Stochastic Optimal Impulse Control of Jump Diffusions with Application to Exchange Rate

by

Sandun C. Perera

A Dissertation Submitted to the Faculty of

The Charles E. Schmidt College of Science

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Florida Atlantic University

Boca Raton, Florida

August 2009

Stochastic Optimal Impulse Control of Jump Diffusions with Application to Exchange Rate

by
Sandun C. Perera

This dissertation was prepared under the direction of the candidate's dissertation advisor, Dr. Hongwei Long, Department of Mathematical Sciences, and it has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

SUPERVISORY COMMITTEE:

Hongwei Long, Ph.D.
Dissertation Advisor

Lianfen Qian, Ph.D.

Dragan Radulovic, Ph.D.

Fred Richman, Ph.D.

Jiongmin Yong, Ph.D.

Lee Klingler, Ph.D.
Chair, Department of Mathematical Sciences

Gary W. Perry, Ph.D.
Dean, The Charles E. Schmidt College of Science

Barry T. Rosson, Ph.D.
Dean, Graduate College

July 8, 2009
Date

Acknowledgements

I am, and I will always be grateful to my advisor Dr. Hongwei Long for his thoughtful supervision throughout my graduate studies. His valuable guidance allowed me to study a very rigorous sequence of courses within a considerably short period of time. I sincerely thank him for his understanding, encouragement, dedication and guidance that definitely helped me to successfully complete this dissertation.

I also sincerely thank the other members of my dissertation committee, Professors Lianfen Qian, Dragan Radulovic, Fred Richman and Jiongmin Yong for their contributions. I also would like to thank all the other faulty members at the Department of Mathematical Sciences; especially, Professors Heinrich Niederhausen, Spyros Magliveras, Xiao-Dong Zhang, Stephen Locke, Yuandan Lin, William Kalies. I also thank Beth Broer (our secretary), Emily Cimillo (our budget coordinator), and Gail Skeldon (NSF Grant Coordinator) for assisting me in various ways.

I am still grateful to the entire faculty at the Department of Mathematics, University of Colombo, Sri Lanka for helping me to understand the basic concepts of mathematics while I was an undergraduate student there. I also would like to extend my gratitude to Professors Romaine Jayewardene, Kirthi Premadasa and Chanakya Wijeratne for assisting me to start my graduate studies in USA.

Very special thanks go to Professor Bernt Øksendal, Department of Mathematics, University of Oslo for his prompt answers to some of my research-related questions.

I also thank him and all the other authors of the references of this dissertation for their valuable work in the area of Stochastic Analysis.

I am also grateful to Ms. Ingrid Jones (former Director of International Students Scholar Services, FAU) and Mr. William Fabricant for helping me to overcome some of the financial problems that I had at the beginning of my graduate studies. I also would like to extend my gratitude to the staff of the Graduate College for assisting me to complete this dissertation on time.

I also sincerely thank to all of my fellow students of the graduate program at the Department of Mathematical Sciences. In particular, I am grateful to Dr. Winston Buckley for encouraging me to successfully complete my dissertation on time.

Finally, I would like to thank my loving wife Erandi for everything. Her assistance was invaluable during this period. I lovingly thank her for taking care of me, understanding me and being patient with me during these years. I also thank my daughter Nisangi for bringing the joy to us during this difficult period. I am also forever grateful to my parents, parents-in-law, sister, sister-in-law, brothers-in-law and family members for giving me strength, courage and blessing during this period. They all contributed in various intangible but nonetheless indispensable ways.

Thank you everyone for everything!

Sandun Perera, August 2009

Abstract

Author: Sandun C. Perera

Title: Stochastic Optimal Impulse Control of Jump Diffusions with Application to Exchange Rate

Institution: Florida Atlantic University

Dissertation Advisor: Dr. Hongwei Long

Degree: Doctor of Philosophy

Year: 2009

We generalize the theory of stochastic impulse control of jump diffusions introduced by Øksendal and Sulem (2004) with milder assumptions. In particular, we assume that the original process is affected by the interventions. We also generalize the optimal central bank intervention problem including market reaction introduced by Moreno (2007), allowing the exchange rate dynamic to follow a jump diffusion process. We furthermore generalize the approximation theory of stochastic impulse control problems by a sequence of iterated optimal stopping problems which is also introduced in Øksendal and Sulem (2004). We develop new results which allow us to reduce a given impulse control problem to a sequence of iterated optimal stopping problems even though the original process is affected by interventions.

Dedication

This research is dearly dedicated to my wife for taking care of me and understanding me during this period, my daughter for giving me extra energy to work hard, my parents for instilling the passion for learning in me, and my sister for mentoring me when I was a teenager.

To: Erandi, Nisangi, Indrasena, Shanthilatha and Niranjala

"Nothing splendid has ever been achieved except by those who dared believe that something inside of them was superior to circumstance." ∽ *Bruce Barton*

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Exchange rates have always been considered a key economic variable. In countries dependent on foreign trade and foreign capital, the central bank is usually in charge of exchange rate policy. So the central bank has the ability to intervene in the foreign exchange market in order to keep their exchange rates within a band, or close to a target rate set by the central bank's strategy makers.

The central bank has two control possibilities to intervene in the foreign exchange market. The first of these involves changing the domestic interest rate. In this type of control, the central bank sets a higher domestic interest rate to compensate the investors for a weak national currency, so that these investors get a higher return by investing in the domestic currency. If this policy works, the investors earn enough interest income to compensate them for a depreciating currency. Then the investors will begin to buy the domestic currency. Consequently, the domestic currency becomes more valuable and the exchange rate goes down towards the central parity again. This type of control is called *continuous control*. In the second type of control, the country can use its international reserves to intervene in the foreign exchange market at selected times. The effect of intervening by buying (selling) foreign currency is to make the domestic currency weaker (stronger). Hence the domestic currency becomes less (more) valuable and the exchange rate goes up (down). This type of control is called *impulse control*. A combination of both continuous control and impulse control is called *combined stochastic control*. However, we will only focus on stochastic impulse control of jump diffusions in this dissertation. Therefore we refer the reader to Brekke and Øksendal [12], Mundaca and Øksendal [47], and Øksendal and Sulem [52] for a complete study of combined stochastic control.

The research on target-zone exchange rate regimes has progressed rapidly over the last two decades. In a target-zone regime the exchange rate is allowed to move only within a specified band. The central bank intervenes to prevent the exchange rate moving outside from the band. Many countries in recent history chose target-zone regimes, even though some recent financial crises have induced them to move to a free floating regime. Krugman [38] introduced the standard target-zone model through his working paper in 1987. Target-zone models have been discussed by many authors thereafter. We refer the reader to Bertola [10], Flood and Garber [26], Froot and Obsfeld [27], Garber and Svensson [28], Krugman [39], Miller and Weller [45], and Svensson [60] for a review of the relevant later literature. The first person to apply the theory of stochastic impulse control to this problem was Jeanblanc-Picqué [33], later extended by Korn [36]. In all of these models, the exchange rate intervention level is exogenous and no intramarginal interventions are allowed. This means that the band for the exchange rate is given and no interventions take place when the exchange rate is strictly inside the band.

However, it is often accepted that, in countries that try to keep their exchange rate within a band, most interventions in the foreign exchange rate market take place within the band (See Avesani and Gallo [3], and Bertola and Caballero [11] for some empirical evidence). The rationale of this strategy is that the pressure on the exchange rate increases when the exchange rate is exactly at the boundary since speculative attacks are more likely to be triggered and, as a result, the cost of keeping the exchange rate within the band once the boundary of the band is reached increases. Furthermore, even if the exchange regime of choice is that of a flexible exchange rate (free float), it is not unusual to observe intervention in the foreign exchange rate market when the exchange rate drifts too far away from a given benchmark (See Bekaert and Gray [7], and Dominguez and Frankel [21] for some empirical evidence). Target-zone models could not be used to justify these circumstances.

Mundaca and Øksendal [47] provided a solution for this problem in their important paper in 1998. They showed that it is not necessary to set a target band exogenously. But the correct target band can be derived endogenously as part of the solution to the optimization problem, using a cost function which combines cost of intervention with a running cost given by an increasing function of the distance between the target rate and the current rate. They used a standard Brownian motion model for their underlying exchange rate. However they did not provide an exact analytic solution to the problem. Cadenillas and Zapatero [14], [15] provided analytical solutions to this problem when the underlying exchange rate follows a geometric Brownian motion. Øksendal and Sulem [52] recently discussed stochastic impulse control theory in a very broad setting using a general jump diffusion process. They also provided analytical solutions for many different intervention problems when the underlying exchange rate follows a jump diffusion process. More recently, Moreno [46] added a new regime-switching idea to this intervention problem in the continuous exchange market. This allows us to have much milder assumptions than Øksendal and Mundaca [47] and Cadenillas and Zapatero [14], [15].

In this dissertation, we generalize the theory of optimal central bank intervention in the foreign exchange market in a very general setting. We first generalize the idea of Moreno [46] (which is indeed a generalization of Cadenillas and Zapatero [14], [15]) allowing the exchange rate dynamic to follow a jump diffusion process. Since we use the same broad setting that Øksendal and Sulem [52] used in their book, our main results can be applied not only for optimal central bank intervention problems but also for many other impulse control problems when the original process is affected by interventions.

We furthermore generalize the approximation theory of stochastic impulse control problems by a sequence of iterated optimal stopping problems which is also introduced in Øksendal and Sulem [52]. In particular, we develop new results which allow us to reduce a given impulse control problem to a sequence of iterated optimal stopping problems even though the original process is affected by interventions.

Organization

The dissertation is organized as follows:

Chapter 2 reviews important background material for our research. We first introduce the basic definitions and results on Markov processes, stopping times, martingales and Lévy processes. Then we present the Itô formula and discuss Lévy stochastic differential equations. Moreover, we give a review of optimal stopping and impulse control of jump diffusions. Important verification theorems are also introduced in this chapter.

Chapter 3 extends the theory of optimal central bank intervention in the foreign exchange market allowing the process driving the exchange-rate dynamics to change temporarily for a bounded, constant length of time $T > 0$ after an intervention. Then the process reverts back to the pre-intervention process at the end of this period of time. This means that we particularly assume that the process driving the exchange rate dynamics is affected by interventions. For this, we first introduce the definition of reacted intervention operator and then use it to generalize the verification theorem with quasi-integrovariational inequalities for impulse control introduced by Øksendal and Sulem [52]. We conclude this chapter with an application of our new verification theorem to optimal central bank intervention problem in the foreign exchange market. Hence this chapter also generalizes the idea of Moreno [46] allowing the exchange rate dynamic to follow a jump diffusion process.

Chapter 4 generalizes our own results in Chapter 3. In this chapter, we assume that the process driving the exchange-rate dynamics changes temporarily for a bounded, *random* length of time $T_i > 0$ after an intervention. Then the process reverts back to the pre-intervention process at the end of this period of time. Moreover, we assume that T_i's are independent and identically distributed with the distribution function F_T. We first generalize the definition of reacted intervention operator and then use it to generalize the verification theorem for this case. We conclude this chapter with an application to optimal central bank interventions.

Chapter 5 presents new results which allow us to reduce a given impulse control problem to a sequence of iterated optimal stopping problems even though the original process is affected by interventions. We use the definitions of the reacted intervention operators that we introduced in Chapters 3 and 4 to generalize the approximation theory of optimal impulse control of jump diffusions using iterated optimal stopping problems introduced by Øksendal and Sulem [52]. The main results of this chapter are proved for both constant and random market reaction periods. We conclude this chapter with an application of our main findings of the chapter to optimal central bank interventions in the foreign exchange market.

Chapter 6 presents some numerical simulation results with graphs, and outlines some possible future research in this area.

Chapter 2

Review of Stochastic Calculus with Jump diffusions and Impulse Control Theory

Part I: Stochastic Calculus with Jump Diffusions

In this part of the chapter we introduce some basic concepts and results about stochastic calculus of jump diffusions, which will be used in this dissertation. Most of the definitions and results in this part are abstracted from Øksendal and Sulem [52].

2.1 Markov processes and martingales

We present the definitions of Markov processes, stopping times, martingales and some results related to these definitions in this section. We refer to Applebaum [2], Cont and Tankov [18], Ethier and Kurtz [24], Protter [55] and Øksendal [49] for a complete study.

Definition 2.1. ***(Markov Process)*** *Let $X = \{X(t); t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t^X = \sigma(X(s); s \leq t)$. Then X is a*

Markov process if

$$P\left\{X(t+s)\in A\mid\mathcal{F}_t^X\right\}=P\left\{X(t+s)\in A\mid X(t)\right\}$$

for all $s,t\geq 0$ *and* $A\in\mathcal{B}(\mathbb{R}^n)$*,*

i.e. $E\left[f\left(X(t+s)\right)\mid\mathcal{F}_t^X\right]=E\left[f\left(X(t+s)\right)\mid X(t)\right]$ *for all* $s,t\geq 0$ *and* $f:\mathbb{R}^n\to\mathbb{R}$ *Borel measurable functions.*

Remark 2.1. *If* $(\mathcal{G}_t)$ *is a filtration such that* $\mathcal{F}_t^X\subset\mathcal{G}_t$ *for all* $t\geq 0$*, then* X *is a Markov process with respect to* $(\mathcal{G}_t)$ *if*

$$P\left\{X(t+s)\in A\mid\mathcal{G}_t\right\}=P\left\{X(t+s)\in A\mid X(t)\right\}$$

for all $s,t\geq 0$ *and* $A\in\mathcal{B}(\mathbb{R}^n)$*.*

Note that if X *is a Markov process with respect to* $(\mathcal{G}_t)$*, then it is a Markov process.*

Definition 2.2. ***(Stopping Time)*** *Let* $\mathcal{F}_t$ *be an increasing family of* σ*-algebras (of subsets of* Ω*). A random variable* $\tau:\Omega\to[0,\infty)$ *is an* $(\mathcal{F}_t)$*-stopping time if* $\{\tau\leq t\}\in\mathcal{F}_t$ *for every* $t\geq 0$*.*

This means that we are able to decide whether or not $\tau\leq t$ has taken place with our information $\mathcal{F}_t$ at time t. We have many interesting examples of stopping times.

Example 2.1. *Let* X *be an* $(\mathcal{F}_t)$*-adapted càdlàg (right continuous with left limits) process and* A *be open. We define the first hitting time by*

$$T_A=\inf\left\{t>0;X(t)\in A\right\}.$$

We only need to know the past positions of X *to determine whether the set* A *has been reached or not. Therefore* T_A *is a* $(\mathcal{F}_t)$*-stopping time.*

Example 2.2. *Let U be open. Then the first exit time*

$$T_U = \inf \{t > 0; X(t) \notin U\}$$

is a $\left(\mathcal{F}_t^X\right)$-stopping time.

Let $(\Omega, \mathcal{F}, P)$ be equipped with a filtration $(\mathcal{F}_t)$. Suppose that X is an adapted process and τ is a stopping time with respect to $(\mathcal{F}_t)$, then the stopped random variable $X(\tau)$ is defined by

$$X(\tau)(\omega) = X(\tau(\omega))(\omega)$$

and the stopped σ-algebra $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}; A \cap \{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0\}.$$

If X is càdlàg, then $X(\tau)$ is $\mathcal{F}_\tau$-measurable (see Kunita [41]).

Definition 2.3. ***(Strong Markov Process)*** *Let $X = \{X(t); t \geq 0\}$ be a Markov process with respect to a filtration $(\mathcal{F}_t)$, defined on a probability space $(\Omega, \mathcal{F}, P)$ such that X is $(\mathcal{F}_t)$-progressive (i.e. for each $t \geq 0$ the restriction of X to $[0,t] \times \Omega$ is $\mathcal{B}[0,t] \times (\mathcal{F}_t)$-measurable). Now let f be a bounded Borel function on $\mathbb{R}^n$ and τ be a $(\mathcal{F}_t)$-stopping time such that $\tau < \infty$ a.s., then X is strong Markov at τ if*

$$E\left[f\left(X(\tau + t)\right) \mid \mathcal{F}_\tau\right] = E^{X_\tau}\left[f\left(X(t)\right)\right]$$

for all $t \geq 0$. X is a strong Markov process with respect to $(\mathcal{F}_t)$ if X is strong Markov at τ for all $(\mathcal{F}_t)$-stopping times τ such that $\tau < \infty$ a.s.

Definition 2.4. ***(Martingales)*** *A real valued, $(\mathcal{F}_t)$-adapted process X with $E\left[|X(t)|\right] < \infty$ for all $t \geq 0$ is an*

(i). $(\mathcal{F}_t)$-martingale if $E\left[X(s) \mid \mathcal{F}_t\right] = X(t)$ a.s. for all $0 \leq t < s < \infty$,

(ii). $(\mathcal{F}_t)$-submartingale if $E\left[X(s) \mid \mathcal{F}_t\right] \geq X(t)$ a.s. for all $0 \leq t < s < \infty$,

(iii) $(\mathcal{F}_t)$*-supermartingale if* $E[X(s) \mid \mathcal{F}_t] \leq X(t)$ *a.s. for all* $0 \leq t < s < \infty$ *(or* $-X$ *is a submartingale).*

Note: X is a martingale if both X and $-X$ are submartingales.

Remark 2.2. *If* X *is a martingale, then the map* $t \longrightarrow E[X(t)]$ *is constant.*

Theorem 2.1. *(**Doob's Optional Sampling Theorem for Submartingales** - Protter [55]) Let* X *be a right continuous submartingale, which is closed by a random variable* X_∞*. Let* τ_1 *and* τ_2 *be two stopping times such that* $\tau_1 \leq \tau_2$ *a.s. Then* τ_1 *and* τ_2 *are integrable and*

$$E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] \geq X_{\tau_1} \text{ a.s.}$$

We have similar versions of the above theorem for supermartingales and martingales. However we only need the above version of the Doob's Optional Sampling Theorem in this dissertation.

Definition 2.5. ***(Local Martingales)*** *A real valued,* $(\mathcal{F}_t)$*-adapted process* X *is an* $(\mathcal{F}_t)$*-local martingale if there exist* $(\mathcal{F}_t)$*-stopping times* $\tau_1 \leq \tau_2 \leq \ldots$ *with* $\lim_{n\to\infty} \tau_n = \infty$ *a.s. such that each of the processes* $\{X(t \wedge \tau_n); t \geq 0\}$ *is a martingale.*

Definition 2.6. ***(Semimartingales)*** *A real valued,* $(\mathcal{F}_t)$*-adapted process* X *is an* $(\mathcal{F}_t)$*-semimartingale if for all* t*,* $X(t) = X(0) + M(t) + C(t)$*, where* M *is a* $(\mathcal{F}_t)$*-local martingale and* C *is an* $(\mathcal{F}_t)$*-adapted process of finite variation.*

2.2 Lévy processes

For a complete study of Lévy processes and it's applications, we refer to Applebaum [2], Bertoin [9], Jacod and Shiryaev [32], Protter [55], and Sato [56]. For more applications to finance, Cont and Tankov [18] and Schoutens [57] are recommended.

Definition 2.7. ***(Lévy Process)*** *Let $X = \{X(t); t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. We say that X has independent increments if for each $n \in \mathbb{N}$ and $0 \leqslant t_1 < t_2 < \cdots < t_{n+1} < \infty$ the random variables $\{X(t_{j+1}) - X(t_j); 1 \leq j \leq n\}$ are independent, and that X has stationary increments if $X(t_{j+1}) - X(t_j) \sim X(t_{j+1} - t_j) - X(0)$ for $j = 1, 2, ..., n$. We say that X is a Lévy process if:*

(***L1***) *$X(0) = 0$ almost surely;*

(***L2***) *X has independent and stationary increments;*

(***L3***) *X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$*

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Remark 2.3. *One can prove in the presence of (**L1**) and (**L2**), that (**L3**) is equivalent to the condition*

$$\lim_{t \downarrow 0} P(|X(t)| > a) = 0$$

for all $a > 0$

Theorem 2.2. *(Sato [56]) Let $X = \{X(t); t \geq 0\}$ be a Lévy process. Then $X(t)$ has a* ***càdlàg version*** *which is also a Lévy process.*

As a result of the above theorem we will from now on assume that the Lévy processes are càdlàg throughout this dissertation.

The *jump* of $X(t)$ at $t \geq 0$ is defined by

$$\triangle X(t) = X(t) - X(t-). \tag{2.2.1}$$

Let $\mathbb{B}_0$ be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\overline{U}$ does not contain 0. For

$U \in \mathbb{B}_0$ we define

$$N(t, U) = N(t, U, \omega) = \sum_{0<s\leq t} I_U(\triangle X(s)). \tag{2.2.2}$$

So $N(t, U)$ is the number of jumps of size $\triangle X(s) \in U$ that occur before or at time t. $N(t, U)$ is called the ***Poisson random measure*** (or ***jump measure***) of $X(\cdot)$. The differential form of this measure is written as $N(dt, dz)$.

Definition 2.8. ***(The Poisson process)*** *The Poisson process $\pi(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ and such that*

$$P[\pi(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \qquad n = 0, 1, 2, ...$$

Theorem 2.3. *(Protter [55])*
(i) The set function $U \to N(t, U, \omega)$ defines a σ-finite measure on $\mathbb{B}_0$ for each fixed t, ω.
(ii) The set function

$$\nu(U) = E[N(1, U)], \tag{2.2.3}$$

where $E = E_P$ denotes expectation with respect to P, also defines a σ-finite measure on $\mathbb{B}_0$, and is called the Lévy measure of $\{X(t)\}$
(iii) Fix $U \in \mathbb{B}_0$. Then the process

$$\pi_U(t) := \pi_U(t, \omega) := N(t, U, \omega)$$

is a Poisson process of intensity $\lambda = \nu(U)$.

Definition 2.9. ***(The compound Poisson process)*** *Let $\{X(n); n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}$ with common distribution $\mu_{X}(1) = \mu_X$ and let $\pi(t)$ be a Poisson process of intensity λ, independent of all the*

*$X(n)$'s. The **compound Poisson process** $Y(t)$ is defined by*

$$Y(t) = X(1) + \cdots + X(\pi(t)); \qquad t \geq 0. \tag{2.2.4}$$

An increment of this process is given by

$$Y(s) - Y(t) = \sum_{k=\pi(t+1)}^{\pi(s)} X(k); \qquad s > t.$$

This is independent of $X(1), \cdots, X(\pi(t))$, and depends only on the difference $s - t$. Thus $Y(t)$ is a Lévy process. The Lévy measure ν of $Y(t)$ is given by $\nu = \lambda\mu_X$.

Remark 2.4. *The Poisson process and the compound Poisson process are two examples of Lévy processes. But we have many examples of Lévy processes which are not used in this dissertation. The reader may read the references mentioned at the beginning of the section for more examples.*

Theorem 2.4. *(**Lévy Decomposition** - Jacod and Shiryaev [32]) Let $\{X(t)\}$ be a Lévy process. Then $X(t)$ has the decomposition*

$$X(t) = \alpha t + \beta B(t) + \int_{|z|<r} z\widetilde{N}(t, dz) + \int_{|z|\geq r} zN(t, dz), \tag{2.2.5}$$

for some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $r \in [0, \infty]$. Here

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt \tag{2.2.6}$$

is the compensated Poisson random measure of $X(\cdot)$ and $B(t)$ is an independent Brownian motion. For each $A \in \mathbb{B}_0$ the process

$$M_t := \widetilde{N}(t, A) \tag{2.2.7}$$

is a martingale. If $\alpha = 0$ and $r = \infty$, we call $X(t)$ a Lévy martingale.

Theorem 2.5. *(Sato [56]) We can always choose $r = 1$.*

If $E[X(t)] < \infty$ for all $t \geq 0$, then we may choose $r = \infty$ and hence write

$$X(t) = \alpha t + \beta B(t) + \int_{\mathbb{R}} z\widetilde{N}(t, dz).$$

Theorem 2.6. *(Protter [55]) A Lévy process is a strong Markov process.*

Note: The Lévy measure ν is defined on $\mathbb{R} \setminus \{0\}$. However, we may define the Lévy measure on $\mathbb{R}$ with the convention $\nu(\{0\}) = 0$. Hence, we will from now on define the Lévy measure on $\mathbb{R}$ with this convention.

Theorem 2.7. *(**The Lévy-Khintchine formula** - Protter [55]) Let $X(t)$ be a Lévy process with Lévy measure ν. Then*

$$\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$$

and

$$E[e^{iuX(t)}] = e^{t\psi(u)}, \qquad u \in \mathbb{R} \tag{2.2.8}$$

where

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{|z|<r} \{e^{iuz} - 1 - iuz\}\nu(dz) + \int_{|z|\geq r} (e^{iuz} - 1)\nu(dz). \tag{2.2.9}$$

Conversely, given constants α, σ^2, and a measure ν on $\mathbb{R}$ s.t.

$$\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty ,$$

there exists a Lévy process $X(t)$ (unique in law) such that (2.2.8)-(2.2.9) hold.

Remark 2.5. *Consider* $\nu(dz) = \dfrac{c}{|z|^{1+\alpha}}\,dz$, *where* c *is a constant and* $0 < \alpha < 1$. *Then*

$$\int_r^\infty z\nu(dz) = \int_r^\infty \frac{c}{z^\alpha}dz = \infty.$$

Therefore, it is possible that

$$\int_{|z|\geq r} |z|\nu(dz) = \infty.$$

Theorem 2.8. *(Applebaum [2]) A Lévy process is a semimartingale.*

Definition 2.10. *Let* $\mathbf{D}_{ucp}$ *denote the space of càdlàg adapted processes, equipped with the topology of uniform convergence on compacts in probability (ucp):*$H_n \to H$ *ucp if for all* $t > 0$, $\sup_{0\leq s\leq t} |H_n - H| \to 0$ *in probability.*
Let $\mathbf{L}_{ucp}$ *denote the space of adapted càglàd processes (left continuous with right limits), equipped with the ucp topology. If* $H(t)$ *is a step function of the form*

$$H(t) = H_0 I_{\{0\}}(t) + \sum_i H_i I_{(T_i,T_{i+1}]}(t),$$

where $H_i \in \mathcal{F}_{T_i}$ *and* $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1} < \infty$ *are* $\mathcal{F}_t$*-stopping times and* X *is càdlàg, we define*

$$J_X H(t) := \int_0^t H_s dX_s = H_0 X_0 + \sum_i H_i (X_{T_{i+1}\wedge t} - X_{T_i \wedge t}); \qquad t \geq 0.$$

Theorem 2.9. *(Protter [55]) Let* X *be a semimartingale. Then the mapping* J_X *can be extended to a continuous linear map*

$$J_X : \mathbf{L}_{ucp} \to \mathbf{D}_{ucp}\,.$$

Using the above construction we can define stochastic integrals of the form

$$\int_0^t H(s)dX(s)$$

for all $H \in \mathbf{L}_{ucp}$ (This will be explained with more details in Remark 2.6). By using Lévy decomposition (2.2.5), we can split this integral into integrals with respect to ds, $dB(s)$, $\widetilde{N}(ds, dz)$ and $N(ds, dz)$. Hence we can consider more general stochastic integrals of the form

$$X(t) = X(0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\bar{N}(ds,dz), \quad (2.2.10)$$

where the integrands satisfy the appropriate conditions for the integrals to exist. For simplicity, we have put

$$\bar{N}(ds,dz) = \begin{cases} N(ds,dz) - \nu(dz)ds, & \text{if } |z| < r, \\ N(ds,dz), & \text{if } |z| \geq r, \end{cases} \quad (2.2.11)$$

with r as in Theorem 2.3. We usually use the following shorthand differential notation for the process $X(t)$ satisfying (2.2.10):

$$dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dz). \quad (2.2.12)$$

We call such processes ***Itô-Lévy processes.***

2.3 The Itô formula and the Itô-Lévy isometry

We introduce the very important Itô formula for Itô-Lévy processes in this section. The reader may consult Applebaum [2], Bensoussan [8], Protter [55], and Cont and

Tankov [18] for a complete study of the Itô formula and its applications.

Theorem 2.10. ***(The 1-dimensional Itô formula*** *- Bensoussan and Lions [8]) Suppose $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dB(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\bar{N}(dt,dz), \tag{2.3.1}$$

where $\bar{N}(ds,dz)$ is defined in (2.2.11).

Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t,X(t))$. Then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned} dY(t) = {} & \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))[\alpha(t,\omega)dt + \beta(t,\omega)dB(t)] \\ & + \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X(t))dt \\ & + \int_{|z|<r} \{f(t,X(t-)+\gamma(t,z)) - f(t,X(t-))) - \frac{\partial f}{\partial x}(t,X(t-))\gamma(t,z)\}\nu(dz)dt \\ & + \int_{\mathbb{R}} \{f(t,X(t-)+\gamma(t,z)) - f(t,X(t-))\}\bar{N}(dt,dz). \end{aligned} \tag{2.3.2}$$

Note: If $r = 0$, then $\bar{N} = N$ everywhere. If $r = \infty$, then $\bar{N} = \widetilde{N}$ everywhere.

Theorem 2.11. ***(The multi-dimensional Itô formula*** *- Øksendal and Sulem [52]) Let $X(t) \in \mathbb{R}^n$ be an Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \sigma(t,\omega)dB(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\bar{N}(dt,dz), \tag{2.3.3}$$

where $\alpha : [0,T] \times \Omega \to \mathbb{R}^n$, $\sigma : [0,T] \times \Omega \to \mathbb{R}^{n\times m}$ and $\gamma : [0,T] \times \mathbb{R}^l \times \Omega \to \mathbb{R}^{n\times l}$ are adapted processes such that the integrals exist. Here $B(t)$ is an m-dimensional

Brownian motion and

$$\bar{N}(dt,dz)^T = (\bar{N}_1(dt,dz_1),\cdots,\bar{N}_l(dt,dz_l)),$$

where $\{N_j\}$ *are independent Poisson random measure with Lévy measures* ν_j *coming from* l *independent (1-dimensional) Lévy processes* $\eta_1,\cdots,\eta_l$.

Note that each column γ^k *of the* $n\times l$ *matrix* $\gamma=[\gamma_{ij}]$ *depends on* z *only through the* k^{th} *coordinate* z_k*, i.e.*

$$\gamma^k(t,z,\omega)=\gamma^k(t,z_k,\omega);\quad z=(z_1,\cdots,z_l)\in\mathbb{R}^l.$$

Thus the integral on the right of (2.3.3) is just a shorthand matrix notation. i.e.

$$\int_{\mathbb{R}}\gamma(t,z,\omega)\bar{N}(dt,dz) = \left(\sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{1j}(t,z_j,\omega)\bar{N}_j(dt,dz_j),\dots,\sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{nj}(t,z_j,\omega)\bar{N}_j(dt,dz_j)\right)^T$$

We can rewrite (2.3.3) in the componentwise form:

$$dX_i(t)=\alpha_i(t,\omega)dt+\sum_{j=1}^{m}\sigma_{ij}(t,\omega)dB_j(t)+\sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{ij}(t,z_j,\omega)\bar{N}_j(dt,dz_j);\qquad 1\le i\le n.$$

Let $f\in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$*. Put* $Y(t)=f(t,X(t))$*. Then*

$$
\begin{aligned}
dY(t) = & \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}[\alpha_i dt + \sigma_i dB(t)] + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}dt \\
& + \sum_{k=1}^{l}\int_{|z_k|<r_k}\{f(t, X(t-) + \gamma^{(k)}(t, z_k)) - f(t, X(t-)) \\
& \qquad - \sum_{i=1}^{n}\gamma_i^{(k)}(t, z_k)\frac{\partial f}{\partial x_i}(X(t-))\}\nu_k(dz_k)dt \\
& + \sum_{k=1}^{l}\int_{\mathbb{R}}\{f(t, X(t-) + \gamma^{(k)}(t, z_k)) - f(t, X(t-))\}\bar{N}_k(dt, dz_k). \qquad (2.3.4)
\end{aligned}
$$

where $\gamma^{(k)} \in \mathbb{R}^n$ is the k^{th} column of the $n \times l$ matrix $\gamma = [\gamma_{ik}]$ and $\gamma_i^{(k)} = \gamma_{ik}$ is the i^{th} coordinate of $\gamma^{(k)}$.

