	1	7	30	82	151	179	112
5	1	6	22	49	67	47	**0**
4	1	5	15	25	20	**0**	
3	1	4	9	9	**0**		
2	1	3	4	**0**			
1	1	2	**0**				
0	1	**1**					
	0	1	2	3	4	5	6 n

Let $S_n(m)$ be the number above.

Let $T_n(m) = S_n(m - n + 1)$

$T_n(m):$

6	1	7							
5	1	6	30						
4	1	5	22	82					
3	1	5	15	49	151				
2	1	3	9	25	67	179			
1	1	2	4	9	20	47	112	273	676
0	1	1	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	9

$$\sum_{n\geq 0} T_n(x)t^n = (1+t)\left(\frac{1-t-2t^2-\sqrt{4t^2(t^2-1)+(t-1)^2}}{2t^3}\right)^x$$
$$= (1+t)\, e^{x \ln \frac{1-t-2t^2-\sqrt{4t^2(t^2-1)+(t-1)^2}}{2t^3}}$$

$$\sum_{n\geq 0} T_n(1)t^n = (1+t)\frac{1-t-2t^2-\sqrt{4t^2(t^2-1)+(t-1)^2}}{2t^3})$$
$$= \frac{(1+t)}{2t^3}(1-t-2t^2-(1-t)\sqrt{1-\frac{4t^2(1+t)}{1-t}})$$
$$= 1+2t+4t^2+9t^3+20t^4+47t^5+...$$

$$\frac{1}{2t^3}(\left(1-t-2t^2\right)(1+t)-\sum_{j=0}^{\infty}\binom{1/2}{j}(-1)^j\,4^j t^{2j}\frac{(1+t)^{j+1}}{(1-t)^{j-1}})$$

$$= \frac{1}{2t^3}(\left(1-t-2t^2\right)(1+t)$$

$$-\sum_{j=0}^{\infty}\binom{1/2}{j}(-1)^j\,4^j\sum_{l=0}^{j+1}\binom{j+1}{l}\sum_{k=0}^{\infty}\binom{-j+1}{k}(-1)^k\,t^{k+l+2j})$$

$$= \frac{\left(1-t-2t^2\right)(1+t)}{2t^3}$$

$$-\frac{1}{2}\sum_{j=0}^{\infty}\binom{1/2}{j}(-1)^j\,4^j\sum_{l=0}^{j+1}\binom{j+1}{l}\sum_{k=0}^{\infty}\binom{-j+1}{k}(-1)^k\,t^{k+l+2j-3})$$

$$= -\frac{1}{2}\sum_{j=0}^{\infty}\binom{1/2}{j}(-1)^j\,4^j\sum_{l=0}^{j+1}\binom{j+1}{l}\binom{n+1-j-l}{n+3-2j-l},\text{ for } n \geq 3$$

2.2 Open Problems

Problem 30. *What's the number of n-step walks, taking steps from $\{\uparrow, \rightarrow\}$, with the numbers of strings of "$\uparrow^b$" = the number of strings of "$\rightarrow^d$", counting overlaps?*

Problem 31. *What is the number of walks from $(0,0)$ to (n,m), weakly above the diagonal ($y = x$ axis), taking steps from $\{\uparrow, \longrightarrow, \nearrow\}$, avoiding $\uparrow^i, \longrightarrow^j, \nearrow^k$?*

Example 32. *The number of walks from $(0,0)$ to (n,m), weakly above the diagonal ($y = x$ axis), taking steps from $\{\uparrow, \longrightarrow, \nearrow\}$:*

↑ m							
8	1	16	126	430	2490	7432	
7	1	14	96	264	1408	3534	
6	1	12	70	264	714	1412	
5	1	10	48	146	304	394	
4	1	8	30	68	90		
3	1	6	16	22			
2	1	4	6				
1	1	2					
0							
	0	1	2	3	4	5	→ n

$d\,(n,m)$

$$d(n,m) = d(n-1,m) + d(n,m-1) + d(n-1,m-1).$$

Using finite operator calculus

$$I = E^{-1} + B + BE^{-1}$$

$$E = I + BE + B$$

$$E = \frac{I+B}{I-B}$$

$$b_n(x) = x\sum_{i=1}^{n} [B^n]\,(B + BE^{-1})^i \frac{1}{x}\binom{i-1+x}{i}$$

$$\sum_{n\geq 0} b_n(x)t^n = (\frac{1+t}{1-t})^x$$

$$s_n(x) = \frac{x-n+1}{x+1} b_n(x+1)$$

Example:

$$s_0(x) = 1$$

$$s_1(x) = 2x$$

$$s_2(x) = 2x^2 - 2$$

$$s_3(x) = \frac{4}{3}x^3 - \frac{10}{3}x - 4$$

$$s_4(x) = (x-3)\,x^2\frac{2+x^2}{x+1}$$

CHAPTER 3

LATTICE PATHS AND WALKS WITH CERTAIN STRETCHES

3.1 Lattice Path with East, North Steps:

Consider the paths from $(0,0)$ to (m,n) with s level-stretches (a level-stretch (up-stretch) is one or some unextendable continues level (north) steps), k right turns and h up-stretches. [39], [18]

Let $f_1(m,n,k)$ denote number of walks from $(0,0)$ to (m,n) with k right turns, then

$$f_1(m,n,k) = \binom{m}{k}\binom{n}{k}.$$

Example: $f_1(1,1,1) = 1$, $f_1(1,1,0) = 1$, $f_1(2,2,1) = 4$.

Let $f_2(m,n,s)$ denote the number of walks from $(0,0)$ to (m,n) with s level-stretches, then

$$f_2(m,n,s) = \binom{m-1}{s-1}\binom{n+1}{s}.$$

Example: $f_2(1,1,1) = 2$, $f_2(1,2,1) = 3$, $f_2(3,2,2)$=6.

Let $f_3(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$f_3(m,n,h) = \binom{m+1}{h}\binom{n+1}{h-1}.$$

Example: $f_3(1,1,1) = 2$, $f_3(1,2,2) = 3$, $f_3(3,3,2) = 24$.

Let $f_4(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t up and level stretches, then $f_4(m,n,t) =$

$$2\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ when } t \text{ is even,}$$
$$\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1} + \binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1} \text{ when } t \text{ is odd.}$$

Example:$f_4(1,1,2) = 2$, $f_4(3,3,4) = 8$, $f_4(2,1,3) = 1$, $f_4(3,4,5) = 9$.

Let $f_5(m,n)$ denote the number of walks from $(0,0)$ to (m,n). It is well known that

$$f_5(m,n) = \binom{m+n}{n} = \binom{m+n}{m}.$$

We also have

$$f_5(m,n) = \sum_{i\geq 1}\binom{m+1}{i}\binom{n-1}{i-1}.$$

Example: $f_5(1,1) = 2$, $f_5(1,2) = 3$, $f_5(2,2) = 6$.

3.2 ENW Walks

Counting walks which start at the origin $(0,0)$ and take unit steps $(1,0)$, $(0,1)$, and $(-1,0)$ with the restriction that no E step immediately follows a W step and vice versa. The restriction has the effect of making the walks self-avoiding. It is a major unsolved problem to enumerate all self-avoiding walks. We start by counting walks which start at the origin $(0,0)$ and take unit steps $(1,0)$,and $(0,1)$. Let $p(m,n)$ denote the number of ENW walks from $(0,0)$ to (m,n). We have

$$p(m,n) = p(m,n-1) + 2\sum_{i>0} p(m-j,n-1).$$

Let $p_1(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$p_1(m,n,h) = 2^{h+1}\binom{m}{h}\binom{n-1}{h-1} + 2^{h-1}\binom{m-1}{h-2}\binom{n-1}{h-1}.$$

Example: $p_1(1,1,1) = 4, p_1(3,4,2) = 78$.

Let $p_2(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t up and level stretches, then $p_2(m,n,t) =$

$$2^{(t/2+1)}\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ when } t \text{ is even;}$$

$$2^{(t+1)/2}\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1} + 2^{(t-1)/2}\binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$

when t is odd.

Example: $p_2(1,1,2) = 4, p_2(4,3,4) = 48, p_2(1,2,3) = 2, p_2(3,4,5) = 48$.

Let $p_2(N,t)$ denote the number of walks of length N and t up and level stretches, then $p_2(N,t) =$

$$\sum_{n=t/2}^{N-t/2} 2^{t/2+1}\binom{N-n+1}{t/2-1}\binom{n}{t/2-1} \text{ for even } t$$

$$\sum_{n=(t-1)/2}^{N-(t+1)/2} 2^{(t+1)/2}\binom{N-n-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1}$$

$$+\sum_{n=(t+1)/2}^{N-(t-1)/2} 2^{(t-1)/2}\binom{N-n-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1} \text{ for odd } t.$$

Example: $p_2(2,2) = 4, p_2(3,2) = 8$, $p_2(1,1) = 3$, $p_2(3,3) = 6$.

Let $p(N)$ be the number of walks of length N, then

$$p(N) = 3 + \sum_{t=1}^{N} p_2(N,t).$$

Let $p(m,n)$ denote the number of walks from $(0,0)$ to (m,n), then

$$\begin{aligned}
p(m,n) &= \sum_{h=1}^{\min\{n,m+1\}} \{2^{h+1}\binom{m}{h}\binom{n-1}{h-1} + 2^{h-1}\binom{m-1}{h-2}\binom{n-1}{h-1}\} \\
&= \sum_{h=0}^{\min\{n-1,m\}} 2^{h+2}\binom{m}{h+1}\binom{n-1}{h} + \sum_{h=0}^{\min\{n-2,m-1\}} 2^{h+1}\binom{m-1}{h}\binom{n-1}{h+1} \\
&= \sum_{h=0}^{\min\{n-1,m\}} 2^{h+1}\{\binom{m}{h+1}\binom{n-1}{h} + \binom{m-1}{h+1}\binom{n-1}{h} + \binom{m-1}{h}\binom{n}{h+1}\} \\
&= \sum_{h=0}^{\min\{n-1,m\}} 2^{h+1}\{\binom{m}{h+1}\binom{n-1}{h} + \binom{m-1}{h+1}\binom{n-1}{h} \\
&+ \binom{m-1}{h}\binom{n-1}{h} + \binom{m-1}{h}\binom{n-1}{h+1}\} \\
&= \sum_{h=0}^{\min\{n-1,m\}} 2^{h+1}\{\binom{m}{h+1}\binom{n-1}{h} + \binom{m}{h+1}\binom{n-1}{h} + \binom{m-1}{h}\binom{n-1}{h+1}\}.
\end{aligned}$$

Example: $p(1,2) = 6$, $p(2,2) = 18$, $p(2,2) = 50$.

Also

$$p(m,n) = \sum_{i=0} 2^{i+1}\binom{m-1}{i}\binom{n+1}{i+1}$$

$$p(m,n) = \frac{n+1}{m}\sum_{i=0}^{m}\binom{m}{i}\binom{n+i}{m-1}.$$

for $N > 3$,

$$p(N) = 3 + \sum_{n=1}^{N-1}(\sum_{h=1}^{\min\{n,N-n+1\}} 2^{h+1}\binom{N-n}{h}\binom{n-1}{h-1})$$
$$+ \sum_{n=2}^{N-1}(\sum_{h=2}^{\min\{n,N-n+1\}} 2^{h-1}\binom{N-n-1}{h-2}\binom{n-1}{h-1})$$
$$= \sum_{n=1}^{N-1} \frac{n+1}{N-n} \sum_{i=0}^{N-n} \binom{N-n}{i}\binom{n+i}{N-n-1} + 3$$
$$= \sum_{n=1}^{k-1}(\sum_{i=0}^{k-n} 2^{i+1}\binom{k-n-1}{i}\binom{n+1}{i+1}) + 3.$$

Example: $p(4) = 41$, $p(5) = 99$, $p(6) = 239$, $p(7) = 577$, $p(8) = 1393$.

3.3 ENW Walks without Ending with a W Step

We now consider ENW walks from $(0,0)$ to (m,n) with the additional restriction no walk ends with a W step. Let $q_1(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$q_1(m,n,h) = 2^{h-1}\binom{n-1}{h-1}(\binom{m}{h-1} + 2\binom{m}{h}).$$

Example: $q_1(1,1,1) = 3, q_1(2,2,1) = 5, q_1(3,4,2) = 54$.

Let $q_2(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t up and level stretches, then $q_2(m,n,t) =$

$$3 \times 2^{t/2-1}\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ for even } t$$
$$2^{(t-1)/2}\binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$
$$+ 2^{(t-1)/2}\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1} \text{ for odd } t.$$

Example:$q_2(1,1,2) = 3, q_2(4,3,4) = 36, q_2(1,2,3) = 2$,$q_2(3,4,5) = 36$.

Let $q_2(N,t)$ denote the number of walks with length N and t up and level stretches, then $q_2(N,t) =$

$$\sum_{n=t/2}^{N-t/2} 3 \times 2^{t/2-1} \binom{N-m-1}{t/2-1} \binom{n-1}{t/2-1} \text{ for even } t$$

$$\sum_{n=(t-1)/2}^{N-(t-1)/2} (2^{(t-1)/2} \binom{N-n-1}{(t+1)/2-2} \binom{n-1}{(t+1)/2-1}$$
$$+ 2^{(t-1)/2} \binom{N-n-1}{(t-1)/2} \binom{n-1}{(t-1)/2-1}) \text{ for odd } t.$$

Example:$q_2(2,2) = 3$, $q_2(3,2) = 6$, $q_2(3,3) = 4$, $q_2(5,3) = 24$.

Let $q(N)$ be the number of walks of length N, then

$$q(N) = \sum_{t=1}^{N} p_2(N,t).$$

Let $q(m,n)$ denote the number of walks from $(0,0)$ to (m,n), then

$$q(m,n) = \sum_{h=1}^{n} 2^{h-1} \binom{n-1}{h-1} (\binom{m}{h-1} + 2\binom{m}{h}).$$

Example:$q(1,1) = 3, q(1,2) = 5, q(3,4) = 129.$

$$q(N) = \sum_{n=1}^{N} (\sum_{h=1}^{n} 2^{h-1} \binom{n-1}{h-1} (\binom{N-n}{h-1} + 2\binom{N-n}{h})) + 1.$$

Example: $q(0) = 1, q(1) = 2, q(2) = 5, q(3) = 12, q(4) = 29, q(5) = 70, q(6) = 169.$

3.4 END Walks

We now consider walks from $(0,0)$ to (m,n) and taking unit steps $(1,0) = E$(east) and $(0,1) = N$(north) and double east steps of length 2 denoted by D.

Let $c(m,n,h,d)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches and d copies of D, then

$$c(m,n,h,d) = \binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1,1,0) = 2$, $c(2,1,1,1) = 2$, $c(6,6,4,2) = 300$, $c(6,6,4,3) = 10$.

Let $c(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$c(m,n,h) = \sum_{d=0}^{m} \binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1,1) = 2, c(2,1,1) = 5, c(4,3,2) = 62$.

Let $c(m,n)$ denote the number of walks from $(0,0)$ to (m,n),then

$$c(m,n) = \sum_{h=0}^{n}\sum_{d=0}^{m} \binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1) = 2, c(1,2) = 3, c(3,3) = 40$.

Also,

$$\begin{aligned} c(m,n) &= \sum_{h=0}^{n}\sum_{d=0}^{m} \binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h} \\ &= \sum_{d=0}^{m}\binom{m-d}{d}\sum_{h=0}^{n}\binom{n-1}{h-1}\binom{m-d+1}{h} \\ &= \sum_{d=0}^{m}\binom{m-d}{d}\binom{n+m-d}{m-d}. \end{aligned}$$

Let $c(N)$ denote the number of walks of length N, then

$$
\begin{aligned}
c(N) &= \sum_{n=0}^{N}\sum_{d=0}^{N-n}\binom{N-n-d}{d}\binom{N-d}{N-n-d} \\
&= \sum_{n=0}^{N}\sum_{d=0}^{N-n}\binom{N-n-d}{d}\binom{N-d}{n} \\
&= \frac{\left(1+\sqrt{2}\right)^{N+1}-\left(1-\sqrt{2}\right)^{N+1}}{2\sqrt{2}}.
\end{aligned}
$$

Example:$c(1) = 2, c(2) = 5, c(3) = 12, c(4) = 29.$

3.5 ENDD Walks

We now consider walks from $(0,0)$ to (m,n) and taking unit steps East, North and double east steps of length 2 denoted by E^D and double north steps of length 2 denoted by N^D. We call this class of walks ENDD walks.

Let $u(m,n)$ denote the number of walks from $(0,0)$ to (m,n),then

$$
\begin{aligned}
u(m,n) &= u(m,n-1)+u(m,n-2)+u(m-1,n)+u(m-2,n) \\
& m \geq 2, n \geq 2
\end{aligned}
$$

with $u(0,1) = 1, u(0,2) = 2, u(1,0) = 1, u(1,1) = 2, u(1,2) = 5, u(2,1) = 5.$

Let $u(m,n,h,a,b)$ denote the number of walks from $(0,0)$ to (m,n) with h up stretches, a copies of N^D's and b copies of E^D's, then

$$
\begin{aligned}
u(m,n,h,a,b) &= \binom{n-1}{h-1}\binom{n-a}{a}\binom{m-b}{b}(\binom{m-b-1}{h-2} \\
&+2\binom{m-b-1}{h-1}+\binom{m-b-1}{h}) \\
&= \binom{n-1}{h-1}\binom{n-a}{a}\binom{m-b}{b}\binom{m-b+1}{h}.
\end{aligned}
$$

Example: $u(1,1,1,0,0) = 2, u(2,1,1,0,0) = 3, u(2,1,1,0,1) = 2$, $u(1,2,1,1,0) = 2, u(2,2,1,1,1) = 2, u(2,2,1,1,0) = 3, u(6,6,3,2,2) = 3600$.

Let $u(m,n,h,t)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches and t copies of N^D's and E^D's, then

$$u(m,n,h,t) = \sum_{a=0}^{\min\{t,\lfloor n/2\rfloor,n-h\}} \binom{n-1}{h-1}\binom{n-a}{a}\binom{m-t+a}{t-a}\binom{m-t+a+1}{h}.$$

Example: $u(0,1,1,0) = 1$, $u(0,2,1,0) = 1$, $u(1,1,1,0) = 2$, $u(2,1,1,1) = 2$, $u(2,2,1,2) = 2$, $u(2,2,2,1) = 1$, $u(2,2,2,0) = 3$, $u(6,6,3,3) = 9390$.

Let $v(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t copies of N^D's and E^D's, then (for $n > 0$)

$$v(m,n,t) = \sum_{h=0}^{n} \sum_{a=0}^{\min\{t,\lfloor n/2\rfloor,n-h\}} \binom{n-1}{h-1}\binom{n-a}{a}\binom{m-t+a}{t-a}\binom{m-t+a+1}{h}.$$

Example: $v(1,1,0) = 2, v(2,1,1) = 2, v(2,2,2) = 2, v(3,1,1) = 6, v(3,3,2) = 36$.

Let $v(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n), then (for $n > 0$)

$$v(m,n) = \sum_{t=0}^{\lfloor m/2\rfloor+\lfloor n/2\rfloor} \sum_{h=0}^{n} \sum_{a=0}^{\min\{t,\lfloor n/2\rfloor,n-h\}} \binom{n-1}{h-1}\binom{n-a}{a}\binom{m-t+a}{t-a}\binom{m-t+a+1}{h}$$

Example: $v(1,1) = 2, v(1,2) = 5, v(2,2) = 14$

Let $v(N)$ denote the number of walks of length N, then

$$\begin{aligned} v(N) = \sum_{n=1}^{N} & \sum_{t=0}^{\lfloor (N-n)/2\rfloor+\lfloor n/2\rfloor} (\sum_{h=0}^{n}(\sum_{a=0}^{\min\{t,\lfloor n/2\rfloor,n-h\}} \binom{n-1}{h-1}\binom{n-a}{a} \\ & \times \binom{N-n-t+a}{t-a}\binom{N-n-t+a+1}{h})) \\ & + \sum_{i=0}^{\lfloor (N-n)/2\rfloor} \binom{N-n-i}{i}. \end{aligned}$$

Example: $v(1)=2$, $v(2)=6$.

Let $V(n,m)$ denote the number of walks of length n and height m, then

$$\begin{aligned}
v(m,n) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{j}{n-j}\sum_{a=0}^{m-i}\binom{j+a-1}{a}\binom{i+a}{m-i-a} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{j}{n-j}\sum_{a=0}^{m-i}\binom{j+a-1}{a}\binom{i+a}{m-i-a} \\
&= \sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}\binom{m+n-i-j}{m-i}\binom{m-i}{i} \\
&= \sum_{j=0}^{n}\binom{j}{n-j}\sum_{i=0}^{n+j}\binom{i}{j}\binom{i-j}{m-i+j}.
\end{aligned}$$

The generating function:

Let

$$\varphi(t,x) = \sum_{m\geq 0, n\geq 0} v(m,n)t^m x^n.$$

Then

$$\varphi(t,x) = \frac{1}{1-t-t^2-x-x^2}$$

and

$$\varphi(t,0) = \frac{1}{1-t-t^2}$$

$$\varphi(0,x) = \frac{1}{1-x-x^2}$$

$\varphi(0,x)$ is the generating function of Fibonacci numbers.

The generating function of $\{c(m,1)\}$ is

$$\frac{x}{(1-x-x^2)^2}.$$

Let $V(N,m)$ denote the number of walks of length N and height m, then the generating function of $\{V(N,m)\}$ is

$$\frac{1}{1-t-t^2-tx-t^2x^2}.$$

Also, $V(m,n)=0$, if $m<n$

$$V(m,0)=F_{m+1}$$

$$V(m,m)=F_{m+1}$$

$$V(m,n)=V(m-1,n)+V(m-2,n)+V(m-1,n-1)+V(m-2,n-2)$$

$$\begin{aligned} v(N) &= \sum_{n=0}^{N}\sum_{i=0}^{\lfloor N/2\rfloor}\sum_{j=0}^{\lfloor N/2\rfloor}\binom{n-j}{j}\binom{N-i-j}{n-j}\binom{N-n-i}{i} \\ &= \sum_{i=0}^{N}2^i\binom{i}{N-i} \end{aligned}$$

The matrix $(V(m,n))_{m,n\geq 0}$ is not an Riordan matrix.

3.6 Riordan Matrix [45], [55]

Consider an infinite matrix $U=(m_{ij})_{,i,j\geq 0}$ with entries in $\mathbb{C}$ ($\mathbb{C}$ the complex numbers). Let $C_i(x)=\sum_{n\geq 0}^{\infty}m_{n,i}x^n$ be the generating function of the $i-$th column of U. We assume that

$$C_i(x)=g(x)[f(x)]^i$$

where

$$g(x)=1+g_1x+g_2x^2+g_3x^3+...$$

$$f(x)=x+f_1x+f_2x^2+f_3x^3+...$$

In this case we write $U = (g(x), f(x))$ and say that $(g(x), f(x))$ is a Riordan matrix.

Example:

$$P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$$

$$= \begin{matrix} 1 & 0 & & & & & . \\ 1 & 1 & 0 & & & & . \\ 1 & 2 & 1 & 0 & & & . \\ 1 & 3 & 3 & 1 & 0 & & . \\ 1 & 4 & 6 & 4 & 1 & 0 & . \\ . & . & . & . & . & . & . \end{matrix}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + ...$$
$$\frac{1}{1-z}\frac{z}{1-z} = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + ...$$
$$\frac{1}{1-z}(\frac{z}{1-z})^2 = z^2 + 3z^3 + 6z^4 + 10z^5 + ...$$
$$\frac{1}{1-z}(\frac{z}{1-z})^3 = z^3 + 4z^4 + 10z^5 + 20z^6 + ...$$

CHAPTER 4

PRUDENT SELF-AVOIDING WALKS

A self-avoiding walk (SAW) is a sequence of moves on a lattice not visiting the same point more than once. A SAW on the square lattice is prudent if it never takes a step towards a vertex it has already visited. Some problems and some sequences arising from prudent walks are discussed in this part.

4.1 Introduction

A well-known long standing problem in combinatorics and statistical mechanics is to find the generating function for self-avoiding walks (SAW) on a two-dimensional lattice, enumerated by perimeter. A SAW is a sequence of moves on a square lattice which does not visit the same point more than once. It has been considered by more than one hundred researchers in the pass one hundred years, including George Polya, Tony Guttmann, Laszlo Lovasz, Donald Knuth, Richard Stanley, Doron Zeilberger, Mireille Bousquet-Mélou, Thomas Prellberg, Neal Madras, Gordon Slade, Agnes Dittel, E.J. Janse van Rensburg, Harry Kesten, Stuart G. Whittington, Lincoln Chayes, Iwan Jensen, Arthur T. Benjamin, and others. More than three hundred papers and a few volumes of books were published in this area. A SAW is interesting for simulations because its properties cannot be calculated analytically. Calculating the number of self-avoiding walks is a common computational problem [20], [22], [29]. In the past few decades, many mathematicians have studied the following two problems:

Problem 1

What is the number of SAWs from $(0,0)$ to $(n-1,n-1)$ in an $n \times n$ grid, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Donald Knuth claimed that the number is between 1.3×10^{24} and 1.6×10^{24} for $n = 11$ and he did not believe that he would ever in his lifetime know the exact answer (which appeared in [27] after 28 years) to this problem in 1975. However, after a few years, Richard Schroeppel pointed out that the exact value is $1,568,758,030,464,750,013,214,100 = 2^2 3^2 5^2 31 \times 115\,422\,379 \times 487\,148\,912\,401$, [3], [26], [27], [25] . It is still an unsolved problem for $n > 20$.