Theorem 2.12. ***(The Itô -Lévy isometry*** *- Øksendal and Sulem [52]) Let $X(t) \in \mathbb{R}^n$ be as in (2.3.3) but with $X(0) = 0$ and $\alpha = 0$. Then*

$$
\begin{aligned}
E[X^2(T))] &= E[\int_0^T \{\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{ij}^2\nu_j(dz_j)\}dt] \\
&= \sum_{i=1}^{n} E[\int_0^T \{\sum_{j=1}^{m}\sigma_{ij}^2(t) + \sum_{j=1}^{l}\int_{\mathbb{R}}\gamma_{ij}^2\nu_j(dz_j)\}dt], \qquad (2.3.5)
\end{aligned}
$$

provided that the right-hand side is finite.

Remark 2.6. *As a special case of Theorem 2.12, if we assume that*

$$
X(t) = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(s, dz) \in \mathbb{R}
$$

with

$$
E[X^2(T)] = T\int_{\mathbb{R}} z^2\nu(dz) < \infty,
$$

then we get the isometry

$$E[(\int_0^T H(t)dX(t))^2] = E[\int_0^T H^2(t)dt] \cdot \int_{\mathbb{R}} z^2 \nu(dz)$$

for all $H \in \mathbf{L}_{ucp}$ *(see Definition 2.10) such that* $H \in L^2([0,T] \times \Omega)$*, i.e.*

$$\|H\|^2_{L^2([0,T]\times\Omega)} := E[\int_0^T H^2(t)dt] < \infty.$$

Using this we can extend the definition of the integral

$$\int_0^T Y(t)dX(t) \in L^2(\Omega)$$

to all processes $Y(t)$ *which are limits in* $L^2([0,T] \times \Omega)$ *of the processes* $H_n(t) \in \mathbf{L}_{ucp} \cap L^2([0,T] \times \Omega)$. *We call such processes Y(t)* ***predictable processes.***

2.4 Stochastic differential equations driven by Lévy processes

The solution of a stochastic differential equation (SDE) driven by Lévy processes is called a *Lévy diffusion*. We discuss the existence and uniqueness of Lévy diffusions and some related results in this section (See Applebaum [2], Protter [55]).

Theorem 2.13. ***(Existence and uniqueness of solutions of Lévy SDEs*** *- Applebaum [2]) Consider the Lévy SDE in* $\mathbb{R}^n$ *such that* $X(0) = x_0 \in \mathbb{R}^n$ *and*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dB(t) + \int_{\mathbb{R}} \gamma(t, X(t-), z)\widetilde{N}(dt, dz), \qquad (2.4.1)$$

where $\alpha : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *and* $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n\times l}$

satisfy the following conditions

(At most linear growth) *There exists a constant $C_1 < \infty$ such that*

$$\|\sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{l} |\gamma^{(k)}(t,x,z_k)|^2 \nu(dz_k) \leq C_1(1+|x|^2)$$

for all $x \in \mathbb{R}^n$.

(Lipschitz continuity) *There exists a constant $C_2 < \infty$ such that*

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + |\alpha(t,x) - \alpha(t,y)|^2 + \sum_{k=1}^{l} \int_{\mathbb{R}} |\gamma^{(k)}(t,x,z_k) - \gamma^{(k)}(t,y,z_k)|^2 \nu(dz_k) \leq C_2|x-y|^2;$$

for all $x, y \in \mathbb{R}^n$. Then there exists a unique càdlàg adapted solution $X(t)$ such that

$$E[|X(t)|^2] < \infty \qquad \textit{for all } t.$$

Note: Solutions of Lévy SDEs in the *time homogeneous* case, i.e. when $\alpha(t,x) = \alpha(x)$, $\sigma(t,x) = \sigma(x)$ and $\gamma(t,x,z) = \gamma(x,z)$, are called ***jump diffusions*** (or ***Lévy diffusions***).

Theorem 2.14. *(Protter [55]) A jump diffusion is a strong Markov process.*

Definition 2.11. *Let $X(t) \in \mathbb{R}^n$ be a jump diffusion. Then the generator A of X is defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ by*

$$Af(x) = \lim_{t \to 0^+} \frac{1}{t}\{E^x[f(X(t))] - f(x)\} \qquad \textit{(if the limit exists),}$$

where $E^x[f(X(t))] = E[f(X^{(x)}(t))]$, $X^{(x)}(0) = x$.

Theorem 2.15. *(Applebaum [2]) Let $C_0^2(\mathbb{R}^n)$ be the functions in $C^2(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. Suppose $f \in C_0^2(\mathbb{R}^n)$. Then $Af(x)$ exists and is given by*

$$\begin{aligned} Af(x) = & \sum_{i=1}^{n} \alpha_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ & + \int_{\mathbb{R}} \sum_{k=1}^{l} \{f(x + \gamma^{(k)}(x,z)) - f(x) - \nabla f(x) \cdot \gamma^{(k)}(x,z)\} \nu_k(dz_k). \end{aligned} \tag{2.4.2}$$

Therefore, for all f such that the partial derivatives of f and the integrals in (2.4.2) exist at x, we can define $Af(x)$ by the expression (2.4.2).

Theorem 2.16. *(**The Dynkin formula I** - Øksendal and Sulem [52]) Let $X(t) \in \mathbb{R}^n$ be a jump diffusion and let $f \in C_0^2(\mathbb{R}^n)$. Let τ be a stopping time such that*

$$E^x[\tau] < \infty.$$

Then

$$E^x[f(X(\tau))] = f(x) + E^x[\int_0^\tau Af(X(s))ds].$$

Since the above version of the Dynkin formula is usually strong enough for applications in the case when there are no jumps ($N = 0$), we now introduce a stronger and localized version of the Dynkin formula for jump diffusions.

Theorem 2.17. *(**The Dynkin formula II** - Øksendal and Sulem [52]) Let $X(t) \in \mathbb{R}^n$ be a jump diffusion, $G \subset \mathbb{R}$ be an open set and let $f \in C^2(G) \cap C(\bar{G})$. Let $\tau < \infty$ be a stopping time. Suppose that*

$$\tau \leq \tau_G := \inf\{t > 0; X(t) \notin G\}, \tag{2.4.3}$$

$$X(\tau) \in \bar{G} \qquad a.s., \tag{2.4.4}$$

and

$$E^x[|f(X(\tau)|+\int_0^\tau \{|Af(X(t))|+|\sigma^T(X(t))\nabla f(X(t))|^2 + \sum_{k=1}^{l}\int_{\mathbb{R}}|f(X(t)+\gamma^{(k)}(X(t),z_k))-f(X(t))|^2\nu_k(dz_k)\}dt]<\infty. \quad (2.4.5)$$

Then

$$E^x[f(X(\tau))]=f(x)+E^x[\int_0^\tau (Af)(X(t))dt].$$

Part II: Optimal Stopping and Impulse Control of Jump Diffusions

An *impulse control* consists of a set of intervention times and intervention amounts chosen in order to minimize (maximize) a certain criterion of performance. For example, consider the situation in which a central bank is aiming to keep the exchange rate close to a predetermined target. Deviations from this target are costly and the central bank can intervene in the market by buying or selling reserves in the foreign currency. An impulse control not only specifies the amount of reserves the central bank should buy or sell per intervention, but also the times at which it is optimal to intervene. Impulse control problems have received special attention lately because of their applications in mathematical finance.

An impulse control can be thought of as a sequence of *optimal stopping problems*. So we can use similar approaches to solve both problems. We first present some of the key results in optimal stopping theory in section 2.5. Then we summarize some of the main results on impulse control theory in section 2.6. For a complete study of the optimal stopping and impulse control, we refer to Bensoussan and Lions [8], Øksendal [49], Øksendal and Sulem [52] and Sethi and Thompson [58]. Most of the definitions and results in this part follow Øksendal and Sulem [52].

2.5 Optimal stopping of jump diffusions

The *optimal stopping problems* can be thought as the problems in which one wants to decide the appropriate time to stop a process in order to minimize (maximize) a certain functional. The theory of optimal stopping is often dealt with in two equivalent formulations: the *martingale approach*, which relies on the concept of finding the Snell envelope, and the *Markovian approach* that allows the problem to be transformed into a free-boundary problem. We will discuss only the Markovian approach in this dissertation. The reader may consult Karatzas and Shreve [35] or Peskir and Shiryaev [54] to study about the martingale approach.

A general formulation and a verification theorem:

Fix an open set $\mathcal{S} \subset \mathbb{R}^k$ (the *solvency region*) and let $Y(t)$ be a jump diffusion in $\mathbb{R}^k$ given by

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}} \gamma(Y(t-), z)\widetilde{N}(dt, dz), \qquad Y(0) = y \in \mathbb{R}^k, \tag{2.5.1}$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times m}$ and $\gamma : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{k\times l}$ are given functions such that a unique solution $Y(t)$ exists (see Theorem 2.13). Let

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}(y, \omega) = \inf\{t > 0; Y(t) \notin \mathcal{S}\}$$

be the *bankruptcy time* and let $\mathcal{T}$ denote the set of all stopping times $\tau \leq \tau_{\mathcal{S}}$.

The result below remains valid, with the natural modifications, if we allow $\mathcal{S}$ to be any Borel set such that $\mathcal{S} \subset \bar{\mathcal{S}}^0$ where $\mathcal{S}^0$ denotes the interior of $\mathcal{S}$, and $\bar{\mathcal{S}}^0$ closure of $\mathcal{S}^0$.

Let $f : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$ be continuous functions satisfying the conditions

$$E^y \left[\int_0^{\tau_{\mathcal{S}}} f^-(Y(t))dt \right] < \infty \qquad \textit{for all } y \in \mathbb{R}^k \tag{2.5.2}$$

and the family $\{g^-(Y(\tau)) \cdot I_{\{\tau<\infty\}}; \tau \in \mathcal{T}\}$ is uniformly integrable, for all $y \in \mathbb{R}^k$. (If x is a real number, then $x^- := \max(-x, 0)$ denotes the negative part of x.)

The general *optimal stopping problem* is the following:

Find $\Phi(y)$ and $\tau^* \in \mathcal{T}$ such that

$$\Phi(y) = \inf_{\tau \in \mathcal{T}} J^\tau(y) = J^{\tau^*}(y), \qquad y \in \mathbb{R}^k,$$

where

$$J^\tau(y) = E^y[\int_0^\tau f(Y(t))dt + g(Y(\tau)) \cdot I_{\{\tau<\infty\}}], \qquad \tau \in \mathcal{T}$$

is the *performance criterion.*

The function Φ is called the *value function* and the stopping time τ^* (if it exists) is called an *optimal stopping time.*

In the following we let A be the integro-differential operator which coincides with the generator of $Y(t)$ on $C_0^2(\mathbb{R}^k)$, i.e.

$$A\phi(y) = \sum_{i=1}^{k} b_i(y)\frac{\partial \phi}{\partial y_i}(y) + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y)\frac{\partial^2 \phi}{\partial y_i \partial y_j}(y)$$
$$+ \sum_{j=1}^{l}\int_{\mathbb{R}}\{\phi(y+\gamma^{(j)}(y,z_j)) - \phi(y) - \nabla\phi(y)\cdot\gamma^{(j)}(y,z_j)\}\nu_j(dz_j) \qquad (2.5.3)$$

for all $\phi : \mathbb{R}^k \to \mathbb{R}$ and $y \in \mathbb{R}^k$ such that (2.5.3) exists (See Theorem 2.15 and Theorem 2.17).

Definition 2.12. *In general, if $\{\psi_m\}_{m=1}^{\infty}$ and g are functions defined on a set $G \subset \mathbb{R}^n$, we say that $\psi_m \to g$ pointwise dominatedly in G if $\psi_m(x) \to g(x)$ for all $x \in G$ and there exists a constant $C < \infty$ such that*

$$|\psi_m(x)| \leq C|g(x)| \qquad \text{for all } x \in G, \quad m = 1, 2, \cdots$$

We will need the following result. A proof of a related result (in the no-jump case) can be found in Øksendal [49].

Theorem 2.18. *(**Approximation theorem**, Øksendal and Sulem [52]) Let D be an open set, $D \subset \mathcal{S}$. Assume that*

$$\partial D \text{ is a Lipschitz surface} \qquad (2.5.4)$$

(i.e. ∂D is locally the graph of a Lipschitz continuous function) and let $\varphi : \bar{\mathcal{S}} \to \mathbb{R}$ be a function with the following properties:

$$\varphi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}}) \qquad (2.5.5)$$

and

$$\varphi \in C^2(\mathcal{S} \setminus \partial D) \tag{2.5.6}$$

and the second order derivatives of φ *are locally bounded near* ∂D. *Then there exists a sequence* $\{\varphi_m\}_{m=1}^{\infty} \subset C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ *such that, with* A *as in (2.5.3),*

$$\varphi_m \to \varphi \text{ pointwise dominatedly in } \bar{\mathcal{S}} \text{ as } m \to \infty, \tag{2.5.7}$$

$$\frac{\partial \varphi_m}{\partial x_i} \to \frac{\partial \varphi}{\partial x_i} \text{ pointwise dominatedly in } \mathcal{S} \text{ as } m \to \infty, \tag{2.5.8}$$

$$\frac{\partial^2 \varphi_m}{\partial x_i \partial x_j} \to \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \text{ and } A\varphi_m \to A\varphi \text{ pointwise dominatedly in } \mathcal{S} \setminus \partial D \text{ as } m \to \infty. \tag{2.5.9}$$

We now introduce a verification theorem for optimal stopping of jump diffusions. In the verification theorem, we formulate a set of sufficient conditions so that a given function ϕ actually coincides with the value function Φ and that a corresponding stopping time, τ_D, actually is optimal.

Theorem 2.19. ***(Integro-variational inequalities for optimal stopping,*** *Øksendal and Sulem [52])*

(a) Suppose we can find a function $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ *such that*

(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$,

(ii) $\phi \leq g$ *on* $\mathcal{S}$.

Define

$$D = \{y \in \mathcal{S}; \phi(y) < g(y)\} \qquad \text{(the continuation region).}$$

Suppose

(iii)

$$E^y \left[\int_0^{\tau_{\mathcal{S}}} I_{\partial D}(Y(t))dt \right] = 0,$$

(iv) ∂D is a Lipschitz surface,

(v) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ with locally bounded derivatives near ∂D,

(vi) $A\phi + f \geq 0$ on $\mathcal{S} \setminus \partial D$,

and

(vii) $Y(\tau_{\mathcal{S}}) \in \partial \mathcal{S}$ a.s. on $\{\tau_{\mathcal{S}} < \infty\}$ and

$$\lim_{t \to \tau_{\mathcal{S}}^-} \phi(Y(t)) = g(Y(\tau_{\mathcal{S}})) \cdot I_{\{\tau_{\mathcal{S}} < \infty\}}$$

and

(viii)

$$E^y[|\phi(Y(\tau))|] + \int_0^{\tau_{\mathcal{S}}} \{|A\phi(Y(t))| + |\sigma^T(Y(t))\nabla\phi(Y(t))|^2 + \sum_{j=1}^{l}[\int_{\mathbb{R}} |\phi(Y(t) + \gamma^{(j)}(Y(t), z)) - \phi(Y(t))|^2 \nu_j(dz_j)]\}dt] < \infty \qquad \text{for all } \tau \in \mathcal{T}.$$

Then $\phi(y) \leq \Phi(y)$ for all $y \in \bar{\mathcal{S}}$.

(b) Moreover, assume

(ix) $A\phi + f = 0$ on D,

(x) $\tau_D := \inf\{t > 0; Y(t) \notin D\} < \infty$ a.s. for all y,

(xi) $\{\phi(Y(\tau)); \tau \in \mathcal{T}\}$ is uniformly integrable, for all y.

Then

$$\phi(y) = \Phi(y)$$

and

$$\tau^* = \tau_D \quad \text{is an optimal stopping time.}$$

The following two propositions are also very useful in optimal stopping theory.

Proposition 2.1. *(Øksendal and Sulem [52]) Suppose the conditions of Theorem 2.23 hold. Suppose* $g \in C^2(\mathbb{R}^k)$ *and that* $\phi = g$ *satisfies (viii). Define*

$$U = \{y \in \mathcal{S}; Ag(y) + f(y) < 0\}.$$

Suppose that for all $y \in U$ *there exists a neighbourhood* W_y *of* y *such that* $\tau_{W_y} := \inf\{t > 0; Y(t) \notin W_y\} < \infty$ *a.s. Then*

$$U \subset \{y \in \mathcal{S}; \Phi(y) < g(y)\} = D.$$

Hence it is never optimal to stop while $Y(t) \in U$*.*

Proposition 2.2. *(Øksendal and Sulem [52]) Let* U *be as in Proposition 2.1. Suppose* $U = \emptyset$*. Then*

$$\Phi(y) = g(y) \text{ and } \tau^* = 0 \text{ is optimal.}$$

We refer the reader to Øksendal and Sulem [52] and Makasu [44] for applications of optimal stopping of jump diffusions.

2.6 Impulse control of jump diffusions

An impulse control problem can be thought of as a sequence of optimal stopping problems. Impulse control problems refer to situations in which the controller has the ability to intervene a process at certain chosen times by forcing a jump of chosen size to the process. In a pure impulse control, the controller does not affect either the drift or the volatility of the process. However this could be done if in addition we consider a continuous stochastic control. In this section, we explain and summarize the main results in impulse control theory.

A general formulation and a verification theorem

When there are no interventions, we assume that the state $Y(t) \in \mathbb{R}^k$ of the system is a jump diffusion process defined by (2.5.1). The generator A of $Y(t)$ is given by (2.5.3), i.e.

$$A\phi(y) = \sum_{i=1}^{k} b_i(y) \frac{\partial \phi}{\partial y_i}(y) + \frac{1}{2} \sum_{i,j=1}^{k} (\sigma\sigma^T)_{ij}(y) \frac{\partial^2 \phi}{\partial y_i \partial y_j}(y)$$
$$+ \sum_{j=1}^{l} \int_{\mathbb{R}} \{\phi(y + \gamma^{(j)}(y, z_j)) - \phi(y) - \nabla\phi(y) \cdot \gamma^{(j)}(y, z_j)\} \nu_j(dz_j).$$

Now suppose that at any time t and any state y we are free to intervene and give the system an impulse $\zeta \in \mathcal{Z} \subset \mathbb{R}^p$, where $\mathcal{Z}$ is a given set (the set of admissible impulse values). Suppose the result of giving the impulse ζ, when the state is y, is that the state jumps immediately from $y = Y(t-)$ to $Y(t) = \Gamma(y, \zeta) \in \mathbb{R}^k$, where $\Gamma : \mathbb{R}^k \times \mathcal{Z} \to \mathbb{R}^k$ is a given function.

An *impulse control* for this system is a double (possibly finite) sequence

$$\nu = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)_{j \le M}\ ; \qquad M \le \infty,$$

where $\tau_1 < \tau_2 < \cdots$ are $\mathcal{F}_t$-stopping times (the *intervention times*) and $\zeta_1, \zeta_2, \ldots$ are corresponding *impulses* at these times. We assume that ζ_j is $\mathcal{F}_{\tau_j}$-measurable for all j.

If $\nu = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)_{j \le M}$ is an impulse control, the corresponding state process $Y_y^{(\nu)}(t)$ with $Y_y^{(\nu)}(0) = y$ is defined by

$$Y_y^{(\nu)}(t) = Y_y(t) \ ; \qquad 0 \le t < \tau_1, \tag{2.6.1}$$

$$Y_y^{(\nu)}(\tau_j) = \Gamma(Y_y^{(\nu)}(\tau_j-) + \triangle_N Y(\tau_j), \zeta_j); \qquad j = 1, 2, ..., M, \tag{2.6.2}$$

$$Y_y^{(\nu)}(t) = Y^{(\nu)}_{Y_y^{(\nu)}(\tau_j)}(t) \qquad \text{for } \tau_j \le t < \tau_{j+1} \wedge \tau^*, \tag{2.6.3}$$

where $\triangle_N Y(t)$ is the jump of Y stemming from the jump of the random measure $N(t, \cdot)$ only and

$$\tau^* = \tau^*(\omega) = \lim_{R\to\infty}(\inf\{t > 0; |Y_y^{(\nu)}(t)| \ge R\}) \le \infty \tag{2.6.4}$$

is the explosion time of $Y_y^{(\nu)}(t)$.

Note that here we must distinguish between the (possible) jump of $Y_y^{(\nu)}(\tau_j)$ stemming from N, denoted by $\triangle_N Y_y(\tau_j)$ and the jump caused by the intervention ν, given by

$$\triangle_\nu Y_y^{(\nu)}(\tau_j) := \Gamma(\check{Y}_y^{(\nu)}(\tau_j-), \zeta) - \check{Y}_y^{(\nu)}(\tau_j-), \tag{2.6.5}$$

where

$$\check{Y}_y^{(\nu)}(\tau_j-) = Y_y^{(\nu)}(\tau_j-) + \triangle_N Y_y(\tau_j). \tag{2.6.6}$$

Let $\mathcal{S} \subset \mathbb{R}^k$ be a fixed open set (the *solvency region*). Define

$$\tau_{\mathcal{S}} = \inf\{t \in (0, \tau^*); Y_y^{(\nu)}(t) \notin \mathcal{S}\}. \tag{2.6.7}$$

Suppose we are given a *cost function* $f : \mathcal{S} \to \mathbb{R}$ and a *bequest function* $g : \mathbb{R}^k \to \mathbb{R}$. Moreover, suppose the cost/utility of making an intervention with impulse $\zeta \in \mathcal{Z}$ is $K(y, \zeta)$ when the state is y, where $K : \mathcal{S} \times \mathcal{Z} \to \mathbb{R}$ is a given function.

We assume that we are given a set $\mathcal{V}$ of *admissible impulse controls* which is included in the set of $\nu = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)_{j \le M}$ such that a unique solution $Y_y^{(\nu)}(t)$ of (2.6.1)-(2.6.3) exists and

$$\tau^* = \infty \qquad \text{a.s.} \tag{2.6.8}$$

and

$$\lim_{j\to\infty} \tau_j = \tau_{\mathcal{S}} \quad \text{a.s.} \qquad (\text{if } M < \infty \text{ we assume } \tau_M = \tau_{\mathcal{S}} \text{ a.s.}) \tag{2.6.9}$$

We also assume that

$$E^y[\int_0^{\tau_{\mathcal{S}}} f^-(Y^{(\nu)}(t))dt] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}, \tag{2.6.10}$$

$$E^y[g^-(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}, \tag{2.6.11}$$

and

$$E^y[\sum_{\tau_j \le \tau_{\mathcal{S}}} K^-(\check{Y}^{(\nu)}(\tau_j-), \zeta_j)] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}. \tag{2.6.12}$$

Now define the performance criterion

$$J^{(\nu)}(y) = E^y[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu)}(t))dt + g(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{\tau_j \le \tau_{\mathcal{S}}} K(\check{Y}^{(\nu)}(\tau_j-), \zeta_j)].$$

The *impulse control* problem is the following:

Find $\Phi(y)$ and $\nu^* \in \mathcal{V}$ such that

$$\Phi(y) = \inf\{J^{(\nu)}(y); \nu \in \mathcal{V}\} = J^{(\nu^*)}(y). \tag{2.6.13}$$

We now define the very important *intervention operator*:

Definition 2.13. *Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$. The intervention operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\mathcal{M}h(y) = \inf\{h(\Gamma(y,\zeta)) + K(y,\zeta);\ \zeta \in \mathcal{Z} \ \ \text{and} \ \ \Gamma(y,\zeta) \in \mathcal{S}\}. \tag{2.6.14}$$

We also put $\mathcal{T} = \{\tau;\ \tau \text{ a stopping time}, 0 \le \tau \le \tau_{\mathcal{S}} \text{ a.s.}\}$.

We can now state the main result of this section, a verification theorem for impulse control problems:

Theorem 2.20. ***(Quasi-integrovariational inequalities for impulse control**, Øksendal and Sulem [52])*

(a) Suppose we can find a function $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ such that

(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$,

(ii) $\phi \le \mathcal{M}\phi$ on $\mathcal{S}$.

Define

$$D = \{y \in \mathcal{S}; \phi(y) < \mathcal{M}\phi(y)\} \qquad \text{(the continuation region).}$$

Suppose

(iii)

$$E^y\left[\int_0^{\tau_{\mathcal{S}}} I_{\partial D}(Y^{(\nu)}(t))dt\right] = 0 \quad \text{for all} \ \ y \in \mathcal{S},\ \nu \in \mathcal{V},$$

(iv) ∂D is a Lipschitz surface,

(v) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ with locally bounded derivatives near ∂D,

(vi) $A\phi + f \ge 0$ on $\mathcal{S} \setminus \partial D$,

(vii) $\phi(Y_y^{(\nu)}(t)) \to g(Y_y^{(\nu)}(\tau_{\mathcal{S}})) \cdot I_{\{\mathcal{S}<\infty\}}$ as $t \to \tau_{\mathcal{S}}-$ a.s., for all $y \in \mathcal{S}$, $\nu \in \mathcal{V}$,

(viii) $\{\phi^-(Y_y^{(\nu)}(\tau)); \tau \in \mathcal{T}\}$ is uniformly integrable, for all $y \in \mathcal{S}$, $\nu \in \mathcal{V}$,

(ix)

$$E^y[|\phi(Y^{(\nu)}(\tau))|] + \int_0^{\tau_{\mathcal{S}}} \{|A\phi(Y^{(\nu)}(t))| + |\sigma^T(Y^{(\nu)}(t))\nabla\phi(Y^{(\nu)}(t))|^2 + \sum_{j=1}^{l}[\int_{\mathbb{R}} |\phi(Y^{(\nu)}(t) + \gamma^{(j)}(Y^{(\nu)}(t), z)) - \phi(Y^{(\nu)}(t))|^2 \nu_j(dz_j)]\}dt] < \infty$$

for all $\tau \in \mathcal{T}$, $y \in \mathcal{S}$, $\nu \in \mathcal{V}$.

Then

$$\phi(y) \leq \Phi(y) \quad \text{for all } \; y \in \mathcal{S}. \tag{2.6.15}$$

(b) Suppose in addition that

(x) $A\phi + f = 0$ *in* D,

(xi) $\hat{\zeta}(y) \in \operatorname{argmin} \{\phi(\Gamma(y, \cdot)) + K(y, \cdot)\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\zeta}(\cdot)$ *is a Borel measurable selection.*

Put $\hat{\tau}_0 = 0$ *and define* $\hat{\nu} = (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j, \dots; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j, \dots)$ *inductively by* $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j; Y_y^{(\hat{\nu}_j)}(t) \notin D\} \wedge \tau_{\mathcal{S}}$ *and* $\hat{\zeta}_{j+1} = \hat{\zeta}(Y_y^{(\hat{\nu}_j)}(\hat{\tau}_{j+1}-))$ *if* $\hat{\tau}_{j+1} < \tau_{\mathcal{S}}$, *where* $Y_y^{(\hat{\nu}_j)}$ *is the result of applying* $\hat{\nu}_j := (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j)$ *to* Y.

Suppose

(xii) $\hat{\nu} \in \mathcal{V}$ *and* $\{\phi(Y^{(\hat{\nu})}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable.*

Then

$$\phi(y) = \Phi(y) \quad \text{and} \quad \hat{\nu} \quad \text{is an optimal impulse control.} \tag{2.6.16}$$

Impulse control theory can be applied in many different areas to solve various problems. In particular, we can apply impulse control theory in optimal central bank intervention problems. This has been addressed in Candenillas and Zapatero [14], Candenillas and Zapatero [15], Jeanblanc-Picqué [33], Korn [36], Mundaca and Øksendal [47] and Øksendal and Sulem [52].

Impulse control theory has many other applications in prototype problems, optimal dividend policy payment problems under transaction costs and inventory control problems. We refer the reader to Buckley and Korn [13], Constantinides and Richard [17], Eastham and Hasting [22], Harrison, Selke and Taylor [29], Korn [36], Jeanblanc-Picqué and Shiryaev [34], Øksendal [48], Øksendal and Sulem [51], Øksendal, Ubøe and Zhang [53], Sato [56], Sulem [59] and Varner [61] for a complete study about these applications.

Recently, there has been research in the intervention cost structure of impulse control. Øksendal [48] and Øksendal, Ubøe and Zhang [53] are excellent resources to study about this. Korn [36] and Øksendal [50] are also very good resources in impulse control theory, which address delayed control actions, and controls with random consequences, respectively. It is also important to mention the dissertation of Egami [23] which tackles impulse control problems using the new methodology proposed by Dayanik and Karatzas [20] for solving optimal-stopping problems.

Chapter 3

Stochastic Optimal Impulse Control of Jump Diffusions with Regime Switching

Øksendal and Mundaca [47] and Cadenillas and Zapatero [14], [15] expressly assumed that investors do not observe or anticipate the interventions of the central bank. Therefore they particularly assumed that the process driving the exchange rate dynamics is not affected by interventions. This is unrealistic, but to do otherwise "would yield different dynamics for the exchange rate and would probably make the model intractable" (See Cadenillas and Zapatero [15]). Moreno [46] was able to relax this assumption for a special type of continuous process allowing for a market reaction to interventions.

In this chapter, we overcome this intractability to solve the problem with a more general setting than all of the above references. We also use a more general jump diffusion processes as our exchange-rate process. However, we still retain the assumption that investors do not anticipate interventions by the central bank. This is much more reasonable than the assumption that investors do not observe and react to interventions. It is usually acceptable that the central bank itself revises the parameters used in the exchange-rate model as time passes, because the optimal solution today would not persist due to the incorporation of new information in the bank's model. Therefore investors are unlikely to have much confidence in forecasting the next bank intervention. Hence it is reasonable to assume that interventions are invisible to the market in order to reasonably ignore the market effect of investors' prediction of future intervention times.

The main results that we develop in this chapter can be applied to more general situations than the central bank intervention problem. So we first develop our theory using a more general setting and then apply it to solve the central bank intervention problem later in the chapter.

3.1 A general formulation

Suppose that, if there are no interventions, the state $Y(t) \in \mathbb{R}^k$ of the system we consider is a jump diffusion of the form

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}} \gamma(Y(t-), z)\widetilde{N}(dt, dz), \qquad (3.1.1)$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times m}$ and $\gamma : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{k\times l}$ are given functions satisfying the conditions for the existence and uniqueness of a solution $Y(t)$ (see Theorem 2.13).

Now suppose that the state of the system changes temporarily after an intervention takes place. If the i^{th} intervention occurs at time τ_i, then the state of the system takes the following jump diffusion for a bounded, constant length of time $T_i \geq 0$.

$$d\tilde{Y}(t) = \tilde{b}(\tilde{Y}(t))dt + \tilde{\sigma}(\tilde{Y}(t))dB(t) + \int_{\mathbb{R}} \tilde{\gamma}(\tilde{Y}(t-), z)\widetilde{N}(dt, dz), \qquad (3.1.2)$$

where $\tilde{b} : \mathbb{R}^k \to \mathbb{R}^k$, $\tilde{\sigma} : \mathbb{R}^k \to \mathbb{R}^{k\times m}$ and $\tilde{\gamma} : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{k\times l}$ are given functions satisfying the conditions for the existence and uniqueness of a solution $\tilde{Y}(t)$ (see Theorem 2.13).

Since T_i is fixed, we have $T_i = T$ for all $i > 0$. However, since there is no reaction at the beginning, we use that $T_0 = 0$ as a convention.

We call the period $(\tau_i, \tau_i + T_i]$, the *reaction period* and the time T_i, the *reaction time*.

The generator A of $Y(t)$ is

$$A\phi(y) = \sum_{i=1}^{k} b_i(y)\frac{\partial \phi}{\partial y_i}(y) + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y)\frac{\partial^2 \phi}{\partial y_i \partial y_j}(y)$$
$$+ \sum_{j=1}^{l}\int_{\mathbb{R}}\{\phi(y + \gamma^{(j)}(y, z_j)) - \phi(y) - \nabla\phi(y)\cdot\gamma^{(j)}(y, z_j)\}\nu_j(dz_j)$$

and the generator $\tilde{A}$ of $\tilde{Y}(t)$ is

$$\tilde{A}\phi(y) = \sum_{i=1}^{k} \tilde{b}_i(y)\frac{\partial \phi}{\partial y_i}(y) + \frac{1}{2}\sum_{i,j=1}^{k}(\tilde{\sigma}\tilde{\sigma}^T)_{ij}(y)\frac{\partial^2 \phi}{\partial y_i \partial y_j}(y)$$
$$+ \sum_{j=1}^{l}\int_{\mathbb{R}}\{\phi(y + \tilde{\gamma}^{(j)}(y, z_j)) - \phi(y) - \nabla\phi(y)\cdot\tilde{\gamma}^{(j)}(y, z_j)\}\nu_j(dz_j).$$

Now suppose that at any time t and any state y we are free to intervene and give the system an impulse $\zeta \in \mathcal{Z} \subset \mathbb{R}^P$, where $\mathcal{Z}$ is a given set of admissible impulse values.

Suppose the result of giving the impulse ζ, when the state is y, is that the state jumps immediately from $y = Y(t-)$ to $Y(t) = \Gamma(y, \zeta) \in \mathbb{R}^k$, where $\Gamma : \mathbb{R}^k \times \mathcal{Z} \to \mathbb{R}^k$ is a given function.

An *impulse control* for this system is a double (possibly finite) sequence

$$\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)_{j \leq M}\ ; \qquad M \leq \infty,$$

where $\tau_1 < \tau_2 < \cdots$ are $\mathcal{F}_t$-stopping times (the *intervention times*) and $\zeta_1, \zeta_2, \dots$ are the corresponding *impulses* at these times. We assume that ζ_j is $\mathcal{F}_{\tau_j}$-measurable for all j.

If $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)_{j \leq M}$ is an impulse control, the corresponding state process $Y_y^{(\nu)}(t)$ with $Y_y^{(\nu)}(0) = y$ is defined by

$$Y_y^{(\nu)}(t) = Y_y(t)\ ; \qquad 0 \leq t < \tau_1, \tag{3.1.3}$$

$$Y_y^{(\nu)}(\tau_j) = \Gamma(Y_y^{(\nu)}(\tau_j -) + \triangle_N Y(\tau_j), \zeta_j); \qquad j = 1, 2, ..., M, \tag{3.1.4}$$

$$Y_y^{(\nu)}(t) = \tilde{Y}_{Y_y^{(\nu)}(\tau_j)}(t) \qquad \text{for } \tau_j \leq t \leq \tau_j + T_j; \quad j = 1, 2, ..., M, \tag{3.1.5}$$

$$Y_y^{(\nu)}(t) = Y_{Y_y^{(\nu)}(\tau_j + T_j)}(t) \qquad \text{for } \tau_j + T_j < t < \tau_{j+1} \wedge \tau^*; \quad j = 1, 2, ..., M, \tag{3.1.6}$$

where $\triangle_N Y(t)$ is the jump of Y stemming from the jump of the random measure $N(t, \cdot)$ only and

$$\tau^* = \tau^*(\omega) = \lim_{R \to \infty} (\inf\{t > 0; |Y_y^{(\nu)}(t)| \geq R\}) \leq \infty \tag{3.1.7}$$

is the explosion time of $Y_y^{(\nu)}(t)$.

Note that here we must distinguish between the (possible) jump of $Y_y^{(\nu)}(\tau_j)$ stemming from N, denoted by $\triangle_N Y_y(\tau_j)$ and the jump caused by the intervention ν, given by

$$\triangle_\nu Y_y^{(\nu)}(\tau_j) := \Gamma(\check{Y}_y^{(\nu)}(\tau_j-), \zeta) - \check{Y}_y^{(\nu)}(\tau_j-), \tag{3.1.8}$$

where

$$\check{Y}_y^{(\nu)}(\tau_j-) = Y_y^{(\nu)}(\tau_j-) + \triangle_N Y_y(\tau_j). \tag{3.1.9}$$

Let $\mathcal{S} \subset \mathbb{R}^k$ be a fixed open set (the *solvency region*). Define

$$\tau_{\mathcal{S}} = \inf\{t \in (0, \tau^*); Y_y^{(\nu)}(t) \notin \mathcal{S}\}. \tag{3.1.10}$$

Suppose we are given a *cost function* $f : \mathcal{S} \to \mathbb{R}$ and a *bequest function* $g : \mathbb{R}^k \to \mathbb{R}$. Moreover, suppose the cost/utility of making an intervention with impulse $\zeta \in \mathcal{Z}$ is $K(y, \zeta)$ when the state is y, where $K : \mathcal{S} \times \mathcal{Z} \to \mathbb{R}$ is a given function.

We assume that we are given a set $\mathcal{V}$ of *admissible impulse controls* which is included in the set of $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)_{j \leq M}$ such that a unique solution $Y_y^{(\nu)}(t)$ of (3.1.3)-(3.1.6) exists and

$$\tau^* = \infty \qquad \text{a.s.} \tag{3.1.11}$$

and

$$\lim_{j\to\infty} \tau_j = \tau_{\mathcal{S}} \quad \text{a.s.} \qquad (\text{if } M < \infty \text{ we assume } \tau_M = \tau_{\mathcal{S}} \text{ a.s.}). \tag{3.1.12}$$

We also assume that

$$E^y \left[\int_0^{\tau_{\mathcal{S}}} f^-(Y^{(\nu)}(t)) \right] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}, \tag{3.1.13}$$

$$E^y \left[g^-(Y^{(\nu)}(\tau_{\mathcal{S}})) I_{\{\tau_{\mathcal{S}} < \infty\}} \right] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}, \tag{3.1.14}$$

$$E^y[\sum_{\tau_j \le \tau_{\mathcal{S}}} K^-(\check{Y}^{(\nu)}(\tau_j-), \zeta_j)] < \infty \qquad \text{for all} \quad y \in \mathbb{R}^k,\ \nu \in \mathcal{V}, \tag{3.1.15}$$

and

$$\tau_{j+1} - \tau_j \ge T_j \qquad \text{for all } j. \tag{3.1.16}$$

Condition (3.1.16) means that no interventions are allowed during the reaction period.

Now we define the performance criterion

$$J^{(\nu)}(y) = E^y \left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu)}(t))dt + g(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{\tau_i \le \tau_{\mathcal{S}}} K(\check{Y}^{(\nu)}(\tau_j-), \zeta_j) \right].$$

The *impulse control* problem is the following:

Find $\Phi(y)$ and $\nu^* \in \mathcal{V}$ such that

$$\Phi(y) = \inf \left\{ J^{(\nu)}(y); \nu \in \mathcal{V} \right\} = J^{(\nu^*)}(y). \tag{3.1.17}$$

3.2 The reacted intervention operator

We now define two very crucial concepts for our main result.

Definition 3.1. *Let $f : \mathcal{S} \to \mathbb{R}$ be a cost function. Then we define the running cost during the reaction period by*

$$R(\tilde{y}) = E^{\tilde{y}} \left[\int_0^T f\left(\tilde{Y}(t)\right) dt \right] \tag{3.2.1}$$

where $\tilde{y}$ is the state of the process after an intervention.

Definition 3.2. *Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$. The* ***reacted intervention operator*** *$\mathcal{M}_r : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\mathcal{M}_r h(y) = \inf \left\{ K(y,\zeta) + R\left(\Gamma(y,\zeta)\right) + E^{\Gamma(y,\zeta)}\left[h\left(\tilde{Y}(T)\right)\right] ; \ \zeta \in \mathcal{Z} \ \ and \ \ \Gamma(y,\zeta) \in \mathcal{S} \right\}. \tag{3.2.2}$$

We also put $\mathcal{T} = \{\tau;\ \tau \text{ stopping time }, 0 \le \tau \le \tau_{\mathcal{S}} \text{ a.s.}\}$ and introduce the following notation before we state the main result of this chapter.

Notation:

$$\bar{A}\phi(Y_y^{(\nu)}(t)) = \begin{cases} \tilde{A}\phi(Y_y^{(\nu)}(t)) & ;\ t \in (\tau_i, \tau_i + T_i] \quad \text{for all } i, \\ A\phi(Y_y^{(\nu)}(t)) & ;\ \text{otherwise.} \end{cases}$$

3.3 A verification theorem with regime switching

We can now state the main result of this chapter, a verification theorem for impulse control problems with *regime switching*:

Theorem 3.1. *(**Quasi-integrovariational inequalities for impulse control with regime switching :- constant reaction time**)*
(a) Suppose we can find a function $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ such that
(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$,
(ii) $\phi \le \mathcal{M}_r\phi$ on $\mathcal{S}$.
Define

$$D = \{y \in \mathcal{S}; \phi(y) < \mathcal{M}_r\phi(y)\} \qquad \textit{(the continuation region).}$$

Suppose

(iii)

$$E^y\left[\int_0^{\tau_{\mathcal{S}}} I_{\partial D}(Y^{(\nu)}(t))dt\right] = 0 \quad \textit{for all} \ \ y \in \mathcal{S},\ \nu \in \mathcal{V},$$

(iv) ∂D *is a Lipschitz surface,*

(v) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ *with locally bounded derivatives near* ∂D*,*

(vi) $\bar{A}\phi + f \geq 0$ *on* $\mathcal{S} \setminus \partial D$*,*

(vii) $\phi(Y_y^{(\nu)}(t)) \to g(Y_y^{(\nu)}(\tau_{\mathcal{S}})) \cdot I_{\{\mathcal{S}<\infty\}}$ *as* $t \to \tau_{\mathcal{S}}-$ *a.s., for all* $y \in \mathcal{S}$*,* $\nu \in \mathcal{V}$*,*

(viii) $\{\phi^-(Y_y^{(\nu)}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable, for all* $y \in \mathcal{S}$*,* $\nu \in \mathcal{V}$*,*

(ix)

$$E^y[|\phi(Y^{(\nu)}(\tau))| + \int_0^{\tau_s} \{|\bar{A}\phi(Y^{(\nu)}(t))| + |\sigma^T(Y^{(\nu)}(t))\nabla\phi(Y^{(\nu)}(t))|^2 + \sum_{j=1}^{l} \int_{\mathbb{R}} |\phi(Y^{(\nu)}(t) + \gamma^{(j)}(Y^{(\nu)}(t), z_j)) - \phi(Y^{(\nu)}(t))|^2 \nu_j(dz_j)]\}dt] < \infty$$

for all $\tau \in \mathcal{T}$*,* $y \in \mathcal{S}$*,* $\nu \in \mathcal{V}$*.*

Then

$$\phi(y) \leq \Phi(y) \quad \text{for all } \; y \in \mathcal{S}. \tag{3.3.1}$$

(b) Suppose in addition that

(x) $A\phi + f = 0$ *in* D*,*

(xi) $\hat{\zeta}(y) \in \text{argmin} \left\{K(y,\cdot) + R(\Gamma(y,\cdot)) + E^{\Gamma(y,\cdot)}\left[\phi\left(\tilde{Y}(T)\right)\right]\right\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\zeta}(\cdot)$ *is a Borel measurable selection.*

Put $\hat{\tau}_0 = 0$ *and define* $\hat{\nu} = (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j, \dots; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j, \dots)_{j \leq M}$ *inductively by* $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j + T_j; Y_y^{(\hat{\nu}_j)}(t-) \notin D\} \wedge \tau_{\mathcal{S}}$ *and* $\hat{\zeta}_{j+1} = \hat{\zeta}(Y_y^{(\hat{\nu}_j)}(\hat{\tau}_{j+1}-))$ *if* $\hat{\tau}_{j+1} < \tau_{\mathcal{S}}$*, where* $Y_y^{(\hat{\nu}_j)}$ *is the result of applying* $\hat{\nu}_j := (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j)$ *to* Y*.*

Suppose

(xii) $\hat{\nu} \in \mathcal{V}$ *and* $\{\phi(Y_y^{(\hat{\nu})}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable.*

Then

$$\phi(y) = \Phi(y) \quad \text{and} \quad \hat{\nu} \quad \text{is an optimal impulse control.} \tag{3.3.2}$$

Proof:

(a) By the approximation Theorem 2.18 and (i), $(iii)-(v)$, we may assume that $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$.

Choose $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)_{j \le M} \in \mathcal{V}$ and set $\tau_0 = 0$. By another approximation argument and (ix), we may assume that we can apply the Dynkin Theorem II (Theorem 2.17) to the stopping times τ_i and $\tau_i + T_i$.

Then, for $i = 1, 2, ..., M$, we have

$$E^y\left[\phi(Y^{(\nu)}(\tau_i))\right] - E^y\left[\phi(Y^{(\nu)}(\tau_i + T_i))\right] = -E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi(Y^{(\nu)}(t))dt\right], \quad (3.3.3)$$

and, for $i = 0, 1, 2, ..., M$, we have

$$E^y\left[\phi(Y^{(\nu)}(\tau_i + T_i)\right] - E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{i+1}-))\right] = -E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} A\phi(Y^{(\nu)}(t))dt\right], \quad (3.3.4)$$

where $\check{Y}_y^{(\nu)}(\tau_{i+1}-) = Y_y^{(\nu)}(\tau_{i+1}-) + \triangle_N Y_y(\tau_{i+1})$.

Summing equation (3.3.3) from $i = 1$ to $i = m < M$, we get

$$\sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i))\right] - \sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i + T_i))\right] = -\sum_{i=1}^{m} E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi(Y^{(\nu)}(t))dt\right]. \quad (3.3.5)$$

Summing equation (3.3.4) from $i = 0$ to $i = m < M$, we get

$$\sum_{i=0}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i + T_i)\right] - \sum_{i=0}^{m} E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{i+1}-))\right] = -\sum_{i=0}^{m} E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} A\phi(Y^{(\nu)}(t))dt\right].$$

Equivalently, we have

$$\phi(y) + \sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i + T_i)\right] - \sum_{i=1}^{m} E^y\left[\phi(\check{Y}^{(\nu)}(\tau_i -))\right] - E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1} -))\right]$$
$$= -\sum_{i=0}^{m} E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} A\phi(Y^{(\nu)}(t))dt\right]. \qquad (3.3.6)$$

Adding equation (3.3.5) to (3.3.6) and using condition (vi), we get

$$\phi(y) + \sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i -))\right] - E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1} -))\right]$$
$$= -\sum_{i=1}^{m} E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi(Y^{(\nu)}(t))dt\right] - \sum_{i=0}^{m} E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} A\phi(Y^{(\nu)}(t))dt\right]$$
$$\leq -\sum_{i=1}^{m} E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi(Y^{(\nu)}(t))dt\right] + \sum_{i=0}^{m} E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} f(Y^{(\nu)}(t))dt\right].$$

Therefore, we have

$$\phi(y) + \sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i -))\right] - E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1} -))\right]$$
$$\leq -\sum_{i=1}^{m} E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi(Y^{(\nu)}(t))dt\right] + \sum_{i=0}^{m} E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} f(Y^{(\nu)}(t))dt\right]. \qquad (3.3.7)$$

By the definition of $\mathcal{M}_r$, we have for $\Gamma(y, \zeta) \in \mathcal{S}$

$$\mathcal{M}_r\phi(y) \leq K(y, \zeta) + E^{\Gamma(y,\zeta)}\left[\int_0^T f\left(\tilde{Y}(t)\right) dt\right] + E^{\Gamma(y,\zeta)}\left[\phi\left(\tilde{Y}(T)\right)\right].$$

However, if $\tau_i < \tau_{\mathcal{S}}$ then we have $Y_y^{(\nu)}(\tau_i) = \Gamma(\check{Y}_y^{(\nu)}(\tau_i -), \zeta_i) \in \mathcal{S}$. Therefore, we can plug in $\check{Y}_y^{(\nu)}(\tau_i -)$ for y in the above equation. Then by the strong Markov property, we get

$$\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-)) \leq K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i) + E^y\left[\int_{\tau_i}^{\tau_i+T_i} f\left(\tilde{Y}_{Y^{(\nu)}(\tau_i)}(t)\right)dt|\mathcal{F}_{\tau_i}\right] + E^y\left[\phi\left(\tilde{Y}_{Y^{(\nu)}(\tau_i)}(\tau_i+T_i)\right)|\mathcal{F}_{\tau_i}\right].$$

But $Y_y^{(\nu)}(t) = \tilde{Y}_{Y_y^{(\nu)}(\tau_i)}(t)$ for $\tau_i < t \leq \tau_i + T_i$. Therefore, we have

$$\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-)) \leq K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i) + E^y\left[\int_{\tau_i}^{\tau_i+T_i} f\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right] + E^y\left[\phi\left(Y^{(\nu)}(\tau_i+T_i)\right)|\mathcal{F}_{\tau_i}\right]. \tag{3.3.8}$$

We now simplify $E^y\left[\phi(Y^{(\nu)}(\tau_i+T_i))|\mathcal{F}_{\tau_i}\right]$. For this, we apply the multidimensional Itô formula in Theorem 2.11 with $f(t, Y_y^{(\nu)}(t)) = \phi\left(Y_y^{(\nu)}(t)\right)$ from τ_i to $\tau_i + T_i$. Then, we get

$$\begin{aligned}
\phi\left(Y_y^{(\nu)}(\tau_i+T_i)\right) &= \phi\left(Y_y^{(\nu)}(\tau_i)\right) \\
&+ \int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{k}\frac{\partial\phi}{\partial y_p}\left(Y_y^{(\nu)}(t)\right)[\tilde{b}_p\left(Y_y^{(\nu)}(t)\right)dt + \tilde{\sigma}_p\left(Y_y^{(\nu)}(t)\right)dB(t)] \\
&+ \frac{1}{2}\int_{\tau_i}^{\tau_i+T_i}\sum_{p,q=1}^{k}(\tilde{\sigma}\tilde{\sigma}^T)_{pq}\left(Y_y^{(\nu)}(t)\right)\frac{\partial^2\phi}{\partial y_p\partial y_q}\left(Y_y^{(\nu)}(t)\right)dt \\
&+ \int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{l}\int_{\mathbb{R}}\{\phi\left(Y_y^{(\nu)}(t-) + \tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t-),z_p)\right) - \phi\left(Y_y^{(\nu)}(t-)\right) \\
&\qquad - \sum_{q=1}^{k}\tilde{\gamma}_q^{(p)}(Y_y^{(\nu)}(t-),z_p)\frac{\partial\phi}{\partial y_q}\left(Y_y^{(\nu)}(t-)\right)\}\nu_p(dz_p)dt \\
&+ \int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{l}\int_{\mathbb{R}}\{\phi\left(Y_y^{(\nu)}(t-) + \tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t-),z_p)\right) \\
&\qquad - \phi\left(Y_y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt,dz_p).
\end{aligned}$$

By rearranging the terms in the above equation, we get

$$\begin{aligned}\phi\left(Y_y^{(\nu)}(\tau_i+T_i)\right) = \phi\left(Y_y^{(\nu)}(\tau_i)\right) &+ \int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{k}\frac{\partial\phi}{\partial x_p}\left(Y_y^{(\nu)}(t)\right)\tilde{\sigma}_p\left(Y_y^{(\nu)}(t)\right)dB(t)\\ &+ \int_{\tau_i}^{\tau_i+T_i}\tilde{A}\phi\left(Y_y^{(\nu)}(t)\right)dt\\ &+ \int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{l}\int_{\mathbb{R}}\{\phi\left(Y_y^{(\nu)}(t-)+\tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t),z_p)\right)\\ &\qquad\qquad - \phi\left(Y_y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt,dz_p).\end{aligned}$$

Now we take the conditional expectation of both sides of the above equation. Then, we get

$$E^y\left[\phi\left(Y^{(\nu)}(\tau_i+T_i)\right)|\mathcal{F}_{\tau_i}\right] = E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)|\mathcal{F}_{\tau_i}\right] + E^y\left[\int_{\tau_i}^{\tau_i+T_i}\tilde{A}\phi\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right], \tag{3.3.9}$$

since

$$E^y\left[\int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{k}\frac{\partial\phi}{\partial x_p}\left(Y^{(\nu)}(t)\right)\tilde{\sigma}_p\left(Y^{(\nu)}(t)\right)dB(t)|\mathcal{F}_{\tau_i}\right] = 0$$

and

$$E^y\left[\int_{\tau_i}^{\tau_i+T_i}\sum_{p=1}^{l}\int_{\mathbb{R}}\{\phi\left(Y^{(\nu)}(t-)+\tilde{\gamma}^{(p)}(Y^{(\nu)}(t-),z_p)\right) - \phi\left(Y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt,dz_p)|\mathcal{F}_{\tau_i}\right] = 0.$$

Substituting (3.3.9) in (3.3.8), we get

$$\begin{aligned}\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-)) \le K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i) &+ E^y\left[\int_{\tau_i}^{\tau_i+T_i} f\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right]\\ &+ E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)|\mathcal{F}_{\tau_i}\right] + E^y\left[\int_{\tau_i}^{\tau_i+T_i}\tilde{A}\phi\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right].\end{aligned}$$

Subtracting $\phi(\check{Y}_y^{(\nu)}(\tau_i-))$ from the both sides, we get

$$\begin{aligned}\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-)) - \phi(\check{Y}_y^{(\nu)}(\tau_i-)) \le K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i) &+ E^y\left[\int_{\tau_i}^{\tau_i+T_i} f\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right] \\ &+ E^y\left[\int_{\tau_i}^{\tau_i+T_i} \tilde{A}\phi\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right] \\ &+ E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)|\mathcal{F}_{\tau_i}\right] - \phi(\check{Y}_y^{(\nu)}(\tau_i-)). \qquad (3.3.10)\end{aligned}$$

Hence, by (3.3.7) and (3.3.10) we have

$$\begin{aligned}&\phi(y) + \sum_{i=1}^m E^y\left[\mathcal{M}_r\phi(\check{Y}^{(\nu)}(\tau_i-)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\ &\quad\le E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right] - \sum_{i=1}^m E^y\left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\ &\quad - \sum_{i=1}^m E^y\left[\int_{\tau_i}^{\tau_i+T_i}\tilde{A}\phi(Y^{(\nu)}(t))dt\right] + \sum_{i=0}^m E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} f(Y^{(\nu)}(t))dt\right] \\ &\quad + E^y\left[\sum_{i=1}^m K(\check{Y}^{(\nu)}(\tau_i-),\zeta_i)\right] + \sum_{i=1}^m E^y\left[\int_{\tau_i}^{\tau_i+T_i} f\left(Y^{(\nu)}(t)\right)dt\right] \\ &\quad + \sum_{i=1}^m E^y\left[\int_{\tau_i}^{\tau_i+T_i}\tilde{A}\phi(Y^{(\nu)}(t))dt\right] + \sum_{i=1}^m E^y\left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\ &\quad = E^y\left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt\right] + E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right] + E^y\left[\sum_{i=1}^m K(\check{Y}^{(\nu)}(\tau_i-),\zeta_i)\right] \\ &\quad = E^y\left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt + \phi(\check{Y}^{(\nu)}(\tau_{m+1}-)) + \sum_{i=1}^m K(\check{Y}^{(\nu)}(\tau_i-),\zeta_i)\right].\end{aligned}$$

Namely, we have

$$\begin{aligned}&\phi(y) + \sum_{i=1}^m E^y\left[\mathcal{M}_r\phi(\check{Y}^{(\nu)}(\tau_i-)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\ &\qquad\le E^y\left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt + \phi(\check{Y}^{(\nu)}(\tau_{m+1}-)) + \sum_{i=1}^m K(\check{Y}^{(\nu)}(\tau_i-),\zeta_i)\right].\end{aligned}$$

Letting $m \to M$ and using conditions $(vii) - (viii)$, we get

$$\phi(y) \leq E^y \left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu)}(t))dt + g(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{i=1}^{M} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i) \right] = J^{(\nu)}(y). \tag{3.3.11}$$

Hence

$$\phi(y) \leq \Phi(y).$$

(b) We now assume that $(x) - (xii)$ is also hold. We then apply the above argument to $\hat{\nu} = (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j, \dots; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j, \dots)_{j \leq M}$. Then by (x) we get the equality in (3.3.7) and by our choice of $\zeta_i = \hat{\zeta}_i$, we also have the equality in (3.3.8) and (3.3.10). Hence we get the desired equality in (3.3.11). Therefore we have,

$$\phi(y) = J^{(\hat{\nu})}(y). \qquad \square$$

3.4 Optimal central bank intervention when the interventions affect market dynamics

We now apply Theorem 3.1 to solve the central bank intervention in the foreign exchange market with jumps when the interventions affect the market dynamics. This means that we assume that the process driving the exchange-rate dynamics is affected by interventions. For this, we assume that the exchange-rate dynamics changes to a different process for a constant period of time T after each intervention and then reverts back to the pre-intervention process at the end of this period of time. Hence our solution is more general than Øksendal and Mundaca [47], Cadenillas and Zapatero [14], [15] and Øksendal and Sulem [52] as they expressly assumed that investors do not observe the interventions of the central bank.

Our analysis currently requires us to impose the restriction that the central bank is not allowed to intervene during the reaction period. However, this restriction is reasonable from the perspective of the central bank policy. The reaction periods are intended to model short market re-adjustment periods, so the restriction is short lived. Additionally, after they have reset the rate to desired target, central banks will want to wait for a period of time to observe the medium-term effects of their action. If they think the market is still in a temporary reaction mode with uncertain parameters, but will soon revert to the previous dynamic, it is reasonable for them to preserve their capital and wait for the re-establishment of the long term dynamical rate parameters before contemplating another intervention.

3.4.1 Mathematical formulation of the problem

Let $X_x(t)$ be the exchange rate process. Suppose in the absence of interventions, that $X_x(t)$ follows a jump diffusion of the form

$$X_x(t) = x + \mu t + \sigma B(t) + \int_0^t \int_{\mathbb{R}} \theta z \widetilde{N}(ds, dz) \; ; \quad B(0) = 0, \tag{3.4.1}$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$, $\sigma > 0$, $\theta \geq 0$ are constants.

Suppose that without interventions the system has the form

$$Y_y(t) = \begin{bmatrix} s + t \\ X_x(t) \end{bmatrix} \in \mathbb{R}^2 \; ; \qquad Y_y(0) = y = \begin{bmatrix} s \\ x \end{bmatrix}.$$

Now suppose that the state of the system changes temporarily after an intervention takes place. If the i^{th} intervention occurs at time τ_i, then the state of the system takes a different jump diffusion for a bounded, constant length of time $T \geq 0$.

Let $\tilde{X}_x(t)$ be the exchange rate process followed during this reaction period of duration T after each intervention.

$$\tilde{X}_{\tilde{x}}(t) = \tilde{x} + \tilde{\mu}t + \tilde{\sigma}B(t) + \int_0^t \int_{\mathbb{R}} \tilde{\theta} z \widetilde{N}(ds, dz) \ ; \quad B(0) = 0, \tag{3.4.2}$$

where $\tilde{x} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}$, $\tilde{\sigma} > 0$, $\tilde{\theta} \geq 0$ are constants.

For simplicity we impose the restriction that new interventions are not allowed during this reaction period.

Now suppose that we are only allowed to give the system impulses ζ with values in $\mathcal{Z} := (0, \infty)$ and if we apply an impulse control $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)$ to $Y_y(t)$, then it gets the form

$$Y_y^{(\nu)}(t) = \begin{bmatrix} s + t \\ X_x^{(v)}(t) \end{bmatrix},$$

where the exchange rate process $X_x^{(\nu)}$, starting at value x, can be written as

$$X_x^{(\nu)}(t) = X_x(t) \ ; \qquad 0 \leq t < \tau_1, \tag{3.4.3}$$

$$X_x^{(\nu)}(\tau_j) = X_x^{(\nu)}(\tau_j^-) + \triangle_N X_x(\tau_j) - \zeta_j; \qquad j = 1, 2, ..., \tag{3.4.4}$$

$$X_x^{(\nu)}(t) = \tilde{X}^{(\nu)}_{X_x^{(\nu)}(\tau_j)}(t) \qquad \text{for } \tau_j \leq t \leq \tau_j + T, \tag{3.4.5}$$

$$X_x^{(\nu)}(t) = X^{(\nu)}_{X_x^{(\nu)}(\tau_j+T)}(t) \qquad \text{for } \tau_j + T < t < \tau_{j+1} \wedge \tau^*. \tag{3.4.6}$$

Here τ_i is the time of the i-th intervention, and ζ_i is the intensity of the i-th intervention. Since we are only allowed to give the system positive impulses, the central bank only pushes the exchange rate downwards. Here we also assume that the target exchange rate ρ to be 0 for the simplicity of the solution.

Suppose that the cost rate $f(t, x)$ (the running cost incurred by deviation from the target exchange rate ρ) if $X_x^{(\nu)} = x$ at time t is given by

$$f(t, x) = e^{-rt} x^2, \tag{3.4.7}$$

where $r > 0$ is the constant discount rate.

Now let C and λ be the fixed cost and the proportional cost per intervention respectively. So the cost of an intervention of size $\zeta > 0$ at time t is

$$K(t, x, \zeta) = K(t, \zeta) = e^{-rt}(C + \lambda\zeta), \tag{3.4.8}$$

where $C > 0$, $\lambda \geq 0$ are constants.

Then the expected total discounted cost associated to a given impulse control is

$$J^{(\nu)}(s, x) = E^x[\int_0^\infty e^{-r(s+t)}(X^{(v)})^2 dt + \sum_{k=1}^{\infty} e^{-r(s+\tau_k)}(C + \lambda\zeta_k)]. \tag{3.4.9}$$

We seek $\Phi(s, x)$ and $\nu^* = (\tau_1^*, \tau_2^*, \dots, \tau_j^*, \dots; \zeta_1^*, \zeta_2^*, \dots, \zeta_j^*, \dots)$ such that

$$\Phi(s, x) = \inf_\nu J^{(\nu)}(s, x) = J^{(\nu^*)}(s, x). \tag{3.4.10}$$

3.4.2 An analytical solution of the problem

The above problem is an impulse-control problem with market reaction of the type described by Theorem 3.1. Using the notation of this chapter, here we have:

$$Y_y^{(\nu)}(t) = \begin{bmatrix} s+t \\ X_x^{(\nu)}(t) \end{bmatrix} \quad ; \quad t \geq 0 \qquad \text{and} \qquad Y_y^{(\nu)}(0^-) = \begin{bmatrix} s \\ x \end{bmatrix} = y \in \mathbb{R}^2,$$

$$\Gamma(y,\zeta) = \Gamma(s,x,\zeta) = \begin{bmatrix} s \\ x-\zeta \end{bmatrix} \quad ; \quad (s,x,\zeta) \in \mathbb{R}^3,$$

$$K(y,\zeta) = K(s,x,\zeta) = e^{-rs}(C+\lambda\zeta), \quad f(y) = f(s,x) = e^{-rs}x^2 \text{ and } g(y) = 0.$$

Note that it is not optimal to move $X_x(t)$ downwards if $X_x(t)$ is already below 0. Hence we may restrict ourselves to consider impulse controls $\nu = (\tau_1, \tau_2, ...; \zeta_1, \zeta_2, ...)$ such that

$$\sum_{j=1}^{\tau_k} \zeta_j \leq X_x(\tau_k) \quad \text{for all } k. \tag{3.4.11}$$

Since we are not allowed to intervene during the reaction period, we also have:

$$\tau_{i+1} - \tau_i \geq T \qquad \text{for all } i \tag{3.4.12}$$

Let $\mathcal{V}$ be set of impulse controls which satisfies both (3.4.11) and (3.4.12).

We guess that the optimal strategy is to wait until the level of $X_x(t)$ reaches an (unknown) value $\bar{x} > 0$. At this time, τ_1, we intervene and give $X_x(t)$ an impulse ζ_1, which brings it down to a lower value $\hat{x} > 0$. Then we do nothing until the next time, $\tau_2 > \tau_1 + T_1$ that $X_x(t)$ reaches the level $\bar{x}$ etc. So the continuation region D in

Theorem 3.1 has the form

$$D = \{(s,x);\ x < \bar{x} \text{ or } s \in (\tau_i, \tau_i + T] \text{ for some } i\} \tag{3.4.13}$$

for some $\bar{x} > 0$ to be determined.