Problem 2

What is the number $f(n)$ of n-step SAWs, on the square lattice, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

The number $f(n)$ is known for $n \leq 91$ [3], [9], [25], [24].

It is clear that

$$2^n \leq f(n) \leq 4 \times 3^{n-1}$$

$$f(m+n) \leq f(m)f(n)$$

There exists a constant C such that

$$\lim_{n \to \infty} f(n)^{1/n} = \inf_n [f(n)]^{1/n} = C.$$

$C = 2.64$ (up to 71 steps have been counted).

$C = 2.638$ (up to 91 steps have been counted).

$$f(n) \approx 2.638^n$$

The number of SAWs/ the number of total walks:

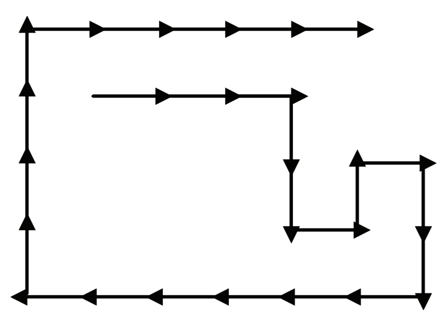
$\frac{1}{1200}$ for $n = 20$

$\frac{1}{2.4\times10^8}$ for $n = 50$

A recently proposed model called prudent self-avoiding walks (PSAW) was first introduced to the mathematics community in an unpublished manuscript of Préa, who called them *exterior walks*. A prudent walk is a connected path on square lattice such that, at each step, the extension of that step along its current trajectory will never intersect any previously occupied vertex. Such walks are clearly self-avoiding [4], [10], [11], [19], [40], [16]. We will talk about around 30 integer sequences arising from PSAWs in the following. Three theorems are proved and other theorems could be proved similarly in this paper.

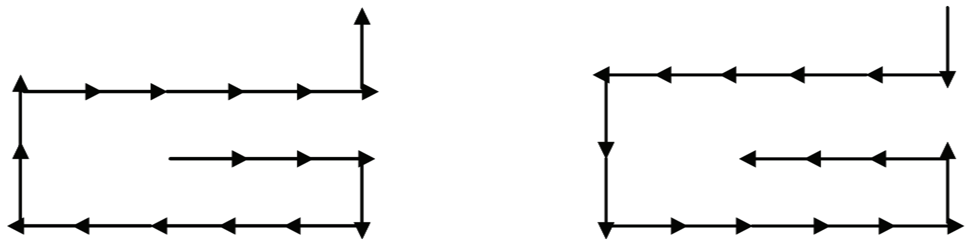
4.2 Prudent Self-Avoiding Walks: Definitions and Examples

A PSAW is a proper subset of SAWs on the square lattice. The walk starts at $(0, 0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. Hence the name prudent. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex. The following walk is a PSAW.



4.3 Properties of a PSAW

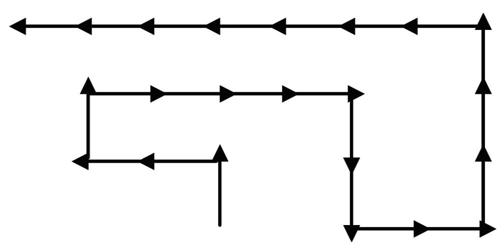
Unlike SAWs, PSAWs are usually not reversible. There is such an example in the following figure.



Each PSAW possesses a minimum bounding rectangle, which we call *box*. Less obviously, the endpoint of a prudent walk is always a point on the boundary of the box. Each new step either inflates the box or walks (prudently) along the border. After an inflating step, there are 3 possibilities for a walk to go on. Otherwise, only 2.

In a *one-sided* PSAW, the endpoint lies always on the top side of the box. The walk is *partially directed*.

A prudent walk is *two-sided* if its endpoint lies always on the top side, or on the right side of the box. The walk in the following figure is a two-sided PSAW.



4.4 One-sided PSAWs

Theorem 33. *The generating function of the number, say $f(n,k)$, of the one-sided n-step prudent walks, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding k or more consecutive east steps, $\rightarrow^{\geq k}$ satisfies*

$$\frac{1+t-t^k}{1-2t-t^2+t^{k+1}},$$

and for $k \geq 2$,

$$f(n,k) = \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{\frac{n-j-i}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i}{k-1}}$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-1}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-i-j-1}{k-1}}$$
$$-\sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-k}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i-k}{k-1}}$$

$$f(n,1) = 2^n.$$

Proof. Let $F(t)$ denote the length generating function of the number of one-sided prudent walks, avoiding k or more consecutive east steps. We have the following three cases.

(1) For the walks which do not contain North steps, they can be empty walk, walks with only west steps, walks with only east steps with length at least one and at most $k-1$, the contributions are 1, $\frac{t}{1-t}$, $\frac{t(1-t^{k-1})}{1-t}$ respectively.

(2) For the walks obtained by concatenating a one-sided walk, a North step, and then a West walk, the contribution is

$$F(t)\frac{t}{1-t}.$$

(3) For the walks obtained by concatenating a one-sided walk, a North step, and then a East walk with at least 1 step and at most $k-1$ steps, the contribution is

$$F(t)\frac{t^2(1-t^{k-1})}{1-t}.$$

Adding these three contributions give the equation

$$F(t) = 1 + \frac{t}{1-t} + \frac{t(1-t^{k-1})}{1-t} + F(t)\frac{t}{1-t} + F(t)\frac{t^2(1-t^{k-1})}{1-t}.$$

Thus,

$$F(t) = \frac{1+t-t^k}{1-2t-t^2+t^{k+1}}.$$

Now, let $[t^n]F(t)$ denote the coefficient of t^n in the power series expansion of $F(t)$.

$$\begin{aligned}
&[t^n]\frac{1+t-t^k}{1-2t-t^2+t^{k+1}} \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}(2t+t^2-t^{k+1})^i \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(2t)^{i-j}(t^2-t^{k+1})^j \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(2t)^{i-j}\sum_{l=0}^{j}\binom{j}{l}(t^2)^{j-l}(-1)^l t^{(k+1)l} \\
&= [t^n](1+t-t^k)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}(-1)^{-l}\, t^{i+j-l+lk}2^{i-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{\frac{n-j-i}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i}{k-1}} \\
&+ \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-1}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-i-j-1}{k-1}} \\
&- \sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^{\frac{n-j-i-k}{k-1}}2^{i-j}\binom{i}{j}\binom{j}{\frac{n-j-i-k}{k-1}}.
\end{aligned}$$

□

If $k = 2$, we obtain sequence $[58, A006356]$:

$1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, 4004, 8997, 20216, \ldots$

It also counts the number of paths for a ray of light that enters two layers of glass and then is reflected exactly n times before leaving the layers of glass.

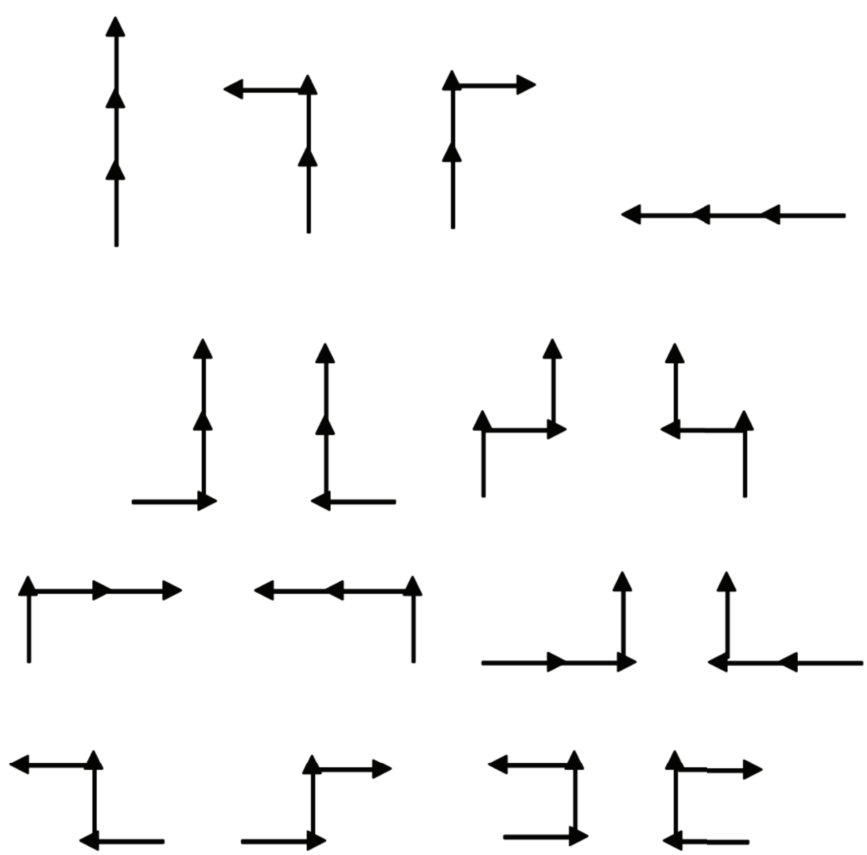
If $k = 3$, we obtain sequence [58, A052967]:

$$1, 3, 7, 16, 38, 89, 209, 491, 1153, 2708, 6360, ...$$

If $k = 4$, we obtain sequence [58, A190360]:

$$1, 3, 7, 17, 40, 96, 229, 547, 1306, 3119, 3119, 7448, ...$$

Example 34. *For the case $k = 3$ in the above theorem, there are* 16 *walks as follows:*



Theorem 35. *The number, say $f(n)$, of one-sided n-step prudent walks, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, equals*

$$\begin{aligned} f(n) &= 2f(n-1) + f(n-2) \\ &= \frac{(1-\sqrt{2})^n + (1+\sqrt{2})^n}{2} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \sum_{i=0}^{n} \binom{i}{n-i} 2^{2i-n} + \sum_{i=0}^{n-1} \binom{i}{n-i-1} 2^{2i-n+1} \\ &= \sum_{j=0}^{n-1} \frac{3n-4j}{n-j} 2^{n-2j-1} \binom{n-j}{j} \quad \textit{for } n > 0. \end{aligned}$$

The generating function of $f(n)$ satisfies [4], [46]:

$$\begin{aligned} \sum_{n \geq 0} f(n)\, t^n &= \frac{1+t}{1-2t-t^2} \\ &= 1 + 3t + 7t^2 + 17t^3 + 41t^4 + 99t^5 + 239t^6 + ... \end{aligned}$$

The above five formulas for $f(n)$ could be derived from the generating function which could be obtained in the similar way as in the previous theorem, or simply set $k = +\infty$.

Example:

$$
\begin{aligned}
&[t^n]\frac{1+t}{1-2t-t^2} \\
&= [t^n](1+t)\sum_{i=0}^{\infty} t(2+t)^i \\
&= [t^n](1+t)\sum_{i=0}^{\infty} t^i \sum_{j=0}^{i} \binom{i}{j} 2^{i-j} t^j \\
&= [t^n](1+t)\sum_{i=0}^{\infty}\sum_{j=0}^{i} \binom{i}{j} 2^{i-j} t^{i+j} \\
&= \sum_{i=0}^{n} \binom{i}{n-i} 2^{2i-n} + \sum_{i=0}^{n-1} \binom{i}{n-i-1} 2^{2i-n+1}
\end{aligned}
$$

We obtain sequence [58, A001333]:

$$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, \ldots$$

Theorem 36. *The number of one-sided n-step prudent walks, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$ is*

$$1 + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \sum_{i=1}^{\min\{n-2k,k\}} \binom{n-2k+1}{i} \binom{k-1}{k-i} \binom{n-k-i}{k}.$$

Proof. In our proof, we will use the following two results which could be found in some mathematics books such as [41]:

The number of ways of putting n like objects into r different cells is

$$\binom{n+r-1}{n} = \binom{n+r-1}{r-1}.$$

It is also the number of nonnegative integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

The number of ways of putting n like objects into r different cells with no empty cell is

$$\binom{n-1}{r-1}.$$

It is also the number of positive integer solutions to the equation

$$\sum_{i=1}^{r} x_i = n.$$

Without loss of generality, we assume that there are k East steps, k West steps and $n-2k$ North steps in a one-sided n-step prudent walks, starting from $(0,0)$ and ending on the y-axis. We also assume that $k > 0$ since there is only one such walk for $k = 0$. It is easy to see that $k \leq \lfloor (n-1)/2 \rfloor$. The $n-2k$ North steps provide $n-2k+1$ positions (we can say $n-2k+1$ different cells) for k East steps and k West steps to be inserted. Suppose that we put k East steps into i $(1 \leq i \leq \min\{n-2k, k\})$ cells with no empty cell. Then there are $\binom{k-1}{k-1}$ ways of putting k East steps into i cells and $\binom{n-2k+1}{i}$ ways of choosing i cells. Now we distribute k West steps into the remaining $n-2k+1-i$ cells, which give us $\binom{n-k-i}{k}$.