We guess that the value function is of the form

$$\phi(s,x) = e^{-rs}\psi(x). \tag{3.4.14}$$

The reaction period is known to be very small and no interventions are allowed during this period. Hence, in the continuation region D, we have by Theorem 3.1 (x).

$$A\phi + f = 0,$$

where A is the generator of Y_y given by

$$dY_y(t) = \begin{bmatrix} dt \\ dX_x(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \mu \end{bmatrix} dt + \begin{bmatrix} o \\ \sigma \end{bmatrix} dB(t) + \begin{bmatrix} 0 \\ \int_{\mathbb{R}} \theta z \widetilde{N}(dt,dz) \end{bmatrix};\ Y_y(0) = \begin{bmatrix} s \\ x \end{bmatrix}.$$

Therefore

$$\begin{aligned} A\phi(s,x) + f(s,x) &= \frac{\partial \phi}{\partial s} + \mu \frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial x^2} \\ &+ \int_{\mathbb{R}} \{\phi(s, x + \theta z) - \phi(s,x) - \theta z \frac{\partial \phi}{\partial x}(s,x)\}\nu(dz) + e^{-rs}x^2 = 0. \end{aligned} \tag{3.4.15}$$

By substituting (3.4.14) into (3.4.15), we get

$$-re^{-rs}\psi(x) + e^{-rs}\mu\frac{\partial\psi}{\partial x} + \frac{1}{2}e^{-rs}\sigma^2\frac{\partial^2}{\partial x^2}(\psi(x)) + \int_{\mathbb{R}} e^{-rs}\{\psi(x+\theta z) - \psi(x) - \theta z\frac{\partial\psi(x)}{\partial x}\}\nu(dz) + e^{-rs}x^2 = 0.$$

Therefore we have

$$-r\psi(x)+\mu\psi'(x)+\frac{\sigma^2}{2}\psi''(x)+\int_{\mathbb{R}}\{\psi(x+\theta z)-\psi(x)-\theta z\psi'(x)\}\nu(dz)+x^2 = 0. \quad (3.4.16)$$

We try a solution of the form,

$$\psi(x) = C_1e^{\alpha_1 x} + C_2e^{\alpha_2 x} + \frac{1}{r}x^2 + \frac{1}{r^2}(2\mu x + b) + \frac{2\mu^2}{r^3}, \quad (3.4.17)$$

where C_1, C_2 are constants (to be determined),

$$b = \sigma^2 + \int_{\mathbb{R}}\theta^2z^2\nu(dz), \quad (3.4.18)$$

and $\alpha_1 > 0$, $\alpha_2 < 0$ are roots of the function

$$F(\alpha) = -r + \mu\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{\mathbb{R}}\{e^{\alpha\theta z} - 1 - \alpha\theta z\}\nu(dz). \quad (3.4.19)$$

Note that if we make no intervention at all, then the cost is

$$\begin{aligned}
e^{-rs}\psi_s(x) = J^{(\nu)}(s,x) &= e^{-rs}E^x[\int_0^\infty e^{-rt}(X_x^{(\nu)}(t))^2dt] \\
&= e^{-rs}E^x[\int_0^\infty e^{-rt}(x + \mu t + \sigma B(t) + \int_0^t\int_{\mathbb{R}}\theta z\widetilde{N}(ds,dz))^2dt] \\
&= e^{-rs}\int_0^\infty e^{-rt}\left(x^2 + \mu^2t^2 + (2\mu x + b)\,t\right)dt \\
&= e^{-rs}\left(\frac{x^2}{r} + \frac{b + 2\mu x}{r^2} + \frac{2\mu^2}{r^3}\right).
\end{aligned}$$

By the definition of $\phi(x)$, we have

$$0 \leq \phi(x) \leq e^{-rs}\psi_s(x), \qquad \text{for all} \quad x \leq \bar{x}.$$

Then we get

$$0 \leq e^{-rs}\psi(x) \leq e^{-rs}\left(\frac{x^2}{r} + \frac{b+2\mu x}{r^2} + \frac{2\mu^2}{r^3}\right) \qquad \text{for all} \quad x \leq \bar{x}.$$

It follows that

$$0 \leq \psi(x) \leq \left(\frac{x^2}{r} + \frac{b+2\mu x}{r^2} + \frac{2\mu^2}{r^3}\right) \qquad \text{for all} \quad x \leq \bar{x}.$$

Therefore by (3.4.17), we have

$$0 \leq C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \frac{1}{r}x^2 + \frac{1}{r^2}(2\mu x + b) + \frac{2\mu^2}{r^3} \leq \left(\frac{x^2}{r} + \frac{b+2\mu x}{r^2} + \frac{2\mu^2}{r^3}\right) \text{ for all } x \leq \bar{x}. \tag{3.4.20}$$

From the right side of the above inequality, we get

$$C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} \leq 0 \qquad \text{for all} \quad x \leq \bar{x}. \tag{3.4.21}$$

Letting $x \to -\infty$ in (3.4.21) , we conclude that $C_2 \leq 0$. However, from the left side of the inequality (3.4.20) we eliminate that $C_2 < 0$. Hence $C_2 = 0$ and $C_1 < 0$.

We let $C_1 = -a$, where $a > 0$ and define

$$\psi_0(x) = -ae^{\alpha_1 x} + \frac{x^2}{r} + \frac{b+2\mu x}{r^2} + \frac{2\mu^2}{r^3}, \tag{3.4.22}$$

and put

$$\psi(x) = \psi_0(x) \; ; \qquad x \in D. \tag{3.4.23}$$

We now consider the running cost during the constant reaction period.

$$\begin{aligned}
R(\tilde{x}) &= E^{\tilde{x}}\left[\int_0^T f(\tilde{X}(t)dt\right] = E^{\tilde{x}}\left[\int_0^T e^{-rt}\left(\tilde{X}(t)\right)^2 dt\right] \\
&= E\left[\int_0^T e^{-rt}\left\{\tilde{x} + \tilde{\mu}t + \tilde{\sigma}B(t) + \int_0^t \int_{\mathbb{R}} \tilde{\theta} z \widetilde{N}(ds, dz)\right\}^2 dt\right] \\
&= \int_0^T e^{-rt}\left(\tilde{x}^2 + \left(2\tilde{\mu}\tilde{x} + \tilde{\sigma}^2 + \int_{\mathbb{R}} \tilde{\theta}^2 z^2 \nu(dz)\right) t + \tilde{\mu}^2 t^2\right) dt \\
&= \int_0^T e^{-rt}\left(\tilde{x}^2 + \left(2\tilde{\mu}\tilde{x} + \tilde{b}\right) t + \tilde{\mu}^2 t^2\right) dt,
\end{aligned}$$

where $\tilde{b} = \tilde{\sigma}^2 + \int_{\mathbb{R}} \tilde{\theta}^2 z^2 \nu(dz)$.

We recall that

$$\begin{aligned}
D &= \{(s,x);\ \phi(s,x) < \mathcal{M}_r \phi(s,x)\} \\
&= \{(s,x);\ \psi(x) < \mathcal{M}_r \psi(x)\}
\end{aligned}$$

and the intervention operator in this case is

$$\mathcal{M}_r \psi(x) = \inf\left\{C + \lambda\zeta + R\left(x - \zeta\right) + E^{x-\zeta}\left[\psi(\tilde{X}(T))\right];\ \zeta > 0\right\}.$$

The first-order condition for a minimum $\hat{\zeta} = \hat{\zeta}(x)$ of the function

$$G(\zeta) = C + \lambda\zeta + R\left(x - \zeta\right) + E^{x-\zeta}\left[\psi(\tilde{X}(T))\right], \qquad \zeta > 0$$

is

$$G'(\hat{\zeta}) = 0. \tag{3.4.24}$$

Therefore we look for points $\hat{x}$, $\bar{x}$ such that $0 < \hat{x} < \bar{x}$ and $G'(\hat{\zeta}) = 0$.

We also have that $x - \hat{\zeta} = \hat{x}$ (i.e. $\hat{\zeta} = x - \hat{x} = \hat{\zeta}(x)$).

By Theorem 3.1-(ii), we have $\psi \leq \mathcal{M}_r\psi$ on $\mathcal{S}$. So if $x \geq \bar{x}$, we have $\psi(x) = \mathcal{M}_r\psi(x)$, because $x \notin D$. Thus, if $x \geq \bar{x}$ we have

$$\psi(x) = C + \lambda\hat{\zeta} + R\left(x - \hat{\zeta}\right) + E^{x-\hat{\zeta}}\left[\psi(\tilde{X}(T))\right].$$

That is

$$\psi(x) = C + \lambda(x - \hat{x}) + R\left(\hat{x}\right) + E^{\hat{x}}\left[\psi(\tilde{X}(T))\right].$$

In particular,

$$\psi'(\bar{x}) = \lambda$$

and

$$\psi(\bar{x}) = C + \lambda(\bar{x} - \hat{x}) + R\left(\hat{x}\right) + E^{\hat{x}}\left[\psi(\tilde{X}(T))\right].$$

Here we should also note that $\tilde{X}_{\hat{x}}(T) \in D$. Therefore, we have $\psi(\tilde{X}_{\hat{x}}(T)) = \psi_0(\tilde{X}_{\hat{x}}(T))$. Then we have

$$\psi(x) = \begin{cases} \psi_0(x), & \text{if } x < \bar{x}, \\ C + \lambda(x - \hat{x}) + R\left(\hat{x}\right) + E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right], & \text{if } x \geq \bar{x}. \end{cases} \tag{3.4.25}$$

Differentiating $\psi(x)$ at $x = \bar{x}$ gives the equation

$$\psi_0'(\bar{x}) = \lambda,$$

which is equivalent to

$$a\alpha_1 e^{\alpha_1 \bar{x}} = \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right). \tag{3.4.26}$$

We also have that

$$G'(\hat{\zeta}) = 0.$$

Equivalently, we have

$$R'(\hat{x}) + E^{\hat{x}}\left[\psi_0'(\tilde{X}(T))\right] = \lambda. \tag{3.4.27}$$

Continuity of $\psi(x)$ at $x = \bar{x}$ gives the equation

$$C + \lambda(\bar{x} - \hat{x}) + R(\hat{x}) + E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right] = \psi_0(\bar{x}),$$

i.e.

$$C + \lambda(\bar{x} - \hat{x}) + R(\hat{x}) + E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right] = -ae^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3}. \tag{3.4.28}$$

The solution of the problem is determined by the system (3.4.26)-(3.4.28). We need to solve the system for $\bar{x}$, $\hat{x}$ and a. Therefore we simplify the system to get more explicit expressions with $\bar{x}$, $\hat{x}$ and a. For this, we compute each term in the system separately and then plug into the system.

We first compute $R(\hat{x})$,

$$R(\hat{x}) = \int_0^T e^{-rt}\left(\hat{x}^2 + \left(2\tilde{\mu}\hat{x} + \tilde{b}\right)t + \tilde{\mu}^2 t^2\right)dt, \qquad \text{where} \quad \tilde{b} = \tilde{\sigma}^2 + \int_{\mathbb{R}} \tilde{\theta}^2 z^2 \nu(dz).$$

By applying integration by parts twice, we get

$$R(\hat{x}) = \frac{1}{r}\left[\hat{x}^2 - e^{-rT}\left\{\hat{x}^2 + \left(2\tilde{\mu}\hat{x} + \tilde{b}\right)T + \tilde{\mu}^2T^2\right\}\right] + \frac{1}{r^2}\left[\left(2\tilde{\mu}\hat{x} + \tilde{b}\right) - e^{-rT}\left\{\left(2\tilde{\mu}\hat{x} + \tilde{b}\right) + 2\tilde{\mu}^2T\right\}\right] + \frac{2\tilde{\mu}^2}{r^3}\left[1 - e^{-rT}\right].$$

By rearranging the terms, we get

$$R(\hat{x}) = A_R\,\hat{x}^2 + B_R\,\hat{x} + C_R, \tag{3.4.29}$$

where $A_R = \frac{1}{r}\left(1 - e^{-rT}\right)$,

$$B_R = \frac{2\tilde{\mu}}{r^2}\left(1 - e^{-rT}\right) - \frac{2\tilde{\mu}T}{r}\,e^{-rT},$$

$$C_R = \frac{2\tilde{\mu}^2}{r^3}\left(1 - e^{-rT}\right) + \frac{1}{r^2}\left\{\tilde{b}\left(1 - e^{-rT}\right) - 2\tilde{\mu}^2Te^{-rT}\right\} - \frac{1}{r}\left(\tilde{b}T + \tilde{\mu}^2T^2\right)e^{-rT}.$$

By differentiating (3.4.29), we get $R'(\hat{x})$

$$R'(\hat{x}) = 2A_R\,\hat{x} + B_R. \tag{3.4.30}$$

We now simplify $E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right]$ and get

$$\begin{aligned} E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right] &= E^{\hat{x}}\left[-ae^{\alpha_1(\tilde{X}(T))} + \frac{(\tilde{X}(T))^2}{r} + \frac{b + 2\mu(\tilde{X}(T))}{r^2} + \frac{2\mu^2}{r^3}\right] \\ &= -aE\left[e^{\alpha_1\left\{\hat{x} + \tilde{\mu}T + \tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}}\tilde{\theta}z\widetilde{N}(ds,dz)\right\}}\right] \\ &+ \frac{1}{r}E\left[\left\{\hat{x} + \tilde{\mu}T + \tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}}\tilde{\theta}z\widetilde{N}(ds,dz)\right\}^2\right] + \frac{b}{r^2} \\ &+ \frac{2\mu}{r^2}E\left[\hat{x} + \tilde{\mu}T + \tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}}\tilde{\theta}z\widetilde{N}(ds,dz)\right] + \frac{2\mu^2}{r^3}. \end{aligned}$$

Therefore, we have

$$E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right] = -ae^{\alpha_1\hat{x}}E\left[e^{\alpha_1\left\{\tilde{\mu}T+\tilde{\sigma}B(T)+\int_0^T\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right]$$
$$+\frac{1}{r}\left\{\hat{x}^2+\left(2\tilde{\mu}\hat{x}+\tilde{b}\right)T+\tilde{\mu}^2T^2\right\}+\frac{b}{r^2}+\frac{2\mu\hat{x}}{r^2}+\frac{2\mu\tilde{\mu}T}{r^2}+\frac{2\mu^2}{r^3}.$$

By rearranging the terms, we get

$$E^{\hat{x}}\left[\psi_0(\tilde{X}(T))\right] = -aA_0\,e^{\alpha_1\hat{x}}+B_0\,\hat{x}^2+C_0\,\hat{x}+D_0, \tag{3.4.31}$$

where

$$\begin{aligned}
A_0 &= E\left[e^{\alpha_1\left\{\tilde{\mu}T+\tilde{\sigma}B(T)+\int_0^T\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right],\\
B_0 &= \frac{1}{r},\\
C_0 &= \frac{2\tilde{\mu}T}{r}+\frac{2\mu}{r^2},\\
D_0 &= \frac{\tilde{b}T+\tilde{\mu}^2T^2}{r}+\frac{b+2\mu\tilde{\mu}T}{r^2}+\frac{2\mu^2}{r^3}.
\end{aligned}$$

By differentiating (3.4.31), we get $E^{\hat{x}}\left[\psi_0'(\tilde{X}(T))\right]$

$$E^{\hat{x}}\left[\psi_0'(\tilde{X}(T))\right] = -a\alpha_1A_0\,e^{\alpha_1\hat{x}}+2B_0\,\hat{x}+C_0. \tag{3.4.32}$$

Combining (3.4.27), (3.4.30) and (3.4.32), we get

$$2A_R\,\hat{x}+B_R-a\alpha_1A_0\,e^{\alpha_1\hat{x}}+2B_0\,\hat{x}+C_0=\lambda.$$

Therefore we have

$$-a\alpha_1A_0\,e^{\alpha_1\hat{x}}+2\left(A_R+B_0\right)\hat{x}+\left(B_R+C_0-\lambda\right)=0. \tag{3.4.33}$$

Combining (3.4.28), (3.4.29) and (3.4.31), we get

$$C + \lambda(\bar{x} - \hat{x}) + (A_R + B_0)\,\hat{x}^2 + (B_R + C_0)\,\hat{x} - aA_0\, e^{\alpha_1 \hat{x}} = \\ - ae^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3} - C_R - D_0. \quad (3.4.34)$$

We now recall (3.4.26),

$$a\alpha_1 e^{\alpha_1 \bar{x}} = \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right). \quad (3.4.35)$$

Since the system (3.4.33)-(3.4.35) is equivalent to the system (3.4.26)-(3.4.28), the solution is also determined by the system (3.4.33)-(3.4.35). We can use any standard method to solve the system (3.4.33)-(3.4.35) of nonlinear equations. In particular, we can use Newton's method for solving a system of nonlinear equations. We have numerically simulated these equations in Chapter 6.

Chapter 4

Stochastic Optimal Impulse Control of Jump Diffusions with Random Reaction Periods

In Chapter 3, we assumed that the reaction period is constant. Hence it is available to both the central bank and the market. However, this assumption may not be a perfect assumption for some exchange markets. In particular, when the reaction time is known, the central bank can always plan their intervention ahead of time after they identify the parameters during the reaction period. On the other hand, if the market knows the reaction period, then the investors would certainly be acquainted with that there will not be any intervention during this fixed time period. This obviously reduces the uncertainty of the market. Consequently, the optimal impulse control with a constant reaction period may not be perfect for some active markets.

In this chapter, we improve our results in Chapter 3 with an inclusion of a random reaction period. This will successfully cover the above circumstances. For this, we assume that the exchange-rate dynamics changes to a different process for a random period of time T after each intervention and then reverts back to the pre-intervention process at the end of this period of time. We again prove our results in a more general setting and then apply for the optimal central bank interventions.

4.1 A general formulation

The formulation of this section is the same as section 3.1 except we now have a random reaction period instead of a constant reaction period. So we refer the reader to section 3.1 for the basic formulation.

We now assume that $T_i \sim T$ for all i, where T is a known positive random variable with the distribution function F_T. For the simplicity, we also assume that T_i's are *iid* and independent of both B_t and $\widetilde{N}_t$. (i.e. T is independent of the jump diffusion process.).

4.2 The reacted intervention operator for a random reaction period

We now modify the definitions of the running cost during the reaction period and the reacted intervention operator for a random reaction period.

Definition 4.1. *Let $f : \mathcal{S} \to \mathbb{R}$ be a cost function. Then we define the running cost during the random reaction period by*

$$\tilde{R}(\tilde{y}) = \int_0^\infty E^{\tilde{y}} \left[\int_0^T f\left(\tilde{Y}(t)\right) dt \,\middle|\, T = u \right] dF_T(u), \tag{4.2.1}$$

where $\tilde{y}$ is the state of the process after an intervention.

Definition 4.2. *Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$ and*

$$M(y,\zeta) = K(y,\zeta) + \tilde{R}\left(\Gamma(y,\zeta)\right) + \int_0^\infty E^{\Gamma(y,\zeta)}\left[h\left(\tilde{Y}(T)\right)\Big| T = u\right] dF_T(u). \quad (4.2.2)$$

Then the ***reacted intervention operator*** *$\mathcal{M}_r : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\mathcal{M}_r h(y) = \inf\left\{M(y,\zeta);\ \zeta \in \mathcal{Z} \ \text{ and } \ \Gamma(y,\zeta) \in \mathcal{S}\right\}. \quad (4.2.3)$$

4.3 A verification theorem for a random reaction period

We can now state the main result of this chapter, a verification theorem for impulse-control problems with *a random reaction period*:

Theorem 4.1. ***(Quasi-integrovariational inequalities for impulse control with regime switching :- random reaction time)***
(a) Suppose we can find a function $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ such that
(i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$,
(ii) $\phi \le \mathcal{M}_r\phi$ on $\mathcal{S}$.
Define

$$D = \{y \in \mathcal{S}; \phi(y) < \mathcal{M}_r\phi(y)\} \qquad \text{(the continuation region).}$$

Suppose
(iii)

$$E^y\left[\int_0^{\tau_{\mathcal{S}}} I_{\partial D}(Y^{(\nu)}(t))dt\right] = 0 \quad \text{for all } \ y \in \mathcal{S},\ \nu \in \mathcal{V},$$

(iv) ∂D is a Lipschitz surface,
(v) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ with locally bounded derivatives near ∂D,

(vi) $\bar{A}\phi + f \geq 0$ *on* $\mathcal{S} \setminus \partial D$,

(vii) $\phi(Y_y^{(\nu)}(t)) \to g(Y_y^{(\nu)}(\tau_{\mathcal{S}})) \cdot I_{\{\mathcal{S}<\infty\}}$ *as* $t \to \tau_{\mathcal{S}}-$ *a.s., for all* $y \in \mathcal{S}$, $\nu \in \mathcal{V}$,

(viii) $\{\phi^-(Y_y^{(\nu)}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable, for all* $y \in \mathcal{S}$, $\nu \in \mathcal{V}$,

(ix)

$$\int_0^\infty \{\ E^y[|\phi(Y^{(\nu)}(\tau+u))| + \int_0^{\tau+u} \{|\bar{A}\phi(Y^{(\nu)}(t))| + |\sigma^T(Y^{(\nu)}(t))\nabla\phi(Y^{(\nu)}(t))|^2 + \sum_{j=1}^{l} \int_{\mathbb{R}} |\phi(Y^{(\nu)}(t) + \gamma^{(j)}(Y^{(\nu)}(t), z_j)) - \phi(Y^{(\nu)}(t))|^2 \nu_j(dz_j)]\} dt]\ \} dF_T(u) < \infty$$

for all $\tau \in \mathcal{T}$, $y \in \mathcal{S}$, $\nu \in \mathcal{V}$.

Then

$$\phi(y) \leq \Phi(y) \quad \text{for all} \;\; y \in \mathcal{S}. \tag{4.3.1}$$

(b) Suppose in addition that

(x) $A\phi + f = 0$ *in* D,

(xi) $\hat{\zeta}(y) \in$ *argmin* $\left\{K(y,\cdot) + \tilde{R}(\Gamma(y,\cdot)) + \int_0^\infty E^{\Gamma(y,\cdot)}\left[h\left(\tilde{Y}(T)\right)\middle| T = u\right] F_T(u)\right\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\zeta}(\cdot)$ *is a Borel measurable selection.*

Put $\hat{\tau}_0 = 0$ *and define* $\hat{\nu} = (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j, \dots; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j, \dots)_{j \leq M}$ *inductively by* $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j + T_j; Y_y^{(\hat{\nu}_j)}(t-) \notin D\} \wedge \tau_{\mathcal{S}}$ *and* $\hat{\zeta}_{j+1} = \hat{\zeta}(Y_y^{(\hat{\nu}_j)}(\hat{\tau}_{j+1}-))$ *if* $\hat{\tau}_{j+1} < \tau_{\mathcal{S}}$, *where* $Y_y^{(\hat{\nu}_j)}$ *is the result of applying* $\hat{\nu}_j := (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j)$ *to* Y.

Suppose

(xii) $\hat{\nu} \in \mathcal{V}$ *and* $\{\phi(Y_y^{(\hat{\nu})}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable.*

Then

$$\phi(y) = \Phi(y) \quad \text{and} \quad \hat{\nu} \quad \text{is an optimal impulse control.} \tag{4.3.2}$$

Proof:

(a) Choose $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)_{j \leq M} \in \mathcal{V}$ and set $\tau_0 = 0$.

By approximation Theorem 2.18 and (i), $(iii)-(v)$, we may assume that $\phi \in C^2(\mathcal{S}) \cap$

$C(\bar{\mathcal{S}})$. Therefore, we can apply the multidimensional Itô 's formula in Theorem 2.11 with $f(t, Y_y^{(\nu)}(t)) = \phi\left(Y_y^{(\nu)}(t)\right)$ from τ_i to $\tau_i + u$ given that $T_i = u$ for $u \in [0, \infty)$. Then, we get

$$
\begin{aligned}
\phi\left(Y_y^{(\nu)}(\tau_i + u)\right) = \phi\left(Y_y^{(\nu)}(\tau_i)\right) \\
&+ \int_{\tau_i}^{\tau_i+u} \sum_{p=1}^{k} \frac{\partial \phi}{\partial x_p}\left(Y_y^{(\nu)}(t)\right)\left[\tilde{b}_p\left(Y_y^{(\nu)}(t)\right) dt + \tilde{\sigma}_p\left(Y_y^{(\nu)}(t)\right) dB(t)\right] \\
&+ \frac{1}{2}\int_{\tau_i}^{\tau_i+u} \sum_{p,q=1}^{k} (\tilde{\sigma}\tilde{\sigma}^T)_{pq}\left(Y_y^{(\nu)}(t)\right) \frac{\partial^2 \phi}{\partial x_p \partial x_q}\left(Y_y^{(\nu)}(t)\right) dt \\
&+ \int_{\tau_i}^{\tau_i+u} \sum_{p=1}^{l} \int_{\mathbb{R}} \{\phi\left(Y_y^{(\nu)}(t-) + \tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t-), z_p)\right) - \phi\left(Y_y^{(\nu)}(t-)\right) \\
&\qquad - \sum_{q=1}^{k} \tilde{\gamma}_q^{(p)}(Y_y^{(\nu)}(t-), z_p) \frac{\partial \phi}{\partial x_q}\left(Y_y^{(\nu)}(t-)\right)\}\nu_p(dz_p)dt \\
&+ \int_{\tau_i}^{\tau_i+u} \sum_{p=1}^{l} \int_{\mathbb{R}} \{\phi\left(Y_y^{(\nu)}(t-) + \tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t-), z_p)\right) \\
&\qquad\qquad - \phi\left(Y_y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt, dz_p).
\end{aligned}
$$

By rearranging the terms in the above equation, we get

$$
\begin{aligned}
\phi\left(Y_y^{(\nu)}(\tau_i + u)\right) = \phi\left(Y_y^{(\nu)}(\tau_i)\right) + \int_{\tau_i}^{\tau_i+u} \sum_{p=1}^{k} \frac{\partial \phi}{\partial x_p}\left(Y_y^{(\nu)}(t)\right) \tilde{\sigma}_p\left(Y_y^{(\nu)}(t)\right) dB(t) \\
&+ \int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y_y^{(\nu)}(t)\right) dt \\
&+ \int_{\tau_i}^{\tau_i+u} \sum_{p=1}^{l} \int_{\mathbb{R}} \{\phi\left(Y_y^{(\nu)}(t-) + \tilde{\gamma}^{(p)}(Y_y^{(\nu)}(t-), z_p)\right) \\
&\qquad\qquad - \phi\left(Y_y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt, dz_p).
\end{aligned}
$$

Now we take the expectation of the both side of the above equation. Then, we get

$$E^y\left[\phi\left(Y^{(\nu)}(\tau_i+T_i)\right) \mid T_i=u\right] = E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right] + E^y\left[\int_{\tau_i}^{\tau_i+T_i} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt \mid T_i=u\right], \quad (4.3.3)$$

since

$$E^y\left[\int_{\tau_i}^{\tau_i+u}\sum_{p=1}^{k}\frac{\partial\phi}{\partial x_p}\left(Y^{(\nu)}(t)\right)\tilde{\sigma}_p\left(Y^{(\nu)}(t)\right)dB(t)\right]=0$$

and

$$E^y\left[\int_{\tau_i}^{\tau_i+u}\sum_{p=1}^{l}\int_{\mathbb{R}}\{\phi(Y^{(\nu)}(t-)+\tilde{\gamma}^{(p)}(Y^{(\nu)}(t),z_p))-\phi\left(Y^{(\nu)}(t-)\right)\}\widetilde{N}_p(dt,dz_p)\right]=0.$$

Integrating (4.3.3) with respect to the probability measure induced by T on $(0,\infty)$, we get

$$\int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i+u)\right)\right]dF_T(u) = E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right] + \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u}\bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u), \quad (4.3.4)$$

since

$$\int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right]dF_T(u) = E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right]\int_0^\infty dF_T(u) = E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right].$$

Therefore, we have

$$E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right] - \int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i+u)\right)\right]dF_T(u) = -\int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u}\bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u). \quad (4.3.5)$$

Using a similar argument, we can also derive

$$\int_0^\infty E^y[\phi(Y^{(\nu)}(\tau_i+u)]dF_T(u) - E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_{i+1}-)\right)\right]$$
$$= -\int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u), \tag{4.3.6}$$

where $\check{Y}_y^{(\nu)}(\tau_{i+1}-) = Y_y^{(\nu)}(\tau_{i+1}-) + \triangle_N Y_y(\tau_{i+1})$.

Summing equation (4.3.5) from $i=1$ to $i=m$, we get

$$\sum_{i=1}^m E^y\left[\phi\left(Y^{(\nu)}(\tau_i)\right)\right] - \sum_{i=1}^m \int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i+u)\right)\right]dF_T(u)$$
$$= -\sum_{i=1}^m \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u). \tag{4.3.7}$$

Summing equation (4.3.6) from $i=1$ to $i=m$, we get

$$\sum_{i=1}^m \int_0^\infty E^y[\phi(Y^{(\nu)}(\tau_i+u))]dF_T(u) - \sum_{i=1}^m E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_{i+1}-)\right)\right]$$
$$= -\sum_{i=1}^m \int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u). \tag{4.3.8}$$

Adding equation (4.3.7) to (4.3.8), we get

$$\sum_{i=1}^m E^y[\phi(Y^{(\nu)}(\tau_i))] - \sum_{i=1}^m E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_{i+1}-)\right)\right]$$
$$= -\sum_{i=1}^m \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u)$$
$$-\sum_{i=1}^m \int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u). \tag{4.3.9}$$

Now by applying the multidimensional Itô 's formula again with $f(t, Y_y^{(\nu)}(t)) = \phi\left(\check{Y}_y^{(\nu)}(t)\right)$ from 0 to τ_1 and then taking the expectation we get,

$$E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_1-)\right)\right] = \phi\left(y\right) + E^y\left[\int_0^{\tau_1} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right].$$

Equivalently, we have

$$\phi\left(y\right) - E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_1-)\right)\right] = -E^y\left[\int_0^{\tau_1} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]. \tag{4.3.10}$$

Adding equation (4.3.9) to (4.3.10) we get,

$$\begin{aligned}
\phi\left(y\right) + \sum_{i=1}^{m} E^y[\phi(Y^{(\nu)}(\tau_i))] - E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_1-)\right)\right] - \sum_{i=1}^{m} E^y\left[\phi\left(\check{Y}^{(\nu)}(\tau_{i+1}-)\right)\right] \\
= -\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u) - E^y\left[\int_0^{\tau_1} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right] \\
-\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u).
\end{aligned}$$

Then by using condition (vi), we have

$$\begin{aligned}
\phi(y) + \sum_{i=1}^{m} E^y\left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] - E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right] \\
= -\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u) - E^y\left[\int_0^{\tau_1} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right] \\
-\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u) \\
\leq -\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u) + E^y\left[\int_0^{\tau_1} f\left(Y^{(\nu)}(t)\right)dt\right] \\
+\sum_{i=1}^{m}\int_0^{\infty} E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} f\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u).
\end{aligned}$$

Hence we have

$$\phi(y)+\sum_{i=1}^{m}E^y\left[\phi(Y^{(\nu)}(\tau_i))-\phi(\check{Y}^{(\nu)}(\tau_i-))\right]-E^y\left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right]$$
$$\leq -\sum_{i=1}^{m}\int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u}\bar{A}\phi\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u)+E^y\left[\int_0^{\tau_1}f\left(Y^{(\nu)}(t)\right)dt\right]$$
$$+\sum_{i=1}^{m}\int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}}f\left(Y^{(\nu)}(t)\right)dt\right]dF_T(u). \tag{4.3.11}$$

Now by the definition of $\mathcal{M}_r$, we have for $\Gamma(y,\zeta)\in\mathcal{S}$

$$\mathcal{M}_r\phi(y)\leq K(y,\zeta)+\int_0^\infty E^{\Gamma(y,\zeta)}\left[\int_0^u f\left(\tilde{Y}(t)\right)dt\right]dF_T(u)$$
$$+\int_0^\infty E^{\Gamma(y,\zeta)}\left[\phi\left(\tilde{Y}(u)\right)\right]dF_T(u).$$

However, if $\tau_i<\tau_{\mathcal{S}}$ then we have $Y_y^{(\nu)}(\tau_i)=\Gamma(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i)\in\mathcal{S}$. Therefore, we can plug in $\check{Y}_y^{(\nu)}(\tau_i-)$ for y in the above equation. Then by the strong Markov property, we get

$$\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-))\leq K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i)+\int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u}f\left(\tilde{Y}_{Y^{(\nu)}(\tau_i)}(t)\right)dt|\mathcal{F}_{\tau_i}\right]dF_T(u)$$
$$+\int_0^\infty E^y\left[\phi\left(\tilde{Y}_{Y^{(\nu)}(\tau_i)}(\tau_i+u)\right)|\mathcal{F}_{\tau_i}\right]dF_T(u).$$

But $Y_y^{(\nu)}(t)=\tilde{Y}_{Y_y^{(\nu)}(\tau_j)}(t)$ for $\tau_j\leq t\leq\tau_j+u$. Therefore, we have

$$\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-))\leq K(\check{Y}_y^{(\nu)}(\tau_i-),\zeta_i)+\int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u}f\left(Y^{(\nu)}(t)\right)dt|\mathcal{F}_{\tau_i}\right]dF_T(u)$$
$$+\int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i+u)\right)|\mathcal{F}_{\tau_i}\right]dF_T(u). \tag{4.3.12}$$

By applying the multidimensional Itô 's formula again with $f(t, Y_y^{(\nu)}(t)) = \phi\left(Y_y^{(\nu)}(t)\right)$ from τ_i to $\tau_i + u$ given that $T = u$ for $u \in [0, \infty)$, and then taking the conditional expectation, we get

$$E^y\left[\phi\left(Y^{(\nu)}(\tau_i + u)\right)|\mathcal{F}_{\tau_i}\right] = \phi\left(Y_y^{(\nu)}(\tau_i)\right) + E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right) dt|\mathcal{F}_{\tau_i}\right]. \quad (4.3.13)$$

Integrating (4.3.13) with respect to the probability measure induced by T on $(0, \infty)$, we get

$$\begin{aligned}\int_0^\infty E^y\left[\phi\left(Y^{(\nu)}(\tau_i + u)\right)|\mathcal{F}_{\tau_i}\right] dF_T(u) &= \phi\left(Y_y^{(\nu)}(\tau_i)\right) \\ &+ \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right) dt|\mathcal{F}_{\tau_i}\right] dF_T(u).\end{aligned} \quad (4.3.14)$$

Substituting (4.3.14) in (4.3.12) and subtracting $\phi(\check{Y}_y^{(\nu)}(\tau_i-))$ from the both sides, we get

$$\begin{aligned}\mathcal{M}_r\phi(\check{Y}_y^{(\nu)}(\tau_i-)) - \phi(\check{Y}_y^{(\nu)}(\tau_i-)) &\leq K(\check{Y}_y^{(\nu)}(\tau_i-), \zeta_i) \\ &+ \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u} f\left(Y^{(\nu)}(t)\right) dt|\mathcal{F}_{\tau_i}\right] dF_T(u) \\ &+ \int_0^\infty E^y\left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right) dt|\mathcal{F}_{\tau_i}\right] dF_T(u) \\ &+ \phi\left(Y_y^{(\nu)}(\tau_i)\right) - \phi(\check{Y}_y^{(\nu)}(\tau_i-)).\end{aligned} \quad (4.3.15)$$

Hence, by (4.3.11) and (4.3.15) we have

$$\begin{aligned}
&\phi(y) + \sum_{i=1}^{m} E^y \left[\mathcal{M}_r\phi(\check{Y}^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\
&\quad\le E^y \left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right] - \sum_{i=1}^{m} E^y \left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\
&\quad - \sum_{i=1}^{m} \int_0^\infty E^y \left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right) dt\right] dF_T(u) + E^y \left[\int_0^{\tau_1} f\left(Y^{(\nu)}(t)\right) dt\right] \\
&\quad + \sum_{i=1}^{m} \int_0^\infty E^y \left[\int_{\tau_i+u}^{\tau_{i+1}} f\left(Y^{(\nu)}(t)\right) dt\right] dF_T(u) \\
&\quad + E^y \left[\sum_{i=1}^{m} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i)\right] + \sum_{i=1}^{m} \int_0^\infty E^y \left[\int_{\tau_i}^{\tau_i+u} f\left(Y^{(\nu)}(t)\right) dt\right] dF_T(u) \\
&\quad + \sum_{i=1}^{m} \int_0^\infty E^y \left[\int_{\tau_i}^{\tau_i+u} \bar{A}\phi\left(Y^{(\nu)}(t)\right) dt\right] dF_T(u) \\
&\quad + \sum_{i=1}^{m} E^y \left[\phi(Y^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\
&\quad = E^y \left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt\right] + E^y \left[\phi(\check{Y}^{(\nu)}(\tau_{m+1}-))\right] + E^y \left[\sum_{i=1}^{m} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i)\right] \\
&\quad = E^y \left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt + \phi(\check{Y}^{(\nu)}(\tau_{m+1}-)) + \sum_{i=1}^{m} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i)\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\phi(y) + \sum_{i=1}^{m} E^y \left[\mathcal{M}_r\phi(\check{Y}^{(\nu)}(\tau_i)) - \phi(\check{Y}^{(\nu)}(\tau_i-))\right] \\
&\qquad \le E^y \left[\int_0^{\tau_{m+1}} f(Y^{(\nu)}(t))dt + \phi(\check{Y}^{(\nu)}(\tau_{m+1}-)) + \sum_{i=1}^{m} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i)\right].
\end{aligned}$$

Letting $m \to M$ and using conditions $(vii) - (viii)$, we get

$$\phi(y) \le E^y \left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu)}(t))dt + g(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{i=1}^{M} K(\check{Y}^{(\nu)}(\tau_i-), \zeta_i)\right] = J^{(\nu)}(y). \tag{4.3.16}$$

Hence

$$\phi(y) \le \Phi(y).$$

(b) We now assume that $(x)-(xii)$ is also hold. We then apply the above argument to $\hat{\nu} = (\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_j, \dots; \hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_j, \dots)_{j \le M}$. Then by (x) we get the equality in (4.3.11) and by our choice of $\zeta_i = \hat{\zeta}_i$, we also have the equality in (4.3.12) and (4.3.15). Hence we get the desired equality in (4.3.16). Therefore we have

$$\phi(y) = J^{(\hat{\nu})}(y). \qquad \square$$

4.4 Optimal central bank intervention with random reaction periods

We now apply Theorem 4.1 to solve the central bank intervention in the foreign exchange market with jumps when the interventions affect the market dynamics for a random period of time. For this, we assume that the exchange rate dynamics changes to a different process for a random period of time T after each intervention and then it reverts back to the pre-interventions process at the end of this period of time. Hence our solution is more general than the solution we obtained in section 3.4. However, we still retain all the other assumptions we used in section 3.4.

4.4.1 Mathematical formulation of the problem

Let $X_x(t)$ be the exchange rate process. Suppose in the absence of interventions $X_x(t)$ follows a jump diffusion of the form

$$X_x(t) = x + \mu t + \sigma B(t) + \int_0^t \int_{\mathbb{R}} \theta z \widetilde{N}(ds, dz) \; ; \quad B(0) = 0, \tag{4.4.1}$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$, $\sigma > 0$, $\theta \geq 0$ are constants.

Suppose that without interventions the system has the form

$$Y_y(t) = \begin{bmatrix} s+t \\ X_x(t) \end{bmatrix} \in \mathbb{R}^2 \; ; \qquad Y_y(0) = y = \begin{bmatrix} s \\ x \end{bmatrix} .$$

Now suppose that the state of the system changes temporarily after an intervention takes place. Therefore if the i-th intervention occurs at time τ_i, then the state of the system takes a different jump diffusion for a bounded, random length of time $T_i \geq 0$. We also assume that $T_i \sim T$ for all i, where T is a known positive random variable with the distribution function F_T. For the simplicity, we also assume that T_i's are *iid* and independent of both B_t and $\widetilde{N}_t$. (i.e. T is independent of the jump diffusion process.).

Let $\tilde{X}_x(t)$ be the exchange rate process followed during this random reaction period of duration T_i after each intervention.

$$\tilde{X}_{\tilde{x}}(t) = \tilde{x} + \tilde{\mu}t + \tilde{\sigma}B(t) + \int_0^t \int_{\mathbb{R}} \tilde{\theta} z \widetilde{N}(ds, dz) \; ; \quad B(0) = 0, \qquad (4.4.2)$$

where $\tilde{x} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}$, $\tilde{\sigma} > 0$, $\tilde{\theta} \geq 0$ are constants.

For simplicity we impose the restriction that new interventions are not allowed during this reaction period.

Now suppose that we are only allowed to give the system impulses ζ with values in $\mathcal{Z} := (0, \infty)$ and if we apply an impulse control $\nu = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)$

to $Y_y(t)$ it gets the form

$$Y_y^{(\nu)}(t) = \begin{bmatrix} s+t \\ X_x^{(v)}(t) \end{bmatrix},$$

where the exchange rate process $X_x^{(\nu)}$, starting at value x, can be written as

$$X_x^{(\nu)}(t) = X_x(t)\ ; \qquad 0 \leq t < \tau_1, \tag{4.4.3}$$

$$X_x^{(\nu)}(\tau_j) = X_x^{(\nu)}(\tau_j^-) + \triangle_N X_x(\tau_j) - \zeta_j; \qquad j = 1, 2, ..., \tag{4.4.4}$$

$$X_x^{(\nu)}(t) = \tilde{X}^{(\nu)}_{X_x^{(\nu)}(\tau_j)}(t) \qquad \text{for } \tau_j \leq t \leq \tau_j + T_j, \tag{4.4.5}$$

$$X_x^{(\nu)}(t) = X^{(\nu)}_{X_x^{(\nu)}(\tau_j+T_j)}(t) \qquad \text{for } \tau_j + T_j < t < \tau_{j+1} \wedge \tau^*. \tag{4.4.6}$$

Here τ_i is the time of the i^{th} intervention, and ζ_i is the intensity of the i^{th} intervention. Since we are only allowed to give the system positive impulses, the central bank only pushes the exchange rate downwards. Here we also assume that the target exchange rate ρ to be 0 for the simplicity of the solution.

Suppose that the cost rate (the running cost incurred by deviation from the target exchange rate ρ) $f(t,x)$ if $X_x^{(\nu)} = x$ at time t is given by

$$f(t,x) = e^{-rt}x^2, \tag{4.4.7}$$

where $r > 0$ is the constant discount rate.

Now let C and λ be fixed cost and proportional cost per intervention respectively. So the cost of an intervention of size $\zeta > 0$ at time t is

$$K(t,x,\zeta) = K(t,\zeta) = e^{-rt}(C + \lambda\zeta), \tag{4.4.8}$$

where $C > 0$, $\lambda \geq 0$ are constants.

Then the expected total discounted cost associated to a given impulse control is

$$J^{(\nu)}(s,x) = E^x[\int_0^\infty e^{-r(s+t)}(X^{(v)})^2 dt + \sum_{k=1}^\infty e^{-r(s+\tau_k)}(C + \lambda\zeta_k)]. \tag{4.4.9}$$

We seek $\Phi(s,x)$ and $\nu^* = (\tau_1^*, \tau_2^*, \ldots, \tau_j^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots, \zeta_j^*, \ldots)$ such that

$$\Phi(s,x) = \inf_\nu J^{(\nu)}(s,x) = J^{(\nu^*)}(s,x). \tag{4.4.10}$$

4.4.2 An analytical solution of the problem

The above problem is an impulse control problem with random reaction periods of the type described by Theorem 4.1. Using the notation of this chapter, here we have:

$$Y_y^{(\nu)}(t) = \begin{bmatrix} s+t \\ X_x^{(\nu)}(t) \end{bmatrix} \quad ; \quad t \geq 0 \qquad \text{and} \qquad Y_y^{(\nu)}(0^-) = \begin{bmatrix} s \\ x \end{bmatrix} = y \in \mathbb{R}^2,$$

$$\Gamma(y,\zeta) = \Gamma(s,x,\zeta) = \begin{bmatrix} s \\ x-\zeta \end{bmatrix} \quad ; \quad (s,x,\zeta) \in \mathbb{R}^3,$$

$$K(y,\zeta) = K(s,x,\zeta) = e^{-rs}(C + \lambda\zeta), \quad f(y) = f(s,x) = e^{-rs}x^2 \text{ and } g(y) = 0.$$

Note that it is not optimal to move $X_x(t)$ downwards if $X_x(t)$ is already below 0. Hence we may restrict ourselves to consider impulse controls $\nu = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ such that

$$\sum_{j=1}^{\tau_k} \zeta_j \leq X_x(\tau_k) \quad \text{for all } k. \tag{4.4.11}$$

Since we are not allowed to intervene during the reaction period, we also have:

$$\tau_{i+1} - \tau_i \geq T_i \qquad \text{for all } i. \tag{4.4.12}$$

Let $\mathcal{V}$ be set of impulse controls which satisfies both (4.4.11) and (4.4.12).

We guess that the optimal strategy is to wait until the level of $X_x(t)$ reaches an (unknown) value $\bar{x} > 0$. At this time, τ_1, we intervene and give $X_x(t)$ an impulse ζ_1, which brings it down to a lower value $\hat{x} > 0$. Then we do nothing until the next time, $\tau_2 > \tau_1 + T_1$ that $X_x(t)$ reaches the level $\bar{x}$ etc. So the continuation region D in Theorem 4.1 has the form

$$D = \{(s, x);\ x < \bar{x} \text{ or } s \in (\tau_i, \tau_i + T_i] \text{ for some } \ i\} \tag{4.4.13}$$

for some $\bar{x} > 0$ to be determined.

We guess that the value function is of the form

$$\phi(s, x) = e^{-rs}\psi(x). \tag{4.4.14}$$

Then, by using a similar argument to section 3.4, we can show that

$$\psi(x) = \psi_0(x)\ ; \qquad x \in D, \tag{4.4.15}$$

where

$$\psi_0(x) = -ae^{\alpha_1 x} + \frac{x^2}{r} + \frac{b + 2\mu x}{r^2} + \frac{2\mu^2}{r^3},\ a > 0 \tag{4.4.16}$$

We now consider the running cost during the random reaction period.

$$\begin{aligned}
\tilde{R}(\tilde{x}) &= \int_0^\infty E^{\tilde{x}}\left[\int_0^T f(\tilde{X}(t)dt \mid T=u\right] dF_T(u) \\
&= \int_0^\infty E^{\tilde{x}}\left[\int_0^T e^{-rt}\left(\tilde{X}(t)\right)^2 dt \mid T=u\right] dF_T(u) \\
&= \int_0^\infty \left(E\left[\int_0^u e^{-rt}\left\{\tilde{x}+\tilde{\mu}t+\tilde{\sigma}B(t)+\int_0^t\int_{\mathbb{R}}\tilde{\theta}z\widetilde{N}(ds,dz)\right\}^2 dt\right]\right) dF_T(u) \\
&= \int_0^\infty \left\{\int_0^u e^{-rt}\left(\tilde{x}^2+\left(2\tilde{\mu}\tilde{x}+\tilde{\sigma}^2+\int_{\mathbb{R}}\tilde{\theta}^2z^2\nu(dz)\right)t+\tilde{\mu}^2t^2\right)dt\right\} dF_T(u) \\
&= \int_0^\infty \left\{\int_0^u e^{-rt}\left(\tilde{x}^2+\left(2\tilde{\mu}\tilde{x}+\tilde{b}\right)t+\tilde{\mu}^2t^2\right)dt\right\} dF_T(u),
\end{aligned}$$

where $\tilde{b}=\tilde{\sigma}^2+\int_{\mathbb{R}}\tilde{\theta}^2z^2\nu(dz)$.

We now recall that

$$\begin{aligned}
D &= \{(s,x);\ \phi(s,x) < \mathcal{M}_r\phi(s,x)\} \\
&= \{(s,x);\ \psi(x) < \mathcal{M}_r\psi(x)\}
\end{aligned}$$

and the intervention operator in this case is

$$\mathcal{M}_r\psi(x) = \inf\left\{C+\lambda\zeta+\tilde{R}\left(x-\zeta\right)+\int_0^\infty E^{x-\zeta}\left[\psi(\tilde{X}(T)) \mid T=u\right] dF_T(u);\ \zeta>0\right\}.$$

The first order condition for a minimum $\hat{\zeta}=\hat{\zeta}(x)$ of the function

$$G(\zeta) = C+\lambda\zeta+\tilde{R}\left(x-\zeta\right)+\int_0^\infty E^{x-\zeta}\left[\psi(\tilde{X}(T)) \mid T=u\right] dF_T(u), \quad \zeta>0$$

is

$$G'(\hat{\zeta}) = 0. \tag{4.4.17}$$

Therefore we look for points $\hat{x}$, $\bar{x}$ such that $0<\hat{x}<\bar{x}$ and $G'(\hat{\zeta})=0$.

We also have that $x - \hat{\zeta} = \hat{x}$. i.e. $\hat{\zeta} = x - \hat{x} = \hat{\zeta}(x)$.

By Theorem 4.1-(ii), we have $\psi \leq \mathcal{M}_r \psi$ on $\mathcal{S}$.

So if $x \geq \bar{x}$, we have $\psi(x) = \mathcal{M}_r \psi(x)$ because $x \notin D$.

Thus, if $x \geq \bar{x}$ we have

$$\psi(x) = C + \lambda \hat{\zeta} + \tilde{R}\left(x - \hat{\zeta}\right) + \int_0^\infty E^{x-\hat{\zeta}} \left[\psi(\tilde{X}(T)) \mid T = u\right] dF_T(u).$$

That is

$$\psi(x) = C + \lambda(x - \hat{x}) + \tilde{R}\left(\hat{x}\right) + \int_0^\infty E^{\hat{x}} \left[\psi(\tilde{X}(T)) \mid T = u\right] dF_T(u).$$

In particular, we have $\psi'(\bar{x}) = \lambda$ and

$$\psi(\bar{x}) = C + \lambda(\bar{x} - \hat{x}) + \tilde{R}\left(\hat{x}\right) + \int_0^\infty E^{\hat{x}} \left[\psi(\tilde{X}(T)) \mid T = u\right] dF_T(u).$$

Here we should also note that $\tilde{X}_{\hat{x}}(T) \in D$. Therefore, we have

$$\psi(\tilde{X}_{\hat{x}}(T)) = \psi_0(\tilde{X}_{\hat{x}}(T)).$$

Then we have,

$$\psi(x) = \begin{cases} \psi_0(x), & \text{if } x < \bar{x}, \\ C + \lambda(x - \hat{x}) + \tilde{R}\left(\hat{x}\right) + \int_0^\infty E^{\hat{x}} \left[\psi_0(\tilde{X}(T)) \mid T = u\right] dF_T(u), & \text{if } x \geq \bar{x}. \end{cases} \tag{4.4.18}$$

Differentiability of $\psi(x)$ at $x = \bar{x}$ gives the equation

$$\psi_0'(\bar{x}) = \lambda,$$

which is equivalent to

$$a\alpha_1 e^{\alpha_1 \bar{x}} = \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right). \tag{4.4.19}$$

We also have that

$$G'(\hat{\zeta}) = 0.$$

Equivalently, we have

$$\tilde{R}'(\hat{x}) + \int_0^\infty E^{\hat{x}} \left[\psi_0'(\tilde{X}(T)) \mid T = u\right] dF_T(u) = \lambda. \tag{4.4.20}$$

Continuity of $\psi(x)$ at $x = \bar{x}$ gives the equation

$$C + \lambda(\bar{x} - \hat{x}) + \tilde{R}\,(\hat{x}) + \int_0^\infty E^{\hat{x}} \left[\psi(\tilde{X}(T)) \mid T = u\right] dF_T(u) = \psi_0(\bar{x}).$$

Therefore, we have

$$C+\lambda(\bar{x}-\hat{x})+\tilde{R}\,(\hat{x})+\int_0^\infty E^{\hat{x}} \left[\psi(\tilde{X}(T)) \mid T = u\right] dF_T(u) = -ae^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3}. \tag{4.4.21}$$

The solution of the problem is determined by the system (4.4.19)-(4.4.21). Moreover, we need to solve the system for $\bar{x}$, $\hat{x}$ and a. Therefore we simplify the system to get more explicit expressions with $\bar{x}$, $\hat{x}$ and a. For this, we compute each term in the system separately and then plug into the system.

We first compute $\tilde{R}(\hat{x})$, which is given by

$$\tilde{R}(\tilde{x}) = \int_0^\infty \left\{ \int_0^u e^{-rt} \left(\tilde{x}^2 + \left(2\tilde{\mu}\tilde{x} + \tilde{b} \right) t + \tilde{\mu}^2 t^2 \right) dt \right\} dF_T(u).$$

By applying integration by parts twice, we get

$$\begin{aligned}\tilde{R}(\hat{x}) = \int_0^\infty (\frac{1}{r} \left[\hat{x}^2 - e^{-ru} \left\{ \hat{x}^2 + \left(2\tilde{\mu}\hat{x} + \tilde{b} \right) u + \tilde{\mu}^2 u^2 \right\} \right] \\ + \frac{1}{r^2} \left[\left(2\tilde{\mu}\hat{x} + \tilde{b} \right) - e^{-ru} \left\{ \left(2\tilde{\mu}\hat{x} + \tilde{b} \right) + 2\tilde{\mu}^2 u \right\} \right] + \frac{2\tilde{\mu}^2}{r^3} \left[1 - e^{-ru} \right]) dF_T(u).\end{aligned}$$

By rearranging the terms, we get

$$\tilde{R}(\hat{x}) = \tilde{A}_R \, \hat{x}^2 + \tilde{B}_R \, \hat{x} + \tilde{C}_R, \tag{4.4.22}$$

where

$$\begin{aligned}\tilde{A}_R &= \frac{1}{r} \left(1 - \int_0^\infty e^{-ru} dF_T(u) \right), \\ \tilde{B}_R &= \frac{2\tilde{\mu}}{r^2} \left(1 - \int_0^\infty e^{-ru} dF_T(u) \right) - \frac{2\tilde{\mu}}{r} \int_0^\infty u e^{-ru} dF_T(u), \\ \tilde{C}_R &= \frac{2\tilde{\mu}^2}{r^3} \left(1 - \int_0^\infty e^{-ru} dF_T(u) \right) \\ &\quad + \frac{1}{r^2} \left\{ \tilde{b} \left(1 - \int_0^\infty e^{-ru} dF_T(u) \right) - 2\tilde{\mu}^2 \int_0^\infty u e^{-ru} dF_T(u) \right\} \\ &\quad - \frac{1}{r} \int_0^\infty \left(\tilde{b}u + \tilde{\mu}^2 u^2 \right) e^{-ru} dF_T(u).\end{aligned}$$

By differentiating (4.4.22), we get

$$\tilde{R}'(\hat{x}) = 2\tilde{A}_R \, \hat{x} + \tilde{B}_R. \tag{4.4.23}$$

By (3.4.31), we have

$$E^{\hat{x}}\left[\psi_0(\tilde{X}(u))\right] = -aA_1\, e^{\alpha_1 \hat{x}} + B_1\, \hat{x}^2 + C_1\, \hat{x} + D_1,$$

where

$$\begin{aligned}
A_1 &= E\left[e^{\alpha_1\left\{\tilde{\mu}u+\tilde{\sigma}B(u)+\int_0^u\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right],\\
B_1 &= \frac{1}{r},\\
C_1 &= \frac{2\tilde{\mu}u}{r} + \frac{2\mu}{r^2},\\
D_1 &= \frac{\tilde{b}u+\tilde{\mu}^2u^2}{r} + \frac{b+2\mu\tilde{\mu}u}{r^2} + \frac{2\mu^2}{r^3}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\int_0^\infty E^{\hat{x}}\left[\psi_0(\tilde{X}(T)) \mid T=u\right] dF_T(u) &= \int_0^\infty E^{\hat{x}}\left[\psi_0(\tilde{X}(u))\right] dF_T(u)\\
&= \int_0^\infty \left[-aA_1\, e^{\alpha_1\hat{x}} + B_1\, \hat{x}^2 + C_1\, \hat{x} + D_1\right] dF_T(u).
\end{aligned}$$

Hence we have

$$\int_0^\infty E^{\hat{x}}\left[\psi_0(\tilde{X}(T)) \mid T=u\right] dF_T(u) = -a\tilde{A}_0\, e^{\alpha_1\hat{x}} + \tilde{B}_0\, \hat{x}^2 + \tilde{C}_0\, \hat{x} + \tilde{D}_0, \qquad (4.4.24)$$

where

$$\begin{aligned}
\tilde{A}_0 &= \int_0^\infty E\left[e^{\alpha_1\left\{\tilde{\mu}u+\tilde{\sigma}B(u)+\int_0^u\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right] dF_T(u),\\
\tilde{B}_0 &= \frac{1}{r},\\
\tilde{C}_0 &= \frac{2\tilde{\mu}}{r}\int_0^\infty u\, dF_T(u) + \frac{2\mu}{r^2},\\
\tilde{D}_0 &= \frac{1}{r}\int_0^\infty \left(\tilde{b}u+\tilde{\mu}^2u^2\right) dF_T(u) + \frac{2\mu\tilde{\mu}}{r^2}\int_0^\infty u\, dF_T(u) + \frac{b}{r^2} + \frac{2\mu^2}{r^3}.
\end{aligned}$$

By differentiating (4.4.24), we get

$$\int_0^\infty E^{\hat{x}}\left[\psi_0'(\tilde{X}(T)) \mid T = u\right] dF_T(u) = -a\alpha_1 \tilde{A}_0\, e^{\alpha_1 \hat{x}} + 2\tilde{B}_0\, \hat{x} + \tilde{C}_0 \qquad (4.4.25)$$

Combining (4.4.20), (4.4.23) and (4.4.25), we get

$$2\tilde{A}_R\, \hat{x} + \tilde{B}_R - a\alpha_1 \tilde{A}_0\, e^{\alpha_1 \hat{x}} + 2\tilde{B}_0\, \hat{x} + \tilde{C}_0 = \lambda.$$

Therefore we have

$$-a\alpha_1 \tilde{A}_0\, e^{\alpha_1 \hat{x}} + 2\left(\tilde{A}_R + \tilde{B}_0\right)\hat{x} + \left(\tilde{B}_R + \tilde{C}_0 - \lambda\right) = 0. \qquad (4.4.26)$$

Combining (4.4.21), (4.4.22) and (4.4.24), we get

$$C + \lambda(\bar{x} - \hat{x}) + \tilde{A}_R\, \hat{x}^2 + \tilde{B}_R\, \hat{x} + \tilde{C}_R - a\tilde{A}_0\, e^{\alpha_1 \hat{x}} + \tilde{B}_0\, \hat{x}^2 + \tilde{C}_0\, \hat{x} + \tilde{D}_0 = \\ - a e^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3},$$

i.e.,

$$C + \lambda(\bar{x} - \hat{x}) + \left(\tilde{A}_R + \tilde{B}_0\right)\hat{x}^2 + \left(\tilde{B}_R + \tilde{C}_0\right)\hat{x} - a\tilde{A}_0\, e^{\alpha_1 \hat{x}} = \\ - a e^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3} - \tilde{C}_R - \tilde{D}_0. \qquad (4.4.27)$$

We now recall (4.4.19),

$$a\alpha_1 e^{\alpha_1 \bar{x}} = \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right). \qquad (4.4.28)$$

Since the system (4.4.26)-(4.4.28) is equivalent to the system (4.4.19)-(4.4.21), the solution is also determined by the system (4.4.26)-(4.4.28). We can use any standard method to solve the system (4.4.26)-(4.4.28) of non-linear equations. In particular, we can use the Newton's method for solving system of non-linear equations. We will numerically simulate these equations in Chapter 6.

Impulse control problems are very difficult to solve analytically, except in very special cases. Therefore we do not find analytical solutions for general situations at the moment. Moreover, there are not many numerical techniques available to solve impulse control problems (See Kushner and Dupuis [42] and Lapeyre, Sulem and Talay [43]). However, we can interpret the solution of a impulse control problem in the sense of *viscosity solutions*. This weak solution concept was first introduced by Crandall and Lions to handle the HJB equations of stochastic control and later extended by them and others to more general equations. We refer the reader to Bardi and Capuzzo-Dolcetta [4], Crandall and Lions [19], Fleming and Soner [25], Ishii [30] and Øksendal and Sulem [52] for a comprehensive study about viscosity solutions.

Chapter 5

Approximating Impulse Control of Jump Diffusions by Iterated Optimal Stopping

In general it is not possible to reduce impulse control to optimal stopping, because the choice of the first intervention time τ_1 and the first impulse ζ_1 will influence the next and so on. However, Øksendal and Sulem [52] proved that if we allow only (up to) a fixed finite number n of interventions, then the corresponding impulse control problem can be solved by solving iteratively n optimal stopping problems. They further showed that if we restrict the number of interventions to (at most) n in a given impulse control problem, then the value function of this restricted problem converge to the value function of the original problem as $n \to \infty$. Hence they proved that it is possible to reduce a given impulse control problem to a sequence of iterated optimal stopping problems. However, we can not directly apply their theory to solve an impulse control problem when the original process is affected by interventions. This is indeed the case when we consider an optimal central bank intervention problem.

Therefore we propose a set of new results which allows us to reduce a given impulse control problem to a sequence of iterated optimal stopping problems even though the original process is affected by interventions. For this, we combine the definition of the reacted intervention operator introduced in Chapter 3 and Chapter 4 with the approximation theory of optimal impulse control of jump diffusions by iterated optimal stopping problems introduced by Øksendal and Sulem [52]. We prove our main results for both constant and random reaction times.

5.1 A general formulation

The formulation of this section is the same as sections 3.1 and 4.1, except we now require our cost function f to be convex for technical reasons. Our results of this chapter are valid for both constant and random reaction times. Therefore we do not distinguish constant and random reaction times in this chapter unless it is necessary to consider these cases separately.

Using the notation of Chapter 3, the impulse control problem is the following:
Find $\Phi(y)$ and $\nu^* \in \mathcal{V}$ such that

$$\Phi(y) = \inf\left\{J^{(\nu)}(y); \nu \in \mathcal{V}\right\} = J^{(\nu^*)}(y), \tag{5.1.1}$$

where $\mathcal{V}$ denotes the set of admissible controls $\nu = (\tau_1, \tau_2, \dots, \tau_j, \dots; \zeta_1, \zeta_2, \dots, \zeta_j, \dots)$ and

$$J^{(\nu)}(y) = E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu)}(t))dt + g(Y^{(\nu)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{\tau_i\le\tau_{\mathcal{S}}} K(\check{Y}^{(\nu)}(\tau_j-), \zeta_j)\right]$$

with

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}^{(\nu)} = \inf\{t \in (0, \tau^*); Y_y^{(\nu)}(t) \notin \mathcal{S}\}.$$

Now, for $n = 1, 2, ...$, let $\mathcal{V}_n$ denote the set of all $\nu \in \mathcal{V}$ such that $\nu = (\tau_1, \tau_2, ..., \tau_n, \tau_{n+1}; \zeta_1, \zeta_2, ..., \zeta_n)$ with $\tau_{n+1} = \tau_{\mathcal{S}}$ a.s. In other words, $\mathcal{V}_n$ is the set of all admissible controls *with at most* n *interventions*. Then

$$\mathcal{V}_n \subseteq \mathcal{V}_{n+1} \subseteq \mathcal{V} \qquad \text{for all } n \tag{5.1.2}$$

Define

$$\Phi_n(y) = \inf\{J^{(\nu)}(y); \nu \in \mathcal{V}_n\};\ n = 1, 2, ... \tag{5.1.3}$$

Since $\mathcal{V}_n \subseteq \mathcal{V}_{n+1} \subseteq \mathcal{V}$, we have $\Phi_n(y) \geq \Phi_{n+1}(y) \geq \Phi(y)$. Moreover, we have

Lemma 5.1. *Suppose* $g \geq 0$. *Then*

$$\lim_{n\to\infty} \Phi_n(y) = \Phi(y) \qquad \textit{for all } y \in \mathcal{S}$$

Proof: This proof follows with obvious modification to the proof of Lemma 7.1 of Øksendal and Sulem [52]. So we will refer the reader to Øksendal and Sulem [52] for the proof. □

5.2 Iterative scheme

We now introduce the iterative procedure as follows:

Let $Y_y(t) = Y_y^{(0)}(t)$ be the process (3.1.1) without interventions. Define

$$\varphi_0(y) = E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t))dt + g(Y(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right] \tag{5.2.1}$$

and inductively, for $j = 1, 2, ..., n$,

$$\varphi_j(y) = \inf_{\tau \in \mathcal{T}} E^y \left[\int_0^\tau f(Y(t))dt + \mathcal{M}_r \varphi_{j-1}(Y(\tau)) \right], \tag{5.2.2}$$

where $\mathcal{M}_r$ is the reacted intervention operator that we introduced in Chapter 3 and Chapter 4 (Definition 3.2 and Definition 4.2), and

$$\mathcal{T} = \{\tau;\ \tau \text{ stopping time }, 0 \le \tau \le \tau_{\mathcal{S}} \text{ a.s.}\}$$

with

$$\tau_{\mathcal{S}} = \inf\{t \in (0, \tau^*); Y_y^{(\nu)}(t) \notin \mathcal{S}\}.$$

Definition 5.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a function. We say that f is of at most polynomial growth if there exists constants C and $m = m(f)$ such that*

$$|f(y)| \le C\left(1 + |y|^m\right) \qquad \textit{for all} \quad y \in \mathbb{R}^k. \tag{5.2.3}$$

Now let $\mathcal{P}(\mathbb{R}^k)$ denote the set of all functions $f : \mathbb{R}^k \to \mathbb{R}$ of at most polynomial growth.