Therefore, we get the number:

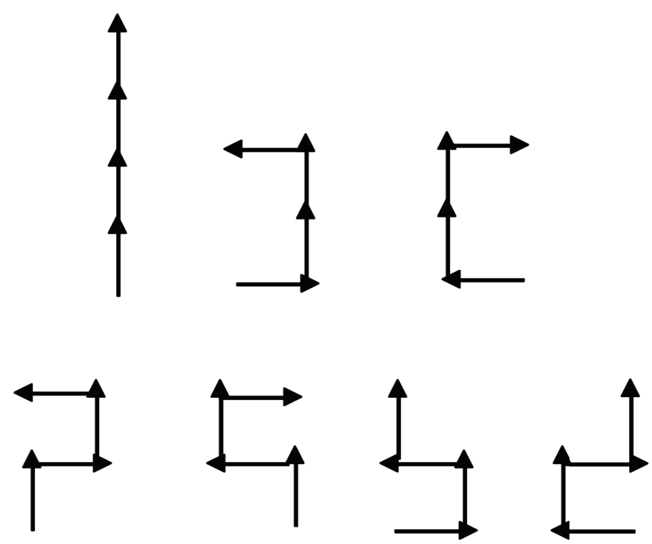
$$1 + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \sum_{i=1}^{\min\{n-2k,k\}} \binom{n-2k+1}{i} \binom{k-1}{k-i} \binom{n-k-i}{k}.$$

□

We obtain sequence [58, A136029]:

$$1, 1, 1, 3, 7, 15, 33, 75, 171, 391, 899, 2077, 4815, 11195, 26097, \ldots$$

Example 37. *For $n = 4$ in the above theorem, we have 7 such walks as follows:*



Theorem 38. *The number of one-sided prudent walks starting from $(0, 0)$ to (x, y), taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, with $k + x$ East Steps, k West steps and y North steps, is*

$$\sum_{i=1}^{\min\{y,k+x\}} \binom{y+1}{i} \binom{k+x-1}{k+x-i} \binom{y+k-i}{k}. \tag{4.1}$$

Proof. The y North steps in a walk provide $y+1$ positions (we can say $y+1$ different cells) for $k + x$ East steps and k West steps to be distributed. Suppose that we put $k + x$ East steps into i cells ($1 \leq i \leq \min\{y, k + x\}$) with no empty cells, which gives $\binom{k+x-1}{k+x-i}$. We have $\binom{y+1}{i}$ ways to choose i cells from $y + 1$ cells. And there are $\binom{y+k-i}{k}$

ways of putting k West steps into the remaining $y+1-i$ cells. Then we get (4.1). $\square$

For $k=2$ and $x=y=n$, the number is

$$\sum_{i=1}^{n}\binom{n+1}{i}\binom{n+1}{i-1}\binom{n+2-i}{2}$$
$$=\frac{(n+1)\,(2n-1)!}{(n-1)!^2}.$$

Associated with the sequence [58, A119578]:

$$0, 2, 18, 120, 700, 3780, 19404, 96096, 463320, 2187900, ...$$

For $k=3$ and $x=y=n$, the number is

$$\frac{(n+1)\,(2+n)\,(2n-1)!}{3(n-1)!^2},$$

associated with the sequence [58, A189769]:

$$2, 24, 200, 1400, 8820, 51\,744, 288\,288, 1544\,400, ...$$

For $k=4$ and $x=y=n$, the number is

$$\frac{(n+1)\,(2+n)\,(n+3)\,(2n)!}{24n\,(n-1)!^2},$$

associated with the sequence [58, A189770]:

$$2, 30, 300, 2450, 17640, 116\,424, 720\,720, ...$$

For $k=\lfloor n/2\rfloor$ and $x=y=n$, the number is

$$\sum_{i=1}^{n}\binom{n+1}{i}\binom{\lfloor n/2\rfloor+n-1}{i-1}\binom{n+\lfloor n/2\rfloor-i}{\lfloor n/2\rfloor},$$

associated with the sequence [58, A190425]:

$$2, 12, 60, 700, 3780, 51\,744, 288\,288, 4247\,100, 24\,066\,900, \ldots$$

Theorem 39. *The number of one-sided n-step prudent walks, from $(0,0)$ to (x,y), ($n-x-y$ is even) taking steps from $\{\uparrow,\leftarrow,\rightarrow\}$ is*

$$\sum_{i=0}^{\min\{y,\frac{n+x-y}{2}\}}\binom{y+1}{i}\binom{\frac{n+x-y}{2}-1}{\frac{n+x-y}{2}-i}\binom{\frac{n-x+y}{2}-i}{\frac{n-x-y}{2}}. \tag{4.2}$$

Proof. We assume that there are $\frac{n-x-y}{2}$ West steps, $\frac{n+x-y}{2}$ East steps and y North steps in a walk from $(0,0)$ to (x,y). The y North steps in a walk provide $y+1$ positions (i.e., $y+1$ cells) for $\frac{n+x-y}{2}$ East steps, and $\frac{n-x-y}{2}$ West steps are to be distributed. Suppose that we put $\frac{n+x-y}{2}$ East steps into i cells ($1\le i\le \min\{y,\frac{n+x-y}{2}\}$) with no empty cells, which gives $\binom{\frac{n+x-y}{2}-1}{\frac{n+x-y}{2}-i}$. We have $\binom{y+1}{i}$ ways to choose i cells from $y+1$ cells. And there are $\binom{\frac{n-x+y}{2}-i}{\frac{n-x-y}{2}}$ ways of putting $\frac{n-x-y}{2}$ West steps into the remaining $y+1-i$ cells. Then we get (4.2). □

For $x=y=3$, we obtain the sequence [58, A163761]:

$$0, 20, 60, 120, 200, 300, 420, 560, 720, 900, 1100, 1320, \ldots$$

Theorem 40. *The number, say $f(n)$, of generalized one-sided n-step prudent walks, taking* *steps from* $\{\uparrow, \leftarrow, \rightarrow, \nearrow\}$ *equals*

$$\sum_{i=0}^{n} \binom{i}{n-i} 2^{n-i}(3)^{2i-n} + \sum_{i=0}^{n-1} \binom{i}{n-i-1} 2^{n-i-1}(3)^{2i-n+1}$$
$$= \frac{5+\sqrt{17}}{2\sqrt{17}} \left(\frac{3+\sqrt{17}}{2}\right)^n - \frac{5-\sqrt{17}}{2\sqrt{17}} \left(\frac{3-\sqrt{17}}{2}\right)^n,$$

with generating function

$$\frac{1+t}{1-3t-2t^2}.$$

Proof. Let $P(t)$ denote the length generating function of generalized one

-sided prudent walks.

The contribution in $P(t)$ of walks that do not contain North steps or Northeast steps (horizontal walks) is

$$\frac{1+t}{1-t}.$$

The contribution of walks obtained by concatenating a generalized one-sided walk, a North step or Northeast step, then a horizontal walk is

$$\frac{2t(1+t)}{1-t} P(t).$$

Adding these two contributions gives a linear equation for $P(t)$:

$$P(t) = \frac{1+t}{1-t} + \frac{2t(1+t)}{1-t} P(t).$$

Therefore,

$$\begin{aligned}
P(t) &= \frac{1+t}{1-3t-2t^2} \\
&= (1+t)\sum_{i=0}^{+\infty}(3t+2t^2)^i \\
&= (1+t)\sum_{i=0}^{+\infty}\binom{i}{j}(3t)^{i-j}(2t^2)^j \\
&= (1+t)\sum_{i=0}^{+\infty}\sum_{j=0}^{i}\binom{i}{j}2^j(3)^{i-j}t^{i+j}
\end{aligned}$$

$$\begin{aligned}
f(n) &= [t^n]P(t) \\
&= \sum_{i=0}^{n}\binom{i}{n-i}2^{n-i}(3)^{2i-n} + \sum_{i=0}^{n-1}\binom{i}{n-i-1}2^{n-i-1}(3)^{2i-n+1}.
\end{aligned}$$

The second formula of $f(n)$ can be easily derived from the length generating function. □

We obtain sequence [58, A055099]:

$$1, 4, 14, 50, 178, 634, 2258, 8042, 28642, 102010, 363314, ...$$

Example 41. *For $n = 2$ in the above theorem, we have 14 such walks:*
EN, NE, WN, NW, $N(NE)$, $(NE)N$, $E(NE)$, $(NE)E$, $(NE)W$, $W(NE)$, NN, WW, EE, $(NE)(NE)$.

Theorem 42. *The generating function of the number, say $f(n)$, of generalized one-sided n-step prudent walks, taking steps from* $\{\rightarrow, \leftarrow, \uparrow, \nearrow, \nwarrow\}$ *is*

$$\begin{aligned}&\frac{1+t}{1-4t-3t^2}\\ &= 1+5t+23t^2+107t^3+497t^4+2309t^5+10\,727t^6+49\,835t^7+...\\ f(n) &= [t^n]\,(1+t)\sum_{k\geq 0} t^k\,(4+3t)^k\\ &= [t^n]\,(1+t)\sum_{k\geq 0}\sum_{m=0}^{k}\binom{k}{m}4^{k-m}3^m t^{m+k}\\ &= \sum_{k=0}^{n}\left[\binom{k+1}{n-k}3+\binom{k}{n-1-k}\right]4^{2k-n}3^{n-1-k}.\end{aligned}$$

Proof. Let $P(t)$ denote the length generating function of generalized one

-sided prudent walks. The contribution in $P(t)$ of walks that do not contain North steps or Northeast steps, or Northwest step (horizontal walks) is

$$\frac{1+t}{1-t}.$$

The contribution of walks obtained by concatenating a generalized one-sided walk, a North step or Northeast step or a Northwest step, then a horizontal walk is

$$\frac{3t(1+t)}{1-t}P(t).$$

Adding these two contributions gives a linear equation for $P(t)$, from which we can get $P(t)$. □

We obtain sequence [58, A126473].

Theorem 43. *The generating function of the number, say $f(n)$, of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y-axis,* *taking steps from* $\{\uparrow, \leftarrow, \rightarrow\}$ *equals*

$$\frac{1}{2t^3}((1+t)(1-t)^2 - \sqrt{(1-t^4)(1-2t-t^2)}),\ \textit{and}$$

$$f(n) = -\sum_{i=0}^{n+3}\binom{1/2}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}$$
$$\binom{l}{(n-2k-i-j+3-l)}(-1)^l\, 2^{2l-n+2k+2i-4}.$$

With the associated sequence [58, A004149]:

$$1, 1, 1, 2, 4, 8, 16, 33, 69, 146, 312, 673, 1463, ...$$

A proof of this theorem involving Finite Operator Calculus [15] is in [17].

We can get $f(n)$ from the generating function as follows:

$$\frac{1}{2t^3}((1+t)(1-t)^2 - \sqrt{(1-t^4)(1-2t-t^2)}).$$
$$= \frac{1}{2t^3}((1+t)(1-t)^2 - \sqrt{1-2t-t^2-t^4+2t^5+t^6})$$
$$= \frac{1}{2t^3}((1+t)(1-t)^2 - \sum_{i=0}^{\infty}\binom{1/2}{i}(-1)^i t^i\left(2+t+t^3-2t^4-t^5\right)^i)$$
$$= \frac{1}{2t^3}(1+t)(1-t)^2$$
$$-\sum_{i=0}^{\infty}\binom{1/2}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}$$
$$(-1)^l\, 2^{l-m+i-j-1} t^{m+2k+i+j-3+l}$$
$$f(n) = [t^n](\frac{1}{2t^3}(1+t)(1-t)^2$$
$$-\sum_{i=0}^{\infty}\binom{1/2}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}$$
$$(-1)^l\, 2^{l-m+i-j-1} t^{m+2k+i+j-3+l})$$

For $n > 0$, we have $f(n) =$

$$-\sum_{i=0}^{n+3} \binom{1/2}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \binom{l}{n-2k-i-j+3-l}$$

$\times (-1)^l \, 2^{2l-n+2k+2i-4}$.