We now state the main result of this chapter.

Theorem 5.1. *Suppose f, g and $\mathcal{M}_r\varphi_{j-1} \in \mathcal{P}(\mathbb{R}^k)$ for $j = 1, 2, ..., n$. Then $\varphi_n = \Phi_n$.*

Since we do not have any smoothness conditions on φ, we first need to establish an important corollary before we prove the above theorem. Moreover, we need a *dynamic programming principle* (or *Bellman principle*) to derive our corollary. Therfore we first state the Bellman principle. This principle is due to Krylov [40] (Theorem 9 and

Theorem 11, page 134) when there is no jumps. The proof of this principle for jump processes can be found in Ishikawa [31] (Section 4).

Lemma 5.2. *Suppose* $G \in \mathcal{P}(\mathbb{R}^k)$*. Define*

$$\psi(y) = \inf_{\tau \in \mathcal{T}} E^y \left[\int_0^\tau f(Y(s))\, ds + G(Y(\tau)) \right]. \tag{5.2.4}$$

(a). Then for all stopping times β *we have*

$$\psi(y) = \inf_{\tau \in \mathcal{T}} E^y \left[\int_0^{\tau \wedge \beta} f(Y(s))\, ds + G(Y(\tau)) I_{\{\tau \le \beta\}} + \psi(Y(\beta)) I_{\{\tau > \beta\}} \right]. \tag{5.2.5}$$

(b). For $\varepsilon > 0$ *define*

$$D^{(\varepsilon)} = \{y \in \mathcal{S}; \psi(y) < G(y) - \varepsilon\}$$

and put

$$\tau^{(\varepsilon)} = \inf\left\{t > 0; Y(t) \notin D^{(\varepsilon)}\right\}.$$

Then if β *is a stopping time such that* $\beta \le \tau^{(\varepsilon)}$ *for some* $\varepsilon > 0$ *we have*

$$\psi(y) = E^y \left[\int_0^\beta f(Y(t))\, dt + \psi(Y(\beta)) \right]. \tag{5.2.6}$$

Now we introduce the corollary that we need to prove our main result.

Corollary 5.1. *(a). For each given* $\tau \in \mathcal{T}$ *the process*

$$U(t) := \int_0^{t \wedge \tau} f(Y(s))\, ds + \psi(Y(t \wedge \tau));\ t \ge 0 \tag{5.2.7}$$

is a submartingale. In particular, if $\tau_1 \le \tau_2 \le \tau_{\mathcal{S}}$ *are stopping times, then*

$$E^y[\psi(Y(\tau_1))] \le E^y \left[\int_{\tau_1}^{\tau_2} f(Y(s))\, ds + \psi(Y(\tau_2)) \right]. \tag{5.2.8}$$

(b). For $\varepsilon > 0$, let $\tau^{(\varepsilon)}$ be as in Lemma 5.2.(b) and let $\beta_1 \leq \beta_2 \leq \tau^{(\varepsilon)}$ be stopping times. Then

$$E^y\left[\psi\left(Y(\beta_1)\right)\right] = E^y\left[\int_{\beta_1}^{\beta_2} f\left(Y(s)\right) ds + \psi\left(Y(\beta_2)\right)\right]. \tag{5.2.9}$$

Proof:

(a). Let $\tau \in \mathcal{T}$ and $t \geq 0$. Define

$$R(t) = \left(Y\left(t \wedge \tau\right), \int_0^{t\wedge\tau} f\left(Y(r)\right) dr\right) \in \mathbb{R}^{k+1}$$

and put

$$H(y,u) = \psi(y) + u;\ (y,u) \in \mathbb{R}^k \times \mathbb{R}.$$

Now for $t > s$, by the Markov property, we have

$$\begin{aligned} E^y\left[U(t) \mid \mathcal{F}_s\right] &= E^y\left[\int_0^{t\wedge\tau} f\left(Y(r)\right) dr + \psi\left(Y(t\wedge\tau)\right) \mid \mathcal{F}_s\right] \\ &= E^y\left[H\left(Y\left(t\wedge\tau\right), \int_0^{t\wedge\tau} f\left(Y(r)\right) dr\right) \mid \mathcal{F}_s\right] \\ &= E^y\left[H(R(t)) \mid \mathcal{F}_s\right] = E^{R(s)}\left[H(R(t-s))\right]. \end{aligned}$$

Therefore we have

$$E^y\left[U(t) \mid \mathcal{F}_s\right] = E^{R(s)}\left[H(R(t-s))\right]. \tag{5.2.10}$$

Now we choose $\beta = (t-s) \wedge \tau$ in Lemma 5.2.(a) and $R(s) = (y, u)$, then we have

$$\begin{aligned}
E^{R(s)}\left[H(R(t-s))\right] &= E^{(y,u)}\left[H\left(Y\left((t-s)\wedge\tau\right), \int_0^{(t-s)\wedge\tau} f\left(Y(r)\right)dr\right)\right] \\
&= E^{y}\left[H\left(Y\left((t-s)\wedge\tau\right), u + \int_0^{(t-s)\wedge\tau} f\left(Y(r)\right)dr\right)\right] \\
&= E^{y}\left[\psi\left(Y\left((t-s)\wedge\tau\right)\right) + u + \int_0^{(t-s)\wedge\tau} f\left(Y(r)\right)dr\right] \\
&= E^{y}\left[\psi\left(Y\left(\beta\right)\right) + \int_0^{\beta} f\left(Y(r)\right)dr\right] + u \\
&\geq \inf_{\beta} E^{y}\left[\int_0^{\beta} f\left(Y(r)\right)dr + \psi\left(Y\left(\beta\right)\right)\right] + u \\
&= \inf_{\tau\geq\beta} E^{y}\left[\int_0^{\tau\wedge\beta} f\left(Y(r)\right)dr + \psi\left(Y\left(\tau\wedge\beta\right)\right)\right] + u \\
&\geq \inf_{\tau} E^{y}\left[\int_0^{\tau\wedge\beta} f\left(Y(r)\right)dr + \psi\left(Y\left(\tau\right)\right)I_{\{\tau\leq\beta\}} + \psi\left(Y\left(\beta\right)\right)I_{\{\tau>\beta\}}\right] + u \\
&= \psi(y) + u = H(y,u) = H\left(R(s)\right) \\
&= H\left(Y\left(s\wedge\tau\right), \int_0^{s\wedge\tau} f\left(Y(r)\right)dr\right) \\
&= \int_0^{s\wedge\tau} f\left(Y(r)\right)dr + \psi\left(Y(s\wedge\tau)\right) \\
&= U(s). \qquad (5.2.11)
\end{aligned}$$

By (5.2.10) and (5.2.11), we have

$$E^{y}\left[U(t) \mid \mathcal{F}_s\right] \geq U(s) \qquad \text{for all} \quad t > s$$

Hence U is a submartingale.

Now since $\tau_1 \leq \tau_2 \leq \tau_{\mathcal{S}}$ are stopping times, by the *Doob's optional sampling theorem* we get

$$E^y \left[U(\tau_2) \mid \mathcal{F}_{\tau_1} \right] \geq U(\tau_1),$$

$$\text{i.e.} \quad E^y \left[\int_0^{\tau_2} f\left(Y(s)\right) ds + \psi\left(Y(\tau_2)\right) \mid \mathcal{F}_{\tau_1} \right] \geq \int_0^{\tau_1} f\left(Y(s)\right) ds + \psi\left(Y(\tau_1)\right).$$

Taking the expectation on both sides and using the double expectation formula, we get

$$E^y \left[\int_0^{\tau_2} f\left(Y(s)\right) ds + \psi\left(Y(\tau_2)\right) \right] \geq E^y \left[\int_0^{\tau_1} f\left(Y(s)\right) ds + \psi\left(Y(\tau_1)\right) \right].$$

It follows that

$$E^y \left[\psi\left(Y(\tau_1)\right) \right] \leq E^y \left[\int_{\tau_1}^{\tau_2} f\left(Y(s)\right) ds + \psi\left(Y(\tau_2)\right) \right].$$

(b). Now let $\varepsilon > 0$, $\tau^{(\varepsilon)}$ be as in Lemma 5.2.(b) and $\beta_1 \leq \beta_2 \leq \tau^{(\varepsilon)}$ be stopping times. Then we have

$$\begin{aligned}
& E^y \left[\int_{\beta_1}^{\beta_2} f\left(Y(t)\right) dt + \psi\left(Y(\beta_2)\right) \right] \\
& = E^y \left[\int_0^{\beta_2} f\left(Y(t)\right) dt + \psi\left(Y(\beta_2)\right) \right] - E^y \left[\int_0^{\beta_1} f\left(Y(t)\right) dt \right] \\
& = \psi(y) - E^y \left[\int_0^{\beta_1} f\left(Y(t)\right) dt + \psi\left(Y(\beta_1)\right) \right] + E^y \left[\psi\left(Y(\beta_1)\right) \right] \\
& = \psi(y) - \psi(y) + E^y \left[\psi\left(Y(\beta_1)\right) \right] \\
& = E^y \left[\psi\left(Y(\beta_1)\right) \right] \qquad \square
\end{aligned}$$

Now we prove the main result of this chapter.

Proof of Theorem 5.1:

Case I: Constant reaction time

Choose $\nu_n = (\tau_1, \tau_2, ..., \tau_n; \zeta_1, \zeta_2, ..., \zeta_n)$ with $\tau_n \leq \tau_{\mathcal{S}}$ and set $\tau_{n+1} = \tau_{\mathcal{S}}$.

Then by Corollary 5.1.(a), we have

$$E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i + T_i)\right)\right] \leq E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} f\left(Y^{(\nu_n)}(t)\right)dt + \varphi_{n-i}\left(\check{Y}^{(\nu_n)}(\tau_{i+1}-)\right)\right]. \tag{5.2.12}$$

Choosing $\tau = 0$ in (5.2.2), we get

$$\varphi_{n-i}(y) \leq \mathcal{M}_r\varphi_{n-i-1}(y) \qquad \text{if} \quad n-i \geq 1. \tag{5.2.13}$$

By the definition of $\mathcal{M}_r$, we have for $\Gamma(y,\zeta) \in \mathcal{S}$

$$\mathcal{M}_r\varphi_{n-i-1}(y) \leq K(y,\zeta) + E^{\Gamma(y,\zeta)}\left[\int_0^T f\left(\tilde{Y}(t)\right)dt\right] + E^{\Gamma(y,\zeta)}\left[\varphi_{n-i-1}\left(\tilde{Y}(T)\right)\right].$$

However, if $\tau_{i+1} < \tau_{\mathcal{S}}$, then we have $Y_y^{(\nu_n)}(\tau_{i+1}) = \Gamma(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1}) \in \mathcal{S}$. Therefore, we can plug in $\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)$ for y in the above equation. Then by the strong Markov property, we get

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \leq\ & K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1}) \\ & + E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+T_{i+1}} f\left(\tilde{Y}_{Y^{(\nu_n)}(\tau_{i+1})}(t)\right)dt | \mathcal{F}_{\tau_{i+1}}\right] \\ & + E^y\left[\varphi_{n-i-1}\left(\tilde{Y}_{Y^{(\nu_n)}(\tau_{i+1})}(\tau_{i+1}+T_{i+1})\right)|\mathcal{F}_{\tau_{i+1}}\right].\end{aligned}$$

But $Y_y^{(\nu_n)}(t) = \tilde{Y}_{Y_y^{(\nu_n)}(\tau_{j+1})}(t)$ for $\tau_{j+1} \le t \le \tau_{j+1} + T_{j+1}$. Therefore, we have

$$\begin{aligned} \mathcal{M}_r \varphi_{n-i-1}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \le K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1}) \\ + E^y \left[\int_{\tau_{i+1}}^{\tau_{i+1}+T_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt | \mathcal{F}_{\tau_{i+1}} \right] \\ + E^y \left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+T_{i+1})\right) | \mathcal{F}_{\tau_{i+1}} \right]. \end{aligned} \tag{5.2.14}$$

By (5.2.13) and (5.2.14), we have

$$\begin{aligned} \varphi_{n-i}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \le K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1}) \\ + E^y \left[\int_{\tau_{i+1}}^{\tau_{i+1}+T_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt | \mathcal{F}_{\tau_{i+1}} \right] \\ + E^y \left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+T_{i+1})\right) | \mathcal{F}_{\tau_{i+1}} \right]. \end{aligned}$$

Taking expectation and using the double expectation formula, we get

$$\begin{aligned} E^y\left[\varphi_{n-i}(\check{Y}^{(\nu_n)}(\tau_{i+1}-))\right] \le E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1})\right] \\ + E^y \left[\int_{\tau_{i+1}}^{\tau_{i+1}+T_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt \right] \\ + E^y \left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+T_{i+1})\right) \right]. \end{aligned} \tag{5.2.15}$$

Combining (5.2.12) and (5.2.15), we obtain

$$\begin{aligned} E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i+T_i)\right)\right] \le E^y \left[\int_{\tau_i+T_i}^{\tau_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt \right] + E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1})\right] \\ + E^y \left[\int_{\tau_{i+1}}^{\tau_{i+1}+T_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt \right] \\ + E^y \left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+T_{i+1})\right) \right]. \end{aligned}$$

Then we have

$$E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i+T_i)\right)\right]-E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+T_{i+1})\right)\right]\leq E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}+T_{i+1}} f\left(Y^{(\nu_n)}(t)\right)dt\right]+E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-),\zeta_{i+1})\right].$$

Summing from $i=0$ to $i=n-1$ we get,

$$E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right]-E^y\left[\varphi_0\left(Y^{(\nu_n)}(\tau_n+T_n)\right)\right]\leq E^y\left[\int_0^{\tau_n+T_n} f\left(Y^{(\nu_n)}(t)\right)dt\right]+\sum_{i=1}^n E^y\left[K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right]. \tag{5.2.16}$$

However, we have

$$E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right]=E^y\left[\varphi_n(y)\right]=\varphi_n(y) \tag{5.2.17}$$

and by the strong Markov property

$$\begin{aligned}&E^y\left[\varphi_0\left(Y^{(\nu_n)}(\tau_n+T_n)\right)\right]\\&=E^y\left[E^{Y^{(\nu_n)}(\tau_n+T_n)}\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu_n)}(t))dt+g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right]\right]\\&=E^y\left[\int_{\tau_n+T_n}^{\tau_{\mathcal{S}}} f(Y^{(\nu_n)}(t))dt+g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right].\end{aligned} \tag{5.2.18}$$

Combining (5.2.16)-(5.2.18), we get

$$\begin{aligned}\varphi_n(y)&\leq E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu_n)}(t))dt+g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}+\sum_{i=1}^n K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right]\\&=J^{(\nu_n)}(y).\end{aligned} \tag{5.2.19}$$

Since $\nu_n \in \mathcal{V}_n$ is an arbitrary element, we have

$$\varphi_n(y) \leq \inf\{J^{(\nu_n)}(y); \nu_n \in \mathcal{V}_n\} = \Phi_n(y). \tag{5.2.20}$$

Now we let $\varepsilon > 0$ and define an increasing sequence of stopping times $0 = \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_n$ as follows:

Let

$$D_i^{(\varepsilon)} = \{y; \varphi_i(y) < \mathcal{M}_r \varphi_{i-1} - \varepsilon\} \qquad \text{for } \; i = 1, 2, ..., n. \tag{5.2.21}$$

Define

$$\hat{\tau}_1 = \inf\left\{t > 0; Y_y^{(0)}(t) \notin D_n^{(\varepsilon)}\right\}, \tag{5.2.22}$$

where $Y_y^{(0)}(t) = Y_y(t)$ is the process without interventions.

Now we choose $\hat{\zeta}_1 = \bar{\zeta}_1(Y_y(\hat{\tau}_1-))$, where $\bar{\zeta}_1 = \bar{\zeta}_1(y) \in \mathcal{Z}$ is ε-optimal for φ_{n-1}, in the sense that

$$\begin{aligned}\mathcal{M}_r \varphi_{n-1}(y) \geq K(y, \bar{\zeta}_1) &+ E^{\Gamma(y,\bar{\zeta}_1)}\left[\int_0^T f\left(\tilde{Y}(t)\right) dt\right] \\ &+ E^{\Gamma(y,\bar{\zeta}_1)}\left[\varphi_{n-1}\left(\tilde{Y}(T)\right)\right] - \varepsilon. \end{aligned} \tag{5.2.23}$$

Now we inductively define other stopping times.

For this, suppose that $0 = \hat{\tau}_0, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i$ have been chosen, where $i \leq n-1$, we let $Y_y^{(i)}(t)$ be the process obtained by applying $\hat{\nu}_i = \left(\hat{\tau}_1, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i\right)$ to $Y_y(t)$. Define

$$\hat{\tau}_{i+1} = \inf\left\{t > \hat{\tau}_i + T_i \;;\; Y_y^{(i)}(t) \notin D_{n-i}^{(\varepsilon)}\right\}, \tag{5.2.24}$$

and choose $\hat{\zeta}_{i+1} = \bar{\zeta}_{i+1}(Y_y(\hat{\tau}_{i+1}-))$, where $\bar{\zeta}_{i+1} = \bar{\zeta}_{i+1}(y) \in \mathcal{Z}$ is ε-optimal for φ_{n-i-1}, in the sense that

$$\mathcal{M}_r\varphi_{n-i-1}(y) \geq K(y,\bar{\zeta}_{i+1}) + E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\int_0^T f\left(\tilde{Y}(t)\right)dt\right] + E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\varphi_{n-1}\left(\tilde{Y}(T)\right)\right] - \varepsilon. \tag{5.2.25}$$

We now put $\hat{\tau}_{n+1} = \tau_{\mathcal{S}}$ and define

$$\hat{\nu}_n = \left(\hat{\tau}_1, ..., \hat{\tau}_n; \hat{\zeta}_1, ..., \hat{\zeta}_n\right) \in \mathcal{V}_n.$$

Now apply the argument (5.2.12)-(5.2.19) to $\hat{\nu}_n$:

By Corollary 5.1.(b), we have

$$E^y\left[\varphi_{n-i}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_i + T_i)\right)\right] = E^y\left[\int_{\hat{\tau}_i+T_i}^{\hat{\tau}_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt + \varphi_{n-i}\left(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)\right)\right]. \tag{5.2.26}$$

Since $\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-) \notin D_{n-i}^{(\varepsilon)}$, by (5.2.21) we have that

$$\varphi_{n-i}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq \mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) - \varepsilon \tag{5.2.27}$$

and (5.2.25) holds when y is replaced by $\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)$. Then by the strong Markov property, we get

$$\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1}) + E^y\left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(\tilde{Y}_{Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(t)\right)dt|\mathcal{F}_{\hat{\tau}_{i+1}}\right] + E^y\left[\varphi_{n-i-1}\left(\tilde{Y}_{Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(\hat{\tau}_{i+1}+T_{i+1})\right)|\mathcal{F}_{\hat{\tau}_{i+1}}\right] - \varepsilon.$$

But $Y_y^{(\hat{\nu}_n)}(t) = \tilde{Y}_{Y_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(t)$ for $\hat{\tau}_{i+1} \le t \le \hat{\tau}_{i+1} + T_{i+1}$. Therefore, we have

$$\begin{aligned} \mathcal{M}_r \varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \ge\ & K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1}) \\ & + E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt | \mathcal{F}_{\hat{\tau}_{i+1}} \right] \\ & + E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1} + T_{i+1})\right) | \mathcal{F}_{\hat{\tau}_{i+1}} \right] - \varepsilon. \end{aligned} \tag{5.2.28}$$

By (5.2.27) and (5.2.28), we have

$$\begin{aligned} \varphi_{n-i}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \ge\ & K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1}) \\ & + E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt | \mathcal{F}_{\hat{\tau}_{i+1}} \right] \\ & + E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1} + T_{i+1})\right) | \mathcal{F}_{\hat{\tau}_{i+1}} \right] - 2\varepsilon. \end{aligned}$$

Taking expectation and using the double expectation formula, we get

$$\begin{aligned} E^y\left[\varphi_{n-i}(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-))\right] \ge\ & E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1})\right] \\ & + E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt \right] \\ & + E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1} + T_{i+1})\right) \right] - 2\varepsilon. \end{aligned} \tag{5.2.29}$$

Combining (5.2.26) and (5.2.29), we obtain

$$\begin{aligned} E^y\left[\varphi_{n-i}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_i + T_i)\right)\right] \ge\ & E^y\left[\int_{\hat{\tau}_i+T_i}^{\hat{\tau}_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt\right] + E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1})\right] \\ & + E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt \right] \\ & + E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1} + T_{i+1})\right) \right] - 2\varepsilon. \end{aligned}$$

Then, we have

$$E^y\left[\varphi_{n-i}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_i+T_i)\right)\right]-E^y\left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}+T_{i+1})\right)\right]$$
$$\geq E^y\left[\int_{\hat{\tau}_i+T_i}^{\hat{\tau}_{i+1}+T_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right]+E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-),\hat{\zeta}_{i+1})\right]-2\varepsilon.$$

Summing from $i=0$ to $i=n-1$ we get,

$$E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(0)\right)\right]-E^y\left[\varphi_0\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_n+T_n)\right)\right]\geq$$
$$E^y\left[\int_0^{\hat{\tau}_n+T_n} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right]+\sum_{i=1}^n E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_i-),\hat{\zeta}_i)\right]-2n\varepsilon. \tag{5.2.30}$$

However, we have

$$E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(0)\right)\right]=E^y\left[\varphi_n\left(y\right)\right]=\varphi_n\left(y\right) \tag{5.2.31}$$

and by the strong Markov property

$$E^y\left[\varphi_0\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_n+T_n)\right)\right]$$
$$=E^y\left[E^{Y^{(\hat{\nu}_n)}(\hat{\tau}_n+T_n)}\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\hat{\nu}_n)}(t))dt+g(Y^{(\hat{\nu}_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right]\right]$$
$$=E^y\left[\int_{\hat{\tau}_n+T_n}^{\tau_{\mathcal{S}}} f(Y^{(\hat{\nu}_n)}(t))dt+g(Y^{(\hat{\nu}_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right]. \tag{5.2.32}$$

Combining (5.2.30)-(5.2.32), we get

$$\varphi_n\left(y\right)\geq E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\hat{\nu}_n)}(t))dt+g(Y^{(\hat{\nu}_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}+\sum_{i=1}^n K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_i-),\hat{\zeta}_i)\right]-2n\varepsilon$$
$$=J^{(\hat{\nu}_n)}(y)-2n\varepsilon. \tag{5.2.33}$$

Since $\varepsilon > 0$ is an arbitrary number, we have

$$\varphi_n(y) \geq \inf\{J^{(\nu_n)}(y); \nu_n \in \mathcal{V}_n\} = \Phi_n(y). \tag{5.2.34}$$

By (5.2.20) and (5.2.34), we have

$$\varphi_n(y) = \Phi_n(y). \qquad \square$$

Case II: Random reaction time

Choose $\nu_n = (\tau_1, \tau_2, ..., \tau_n; \zeta_1, \zeta_2, ..., \zeta_n)$ with $\tau_n \leq \tau_{\mathcal{S}}$ and set $\tau_{n+1} = \tau_{\mathcal{S}}$.

By Corollary 5.1.(a), we have

$$\begin{aligned} &E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i + T_i)\right) \mid T_i = u\right] \\ &\leq E^y\left[\int_{\tau_i+T_i}^{\tau_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt + \varphi_{n-i}\left(\check{Y}^{(\nu_n)}(\tau_{i+1}-)\right) \mid T_i = u\right]. \end{aligned} \tag{5.2.35}$$

Integrating (5.2.35) with respect to the probability measure induced by T, we get

$$\begin{aligned} \int_0^\infty E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i + u)\right)\right] dF_T(u) \leq \int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} f\left(Y^{(\nu_n)}(t)\right) dt\right] dF_T(u) \\ + E^y\left[\varphi_{n-i}\left(\check{Y}^{(\nu_n)}(\tau_{i+1}-)\right)\right]. \end{aligned} \tag{5.2.36}$$

Choosing $\tau = 0$ in (5.2.2), we get

$$\varphi_{n-i}(y) \leq \mathcal{M}_r \varphi_{n-i-1}(y) \qquad \text{if} \quad n - i \geq 1. \tag{5.2.37}$$

By the definition of $\mathcal{M}_r$, we have for $\Gamma(y,\zeta) \in \mathcal{S}$

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(y) \le K(y,\zeta) &+ \int_0^\infty E^{\Gamma(y,\zeta)}\left[\int_0^u f\left(\tilde{Y}(t)\right)dt\right]dF_T(u)\\ &+ \int_0^\infty E^{\Gamma(y,\zeta)}\left[\varphi_{n-i-1}\left(\tilde{Y}(u)\right)\right]dF_T(u).\end{aligned}$$

However, if $\tau_{i+1} < \tau_{\mathcal{S}}$, then we have $Y_y^{(\nu_n)}(\tau_{i+1}) = \Gamma(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1}) \in \mathcal{S}$. Therefore, we can plug in $\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)$ for y in the above equation. Then by the strong Markov property, we get

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \le\ & K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1})\\ &+ \int_0^\infty E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+u} f\left(\tilde{Y}_{Y^{(\nu_n)}(\tau_{i+1})}(t)\right)dt|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u)\\ &+ \int_0^\infty E^y\left[\varphi_{n-i-1}\left(\tilde{Y}_{Y^{(\nu_n)}(\tau_{i+1})}(\tau_{i+1}+u)\right)|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u).\end{aligned}$$

But $Y_y^{(\nu_n)}(t) = \tilde{Y}_{Y_y^{(\nu_n)}(\tau_{j+1})}(t)$ for $\hat{\tau}_{i+1} \le t \le \hat{\tau}_{i+1} + u$. Therefore, we have

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \le\ & K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1})\\ &+ \int_0^\infty E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+u} f\left(Y^{(\nu_n)}(t)\right)dt|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u)\\ &+ \int_0^\infty E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+u)\right)|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u).\end{aligned} \tag{5.2.38}$$

By (5.2.37) and (5.2.38), we have

$$\begin{aligned}\varphi_{n-i}(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-)) \le\ & K(\check{Y}_y^{(\nu_n)}(\tau_{i+1}-), \zeta_{i+1})\\ &+ \int_0^\infty E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+u} f\left(Y^{(\nu_n)}(t)\right)dt|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u)\\ &+ \int_0^\infty E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+u)\right)|\mathcal{F}_{\tau_{i+1}}\right]dF_T(u).\end{aligned}$$

Taking expectation and using the double expectation formula, we get

$$
\begin{aligned}
E^y\left[\varphi_{n-i}(\check{Y}^{(\nu_n)}(\tau_{i+1}-))\right] \leq & \; E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-),\zeta_{i+1})\right] \\
& + \int_0^\infty E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+u} f\left(Y^{(\nu_n)}(t)\right)dt\right]dF_T(u) \\
& + \int_0^\infty E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+u)\right)\right]dF_T(u). \qquad (5.2.39)
\end{aligned}
$$

Combining (5.2.36) and (5.2.39), we obtain

$$
\begin{aligned}
\int_0^\infty E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i+u)\right)\right]dF_T(u) \leq & \int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}} f\left(Y^{(\nu_n)}(t)\right)dt\right]dF_T(u) \\
& + E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-),\zeta_{i+1})\right] \\
& + \int_0^\infty E^y\left[\int_{\tau_{i+1}}^{\tau_{i+1}+u} f\left(Y^{(\nu_n)}(t)\right)dt\right]dF_T(u) \\
& + \int_0^\infty E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+u)\right)\right]dF_T(u).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\int_0^\infty \{\, E^y\left[\varphi_{n-i}\left(Y^{(\nu_n)}(\tau_i+u)\right)\right] - E^y\left[\varphi_{n-i-1}\left(Y^{(\nu_n)}(\tau_{i+1}+u)\right)\right]\,\}dF_T(u) \\
\leq \int_0^\infty E^y\left[\int_{\tau_i+u}^{\tau_{i+1}+u} f\left(Y^{(\nu_n)}(t)\right)dt\right]dF_T(u) \\
+ E^y\left[K(\check{Y}^{(\nu_n)}(\tau_{i+1}-),\zeta_{i+1})\right].
\end{aligned}
$$

Summing from $i=0$ to $i=n-1$, we get

$$
\begin{aligned}
\int_0^\infty \{\, E^y\left[\varphi_n\left(Y^{(\nu_n)}(u)\right)\right] - E^y\left[\varphi_0\left(Y^{(\nu_n)}(\tau_n+u)\right)\right]\,\}dF_T(u) \\
\leq \int_0^\infty E^y\left[\int_u^{\tau_n+u} f\left(Y^{(\nu_n)}(t)\right)dt\right]dF_T(u) \\
+ \sum_{i=1}^n E^y\left[K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right]. \qquad (5.2.40)
\end{aligned}
$$

Now by Corollary 5.1.(a), we have

$$E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right] \leq E^y\left[\int_0^T f\left(Y^{(\nu_n)}(t)\right)dt + \varphi_n\left(Y^{(\nu_n)}(T)\right) \mid T = u\right]$$

Integrating the above equation with respect to the probability measure induced by T, we get

$$\int_0^\infty \{\ E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right] - E^y\left[\varphi_n\left(Y^{(\nu_n)}(u)\right)\right]\ \} dF_T(u) \leq \int_0^\infty E^y\left[\int_0^u f\left(Y^{(\nu_n)}(t)\right)dt\right] dF_T(u). \tag{5.2.41}$$

Adding (5.2.40) and (5.2.41), we get

$$\int_0^\infty \{\ E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right] - E^y\left[\varphi_0\left(Y^{(\nu_n)}(\tau_n+u)\right)\right]\ \} dF_T(u) \leq \int_0^\infty E^y\left[\int_0^{\tau_n+u} f\left(Y^{(\nu_n)}(t)\right)dt\right] dF_T(u) + \sum_{i=1}^n E^y\left[K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right]. \tag{5.2.42}$$

However, we have

$$E^y\left[\varphi_n\left(Y^{(\nu_n)}(0)\right)\right] = E^y\left[\varphi_n(y)\right] = \varphi_n(y) \tag{5.2.43}$$

and by the strong Markov property

$$\begin{aligned}
&E^y\left[\varphi_0\left(Y^{(\nu_n)}(\tau_n+u)\right)\right] \\
&= E^y\left[E^{Y^{(\nu_n)}(\tau_n+u)}\left[\int_0^{\tau_\mathcal{S}} f(Y^{(\nu_n)}(t))dt + g(Y^{(\nu_n)}(\tau_\mathcal{S}))I_{\{\tau_\mathcal{S}<\infty\}}\right]\right] \\
&= E^y\left[\int_{\tau_n+u}^{\tau_\mathcal{S}} f(Y^{(\nu_n)}(t))dt + g(Y^{(\nu_n)}(\tau_\mathcal{S}))I_{\{\tau_\mathcal{S}<\infty\}}\right].
\end{aligned} \tag{5.2.44}$$

Combining (5.2.42)-(5.2.44), we get

$$\begin{aligned}
&\int_0^\infty \left\{ \varphi_n(y) - E^y\left[\int_{\tau_n+u}^{\tau_{\mathcal{S}}} f(Y^{(\nu_n)}(t))dt\right] \right\} dF_T(u) \\
&\le \int_0^\infty E^y\left[\int_0^{\tau_n+u} f\left(Y^{(\nu_n)}(t)\right)dt\right] dF_T(u) \\
&+ \int_0^\infty E^y\left[g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right] dF_T(u) + \sum_{i=1}^n E^y\left[K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right].
\end{aligned}$$

Therefore, by using the fact that $\int_0^\infty dF_T(u) = 1$, we have

$$\begin{aligned}
\varphi_n(y) \le{}& E^y\left[\int_0^{\tau_{\mathcal{S}}} f\left(Y^{(\nu_n)}(t)\right)dt\right] + E^y\left[g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right] \\
&+ \sum_{i=1}^n E^y\left[K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right]
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi_n(y) &\le E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\nu_n)}(t))dt + g(Y^{(\nu_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{i=1}^n K(\check{Y}^{(\nu_n)}(\tau_i-),\zeta_i)\right] \\
&= J^{(\nu_n)}(y). \qquad (5.2.45)
\end{aligned}$$

Since $\nu_n \in \mathcal{V}_n$ is an arbitrary element, we have

$$\varphi_n(y) \le \inf\{J^{(\nu_n)}(y); \nu_n \in \mathcal{V}_n\} = \Phi_n(y). \qquad (5.2.46)$$

Now we let $\varepsilon > 0$ and define an increasing sequence of stopping times $0 = \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_n$ as follows:

Let

$$D_i^{(\varepsilon)} = \{y; \varphi_i(y) < \mathcal{M}_r\varphi_{i-1} - \varepsilon\} \qquad \text{for } \; i = 1, 2, ..., n. \qquad (5.2.47)$$

Define

$$\hat{\tau}_1 = \inf\left\{t > 0; Y_y^{(0)}(t) \notin D_n^{(\varepsilon)}\right\}, \tag{5.2.48}$$

where $Y_y^{(0)}(t) = Y_y(t)$ is the process without interventions.