Theorem 44. *The generating function of the number, say $f(n,k)$, of one-sided n-step prudent walks* exactly *avoiding $\leftarrow^{=k}$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$ equals*

$$\frac{1+t-t^k+t^{k+1}}{1-2t-t^2+t^{k+1}-t^{k+2}}, \text{ and}$$

$f(n,k) = g(n,k) + g(n-1,k) - g(n-k,k) + g(n-k-1,k)$, *where*

$$g(n,k) = \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{j}{l} \sum_{m=0}^{l} \binom{l}{m} 2^{l-m} \binom{j-l}{n-j-m-k(j-l)} (-1)^{j-l-n+j+m+k(j-l)}.$$

Proof. Let $P(t)$ denote the length generating function of generalized one-sided prudent walks.

The contribution in $P(t)$ of walks that do not contain North steps or Northeast steps (horizontal walks) is

$$\frac{1+t}{1-t} - t^k.$$

The contribution of walks obtained by concatenating a generalized one-sided walk, a North step or Northeast step, then a horizontal walk is

$$\frac{t(1+t)}{1-t} P(t) - t^{k+1} P(t).$$

Adding these two contributions gives a linear equation for $P(t)$:

$$P(t) = \frac{1+t}{1-t} - t^k + \frac{t(1+t)}{1-t} P(t) - t^{k+1} P(t).$$

Therefore,

$$P(t) = \frac{1 + t - t^k + t^{k+1}}{1 - 2t - t^2 + t^{k+1} - t^{k+2}}.$$

Now we derive $f(n, k)$ from the generating function:

$$\begin{aligned}
&[t^n]\left(1 - 2t - t^2 + t^{k+1} - t^{k+2}\right)^{-1} \\
&= [t^n] \sum_{j \geq 0} t^j \left(2 + t - t^k + t^{k+1}\right)^j \\
&= [t^n] \sum_{j \geq 0} t^j \sum_{l=0}^{j} \binom{j}{l} (2 + t)^l \left(-t^k + t^{k+1}\right)^{j-l} \\
&= \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{j}{l} \sum_{m=0}^{l} \binom{l}{m} 2^{l-m} \binom{j-l}{n - j - m - k(j-l)} (-1)^{j-l-n+j+m+k(j-l)} = g(n, k) \\
&[t^n]\frac{1 + t - t^k + t^{k+1}}{1 - 2t - t^2 + t^{k+1} - t^{k+2}} \\
&= [t^n](1 + t - t^k + t^{k+1})\left(1 - 2t - t^2 + t^{k+1} - t^{k+2}\right)^{-1} \\
&= g(n, k) + g(n-1, k) - g(n-k, k) + g(n-k-1, k).
\end{aligned}$$

□

If $k = 1$, we obtain sequence [58, A190512]:

$$1, 2, 5, 11, 24, 53, 117, 258, 569, 1255, 2768, 6105, ...$$

If $k = 2$, we obtain sequence [58, A190525]:

$$1, 3, 6, 15, 34, 80, 185, 431, 1001, 2328, 5411, 12580, ...$$

If $k = 3$, we obtain sequence [58, A190528]:

$$1, 3, 7, 16, 39, 92, 219, 521, 1238, 2944, 6999, 16640, ...$$

Theorem 45. *The generating function of the number, say $f(n,k)$, of one-sided n-step prudent walks exactly avoiding $\leftarrow^{=k}$ and $\uparrow^{=k}$ (both at the same time) satisfies*

$$\frac{1+t-2t^k+2t^{k+1}}{1-2t-t^2+2t^{k+1}-2t^{k+2}}, \text{ and}$$

$$f(n,k) = g(n,k)+g(n-1,k)-2g(n-k,k)+2g(n-k-1,k), \text{ where}$$

$$g(n,k) = \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}\binom{l}{-i-j+l-kl+n}(-1)^{i+j+kl-n}\,2^{i-j+l}.$$

This theorem can be proved in a similarly as the above theorem.

Now, we derive $f(n,k)$ from the generating function:

$$\begin{aligned}
&[t^n]\frac{1}{1-2t-t^2+2t^{k+1}-2t^{k+2}} \\
&= [t^n]\sum_{i=0}^{\infty}(2t+t^2-2t^{k+1}+2t^{k+2})^i \\
&= [t^n]\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{u=0}^{k}\binom{i}{j}\binom{j}{l}\binom{l}{u}(-1)^{l-u}\,2^{i-j+l}t^{i+j-l+kl+u} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}\binom{l}{-i-j+l-kl+n}(-1)^{i+j+kl-n}\,2^{i-j+l} = g(n,k) \\
&[t^n]\frac{1+t-2t^k+2t^{k+1}}{1-2t-t^2+2t^{k+1}-2t^{k+2}} \\
&= g(n,k)+g(n-1,k)-2g(n-k,k)+2g(n-k-1,k) = f(n,k).
\end{aligned}$$

For $k=1$,

$$f(n) = \left(2^{n+2}-(-1)^{\lfloor n/2\rfloor}+2(-1)^{\lfloor (n+1)/2\rfloor}\right)/5,$$

also,

$$f(n) = 2f(n-1) - f(n-2) + 2f(n-3)$$
$$\text{with } f(1) = 1,\ f(2) = 3,\ f(3) = 7.$$

Associated with sequence [58, A190569]:

$$1, 1, 3, 7, 13, 25, 51, 103, 205, 409, 819, 1639, 3277, 6553, ...$$

For $k = 2$, we obtain sequence [58, A190570]:

$$1, 3, 5, 13, 27, 63, 137, 309, 683, 1527, 3393, ...$$

For $k = 3$, we obtain sequence [58, A190571]:

$$1, 3, 7, 15, 37, 85, 199, 467, 1089, 2549, 5959, ...$$

Theorem 46. *The number of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$ and $\rightarrow^{\geq 2}$is*

$$\sum_{j=0}^{\lfloor \frac{n+1}{4} \rfloor} \frac{1}{j+1} \binom{2j}{j} \binom{n-2j+1}{2j}.$$

Proof. We suppose that there are j copies of East steps, j copies of West steps and $n - 2j$ copies of North steps.

Now we arrange j copies of East steps and j copies of West steps according to: the total number of the East steps is greater or equal to the total number of West steps from $(0,0)$ to any lattice point on a walk, which gives C_j, the j-th Catalan number.

The $n - 2j$ North steps in a walk provide $n - 2j + 1$ positions (i.e., cells) for j

East steps and j West steps to be distributed, with each cell containing at most 1 East step or 1 West step. Then $0 \leq j \leq \lfloor \frac{n+1}{4} \rfloor$. There are $\binom{n-2j+1}{2j}$ way to choose $2j$ cells for the j East steps and j West steps.

Therefore,

$$\sum_{j=0}^{\lfloor \frac{n+1}{4} \rfloor} \frac{1}{j+1} \binom{2j}{j} \binom{n-2j+1}{2j}.$$

□

Theorem 47. *The number of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$ and $\rightarrow^{\geq 2}$is*

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{k}{\lfloor k/2 \rfloor} \binom{n-k+1}{k}.$$

Proof. We suppose that the total number of East steps and West steps in a walk is k. Then there are $n-k$ North steps.

Now we arrange those k East steps and West steps according to: the total number of the East steps is greater or equal to the total number of West steps from $(0,0)$ to any lattice point on a walk, which gives $\binom{k}{\lfloor k/2 \rfloor}$.

The $n-k$ North steps in a walk provide $n-k+1$ positions (i.e., cells) for k East steps and West steps to be distributed, with each cell containing at most 1 East step or 1 West step. Then $0 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. There are $\binom{n-k+1}{k}$ way to choose k cells for the k East steps and West steps.

Therefore,

$$f(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{k}{\lfloor k/2 \rfloor} \binom{n-k+1}{k}.$$

□

4.5 Problems Generalized from S. C. Locke's Snakes [51]:

Problem 48. *What is the number of non-self-intersecting walks of length n on square lattice such that at each point the angle has 90 degrees (first angle considered to the left - if allowed to both left and right the number is doubled)? D. Dima and S. C. Locke posted [58, A189722] on April 25-26, 2011:*

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 141, 226, 362, 580, 921, 1468, 2344, 3740, 5922, 9413,$$
$$14978, 23829, 37686, 59770, 94882, 150606, 237947, 376784, 597063, 946086, 1493497$$

It is also the number of snakes composed of n identical segments such that it starts with a left turn and snake's other $(n-2)$ joints are bent in 90-degree angles, either to the left or to the right, in such a way that the snake does not overlap.

The number of lattice paths of length n which contain only single East and North steps is 2.

Theorem 49. *The number of n-step self-avoiding walks which contain only single East, West and North steps is*

$$2^{\lfloor n/2 \rfloor} + 2^{\lfloor (n+1)/2 \rfloor},$$

generating function satisfies

$$\frac{1 + 3t + 2t^2}{1 - 2t^2}.$$

Theorem 50. *The number, say $f(n)$, of one-sided n-step prudent walks in the first quadrant, starting from $(0, 0)$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$, $\uparrow^{\geq 2}$, and $\rightarrow^{\geq 2}$is*

$$f(2n) = 2\binom{n}{\lfloor n/2 \rfloor},$$

$$f(2n+1) = \binom{n+1}{\lfloor (n+1)/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. We suppose that the total number of East steps and West steps in a walk is k. Then there are $n-k$ North steps.

Now we arrange those k East steps and West steps according to the following:

1. The total number of the East steps is greater or equal to the total number of West steps from $(0,0)$ to any lattice point on a walk, which gives $\binom{k}{\lfloor k/2 \rfloor}$.

2. The $n-k$ North steps in a walk provide $n-k+1$ positions (i.e., cells, we say, the first cell, the second cell,..., the $n-k+1$-th cell) for k East steps and West steps that are to be distributed, with each cell except the first and the last cell, containing exactly 1 East step or 1 West step. We have three cases:

Case one: If there is no empty cell, $k = \frac{n+1}{2}$, then the contribution of this case is

$$\binom{(n+1)/2}{(n+1)/4}.$$

Case two: If exactly one cell is empty (the first cell or the last cell), $k = \frac{n}{2}$, then the contribution of this case is

$$2\binom{n/2}{n/4}.$$

Case three: If the first cell or the last cell both are empty, $k = \frac{n}{2}$, then the contribution of this case is

$$\binom{(n-1)/2}{(n-1)/4}.$$

Therefore,

$$f(n) = \binom{(n+1)/2}{(n+1)/4} + 2\binom{n/2}{n/4} + \binom{(n-1)/2}{(n-1)/4}.$$

□

Remark: $f(2n)$ is also the number of $(n+1)$-step walks on a line starting from the origin but not returning to it.

$f(2n+1)$ is also the number of symmetric Dyck $(n+2)$-paths which either start UD or are prime i.e. do not return to ground level until the terminal point. For example, $f(2 \times 1 + 2) = 3$ counts $UUUDDD$, $UUDUDD$, $UDUDUD$.

$f(2n+1)$ is also the number of symmetric Dyck $(n+2)$-paths that first return to ground level either right away or not until the very end, i.e., that remain Dyck paths when either the first two steps or the first and last steps are deleted. For example, $f(2 \times 1 + 1) = 3$ counts $UUUDDD$, $UUDUDD$, $UDUDUD$.

If we only count the walks weakly below $y = x$ in the above theorem, there is only one walk.

Problem 51. *What is the number of one-sided prudent walks in the first quadrant, from $(0,0)$ to $(m.n)$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$, $\uparrow^{\geq 2}$, and $\rightarrow^{\geq 2}$?*

Problem 52. *What is the number of n-step one-sided prudent walks starting from $(0,0)$, weakly above $y = x$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$, $\uparrow^{\geq 2}$, and $\rightarrow^{\geq 2}$?*

Problem 53. *What is the number of one-sided prudent walks from $(0,0)$ to (m,n), weakly above $y = x$, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$, $\uparrow^{\geq 2}$, and $\rightarrow^{\geq 2}$?*

4.6 Two-sided PSAWs

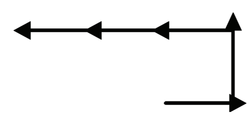
Now, we consider some pattern avoidance problems on two-sided prudent walks. One problem is how to count the number of two-sided, n-step prudent walks ending on the top side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$ (both at the same time), taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?

Theorem 54. *The generating function, say $T(t,u)$, of the above two-sided prudent* *walks ending on the top side of their box satisfies*

$$\left(1 - t^2u - \frac{tu}{u-t}\right) T(t,u) = 1 + tu + T(t,t)t\frac{u-2t}{u-t}, \tag{4.3}$$

where u counts the distance between the endpoint and the north-east (NE) corner of the box.