Now we choose $\hat{\zeta}_1 = \bar{\zeta}_1 \left(Y_y(\hat{\tau}_1-)\right)$, where $\bar{\zeta}_1 = \bar{\zeta}_1 (y) \in \mathcal{Z}$ is ε-optimal for φ_{n-1}, in the sense that

$$\begin{aligned}\mathcal{M}_r\varphi_{n-1}(y) \geq K(y,\bar{\zeta}_1) &+ \int_0^\infty E^{\Gamma(y,\bar{\zeta}_1)}\left[\int_0^u f\left(\tilde{Y}(t)\right)dt\right]dF_T(u)\\ &+ \int_0^\infty E^{\Gamma(y,\bar{\zeta}_1)}\left[\varphi_{n-1}\left(\tilde{Y}(u)\right)\right]dF_T(u) - \varepsilon.\end{aligned} \tag{5.2.49}$$

Now we inductively define other stopping times.

For this, suppose that $0 = \hat{\tau}_0, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i$ have been chosen, where $i \leq n-1$, we let $Y_y^{(i)}(t)$ be the process obtained by applying $\hat{\nu}_i = \left(\hat{\tau}_1, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i\right)$ to $Y_y(t)$. Define

$$\hat{\tau}_{i+1} = \inf\left\{t > \hat{\tau}_i + T_i \; ; \; Y_y^{(i)}(t) \notin D_{n-i}^{(\varepsilon)}\right\}, \tag{5.2.50}$$

and choose $\hat{\zeta}_{i+1} = \bar{\zeta}_{i+1}\left(Y_y(\hat{\tau}_{i+1}-)\right)$, where $\bar{\zeta}_{i+1} = \bar{\zeta}_{i+1}(y) \in \mathcal{Z}$ is ε-optimal for φ_{n-i-1}, in the sense that

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(y) \geq K(y,\bar{\zeta}_{i+1}) &+ \int_0^\infty E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\int_0^u f\left(\tilde{Y}(t)\right)dt\right]dF_T(u)\\ &+ \int_0^\infty E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\varphi_{n-i-1}\left(\tilde{Y}(u)\right)\right]dF_T(u) - \varepsilon.\end{aligned} \tag{5.2.51}$$

We now put $\hat{\tau}_{n+1} = \tau_{\mathcal{S}}$ and define

$$\hat{\nu}_n = \left(\hat{\tau}_1, ..., \hat{\tau}_n; \hat{\zeta}_1, ..., \hat{\zeta}_n\right) \in \mathcal{V}_n.$$

Now apply the argument (5.2.35)-(5.2.45) to $\hat{\nu}_n$:

By Corollary 5.1.(b), we have

$$
\begin{aligned}
&E^y\left[\varphi_{n-i}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_i+T_i)\right) \mid T_i=u\right] \\
&= E^y\left[\int_{\hat{\tau}_i+T_i}^{\hat{\tau}_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt + \varphi_{n-i}\left(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)\right) \mid T_i=u\right]. \qquad (5.2.52)
\end{aligned}
$$

Integrating (5.2.52) with respect to the probability measure induced by T, we get

$$
\begin{aligned}
\int_0^\infty E^y\left[\varphi_{n-i}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_i+u)\right)\right]dF_T(u) = \int_0^\infty E^y\left[\int_{\hat{\tau}_i+u}^{\hat{\tau}_{i+1}} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right]dF_T(u) \\
+ E^y\left[\varphi_{n-i}\left(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)\right)\right]. \qquad (5.2.53)
\end{aligned}
$$

Since $\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-) \notin D_{n-i}^{(\varepsilon)}$, by (5.2.47) we have that

$$
\varphi_{n-i}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq \mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) - \varepsilon \qquad (5.2.54)
$$

and (5.2.51) holds when y is replaced by $\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)$. Then by the strong Markov property, we get

$$
\begin{aligned}
\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq\ & K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-),\hat{\zeta}_{i+1}) \\
&+ \int_0^\infty E^y\left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+u} f\left(\tilde{Y}_{Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(t)\right)dt|\mathcal{F}_{\hat{\tau}_{i+1}}\right]dF_T(u) \\
&+ \int_0^\infty E^y\left[\varphi_{n-i-1}\left(\tilde{Y}_{Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(\hat{\tau}_{i+1}+u)\right)|\mathcal{F}_{\hat{\tau}_{i+1}}\right]dF_T(u) - \varepsilon.
\end{aligned}
$$

But $Y_y^{(\hat{\nu}_n)}(t) = \tilde{Y}_{Y_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1})}(t)$ for $\hat{\tau}_{i+1} < t \leq \hat{\tau}_{i+1} + u$. Therefore, we have

$$\begin{aligned}\mathcal{M}_r\varphi_{n-i-1}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq{}& K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1}) \\ &+ \int_0^\infty E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+u} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt | \mathcal{F}_{\hat{\tau}_{i+1}} \right] dF_T(u) \\ &+ \int_0^\infty E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}+u)\right) | \mathcal{F}_{\hat{\tau}_{i+1}} \right] dF_T(u) - \varepsilon.\end{aligned} \tag{5.2.55}$$

By (5.2.54) and (5.2.55), we have

$$\begin{aligned}\varphi_{n-i}(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-)) \geq{}& K(\check{Y}_y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1}) \\ &+ \int_0^\infty E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+u} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt | \mathcal{F}_{\hat{\tau}_{i+1}} \right] dF_T(u) \\ &+ \int_0^\infty E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}+u)\right) | \mathcal{F}_{\hat{\tau}_{i+1}} \right] dF_T(u) - 2\varepsilon.\end{aligned}$$

Taking expectation and using the double expectation formula, we get

$$\begin{aligned}E^y\left[\varphi_{n-i}(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-))\right] \geq{}& E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}-), \hat{\zeta}_{i+1})\right] \\ &+ \int_0^\infty E^y \left[\int_{\hat{\tau}_{i+1}}^{\hat{\tau}_{i+1}+u} f\left(Y^{(\hat{\nu}_n)}(t)\right) dt \right] dF_T(u) \\ &+ \int_0^\infty E^y \left[\varphi_{n-i-1}\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_{i+1}+u)\right) \right] dF_T(u) - 2\varepsilon.\end{aligned} \tag{5.2.56}$$

Combining (5.2.53) and (5.2.56), we obtain

$$\begin{aligned}
\int_0^\infty E^y\left[\varphi_{n-i}\left(Y^{(\hat\nu_n)}(\hat\tau_i+u)\right)\right]dF_T(u) \geq & \int_0^\infty E^y\left[\int_{\hat\tau_i+u}^{\hat\tau_{i+1}} f\left(Y^{(\hat\nu_n)}(t)\right)dt\right]dF_T(u) \\
& + E^y\left[K(\check Y^{(\hat\nu_n)}(\hat\tau_{i+1}-),\hat\zeta_{i+1})\right] \\
& + \int_0^\infty E^y\left[\int_{\hat\tau_{i+1}}^{\hat\tau_{i+1}+u} f\left(Y^{(\hat\nu_n)}(t)\right)dt\right]dF_T(u) \\
& + \int_0^\infty E^y\left[\varphi_{n-i-1}\left(Y^{(\hat\nu_n)}(\hat\tau_{i+1}+u)\right)\right]dF_T(u) - 2\varepsilon.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\int_0^\infty \{\, E^y\left[\varphi_{n-i}\left(Y^{(\hat\nu_n)}(\hat\tau_i+u)\right)\right] - E^y\left[\varphi_{n-i-1}\left(Y^{(\hat\nu_n)}(\hat\tau_{i+1}+u)\right)\right]\,\}dF_T(u)& \\
\geq \int_0^\infty E^y\left[\int_{\hat\tau_i+u}^{\hat\tau_{i+1}+u} f\left(Y^{(\hat\nu_n)}(t)\right)dt\right]dF_T(u)& \\
+ E^y\left[K(\check Y^{(\hat\nu_n)}(\hat\tau_{i+1}-),\hat\zeta_{i+1})\right] - 2\varepsilon.&
\end{aligned}$$

Summing from $i=0$ to $i=n-1$, we get

$$\begin{aligned}
\int_0^\infty \{\, E^y\left[\varphi_{n}\left(Y^{(\hat\nu_n)}(u)\right)\right] - E^y\left[\varphi_{0}\left(Y^{(\hat\nu_n)}(\hat\tau_{n}+u)\right)\right]\,\}dF_T(u)& \\
\geq \int_0^\infty E^y\left[\int_{u}^{\hat\tau_n+u} f\left(Y^{(\hat\nu_n)}(t)\right)dt\right]& \\
+ \sum_{i=1}^n E^y\left[K(\check Y^{(\hat\nu_n)}(\hat\tau_{i}-),\hat\zeta_{i})\right] - 2n\varepsilon.&
\end{aligned} \tag{5.2.57}$$

Now by Corollary 5.1.(b), we have

$$E^y\left[\varphi_n\left(Y^{(\hat\nu_n)}(0)\right)\right] = E^y\left[\int_0^T f\left(Y^{(\hat\nu_n)}(t)\right)dt + \varphi_n\left(Y^{(\hat\nu_n)}(T)\right) \mid T=u\right].$$

Integrating the above equation with respect to the probability measure induced by

T, we get

$$\int_0^\infty \{\ E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(0)\right)\right] - E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(u)\right)\right]\ \}dF_T(u) \leq \int_0^\infty E^y\left[\int_0^u f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right]dF_T(u). \tag{5.2.58}$$

Adding (5.2.57) and (5.2.58), we get

$$\begin{aligned}\int_0^\infty \{\ E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(0)\right)\right] - E^y\left[\varphi_0\left(Y^{(\hat{\nu}_n)}(\tau_n+u)\right)\right]\ \}dF_T(u) \\ \geq \int_0^\infty E^y\left[\int_0^{\tau_n+u} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right]dF_T(u) \\ + \sum_{i=1}^n E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\tau_i-),\zeta_i)\right] - 2n\varepsilon.\end{aligned} \tag{5.2.59}$$

However, we have

$$E^y\left[\varphi_n\left(Y^{(\hat{\nu}_n)}(0)\right)\right] = E^y\left[\varphi_n\left(y\right)\right] = \varphi_n\left(y\right) \tag{5.2.60}$$

and by the strong Markov property

$$\begin{aligned}&E^y\left[\varphi_0\left(Y^{(\hat{\nu}_n)}(\hat{\tau}_n+u)\right)\right] \\ &= E^y\left[E^{Y^{(\hat{\nu}_n)}(\hat{\tau}_n+u)}\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(\hat{\nu}_n)}(t))dt + g(Y^{(\hat{\nu}_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right]\right] \\ &= E^y\left[\int_{\hat{\tau}_n+u}^{\tau_{\mathcal{S}}} f(Y^{(\hat{\nu}_n)}(t))dt + g(Y^{(\hat{\nu}_n)}(\tau_{\mathcal{S}}))I_{\{\tau_{\mathcal{S}}<\infty\}}\right].\end{aligned} \tag{5.2.61}$$

Combining (5.2.59)-(5.2.61), we get

$$
\begin{aligned}
&\int_0^\infty \left\{ \varphi_n(y) - E^y\left[\int_{\tau_n+u}^{\tau_\mathcal{S}} f(Y^{(\hat{\nu}_n)}(t))dt\right]\right\} dF_T(u)\\
&\geq \int_0^\infty E^y\left[\int_0^{\tau_n+u} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right] dF_T(u) + \int_0^\infty E^y\left[g(Y^{(\hat{\nu}_n)}(\tau_\mathcal{S}))I_{\{\tau_\mathcal{S}<\infty\}}\right] dF_T(u)\\
&+ \sum_{i=1}^n E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\tau_i-),\zeta_i)\right] - 2n\varepsilon.
\end{aligned}
$$

Therefore, by using the fact that $\int_0^\infty dF_T(u) = 1$, we have

$$
\begin{aligned}
\varphi_n(y) \geq E^y&\left[\int_0^{\tau_\mathcal{S}} f\left(Y^{(\hat{\nu}_n)}(t)\right)dt\right] + E^y\left[g(Y^{(\hat{\nu}_n)}(\tau_\mathcal{S}))I_{\{\tau_\mathcal{S}<\infty\}}\right]\\
&+ \sum_{i=1}^n E^y\left[K(\check{Y}^{(\hat{\nu}_n)}(\tau_i-),\zeta_i)\right] - 2n\varepsilon.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\varphi_n(y) &\geq E^y\left[\int_0^{\tau_\mathcal{S}} f(Y^{(\hat{\nu}_n)}(t))dt + g(Y^{(\hat{\nu}_n)}(\tau_\mathcal{S}))I_{\{\tau_\mathcal{S}<\infty\}} + \sum_{i=1}^n K(\check{Y}^{(\hat{\nu}_n)}(\hat{\tau}_i-),\hat{\zeta}_i)\right] - 2n\varepsilon\\
&= J^{(\hat{\nu}_n)}(y) - 2n\varepsilon. \qquad (5.2.62)
\end{aligned}
$$

Since $\varepsilon > 0$ is an arbitrary number, we have

$$\varphi_n(y) \geq \inf\{J^{(\nu_n)}(y); \nu_n \in \mathcal{V}_n\} = \Phi_n(y). \qquad (5.2.63)$$

By (5.2.46) and (5.2.63), we have

$$\varphi_n(y) = \Phi_n(y). \qquad \square$$

Remark 5.1. *In the proof of Theorem 5.1, the construction of the impulse control* $\hat{\nu}_n = \left(\hat{\tau}_1, ..., \hat{\tau}_n; \hat{\zeta}_1, ..., \hat{\zeta}_n\right)$ *also gives us* $2n\varepsilon$*-optimal impulse controls. These impulse controls are inductively defined by (5.2.21)-(5.2.25) (or (5.2.47)-(5.2.51)).*

In particular, if it is possible to choose $\hat{\zeta}_i = \zeta_i^*$ to be optimal (i.e. (5.2.25)/(5.2.51) holds with $\varepsilon = 0$), then $\nu_n^* = (\hat{\tau}_1, ..., \hat{\tau}_n; \zeta_1^*, ..., \zeta_n^*)$ will be an optimal impulse control for Φ_n given by the following procedure:

Let

$$D_i = \{y; \varphi_i(y) < \mathcal{M}_r \varphi_{i-1}(y)\} \qquad \text{for } \; i = 1, 2, ..., n. \tag{5.2.64}$$

Define

$$\hat{\tau}_1 = \inf\left\{t > 0; Y_y^{(0)}(t) \notin D_n\right\}, \tag{5.2.65}$$

and

$$\hat{\zeta}_1 = \bar{\zeta}_1\left(Y_y^{(0)}(\hat{\tau}_1 -)\right), \tag{5.2.66}$$

where $\bar{\zeta}_1 = \bar{\zeta}_1(y)$ is a Borel measurable function such that

$$\begin{aligned}\mathcal{M}_r \varphi_{n-1}(y) = K(y, \bar{\zeta}_1) &+ E^{\Gamma(y,\bar{\zeta}_1)}\left[\int_0^T f\left(\tilde{Y}(t)\right) dt\right] \\ &+ E^{\Gamma(y,\bar{\zeta}_1)}\left[\varphi_{n-1}\left(\tilde{Y}(T)\right)\right] \;;\quad y \in \mathcal{S}.\end{aligned} \tag{5.2.67}$$

Then if $\left(\hat{\tau}_1, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i\right)$ is defined, put

$$\hat{\tau}_{i+1} = \inf\left\{t > \hat{\tau}_i + T_i \;;\; Y_y^{(i)}(t) \notin D_{n-i}\right\}, \tag{5.2.68}$$

and

$$\hat{\zeta}_{i+1} = \bar{\zeta}_{i+1}\left(Y_y^{(j)}(\hat{\tau}_{i+1} -)\right), \tag{5.2.69}$$

where $\bar{\zeta}_{i+1} = \bar{\zeta}_{i+1}(y)$ is a Borel measurable function such that

$$\mathcal{M}_r\varphi_{n-(i+1)}(y) = K(y,\bar{\zeta}_{i+1}) + E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\int_0^T f\left(\tilde{Y}(t)\right)dt\right] + E^{\Gamma(y,\bar{\zeta}_{i+1})}\left[\varphi_{n-1}\left(\tilde{Y}(T)\right)\right] \; ; \quad y \in \mathcal{S},\; i+1 \le n. \tag{5.2.70}$$

As before $Y_y^{(j)}$ denotes the result of applying the impulse control $\hat{\nu}_i = \left(\hat{\tau}_1, ..., \hat{\tau}_i; \hat{\zeta}_1, ..., \hat{\zeta}_i\right)$ to Y_y.

We now state some corollaries due to our main result of this chapter. The proofs of these corollaries are followed with appropriate modifications to the proofs of Øksendal and Sulem [52] (see Corollary (7.6), Corollary(7.7) and Corollary(7.8)). Therefore we state the following results without proofs and refer the reader to Øksendal and Sulem [52] for proofs.

Corollary 5.2. *Assume that $g \ge 0$ and that f, g, $\mathcal{M}_r\varphi_n \in \mathcal{P}(\mathbb{R}^k)$ for $n = 0, 1, 2, ...$, where φ_n is as defined in (5.2.1)-(5.2.2). Then*

$$\varphi_n(y) \downarrow \Phi(y) \quad \text{as} \quad n \to \infty, \text{ for all } \quad y.$$

Corollary 5.3. *Suppose $g \ge 0$ and f, g, $\mathcal{M}_r\varphi_n \in \mathcal{P}(\mathbb{R}^k)$ for $n = 0, 1, 2, ...$ Then Φ is a solution of the following non-linear optimal stopping problem*

$$\Phi(y) = \inf_{\tau \in \mathcal{T}} E^y\left[\int_0^\tau f\left(Y(t)\right)dt + \mathcal{M}_r\Phi\left(Y(\tau)\right)\right]. \tag{5.2.71}$$

Corollary 5.4. *Suppose $g \ge 0$ and f, g, $\mathcal{M}_r\varphi_n \in \mathcal{P}(\mathbb{R}^k)$ for $n = 0, 1, 2, ...$ Moreover, suppose that Φ, $\mathcal{M}_r\Phi$, φ_n and $\mathcal{M}_r\varphi_n$ are continuous for all n, where Φ, φ_n are defined in (5.1.1) and (5.2.1)-(5.2.2) respectively.*

Define

$$D = \{y; \Phi(y) < \mathcal{M}_r \Phi(y)\} \tag{5.2.72}$$

and

$$D_n = \{y; \varphi_n(y) < \mathcal{M}_r \varphi_n(y)\} \qquad \text{for } n = 1, 2, ... \tag{5.2.73}$$

Then

$$D \subseteq D_{n+1} \subseteq D_n \qquad \text{for all} \quad n. \tag{5.2.74}$$

Finally, we can combine the results of this section and summarize the procedure of finding the optimal impulse control $\hat{\nu}_n = \left(\hat{\tau}_1, ..., \hat{\tau}_n; \hat{\zeta}_1, ..., \hat{\zeta}_n\right)$ for Φ_n. (see (5.2.1)-(5.2.2) and (5.2.64)-(5.2.70):

Make the first intervention at the first time $t = \hat{\tau}_1$ that $Y(t) \notin D_n$. Then give the system the impulse $\hat{\zeta}_1$ according to (5.2.66) . Now we have only $n-1$ interventions left, so we wait until the reaction time is over (i.e. $t > \tau_1 + T_1$) and $Y(t)$ exists from the larger set D_{n-1} before making the next intervention, and so on. The last intervention time $\hat{\tau}_n$ is the first time after $\hat{\tau}_{n-1} + T_{n-1}$ that $Y(t) \notin D_1$.

5.3 An approximation of optimal central bank interventions by iterated optimal stopping

We now apply our main results of this section to approximate optimal central bank interventions in the foreign exchange market by iterated optimal stopping. For this we consider the following generalized problem of the problems that we consider in Chapters 3 and 4.

5.3.1 Case I: Constant reaction period

The mathematical formulation of the problem

The formulation of the problem is the same as subsection 3.4.1, except that we are now allowed to give the system both positive and negative impulses. Therefore the central bank can push the exchange rate both downwards and upwards.

An analytical solution of the problem

We now apply our main results of this chapter to solve the above problem. We put $\lambda = \mu = \tilde{\mu} = 0$ and assume that the Lévy measure ν is symmetric (i.e. $\nu(H) = \nu(-H)$, for all $H \subset \mathbb{R} \setminus \{0\}$.) to simplify the computations.
We first define the sequence of value functions as follows:

$$\Phi_n(s,x) = \inf_{\nu \in \mathcal{V}_n} J^{(\nu)}(s,x) = J^{(\nu_n^*)}(s,x);\ n = 1,2,... \tag{5.3.1}$$

We now find $\Psi_n(x) := \Phi_n(0,x)$ by using the iterative procedure (5.2.1)-(5.2.2).
By (5.2.1) we have,

$$\begin{aligned}
\Psi_0(x) &= E^x[\int_0^\infty e^{-rt}(X_x^{(\nu)}(t))^2 dt] \\
&= E[\int_0^\infty e^{-rt}(x + \sigma B(t) + \int_0^t \int_{\mathbb{R}} \theta z \widetilde{N}(ds,dz))^2 dt] \\
&= \int_0^\infty e^{-rt}\left(x^2 + bt\right) dt \\
&= \frac{x^2}{r} + \frac{b}{r^2},
\end{aligned} \tag{5.3.2}$$

where $b = \sigma^2 + \int_{\mathbb{R}} \theta^2 z^2 \nu(dz)$.

Therefore

$$\begin{aligned}\mathcal{M}_r\Psi_0(x) &= \inf\left\{C + R(x+\zeta) + E^{x+\zeta}\left[\Psi_0\left(\tilde{X}(T)\right)\right];\ \zeta \in \mathbb{R}\setminus\{0\}\right\}\\ &= C + R(0) + E^0\left[\Psi_0\left(\tilde{X}_0(T)\right)\right]\\ &= C + C_R^* + D_0^*, \qquad (5.3.3)\end{aligned}$$

where $C_R^* = \frac{\tilde{b}}{r^2}\left(1 - e^{-rT}\right) - \frac{\tilde{b}T}{r}e^{-rT}$ and $D_0^* = \frac{\tilde{b}T}{r} + \frac{b}{r^2}$ with $\tilde{b} = \tilde{\sigma}^2 + \int_{\mathbb{R}}\tilde{\theta}^2 z^2\nu(dz)$ (see (3.4.29) and (3.4.31)).

Hence, we have

$$\mathcal{M}_r\Psi_0(x) = C^* + C + \frac{b}{r^2}, \qquad \text{where} \quad C^* = \frac{\tilde{b}}{r^2}\left(1 + rT\right)\left(1 - e^{-rT}\right). \qquad (5.3.4)$$

Now by (5.2.2), we have

$$\Psi_1(x) = \inf_{\tau>0} E^x\left[\int_0^\tau e^{-rt}(x + \sigma B(t) + \int_0^t\int_{\mathbb{R}}\theta z\widetilde{N}(ds,dz))^2 dt + e^{-r\tau}\left(C^* + C + \frac{b}{r^2}\right)\right]. \qquad (5.3.5)$$

We have to solve the following associated integro-variational inequalities to find the solution of the above optimal stopping problem (see Theorem 2.19).

$$-r\psi_1(x) + \frac{\sigma^2}{2}\psi_1''(x) + \int_{\mathbb{R}}\{\psi_1(x+\theta z) - \psi_1(x) - \theta z\psi_1'(x)\}\nu(dz) + x^2 \geq 0 \quad \text{for all } x, \qquad (5.3.6)$$

$$\psi_1(x) \leq \left(C^* + C + \frac{b}{r^2}\right) \quad \text{for all } x, \qquad (5.3.7)$$

$$-r\psi_1(x) + \frac{\sigma^2}{2}\psi_1''(x) + \int_{\mathbb{R}}\{\psi_1(x+\theta z) - \psi_1(x) - \theta z\psi_1'(x)\}\nu(dz) + x^2 = 0$$
$$\text{on } D_1 := \left\{x; \psi_1(x) < \left(C^* + C + \frac{b}{r^2}\right)\right\}. \qquad (5.3.8)$$

We now guess that D_1 has the following form

$$D_1 = \{(s,x); -\bar{x}_1 < x < \bar{x}_1 \text{ or } s \in (\tau_{n-1}, \tau_{n-1} + T_{n-1}]\} \tag{5.3.9}$$

for some $\bar{x}_1$ to be determined.

Then by using a similar argument to Chapter 3 we can show that

$$\psi_1(x) = \begin{cases} -a_1 \cosh(\alpha x) + \dfrac{x^2}{r} + \dfrac{b}{r^2}, & \text{if } |x| < \bar{x}_1, \\ C^* + C + \dfrac{b}{r^2}, & \text{if } |x| \geq \bar{x}_1, \end{cases} \tag{5.3.10}$$

where $a_1 > 0$ is a constant to be determined and $\alpha > 0$ is the positive solution of the equation

$$F(\alpha) = -r + \frac{\sigma^2}{2}\alpha^2 + \int_{\mathbb{R}} \{e^{\alpha\theta z} - 1 - \alpha\theta z\}\nu(dz). \tag{5.3.11}$$

By differentiability and continuity of $\psi_1(x)$ at $x = \bar{x}_1$, we get

$$-a_1 \cosh(\alpha\bar{x}_1) + \frac{\bar{x}_1^2}{r} + \frac{b}{r^2} = C^* + C + \frac{b}{r^2} \tag{5.3.12}$$

and

$$-a_1\alpha \sinh(\alpha\bar{x}_1) + \frac{2\bar{x}_1}{r} = 0. \tag{5.3.13}$$

Combining (5.3.12) and (5.3.13), we get

$$\tanh(\bar{z}_1) = \frac{2\bar{z}_1}{(\bar{z}_1)^2 - r\alpha^2(C^* + C)} \tag{5.3.14}$$

and

$$a_1 = \frac{2\bar{z}_1}{r\alpha^2 \sinh(\bar{z}_1)}, \tag{5.3.15}$$

where

$$\bar{z}_1 = \alpha \bar{x}_1. \tag{5.3.16}$$

We can easily see that (5.3.14) has a unique solution if $(\bar{z}_1)^2 - r\alpha^2(C^* + C) > 0$ (i.e. $\bar{x}_1 > \sqrt{r(C^* + C)}$). So we can solve for this value of $\bar{x}_1$ and then let a_1 be the corresponding value given by (5.3.15). We can now easily by construction verify that the function ψ_1 given by (5.3.10) satisfies the conditions of the verification Theorem 2.19 and conclude that

$$\Psi_1 = \psi_1. \tag{5.3.17}$$

We now use the iterative procedure (5.2.1)-(5.2.2) again to find Ψ_2:

We first consider

$$\begin{aligned}
\mathcal{M}_r\Psi_1(x) &= \inf\left\{C + R\,(x+\zeta) + E^{x+\zeta}\left[\Psi_1\left(\tilde{X}(T)\right)\right];\ \zeta \in \mathbb{R}\setminus\{0\}\right\} \\
&= C + R\,(0) + E^{0}\left[\Psi_1\left(\tilde{X}(T)\right)\right] \\
&= C + C_R^* + D_0^* - a_1 A_0^*, \qquad (5.3.18)
\end{aligned}$$

where $A_0^* = E\left[e^{\alpha\left\{\tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right]$.

Hence, we have

$$\mathcal{M}_r\Psi_1(x) = C^* + C + \frac{b}{r^2} - a_1 A_0^*. \tag{5.3.19}$$

Now by (5.2.2), we have

$$\Psi_2(x) = \inf_{\tau>0} E^x\left[\int_0^\tau e^{-rt}(x + \sigma B(t) + \int_0^t\int_{\mathbb{R}}\theta z\widetilde{N}(ds,dz))^2 dt + e^{-r\tau}\left(\mathcal{M}_r\Psi_1(x)\right)\right]. \tag{5.3.20}$$

We have to solve the following associated integro-variational inequalities to find the solution of the above optimal stopping problem (see Theorem 2.19).

$$-r\psi_2(x) + \frac{\sigma^2}{2}\psi_2''(x) + \int_{\mathbb{R}} \{\psi_2(x+\theta z) - \psi_2(x) - \theta z \psi_2'(x)\}\nu(dz) + x^2 \geq 0 \quad \text{for all } x, \tag{5.3.21}$$

$$\psi_2(x) \leq \mathcal{M}_r \Psi_1(x) \quad \text{for all } x, \tag{5.3.22}$$

$$-r\psi_1(x) + \frac{\sigma^2}{2}\psi_1''(x) + \int_{\mathbb{R}} \{\psi_1(x+\theta z) - \psi_1(x) - \theta z \psi_1'(x)\}\nu(dz) + x^2 = 0$$

$$\text{on } D_1 := \{x; \psi_1(x) < \mathcal{M}_r \Psi_1(x)\}\,. \tag{5.3.23}$$

We now guess that D_2 has the following form

$$D_2 = \{(s,x); -\bar{x}_2 < x < \bar{x}_2 \text{ or } s \in (\tau_{n-2}, \tau_{n-2} + T_{n-2}]\} \tag{5.3.24}$$

for some $\bar{x}_2$ to be determined.

Then by using a similar argument to Chapter 3 we can show that,

$$\psi_2(x) = \begin{cases} -a_2 \cosh(\alpha x) + \dfrac{x^2}{r} + \dfrac{b}{r^2}, & \text{if } |x| < \bar{x}_2, \\ \mathcal{M}_r \Psi_1(x), & \text{if } |x| \geq \bar{x}_2, \end{cases} \tag{5.3.25}$$

where $a_2 > 0$ is a constant to be determined and $\alpha > 0$ is the positive root of the function

$$F(\alpha) = -r + \frac{\sigma^2}{2}\alpha^2 + \int_{\mathbb{R}} \{e^{\alpha\theta z} - 1 - \alpha\theta z\}\nu(dz). \tag{5.3.26}$$

By differentiability and continuity of $\psi_2(x)$ at $x = \bar{x}_2$, we get

$$-a_2 \cosh(\alpha \bar{x}_2) + \frac{\bar{x}_2^2}{r} = C^* + C - a_1 A_0^*, \tag{5.3.27}$$

and

$$-a_2\alpha\sinh(\alpha\bar{x}_2) + \frac{2\bar{x}_2}{r} = 0. \tag{5.3.28}$$

Combining (5.3.27) and (5.3.28), we get

$$\tanh(\bar{z}_2) = \frac{2\bar{z}_2}{(\bar{z}_2)^2 - r\alpha^2(C^* + C - a_1A_0^*)} \tag{5.3.29}$$

and

$$a_2 = \frac{2\bar{z}_2}{r\alpha^2\sinh(\bar{z}_2)}, \tag{5.3.30}$$

where

$$\bar{z}_2 = \alpha\bar{x}_2. \tag{5.3.31}$$

As above, we can easily see that (5.3.29) has a unique solution if $(\bar{z}_1)^2 - r\alpha^2(C^* + C - a_1A_0^*) > 0$ (i.e. $\bar{x}_1 > \sqrt{r(C^* + C - a_1A_0^*)}$). So we can solve for this value of $\bar{x}_2$ and then let a_2 be the corresponding value given by (5.3.30). Then we can easily by construction verify that the function ψ_2 given by (5.3.25) satisfies the conditions of the verification Theorem 2.19 and conclude that

$$\Psi_2 = \psi_2. \tag{5.3.32}$$

We also observe that $\bar{x}_2 < \bar{x}_1$ and $a_2 > a_1$.