For instance, in the following figure, a walk takes 5 steps, and the distance between the endpoint and the north-east corner is 3. So we can use t^5u^3 to count this walk.



Proof. We consider the following three cases:

Case 1: Neither the top nor the right side has ever moved; the walk consists only of west steps. This case contributes 1 to the generating function.

Case 2: The last inflating step goes east. This implies that the endpoint of the walk was on the right side of the box before that step. After that east step, the walk has made a sequence of north steps to reach the top side of the box. Observe that, by symmetry, the series $T(t,u)$ also counts walks ending on the right side of the box by the length and the distance between the endpoint and the north-east corner. These two observations give the generating function for this class as $T(t,t)$.

Case 3: The last inflating step goes north. After this step, there is either a west step or a bounded sequence of East steps. This gives the generation function for this class as

$$\left(t^2u + \frac{tu}{u-t}\right) T(t,u) - \frac{t^2}{u-t}T(t,t)$$

Putting the three cases together, we get the generating function (4.3) for $T(t,u)$.

□

Now we solve this generating function for $T(t,u)$ using the Kernel Method [52]:

Solution 55. *From*

$$\left(1 - t^2u - \frac{tu}{u-t}\right) T(t,u) = 1 + tu + T(t,t)\left(t - \frac{t^2}{u-t}\right),$$

we can get

$$\begin{aligned}&(1-tu)\left(u - tu - t - t^2u^2 + t^3u\right) T(t,u)\\&= (u-t)(1-tu)(1+tu) - T(t,t)\,(1-tu)\,t\,(2t-u)\end{aligned}$$

Set $(1-tu)\,(u - tu - t - t^2u^2 + t^3u) = 0$*, then there is only one* power series *solution for u [58, A023432]*

$$\begin{aligned}u &= \frac{1}{2t^2}\left(1 - t + t^3 - \sqrt{(1-t-t^3)^2 - 4t^4}\right)\\&= t + t^2 + t^3 + t^4 + 2t^5 + 4t^6 + 7t^7 + 12t^8 + ...\end{aligned}$$

Let U be this solution,

$$U = U(t) = \frac{1}{2t^2}\left(1 - t + t^3 - \sqrt{(1-t-t^3)^2 - 4t^4}\right). \tag{4.4}$$

Set

$$(1+tu)(u-t)(1-tu) + T(t,t)\,(1-tu)\,t\,(u-2t) = 0,$$

and replace u by U:

$$T(t,t) = (1+tU)\,\frac{t-U}{t\,(U-2t)}. \tag{4.5}$$

From

$$(1-tu)\left(u-t-tu-t^2u^2+t^3u\right)T(t,u)$$
$$=(u-t)(1-tu)(1+tu)-T(t,t)\,(1-tu)\,t\,(2t-u)$$

We get

$$T(t,u)=\frac{(t-u)(1-tu)(1+tu)+T(t,t)\,(1-tu)\,t\,(2t-u)}{(1-tu)\,(u-t-tu-t^2u^2+t^3u)}.$$

Replace $T(t,t)$ *by (4.5). Now*

$$T(t,u)=\frac{(1+tu)(u-t)}{u-t-tu-t^2u^2+t^3u}-\frac{(1+tU)\,(U-t)\,(1-tu)\,(u-2t)}{(U-2t)\,(1-tu)\,(u-t-tu-t^2u^2+t^3u)}$$

where $U(t)$ *has been defined in (4.4).*

Summary 56. *Notice that* $T(t,1)$ *is the generating function of the number of two-sided* n*-step prudent walks ending on the top side of their box avoiding both patterns* $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, *taking steps from* $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, *thus [58, A190586]* $T(t,1)=$

$$\frac{(1-2t)\,(1-t)\,\sqrt{(1-t-t^3)^2-4t^4}-(1+t)\,(1-7t+14t^2-11t^3+10t^4-4t^5)}{2t\,(1-2t-t^2+t^3)\,(1-2t-2t^3)}$$
$$=1+3t+6t^2+15t^3+35t^4+83t^5+195t^6+460t^7+1085t^8+...$$

Note that $T(t,0)$ *is the generating function of the number of two-sided* n*-step prudent walks ending at the north-east corner of their box avoiding both patterns* $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, *taking steps from* $\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, *so [58, A190587]* $T(t,0)=$

$$\frac{(1-t)\,\sqrt{(1-t-t^3)^2-4t^4}-1+3t-t^2+t^3+t^4}{(1-2t-2t^3)\,t}$$
$$=1+2t+4t^2+10t^3+24t^4+56t^5+130t^6+304t^7+714t^8+1678t^9+...$$

Furthermore, $2T(t,1) - T(t,0)$ is the generating function of the number of two-sided n-step prudent walks ending on the top side or right side of their box avoiding both patterns $\leftarrow^{\geq 2}$, $\downarrow^{\geq 2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, thus [58, A190589] $2T(t,1) - T(t,0) =$

$$\frac{t(1-t)^2\sqrt{(1-t-t^3)^2 - 4t^4} + 1 - t - 2t^2 - 2t^3 - 2t^4 + 4t^5 - t^6}{(1-2t-t^2+t^3)(1-2t-2t^3)}$$
$$= 1 + 4t + 8t^2 + 20t^3 + 46t^4 + 110t^5 + 260t^6 + 616t^7 + 1456t^8 + 3442t^9 + ...$$

4.7 Open Problems

4.8 A. Meyerowitz's Maple Sequences [53]

Problem 57. *What is the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq k}$, and $\downarrow^{\geq k}$ $(k > 2)$ taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?*

Remark: The generating function:

$$\left(1 - t^2u\frac{1-t^ku^k}{1-tu} - \frac{tu}{u-t}\right)T(t,u)$$
$$= 1 + tu\frac{1-t^ku^k}{1-tu} + \frac{u-2t}{u-t}tT(t,t)$$

where u counts the distance between the endpoint and the north-east corner of the box.

For $k = 3$,

$$\frac{u-t-t^2u^2+t^3u-t^3u^3+t^4u^2-t^4u^4+t^5u^3-tu}{u-t}T(t,u)$$
$$=1+tu+t^2u^2+t^3u^3+\frac{u-2t}{u-t}tT(t,t)$$

i.e.,

$$(-t+\left(1+t^3-t\right)u+\left(t^4-t^2\right)u^2+\left(t^5-t^3\right)u^3+-t^4u^4)T(t,u)$$
$$=(1+tu+t^2u^2+t^3u^3)(u-t)+t\left(u-2t\right)T(t,t).$$

Another *open problem* here is that how to solve for u as a power series of t from [54], [49]

$$-t+\left(1+t^3-t\right)u+t^2\left(t^2-1\right)u^2+t^3\left(t^2-1\right)u^3-t^4u^4=0.$$

If we could get such a solution, we can use Kernel Method to solve the generating function.

We have the first one hundred terms for u, beginning with [58, A177794] given by A. Meyerowitz:

$$U=t+t^2+t^3+t^4+2t^5+4t^6+8t^7+16t^8+33t^9+69t^{10}+145t^{11}+...$$

We use this U as the the solution of $-t+(1+t^3-t)\,u+t^2\,(t^2-1)\,u^2+t^3\,(t^2-1)$ $u^3-t^4u^4=0$, then we can get the following three sequences, i.e., some numbers for the problem.

Sequence [58, A178035]:

$$T(t,1) = 1 + 3t + 7t^2 + 18t^3 + 44t^4 + 110t^5 + 273t^6 + 679t^7 + 1687t^8 + 4191t^9 + 10\,406t^{10} + 25\,830t^{11} + 64\,097t^{12} + 159\,015t^{13} + 394\,391t^{14} + 977\,939t^{15} + 2424\,370t^{16} + 5977\,180t^{17} + 14\,756\,906t^{18} + \ ...$$

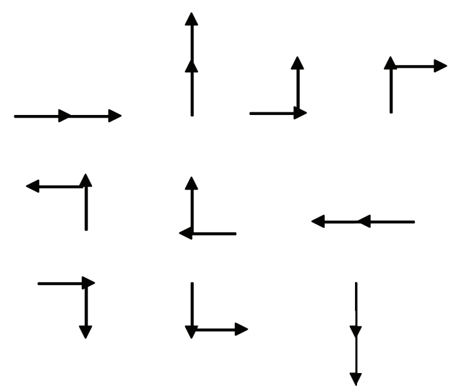
Sequence [58, A178036]:

$$T(t,0) = 1 + 2t + 4t^2 + 10t^3 + 24t^4 + 58t^5 + 140t^6 + 340t^7 + 828t^8 + 2020t^9 + 4934t^{10} + 12\,068t^{11} + 29\,556t^{12} + 72\,468t^{13} + 177\,846t^{14} + 436\,788t^{15} + 1073\,438t^{16} + 2576\,072t^{17} + ...$$

Sequence [58, A178037]:

$$2T(t,1) - T(t,0) = 1 + 4t + 10t^2 + 26t^3 + 64t^4 + 162t^5 + 406t^6 + 1018t^7 + 2546t^8 + 6362t^9 + 15\,878t^{10} + 39\,592t^{11} + 98\,638t^{12} + 245\,562t^{13} + 610\,936t^{14} + 1519\,090t^{15} + 3775\,302t^{16} + 9378\,288t^{17} + \ ...$$

Example 58. *For the above case $k = 3$, we consider the 2-step walks. There are four walks (the first four walks in the following figure) with end point at the northwest corner, seven walks (the first seven walks) with end point on the top side of their* *boxes, and total ten walks with end point on the top side or right side of their boxes.*



Problem 59. *What is the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^{=2}$, $\downarrow^{=2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?*

Remark: The generating function is

$$(1 - \frac{t^2u}{1-tu} - \frac{tu}{u-t} + u^2t^3)T(t,u) = \frac{1}{1-tu} - u^2t^2 + \frac{u-2t}{u-t}tT(t,t).$$

Another *open problem* here is how to solve for u as a formal power series of t from

$$(-t) + \left(1 + t^2 + t^3 - t\right)u + \left(-t - t^4\right)u^2 + \left(t^3 + t^5\right)u^3 + \left(-t^4\right)u^4 = 0.$$

We have the first 100 terms for u to begin with [58, A190591], given by A. Meyerowitz:

$$U = t + t^2 + t^3 + t^4 + 2t^5 + 4t^6 + 7t^7 + 12t^8 + 23t^9 + 47t^{10} + 96t^{11} + 195t^{12}$$
$$+ 402t^{13} + 843t^{14} + 1781t^{15} + 3772t^{16} + 8020t^{17} + 17\,143t^{18} + 36\,810t^{19}$$
$$+ 79\,304t^{20} + 171\,368t^{21} + 371\,450t^{22} + 807\,516t^{23} + ...$$

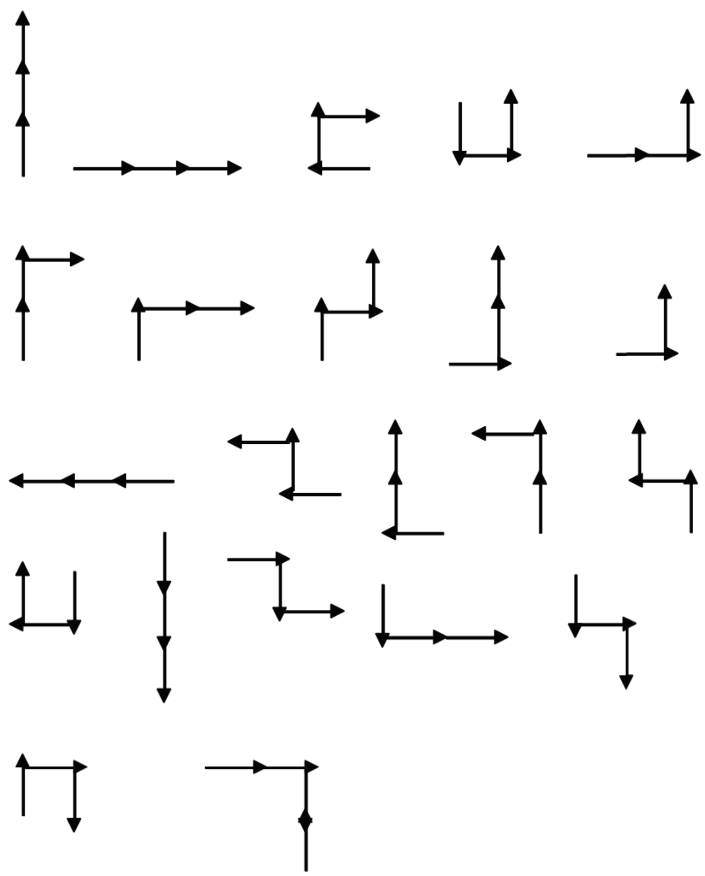
Using this U, we obtain:

$T(t,1) = 1+3t+6t^2+16t^3+38t^4+93t^5+223t^6+540t^7+1305t^8+3157t^9+7629t^{10}+18\,436t^{11}+44\,543t^{12}+107\,607t^{13}+259\,899t^{14}+627\,606t^{15}+1515\,284t^{16}+3657\,904t^{17}+8828\,836t^{18}+\ldots$ [58, A190735]

$T(t,0) = 1+2t+4t^2+10t^3+24t^4+56t^5+130t^6+306t^7+726t^8+1726t^9+4106t^{10}+9784\,t^{11}+23\,356t^{12}+55\,826t^{13}+133\,550t^{14}+319\,716t^{15}+765\,906t^{16}+1835\,856t^{17}+\ldots$ [58, A190794]

$2T(t,1)-T(t,0) = 1+4t+8t^2+22t^3+52t^4+130t^5+316t^6+774t^7+1884t^8+4588t^9+11\,152t^{10}+27\,088t^{11}+65\,730t^{12}+159\,388t^{13}+386\,248t^{14}+935\,496t^{15}+2264\,662t^{16}+5479\,952t^{17}+\ldots$[58, A190795]

Example 60. *We consider the* 3*-step walks in the above problem. There are ten walks (the first ten walk in the following figure) with end point at the northwest corner,* 16 *walks (the first* 16 *walks) with end point on the top side of their boxes, and total* 22 *walks with end point on the top side or right side of their boxes.*



Problem 61. *How to enumerate the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both* $\leftarrow^{\geq i}$*, and* $\downarrow^{\geq j}$ $(i > j > 2)$ *taking steps from* $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$*?*

Problem 62. *How to enumerate the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both* $\leftarrow^{=i}$, $\downarrow^{=j}$ $(i > j > 2)$ *taking steps from* $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$*?*

Problem 63. *What's the number of top-sided-and-down-sided weakly prudent SAWs,* *that could reach the bottom side of the box at least once and at most twice, and each endpoint of such a walk lies always on the top side of the box?*

$\uparrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$											
$\uparrow$		$\downarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\uparrow$
$\uparrow$		$\downarrow$															$\uparrow$
$\uparrow$		$\downarrow$						$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$			$\uparrow$
$\uparrow$		$\downarrow$		$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\uparrow$						$\downarrow$			$\uparrow$
$\uparrow$		$\downarrow$		$\uparrow$										$\downarrow$			$\uparrow$
$\uparrow$		$\downarrow$		$\uparrow$			$\Rightarrow$	$\rightarrow$	$\rightarrow$					$\downarrow$			$\uparrow$
$\uparrow$		$\downarrow$		$\uparrow$					$\downarrow$		$\uparrow$	$\rightarrow$	$\downarrow$	$\downarrow$			$\uparrow$
$\uparrow$		$\downarrow$		$\uparrow$					$\downarrow$	$\rightarrow$	$\uparrow$		$\downarrow$	$\downarrow$			$\uparrow$
$\leftarrow$	$\leftarrow$	$\downarrow$		$\downarrow$									$\downarrow$	$\downarrow$		$\rightarrow$	$\rightarrow$
				$\uparrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\downarrow$	$\rightarrow$	$\uparrow$	

CHAPTER 5

THE TENNIS BALL PROBLEM

The tennis ball problem was presented on pages 304−305 of the book " Sweet Reason: A Field Guide to Modern Logic" by Tom Tymoczko and Jim Henle in 1995. Their presentation deals with adding numbered books to a stack on a table, then removing some, infinitely many times. Motivated by that presentation, Ralph P. Grimaldi and Joseph G. Moser deal with performing the process a finite number of times. Since then more mathematicians have studied the problem, such as C. L. Mallows and L. Shapiro, R.J. Chapman, T.Y. Chow, A. Khetan, D.P. Moulton, R.J. Waters, J. E. Bonin, Anna de Mier, M. Noy, W. Wei, T. Schonbek, J. M. Freeman, H. Niederhausen, J. Fallon, and S. Gao. [2, 8, 32, 33, 13, 14, 21, 23, 30, 31, 44, 47] However, it is still wildly open and might challenge more people in the future. [57] ,[50]

The tennis ball problem can be stated as follows: Given integers r, s, n, with $0 < r < s$, label sn balls $1, 2, \ldots, sn$. Place the first s balls, labeled $1, 2, \ldots, s$ into a bin, then remove r balls from the bin. Repeat this process n times, each time inserting the next s balls in sequence and removing r balls. The question we seek to answer is, "How many different sets of rn balls lie outside the bin after n turns?" We solve for the case $s = 2r$, and derive a generating function for $r = 2$ and $r = 3$.

5.1 Introduction

The tennis ball problem can be viewed as a lattice path enumeration. Consider lattice walks in the plane with East $\langle 1, 0\rangle$ and North $\langle 0, 1\rangle$ steps. We count the number of paths from $(0,1)$ to $((s-r)n+1, rn+1)$ that stay weakly above the boundary $(E^{s-r}N^r)^n$. It is easy to see that, for each $1 \leq i \leq n$, at least ri of the first si steps must be N steps, and for $i = n$ this is an equality. We associate with each of the N steps one of the labeled balls, namely the ball with the label matching the number, in sequence, of the N step as a member of the path to $((s-r)n+1, rn+1)$. Note that this method of counting does not take into consideration the order in which the balls are removed, only the set of labels.

Example 64. $r = 2$, $s = 4$, $n = 3$.

Case 1: Insert balls 1-4, and remove balls 1 and 2. Then insert balls 5-8 and remove balls 3 and 5. Then insert balls 9-12 and remove balls 7 and 9. The set of balls outside the bin is $\{1,2,3,5,7,9\}$, *corresponding to the lattice walk* $NNNENENENENEEE$.

Case 2: Insert balls 1-4, and remove balls 1 and 3. Then insert balls 5-8 and remove balls 5 and 7. Then insert balls 9-12 and remove balls 2 and 9. The set of balls outside the bin is $\{1,2,3,5,7,9\}$, *corresponding to the lattice walk* $NNNENENENENEEE$.

Although the balls removed on each turn differ in the two cases, both the sets of labels and lattice walks were the same. We see, then, that this method of counting avoids redundancy.

Given a boundary $(E^{s-r}N^r)^n$, let $t_n(i)$ represent the number of paths to (n,i) for points above the boundary. The number of walks to (n,i) follows the recursion $t_n(i) = t_{n-1}(i) + t_n(i-1)$. Because $t_0(i) = 1$ for all $i \geq 1$, and $t_n(i) = 0$ at all points (n,i) directly below the boundary $(n > 0)$, we can uniquely extend the values of $t_n(i)$ to polynomials of degree n on points below the boundary. We call the polynomials again $t_n(i)$. Note that $t_0(x) = 1$ for all x, r, s.

i	1	7	28	84	195	381	662	*662*
6	1	6	21	56	111	186	281	0
5	1	5	15	35	55	75	95	-281
4	1	4	10	20	*20*	20	20	-376
3	1	3	6	10	0	0	0	-396
2	1	2	3	4	-10	0	0	-396
1	1	*1*	1	1	-14	-10	0	-396
0	1	0	0	0	-15	4	-10	-396
-1	1	-1	0	0	-15	19	-14	-386
$n:$	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s=6$, $r=3$

In [33], Anna de Mier and Marc Noir showed that the generating function

$$f(z)=\sum_{n\geq 0} t_{(s-r)n+1}(rn+1)z^n$$

satisfies $f(z)=-(1-w_1)\cdots(1-w_{s-r})/z$, where $w_1,\ldots,w_{s-r}$ are the unique fractional power series solutions of $(w-1)^{s-r}-zw^s=0$. Explicit solutions are hard to get from this relationship when $s-r\nmid r$. The case $r=2$, $s=4$ has been solved in [31] by different methods. In the following, we present a theorem for the case $s-r=r$, verify it for $r=2$, and show the resulting generating function for $r=3$ based on this theorem.

As in the introduction, we denote by $t_j(k)$ the number of paths from $(0,1)$ to (j,k) with steps $E=\langle 1,0\rangle$ and $N=\langle 0,1\rangle$, staying weakly above the boundary $(E^rN^r)^n$.

Theorem 65.

$$\frac{1}{r}\sum_{i=0}^{r-1} t_{rn}(rn-i)=C_{rn},$$

where C_n is the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$. [31]

The theorem is trivial for $r = 1$. We will show it now for $r = 2$.

Lemma 66. *If $r = 2$, then $t_{2n}(2n) + t_{2n}(2n-1) = 2C_{2n}$*

Proof. $t_{2n}(2n) - C_{2n}$ is the number of paths that do not stay above the boundary $(EN)^n$, but stay above $(E^2N^2)^n$. These paths stay above the boundary $(EN)^n$ until they reach some point $(2i, 2i-1)$ crossing under this boundary, then stay above the boundary $(E^2N^2)^n$ from then on, so $t_{2n}(2n) - C_{2n} = \sum_{i=1}^{n} C_{2i-1} t_{2(n-i)}(2(n-i))$. In the same way, we see that $t_{2n}(2n-1) = \sum_{i=1}^{n} C_{2i-1} t_{2(n-i)}(2(n-i)-1)$. We now proceed by induction over n. By observation, the lemma holds for $n = 1$. Now assume that it holds for $n-1$. Then $t_{2n}(2n) + t_{2n}(2n-1)$

$$
\begin{aligned}
&= C_{2n} + \sum_{i=1}^{n} C_{2i-1}\left(t_{2(n-i)}(2(n-i)) + t_{2(n-i)}(2(n-i)-1)\right) \\
&= C_{2n} + 2\sum_{i=1}^{n} C_{2i-1} C_{2(n-i)}
\end{aligned}
$$

by our induction hypothesis. Expanding this sum, we see that

$$
\begin{aligned}
&2\sum_{i=1}^{n} C_{2i-1} C_{2(n-i)} \\
&= \sum_{i=1}^{n} C_{2i-1} C_{2(n-i)} + \sum_{i=0}^{n-i} C_{2(n-i)-1} C_{2i} \\
&= \sum_{i=0}^{2n-1} C_i C_{2n-1-i} = C_{2n}.
\end{aligned}
$$

Thus $t_{2n}(2n) + t_{2n}(2n-1) = 2C_{2n}$. □

The numbers $t_{2n}(2n-1) = t_{2n-1}(2n-1)$ are the numbers we desire for the tennis ball problem in the case $s = 4, r = 2$.

5.2 The Tennis Ball Numbers for $r = 2$ and $s = 4$.

For the context of the following definition see "Finite Operator Calculus With Applications to Linear Recursions" by H. Niederhausen [35], or "The Finite Operator Calculus" by Rota, Kahaner, and Odlyzko [43].

Lattice paths with steps N and E describe a Sheffer sequence $(t_i(x))$ for the backwards difference operator ∇, because $\nabla t_n(x) = t_{n-1}(x)$. Since we want to know the values $t_{2n+1}(2n+1)$, we need an operator for the sequence $(t_n(n+x))$, and so we compose the operators E^{-1} and ∇, where $E^{-1}p_i(x) = p_i(x-1)$, so that $E^{-1}\nabla t_n(n+x) = t_{n-1}(n-1+x)$. The operator $E^{-1}\nabla$ has basic polynomials $b_n(x) = \frac{x}{x+n}\binom{2n+x-1}{n}$.

i								
i	1	7	28	80	185	343	554	*554*
6	1	6	21	52	105	158	211	0
5	1	5	15	31	53	*53*	53	-211
4	1	4	10	16	22	0	0	-264
3	1	3	6	*6*	6	-22	0	-264
2	1	2	3	0	0	-28	-22	-264
1	1	*1*	1	-3	0	-28	6	-244
0	1	0	0	-4	-3	-28	34	-250
-1	1	-1	0	-4	1	-25	62	-284
n :	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 4$, $r = 2$

We consider now the tennis ball problem with $s = 4, r = 2$. Because of the boundary given, we get $t_{2n}(2n-3) = 0$ for all $n > 0$, and $t_{2n+1}(2n-2) = -t_{2n}(2n) - t_{2n}(2n-1) = -2C_{2n}$ by Lemma 66 for all $n \geq 0$. By the Binomial Theorem for Sheffer sequences [43],

$$
\begin{aligned}
t_n(n) &= \sum_{i=0}^{n} t_i(i-3)\frac{3}{n+3-i}\binom{2n-2i+2}{n-i} \\
&= \frac{3}{n+3}\binom{2n+2}{n} + \sum_{i=0}^{(n-1)/2} \frac{3t_{2i+1}(2i-2)}{n+2-2i}\binom{2n-4i}{n-2i-1} \\
&= \frac{3}{n+3}\binom{2n+2}{n} - 2\sum_{i=0}^{\lfloor (n-1)/2\rfloor} \frac{3C_{2i}}{n+2-2i}\binom{2n-4i}{n-2i-1}
\end{aligned}
$$

The tennis ball numbers, $t_{2n+1}(2n+1)$, and their generating function are easily obtained from the above closed form.

5.3 The Generating Function for $r = 3$ and $s = 6$

We consider now the case $s = 6, r = 3$. In this case, the tennis ball numbers are $t_{3n+1}(3n+1)$. Note that the t_i are different functions from those used in the $(4, 2)$ case. Using the same operator and basic sequence as before, by the binomial theorem for Sheffer sequences

$$
t_n(n) = \sum_{i=0}^{n} t_i(i-3)\frac{3}{3+n-i}\binom{2(n-i+1)}{n-i}.
$$

The zeroes below the given boundary enforce the condition $t_i(i-3) = 0$ unless $i \equiv 1 \bmod 3$ (see the Table in the Introduction). Therefore this sum simplifies to

$$
t_n(n) = \frac{3}{3+n}\binom{2n+2}{n} - \frac{6}{n+2}\binom{2n}{n-1} + \sum_{i=1}^{\frac{n-1}{3}} \frac{t_{3i+1}(3i-2)}{n-3i+2}\binom{2n-6i}{n-3i+1}.
$$

Let the function $f_{6,3}(z) := \sum_{n\geq 0} t_{3n+1}(3n+1)z^n$ describe the generating function for the tennis ball numbers. The zeroes below the boundary also imply that $t_{3n}(3n-$

$2) + t_{3n}(3n-1) + t_{3n}(3n) = -t_{3n+1}(3n-3)$. Applying the conjecture we get $f_{6,3}(z)$

$$
\begin{aligned}
&= \sum_{n\geq 0} \left(\frac{3}{3n+4}\binom{6n+4}{3n+1} - \frac{2}{n+1}\binom{6n+2}{3n} \right. \\
&\left. + \sum_{i=1}^{n} \frac{t_{3i+1}(3i-2)}{n-i+1}\binom{6n-6i+2}{3n-3i} \right) z^n \\
&= \sum_{n\geq 0} \left(\frac{3}{3n+4}\binom{6n+4}{3n+1} - \frac{2}{n+1}\binom{6n+2}{3n} \right) z^n \\
&+ \left(\sum_{n\geq 1} t_{3n+1}(3n-2) z^n \right) \left(\sum_{n\geq 0} \frac{1}{n+1}\binom{6n+2}{3n} z^n \right) \\
&= \sum_{n\geq 0} \left(\frac{3}{3n+4}\binom{6n+4}{3n+1} - \frac{2}{n+1}\binom{6n+2}{3n} \right) z^n + \sum_{n\geq 0} \frac{z^n}{n+1}\binom{6n+2}{3n} \\
&\times \left(\sum_{n\geq 0} t_{3n+1}(3n+1) z^{n+1} - \sum_{n\geq 0} \frac{6}{3n+4}\binom{6n+5}{3(n+1)} z^{n+1} \right).
\end{aligned}
$$

Solving for $f_{6,3}(z)$, we obtain the "explicit" generating function for the $(6,3)$-tennis ball numbers: $f_{6,3}(z) =$

$$
\frac{\sum_{n\geq 0} \frac{3z^n}{3n+4}\binom{6n+4}{3n+1} z^n + \left(1 - 3\sum_{n\geq 0} C_{3(n+1)} z^{n+1}\right)\left(\sum_{n\geq 0} \frac{z^n}{n+1}\binom{6n+2}{3n}\right)}{1 - \sum_{n\geq 0} \frac{1}{n+1}\binom{6n+2}{3n} z^{n+1}}.
$$

Note that the terms $\frac{1}{n+1}\binom{6n+2}{3n}$ and $\frac{3}{3n+4}\binom{6n+4}{3n+1}$ equal the coefficients of z^m in $\left(2/\left(1+\sqrt{1-4z}\right)\right)^3$ where $m = 0 \bmod 3$, and $m = 1 \bmod 3$, respectively. Therefore, the above generating function can be expressed in term of $2/\left(1+\sqrt{1-4z}\right)$.

CHAPTER 6

GENERATING FUNCTION

The following results could be used for solving many recurrences and generating functions in OEIS and in this dissertation.

$[x^n]f(x)$: the coefficient of x^n in the power series expansion of $f(x)$.

$[x^n]\frac{1}{1-x} = 1$

Theorem 67.

$$[x^n]\frac{1}{1-d_1x^{t_1}} = \sum_{i_1=\lceil n/t_1\rceil}^{\lfloor n/t_1\rfloor} d_1^{i_1}.$$

Proof.

$$[x^n]\frac{1}{1-d_1x^{t_1}} = [x^n]\sum_{i_1=0}^{n}(d_1x^{t_1})^{i_1} = \sum_{i_1=\lceil n/t_1\rceil}^{\lfloor n/t_1\rfloor} d_1^{i_1}.$$

□

Theorem 68.

$$[x^n]\frac{1}{1-(d_1x^{t_1}+d_2x^{t_2})}$$

$$= \sum_{i_1=0}^{\lfloor n/t_1\rfloor} \sum_{i_2=\left\lceil \frac{t_1i_1-n}{t_1-t_2}\right\rceil}^{\left\lfloor \frac{t_1i_1-n}{t_1-t_2}\right\rfloor} \binom{i_1}{\frac{t_1i_1-n}{t_1-t_2}} d_1^{\frac{i_1t_2-n}{t_2-t_1}} d_2^{\frac{t_1i_1-n}{t_1-t_2}}.$$

Proof.

$$
\begin{aligned}
&[x^n]\frac{1}{1-(d_1x^{t_1}+d_2x^{t_2})}\\
&=[x^n]\sum_{i_1=\lceil n/t_1\rceil}^{\lfloor n/t_1\rfloor}\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}d_1^{i_1-i_2}x^{t_1(i_1-i_2)}d_2^{i_2}x^{t_2i_2}\\
&=\sum_{i_1=0}^{\lfloor n/t_1\rfloor}\sum_{i_2=\left\lceil\frac{t_1i_1-n}{t_1-t_2}\right\rceil}^{\left\lfloor\frac{t_1i_1-n}{t_1-t_2}\right\rfloor}\binom{i_1}{\frac{t_1i_1-n}{t_1-t_2}}d_1^{\frac{i_1t_2-n}{t_2-t_1}}d_2^{\frac{t_1i_1-n}{t_1-t_2}}.
\end{aligned}
$$

□

Example 69.

$$
\begin{aligned}
&[t^n]\frac{1+t}{1-4t-3t^2}\\
&=[t^n]\,(1+t)\sum_{k\geq 0}t^k\,(4+3t)^k\\
&=[t^n]\,(1+t)\sum_{k\geq 0}\sum_{m=0}^{k}\binom{k}{m}4^{k-m}3^mt^{m+k}\\
&=\sum_{k=0}^{n}\left[\binom{k+1}{n-k}3+\binom{k}{n-1-k}\right]4^{2k-n}3^{n-1-k}.
\end{aligned}
$$

Theorem 70.

$$
\begin{aligned}
&[x^n]\frac{1}{1-(d_1x^{t_1}+d_2x^{t_2}+d_3x^{t_3})}\\
&=\sum_{i_1=0}^{\lfloor n/t_1\rfloor}\sum_{i_2=0}^{i_1}\sum_{i_3=\left\lceil\frac{t_1i_1-t_1i_2+t_2i_2-n}{t_2-t_3}\right\rceil}^{\left\lfloor\frac{t_1i_1-t_1i_2+t_2i_2-n}{t_2-t_3}\right\rfloor}\binom{i_1}{i_2}\binom{i_2}{\frac{t_1i_1-t_1i_2+t_2i_2-n}{t_2-t_3}}d_1^{i_1-i_2}d_2^{\frac{-t_1i_2+i_2t_3+t_1i_1-n}{t_3-t_2}}d_3^{\frac{t_1i_1-t_1i_2+t_2i_2-n}{t_2-t_3}}.
\end{aligned}
$$

Theorem 71.

$$[x^n]\frac{1}{1-(d_1x^{t_1}+d_2x^{t_2}+d_3x^{t_3}+d_4x^{t_4})}$$
$$=\sum_{i_1=0}^{\lfloor n/t_1\rfloor}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\sum_{i_4=\left\lceil\frac{t_1i_1-t_1i_2+t_2i_2-t_2i_3+t_3i_3-n}{t_3-t_4}\right\rceil}^{\left\lfloor\frac{t_1i_1-t_1i_2+t_2i_2-t_2i_3+t_3i_3-n}{t_3-t_4}\right\rfloor}\binom{i_1}{i_2}\binom{i_2}{i_3}\binom{i_3}{\frac{t_1i_1-t_1i_2+t_2i_2-t_2i_3+t_3i_3-n}{t_3-t_4}}$$
$$\times d_1^{i_1-i_2}d_2^{i_2-i_3}d_3^{\frac{i_3t_4+t_1i_1-t_1i_2+t_2i_2-t_2i_3-n}{t_4-t_3}}d_4^{\frac{t_1i_1-t_1i_2+t_2i_2-t_2i_3+t_3i_3-n}{t_3-t_4}}.$$

Theorem 72.

$$[x^n]\frac{1}{1-(d_1x^{t_1}+d_2x^{t_2}+d_3x^{t_3}+d_4x^{t_4}+d_5x^{t_5})}$$
$$=\sum_{i_1=0}^{\lfloor n/t_1\rfloor}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3}\sum_{i_5=\left\lceil\frac{-t_1i_1+t_1i_2-t_2i_2+t_2i_3-t_3i_3+t_3i_4-t_4i_4+n}{t_5-t_4}\right\rceil}^{\left\lfloor\frac{-t_1i_1+t_1i_2-t_2i_2+t_2i_3-t_3i_3+t_3i_4-t_4i_4+n}{t_5-t_4}\right\rfloor}\binom{i_1}{i_2}\binom{i_2}{i_3}\binom{i_3}{i_4}$$
$$\times\binom{i_4}{\frac{-t_1i_1+t_1i_2-t_2i_2+t_2i_3-t_3i_3+t_3i_4-t_4i_4+n}{t_5-t_4}}d_1^{i_1-i_2}d_2^{i_2-i_3}d_3^{i_3-i_4}$$
$$\times d_4^{\frac{-i_4t_5-t_1i_1+t_1i_2-t_2i_2+t_2i_3-t_3i_3+t_3i_4+n}{t_4-t_5}}d_5^{\frac{-t_1i_1+t_1i_2-t_2i_2+t_2i_3-t_3i_3+t_3i_4-t_4i_4+n}{t_5-t_4}}$$
$$=G(n,d_1,d_2,d_3,d_4,d_5,t_1,t_2,t_3,t_4,t_5).$$

Theorem 73.

$$[x^n]\frac{c_0+c_1x^{l_1}+c_2x^{l_2}+c_3x^{l_3}+c_4x^{l_4}+c_5x^{l_5}}{1-(d_1x^{t_1}+d_2x^{t_2}+d_3x^{t_3}+d_4x^{t_4}+d_5x^{t_5})}$$
$$=\sum_{i=0}^{5}c_iG(n-l_i,d_1,d_2,d_3,d_4,d_5,t_1,t_2,t_3,t_4,t_5).$$

Theorem 74.

$$[x^n](1+c_1x^{l_1}+c_2x^{l_2}+c_3x^{l_3}+c_4x^{l_4}+c_5x^{l_5}+c_6x^{l_6}+c_7x^{l_7}+c_8x^{l_8}+c_9x^{l_9}+c_{10}x^{l_{10}})^s$$
$$=\sum_{i_0=0}^{n}\sum_{i_1=0}^{i_0}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3}\sum_{i_5=0}^{i_4}\sum_{i_6=0}^{i_5}\sum_{i_7=0}^{i_6}\sum_{i_8=0}^{i_7}\binom{s}{i_0}\binom{i_0}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\binom{i_3}{i_4}\binom{i_4}{i_5}\binom{i_5}{i_6}\binom{i_6}{i_7}\binom{i_7}{i_8}$$
$$\times\binom{i_8}{\frac{-l_1i_0+l_1i_1-l_2i_1+l_2i_2-l_3i_2+l_3i_3-l_4i_3+l_4i_4-l_5i_4+l_5i_5-l_6i_5+l_6i_6-l_7i_6+l_7i_7-l_8i_7+l_8i_8-l_9i_8+n}{l_{10}-l_9}}.$$

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