Continuing this inductively we can derive a sequence of functions ψ_n of the form

$$\psi_n(x) = \begin{cases} -a_n\cosh(\alpha x) + \dfrac{x^2}{r} + \dfrac{b}{r^2}, & \text{if } |x| < \bar{x}_n, \\ \mathcal{M}_r\Psi_{n-1}(x), & \text{if } |x| \geq \bar{x}_n, \end{cases} \tag{5.3.33}$$

where

$$\mathcal{M}_r\Psi_{n-1}(x) = C^* + C + \frac{b}{r^2} - a_{n-1}A_0^*, \tag{5.3.34}$$

and a_n and $\bar{x}_n$ are determined by

$$\tanh(\bar{z}_n) = \frac{2\bar{z}_n}{(\bar{z}_n)^2 - r\alpha^2(C^* + C - a_{n-1}A_0^*)} \tag{5.3.35}$$

and

$$a_n = \frac{2\bar{z}_n}{r\alpha^2 \sinh(\bar{z}_n)}, \tag{5.3.36}$$

where

$$\bar{z}_n = \alpha\bar{x}_n. \tag{5.3.37}$$

As above, we can easily see that (5.3.35) has a unique solution if $(\bar{z}_n)^2 - r\alpha^2(C^* + C - a_{n-1}A_0^*) > 0$ (i.e. $\bar{x}_n > \sqrt{r(C^* + C - a_{n-1}A_0^*)}$). So we can solve for this value of $\bar{x}_n$ and then let a_n be the corresponding value given by (5.3.36). Then we can easily by construction verify that the function ψ_n given by (5.3.33) satisfies the conditions of the verification Theorem 2.19 and conclude that

$$\Psi_n = \psi_n. \tag{5.3.38}$$

We also observe that $\bar{x}_n < \bar{x}_{n-1}$ and $a_n > a_{n-1}$.

When there are only three interventions, we use this iterative procedure as follows: The optimal strategy is first to wait until the first time τ_1^* when $X(t) \geq \bar{x}_3$, then move $X(t)$ down to $\hat{x}_3$, next wait until the first time $\tau_2^* > \tau_1^* + T_1$ when $X(t) \geq \bar{x}_2$, then move $X(t)$ down to $\hat{x}_2$ and finally wait until the first time $\tau_3^* > \tau_2^* + T_2$ when $X(t) \geq \bar{x}_1$, before making the last intervention.

5.3.2 Case II: Random reaction period

The mathematical formulation of the problem

The formulation of the problem is the same as subsection 4.4.1, except that we are now allowed to give the system both positive and negative impulses. Therefore the central bank can push the exchange rate both downwards and upwards.

An analytical solution of the problem

We can use a argument similar to the previous section to derive the formulas with a random reaction time. However, we have to replace A_0^*, B^* and C^* by $\tilde{A}_0^*$, $\tilde{B}^*$ and $\tilde{C}^*$ respectively,

where

$$\begin{aligned}
\tilde{A}_0^* &= \int_0^\infty E\left[e^{\alpha\left\{\tilde{\sigma}B(u)+\int_0^u\int_{\mathbb{R}}\tilde{\theta}z\widetilde{N}(ds,dz)\right\}}\right]dF_T(u), \\
\tilde{B}^* &= \frac{1}{r}\left(2-\int_0^\infty e^{-ru}dF_T(u)\right), \\
\tilde{C}^* &= \frac{\tilde{b}}{r^2}\left(1+r\int_0^\infty u dF_T(u)\right)\left(1-\int_0^\infty e^{-ru}dF_T(u)\right).
\end{aligned}$$

Chapter 6

Future Research and Numerical Simulations

6.1 Future research

We may extend our results of this dissertation in the following important ways.

6.1.1 N-regime switching

We successfully extended the theory of stochastic impulse control of jump diffusions introduced by Øksendal and Sulem [52] with milder assumptions. In particular, we assume that the original process is affected by the interventions. For this, we consider two-regime switching and assume that the original process is switched to another process during the reaction period. However, one may consider N-regime switching processes and assume that the original process can take any one of these N-processes with assigned transition probabilities. We still have to assume that the process reverts back to the pre-interventions process at the end of the reaction period.

6.1.2 Stochastic volatility

We consider a non-constant volatility in the form of regime switching to successfully model the stochastic impulse control problem. However one may use a stochastic volatility with jumps to further extend this study.

As an example, one can use the jump-diffusion stochastic volatility model introduced by Bates [6] to formulate the stochastic impulse control problem that we studied in this dissertation. In the original formulation the Bates model has the following form:

$$\begin{aligned} \frac{dS_t}{S_t} &= \left(\mu - \lambda \bar{k}\right) dt + \sqrt{V_t}\, dW_t^S + dZ_t, \\ dV_t &= \zeta\left(\eta - V_t\right) dt + \theta \sqrt{V_t}\, dW_t^V, \end{aligned}$$

where $\left(W_t^S\right)$ and $\left(W_t^V\right)$ are Brownian motions with correlation ρ, driving the main process and the volatility, and Z_t is a compound Poisson process with intensity λ and log-normal distribution of jump sizes such that if k is its jump size then $\ln(1+k) \sim N\left(\ln(1+\bar{k}-\frac{1}{2}\delta^2), \delta^2\right)$.

6.2 Numerical simulations

In this section, we present some numerical simulation for our main results of the dissertation. We use the Kou model [37] as our jump-diffusion process in the simulations. In the Kou model, the distribution of jump sizes is an asymmetric exponential with density of the form

$$\nu_k(dx) = \left[p\gamma_+ e^{-\gamma_+ x} I_{\{x>0\}} + (1-p)\gamma_- e^{-\gamma_-|x|} I_{\{x<0\}}\right] dx, \tag{6.2.1}$$

with $\gamma_+ > 0$, $\gamma_- > 0$ governing the decay of the tails for the distribution of positive and negative jump sizes respectively and $p \in [0,1]$ representing the probability of an upward jump.

Hence the Lévy density for the Kou model is $\nu = \gamma\nu_k$, where γ the intensity of the Poisson process that counts the jumps of the diffusion process.

Since we need to solve (3.4.19) and (5.3.11) for α, we now simplify the following expression using the Lévy density of the Kou model.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{e^{\alpha\theta x}-1-\alpha\theta x\}\nu(dx)\\
&\quad=\int_{-\infty}^{\infty}\{e^{\alpha\theta x}-1-\alpha\theta x\}\left[p\gamma\gamma_+e^{-\gamma_+x}I_{\{x>0\}}+(1-p)\gamma\gamma_-e^{-\gamma_-|x|}I_{\{x<0\}}\right]dx\\
&\quad=\int_{-\infty}^{0}\{e^{\alpha\theta x}-1-\alpha\theta x\}\left[(1-p)\gamma\gamma_-e^{-\gamma_-(-x)}\right]dx\\
&\qquad\qquad+\int_{0}^{\infty}\{e^{\alpha\theta x}-1-\alpha\theta x\}\left[p\gamma\gamma_+e^{-\gamma_+x}\right]dx\\
&\quad=(1-p)\gamma\gamma_-\left[\frac{1}{(\alpha\theta+\gamma_-)}-\frac{1}{\gamma_-}+\frac{\alpha\theta}{(\gamma_-)^2}\right]\\
&\qquad\qquad+p\gamma\gamma_+\left[\frac{1}{(\gamma_+-\alpha\theta)}-\frac{1}{\gamma_+}+\frac{\alpha\theta}{(\gamma_+)^2}\right]
\end{aligned}$$

with $(\alpha\theta+\gamma_-) > 0$ and $(\gamma_+-\alpha\theta) > 0$.

Therefore by (3.4.19) and (5.3.11) (for $\mu = 0$) we have,

$$\begin{aligned}
&(1-p)\gamma\gamma_-\left[\frac{1}{(\alpha\theta+\gamma_-)}-\frac{1}{\gamma_-}+\frac{\alpha\theta}{(\gamma_-)^2}\right]\\
&\qquad+p\gamma\gamma_+\left[\frac{1}{(\gamma_+-\alpha\theta)}-\frac{1}{\gamma_+}+\frac{\alpha\theta}{(\gamma_+)^2}\right]-r+\mu\alpha+\frac{\sigma^2}{2}\alpha^2=0 \qquad (6.2.2)
\end{aligned}$$

for $\alpha=\alpha_1>0$ and $\alpha=\alpha_2<0$.

We also compute the values of b and $\tilde{b}$ w.r.t. the Lévy density of the Kou model. For this, we first compute

$$\begin{aligned}\int_{-\infty}^{\infty} x^2\nu(dx) &= \int_{-\infty}^{\infty} x^2 \left[p\gamma\gamma_+ e^{-\gamma_+ x} I_{\{x>0\}} + (1-p)\gamma\gamma_- e^{-\gamma_-|x|} I_{\{x<0\}}\right] dx \\ &= \int_{-\infty}^{0} x^2 \left[(1-p)\gamma\gamma_- e^{-\gamma_-(-x)}\right] dx + \int_{0}^{\infty} x^2 \left[p\gamma\gamma_+ e^{-\gamma_+ x}\right] dx \\ &= (1-p)\gamma\gamma_- \left[\frac{2}{(\gamma_-)^3}\right] + p\gamma\gamma_+ \left[\frac{2}{(\gamma_+)^3}\right] \\ &= 2\gamma \left[\frac{(1-p)}{(\gamma_-)^2} + \frac{p}{(\gamma_+)^2}\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}b &= \sigma^2 + \theta^2 \int_{\mathbb{R}} x^2\nu(dx) \\ &= \sigma^2 + 2\theta^2\gamma \left[\frac{(1-p)}{(\gamma_-)^2} + \frac{p}{(\gamma_+)^2}\right]. \end{aligned} \tag{6.2.3}$$

By using a similar argument, we get

$$\tilde{b} = \tilde{\sigma}^2 + 2\tilde{\theta}^2\gamma \left[\frac{(1-p)}{(\gamma_-)^2} + \frac{p}{(\gamma_+)^2}\right]. \tag{6.2.4}$$

We now solve each of the systems in chapters 3 and 4 to find the numerical solutions for the optimal central bank intervention problems we discussed in these chapters.

6.2.1 Constant reaction time

We put $\mu = 0.2$, $\sigma = 1.0$, $\theta = 0.8$, $\tilde{\mu} = 0.1$, $\tilde{\sigma} = 1.2$, $\tilde{\theta} = 1$, $C = 0.6$, $\lambda = 0.7$, $\gamma = 0.2$, $\gamma_+ = 6$, $\gamma_- = 2$, $p = 0.8$, $r = 0.1$ and $T = 0.5$ in the simulation.

Then by using (6.2.2) we compute (with Wolfram Mathematica 7) $\alpha_1 = 0.26494 > 0$ and $\alpha_2 = -0.73879 < 0$.

We now recall the system (3.4.33)-(3.4.35)

$$
\begin{aligned}
&-a\alpha_1 A_0\, e^{\alpha_1 \hat{x}} + 2\left(A_R + B_0\right)\hat{x} + \left(B_R + C_0 - \lambda\right) = 0,\\
&C + \lambda(\bar{x} - \hat{x}) + \left(A_R + B_0\right)\hat{x}^2 + \left(B_R + C_0\right)\hat{x} - aA_0\, e^{\alpha_1 \hat{x}} = \\
&\qquad\qquad\qquad - ae^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3} - C_R - D_0,\\
&\qquad\qquad a\alpha_1 e^{\alpha_1 \bar{x}} = \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A_R &= \frac{1}{r}\left(1 - e^{-rT}\right),\\
B_R &= \frac{2\tilde{\mu}}{r^2}\left(1 - e^{-rT}\right) - \frac{2\tilde{\mu}T}{r}\, e^{-rT},\\
C_R &= \frac{2\tilde{\mu}^2}{r^3}\left(1 - e^{-rT}\right) + \frac{1}{r^2}\left\{\tilde{b}\left(1 - e^{-rT}\right) - 2\tilde{\mu}^2 T e^{-rT}\right\} - \frac{1}{r}\left(\tilde{b}T + \tilde{\mu}^2 T^2\right)e^{-rT},\\
A_0 &= E\left[e^{\alpha_1\left\{\tilde{\mu}T + \tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}} \tilde{\theta} z \widetilde{N}(ds,dz)\right\}}\right],\\
B_0 &= \frac{1}{r},\\
C_0 &= \frac{2\tilde{\mu}T}{r} + \frac{2\mu}{r^2},\\
D_0 &= \frac{\tilde{b}T + \tilde{\mu}^2 T^2}{r} + \frac{b + 2\mu\tilde{\mu}T}{r^2} + \frac{2\mu^2}{r^3}.
\end{aligned}
$$

We can easily compute the values above except the value of A_0.

However, since

$$
\int_{|x|\geq 1} e^{\alpha_1 x}\nu(dx) = \frac{(1-p)\gamma\gamma_-}{(\alpha_1 + \gamma_-)} e^{-(\alpha_1 + \gamma_-)} - \frac{p\gamma\gamma_+}{(\alpha_1 - \gamma_+)} e^{(\alpha_1 - \gamma_+)} < \infty
$$

by exponential moments of Lévy processes (see Proposition 3.24, Cont and Tankov [18]), we have

$$
A_0 = E\left[e^{\alpha_1\left\{\tilde{\mu}T + \tilde{\sigma}B(T) + \int_0^T\int_{\mathbb{R}} z \widetilde{N}(ds,dz)\right\}}\right] = e^{T\psi(-i\alpha_1)} \qquad (\text{since } \tilde{\theta} = 1), \qquad (6.2.5)
$$

where ψ is the characteristic exponent of the Lévy process with Lévy triplet $(\tilde{\mu}, \tilde{\sigma}^2, \nu)$. Moreover,

$$\psi(x) = -\frac{1}{2}x^2\tilde{\sigma}^2 + i\tilde{\mu}x + \int_{\mathbb{R}} \left(e^{ixz} - 1 - ixzI_{\{|z|\leq 1\}}\right) \nu(dz).$$

Therefore, we have

$$\psi(-i\alpha_1) = \frac{1}{2}\alpha_1^2\tilde{\sigma}^2 + \alpha_1\tilde{\mu} + \int_{\mathbb{R}} \left(e^{\alpha_1 x} - 1 - \alpha_1 xI_{\{|x|\leq 1\}}\right) \nu(dx). \tag{6.2.6}$$

Now by using the Lévy density of the Kou model, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \{e^{\alpha_1 x} - 1 - \alpha_1 xI_{\{|x|\leq 1\}}\}\nu(dx) \\
&\quad = \int_{-\infty}^{0} \{e^{\alpha_1 x} - 1 - \alpha_1 xI_{\{|x|\leq 1\}}\} \left[(1-p)\gamma\gamma_- e^{-\gamma_-(-x)}\right] dx \\
&\qquad\qquad + \int_{0}^{\infty} \{e^{\alpha_1 x} - 1 - \alpha_1 xI_{\{|x|\leq 1\}}\} \left[p\gamma\gamma_+ e^{-\gamma_+ x}\right] dx \\
&\quad = \gamma\left[\frac{p\gamma_+}{\gamma_+ - \alpha_1} + \frac{(1-p)\gamma_-}{\gamma_- + \alpha_1}\right] - \gamma + \gamma\alpha_1\left(pe^{-\gamma_+} - (1-p)e^{-\gamma_-}\right) \\
&\qquad\qquad - \gamma\alpha_1\left(\frac{(1-p)}{\gamma_-} + \frac{p}{\gamma_+}\right) + \gamma\alpha_1\left(\frac{pe^{-\gamma_+}}{\gamma_+} + \frac{(1-p)e^{-\gamma_-}}{\gamma_-}\right).
\end{aligned} \tag{6.2.7}$$

Now by using (6.2.5)-(6.2.7) we can compute A_0. Therefore we can solve the system (3.4.33)-(3.4.35) for $\hat{x}$, $\bar{x}$ and a.

We use the Newton's method for a system of nonlinear equations to solve our system using statistical software "**R**". Then we get $a = 170.65$, $\hat{x} = 0.85$ and $\bar{x} = 2.95$. Therefore our value function can be written as follows:

$$\psi(x) = \begin{cases} -170.65\, e^{0.265x} + 10x^2 + 40x + 181.85, & \text{if } x < 2.95, \\ 0.3x + 13.18, & \text{if } x \geq 2.95. \end{cases} \tag{6.2.8}$$

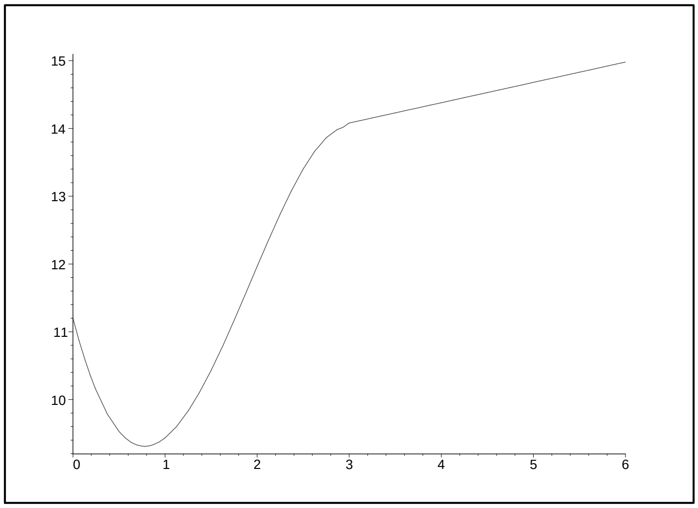

Figure 6.1: *The value function $\psi(x)$ with constant reaction time.*

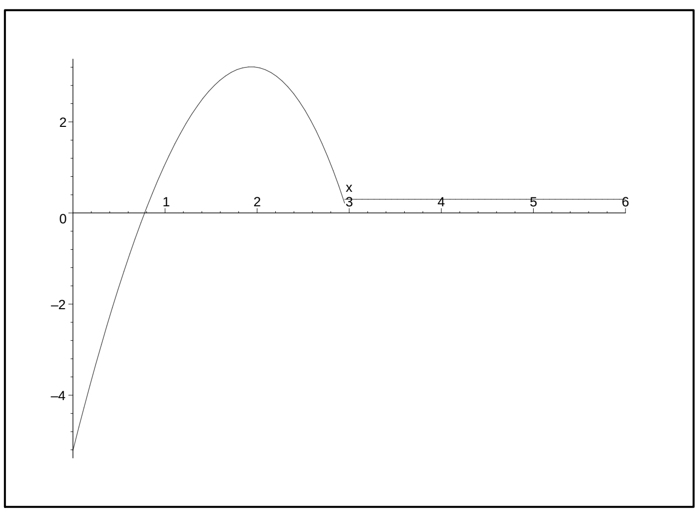

Figure 6.2: *The derivative of the value function $\psi(x)$ with constant reaction time.*

We observe that the value function has the minimum value when $x \approx 0.85 = \hat{x}$ (see Figure 6.1) and that the derivative of the value function is zero when $x \approx 0.85 = \hat{x}$ (see Figure 6.2). This is expected as $x = \hat{x}$ is the minimizer of the value function.

6.2.2 Random reaction time

We put $\mu = 0.2$, $\sigma = 1.0$, $\theta = 0.8$, $\tilde{\mu} = 0.1$, $\tilde{\sigma} = 1.2$, $\tilde{\theta} = 1$, $C = 0.6$, $\lambda = 0.7$, $\gamma = 0.2$, $\gamma_+ = 6$, $\gamma_- = 2$, $p = 0.8$ and $r = 0.1$ in the simulation. We also assume that T is random and $T \sim Exponential(2)$.

Then by using (6.2.2) again we compute (with Wolfram Mathematica 7) $\alpha_1 = 0.26494 > 0$ and $\alpha_2 = -0.73879 < 0$.

We now recall the system (4.4.26)-(4.4.28)

$$\begin{aligned}
-a\alpha_1 \tilde{A}_0\, e^{\alpha_1 \hat{x}} + 2\left(\tilde{A}_R + \tilde{B}_0\right)\hat{x} + \left(\tilde{B}_R + \tilde{C}_0 - \lambda\right) &= 0, \\
C + \lambda(\bar{x} - \hat{x}) + \left(\tilde{A}_R + \tilde{B}_0\right)\hat{x}^2 + \left(\tilde{B}_R + \tilde{C}_0\right)\hat{x} - a\tilde{A}_0\, e^{\alpha_1 \hat{x}} &= \\
- a e^{\alpha_1 \bar{x}} + \frac{\bar{x}^2}{r} + \frac{b + 2\mu\bar{x}}{r^2} + \frac{2\mu^2}{r^3} - \tilde{C}_R - \tilde{D}_0&, \\
a\alpha_1 e^{\alpha_1 \bar{x}} &= \frac{2\bar{x}}{r} - \left(\lambda - \frac{2\mu}{r^2}\right),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{A}_R &= \frac{1}{r}\left(1 - \int_0^\infty e^{-ru} dF_T(u)\right), \\
\tilde{B}_R &= \frac{2\tilde{\mu}}{r^2}\left(1 - \int_0^\infty e^{-ru} dF_T(u)\right) - \frac{2\tilde{\mu}}{r}\int_0^\infty u e^{-ru} dF_T(u), \\
\tilde{C}_R &= \frac{2\tilde{\mu}^2}{r^3}\left(1 - \int_0^\infty e^{-ru} dF_T(u)\right) \\
&\quad + \frac{1}{r^2}\left\{\tilde{b}\left(1 - \int_0^\infty e^{-ru} dF_T(u)\right) - 2\tilde{\mu}^2 \int_0^\infty u e^{-ru} dF_T(u)\right\} \\
&\quad - \frac{1}{r}\int_0^\infty \left(\tilde{b}u + \tilde{\mu}^2 u^2\right) e^{-ru} dF_T(u),
\end{aligned}$$

$$\tilde{A}_0 = \int_0^\infty E\left[e^{\alpha_1\left\{\tilde{\mu}u+\tilde{\sigma}B(u)+\int_0^u\int_{\mathbb{R}}\tilde{\theta}z\tilde{N}(ds,dz)\right\}}\right] dF_T(u),$$
$$\tilde{B}_0 = \frac{1}{r},$$
$$\tilde{C}_0 = \frac{2\tilde{\mu}}{r}\int_0^\infty u\, dF_T(u) + \frac{2\mu}{r^2},$$
$$\tilde{D}_0 = \frac{1}{r}\int_0^\infty \left(\tilde{b}u + \tilde{\mu}^2u^2\right) dF_T(u) + \frac{2\mu\tilde{\mu}}{r^2}\int_0^\infty u\, dF_T(u) + \frac{b}{r^2} + \frac{2\mu^2}{r^3}.$$

Here we use the *Law of large numbers* to compute expectations with respect to T. For this, we compute each of these numbers 10,000 times and then compute the average. We again use the Newton's method for a system of nonlinear equations to solve our system using statistical software "**R**". Then we get $a = 170.23$, $\hat{x} = 0.83$ and $\bar{x} = 2.93$. Therefore our value function can be written as follows:

$$\psi(x) = \begin{cases} -170.23\, e^{0.265x} + 10x^2 + 40x + 181.85, & \text{if } x < 2.93, \\ 0.3x + 13.79, & \text{if } x \geq 2.93. \end{cases} \tag{6.2.9}$$

We observe that the value function has the minimum value when $x \approx 0.83 = \hat{x}$ (see Figure 6.3) and that the derivative of the value function is zero when $x \approx 0.83 = \hat{x}$ (see Figure 6.4). This is expected as $x = \hat{x}$ is the minimizer of the value function.

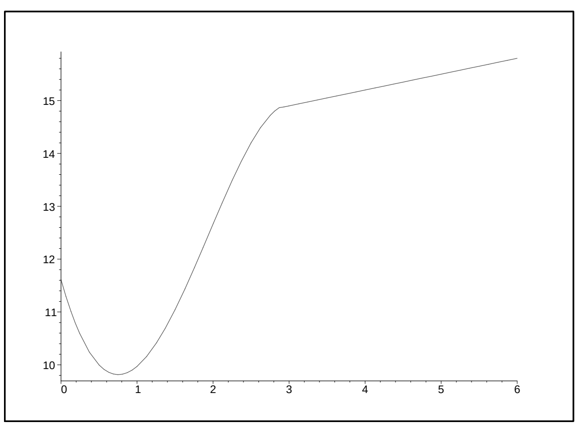

Figure 6.3: *The value function $\psi(x)$ with random reaction time.*

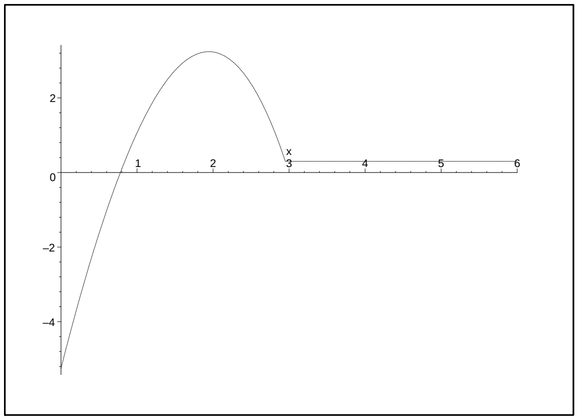

Figure 6.4: *The derivative of the value function $\psi(x)$ with random reaction time.*

6.2.3 Iterative solutions

Constant reaction time

We put $\mu = 0$, $\sigma = 1.0$, $\theta = 0.8$, $\tilde{\mu} = 0$, $\tilde{\sigma} = 1.2$, $\tilde{\theta} = 1$, $C = 0.6$, $\lambda = 0$, $\gamma = 0.2$, $\gamma_+ = 4$, $\gamma_- = 4$, $p = 0.5$, $r = 0.1$ and $T = 0.5$ in the simulation.

Then by using (6.2.2) we compute (with Wolfram Mathematica 7) $\alpha_1 = 0.44365 > 0$ and $\alpha_2 = -0.44365 < 0$.

We solve the system (5.3.35)-(5.3.37) with $a_0 = 0$ to find the values of a_n and $\bar{x}_n$. The results are summarized in Table 6.1.

Random reaction time

We put $\mu = 0$, $\sigma = 1.0$, $\theta = 0.8$, $\tilde{\mu} = 0$, $\tilde{\sigma} = 1.2$, $\tilde{\theta} = 1$, $C = 0.6$, $\lambda = 0$, $\gamma = 0.2$, $\gamma_+ = 4$, $\gamma_- = 4$, $p = 0.5$ and $r = 0.1$ in the simulation. We also assume that T is random and $T \sim Exponential(2)$.

We solve the system again for values of a_n and $\bar{x}_n$. The results are summarized in Table 6.2.

n	a_n	$\bar{x}_n$	ψ_n
1	97.80476	4.805313	3.80
2	101.33450	1.282410	0.27
3	101.51454	0.760730	0.09
4	101.52399	0.723094	0.08
5	101.52448	0.721062	0.08
6	101.52451	0.720956	0.08
7	101.52451	0.720950	0.08
8	101.52451	0.720950	0.08
9	101.52451	0.720950	0.08
10	101.52451	0.720950	0.08

Table 6.1: *Optimal values for constant reaction time.*

n	a_n	$\bar{x}_n$	ψ_n
1	97.79926	4.808877	3.80
2	101.34128	1.266649	0.26
3	101.52295	0.727314	0.08
4	101.53254	0.687244	0.07
5	101.53305	0.685061	0.07
6	101.53307	0.684945	0.07
7	101.53308	0.684940	0.07
8	101.53308	0.684938	0.07
9	101.53308	0.684938	0.07
10	101.53308	0.684938	0.07

Table 6.2: *Optimal values for random reaction time.*

Bibliography

[1] Ait-Sahalia, Y. : *Disentangling Diffusion from Jumps*, Journal of Financial Economics, **74**, 487–528, 2004.

[2] Applebaum, D. : *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2004.

[3] Avesani, R., Gallo, G. : *Jumping in the band: undeclared intervention thresholds in a target zone*, working paper, Economics Department, University of Trento, 1991.

[4] Bardi, M., Capuzzo-Dolcetta, I. : *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, 1997.

[5] Barles, G. : *Solutions de Viscosite des Equations de Hamilton-Jacobi*, Math. and Appl. 17. Springer-Verlag 1994.

[6] Bates, D. : *Jumps and stochastic volatility: the exchange rate processes implicit in Deutschemark options*, Rev. Fin. Studies, 9 (69-107), 1996.

[7] Bekaert, G., Gray, S. : *Target zones and exchange rates: An empirical investigation*, J. Int. Econ., 45 (1-35), 1998.

[8] Bensoussan, A., Lions, J. L. : *Impulse Control and Quasi-Variational Inequalities*, Gauthiers-Villars, Paris, 1984.

[9] Bertoin, J. : *Lévy Processes*, Cambridge University Press, 1996.

[10] Bertola, G. : *Continuous-time models of exchange rate and intervention*, In: van der Ploeg. F. (Ed.), Handbook of International Macroeconomics, Basil Blackwell, London, 1994.

[11] Bertola, G., Caballero, R. : *Target zones and realignments*, Amer. Econ. Rev., 27 (520-536), 1992.

[12] Brekke, K.A., Øksendal , B. : *A verification theorem for combined stochastic control and impulse control*, L. Decreasefond et al (Eds.) Stochastic Analysis and Related Topics Vol. 6, 1997.

[13] Buckley, I. R. C., Korn, R. : *Optimal cash management for equity index tracking in the presence of fixed and proportional costs*, working paper, Centre for Quantitative Finance, Imperial College, 1998.

[14] Cadenillas, A., Zapatero, F. : *Optimal central bank intervention in the foreign exchange market*, J. of Economic Theory, 87(1) (218-242), 1999.

[15] Cadenillas, A., Zapatero, F. : *Classical and impulse stochastic control of the exchange rate using interest rates and reserves*, Mathematical Finance, 10(2) (141-156), 1999.

[16] Chan, T. : *Pricing contingent claims on stocks driven by Lévy processes*, Annals of Appl. Prob., 9 (504-528), 1999.

[17] Constantinides, G. M., Richard, S. F. : *Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time*, Oper. Res., 4 (620-636), 1978.

[18] Cont, R., Tankov, P. : *Financial Modelling with Jump Processes*, Chapman and Hall/CRC, 2004.

[19] Crandall, M.G., Ishii, H., Lions, P.L. : *User?s guide to viscosity solutions of second order partial differential equations*, Bulletin AMS, 27 (1-67), 1992.

[20] Dayanik, S., Karatzas, I. : *On the optimal stopping problem for one-dimensional diffusions*, Stochastic Porocess. Appl., 107 (173-212), 2003.

[21] Dominguez, K., Frankel, J. : *Does Foreign Exchange Intervention Work?*, Institute for International Economics, Washington DC, 1993.

[22] Eastham, J., Hastings, K. : *Optimal impulse control of portfolios*, Math. Oper. Res., 13 (588-605), 1988.

[23] Egami, M. : *Contributions to stochastic optimization applied to financial engineering*, PhD Thesis, 2005.

[24] Ethier, S.N., Kurtz, T.G. : *Markov Processes: Characterization and Convergence*, John Wiley and Sons, 2005.

[25] Fleming, W., Soner, M. : *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, 1993.

[26] Flood, R., Garber, P. : *The linkage between speculative attack and target zone models of exchange rates*, Quarterly Journal of Economics, 106 (1367-1372), 1991.

[27] Froot, K. A., Obstfeld, M. : *Exchange rate dynamics under stochastic regime shifts: A unified approach*, Journal of International Economics, 31 (203-230), 1991.

[28] Garber, P.M., Svensson, L.: *The operation and collapse of fixed exchange rate regime*, In: Rogoff. K., Grossman. G. (Eds.), Handbook of International Economics, North-Holland, Amsterdam, 1996.

[29] Harrison, M. J., Selke, T., Taylor, A. : *Impulse control of a Brownian motion*, Math. Oper. Res., 8 (454-466), 1983.

[30] Ishii, H. : *On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions*, Funkciala Ekvacioj, 38 (101-120), 1995.

[31] Ishikawa, Y. : *Optimal control problem associated with jump processes*, Appl. Math. & Optim., 50 (21-65), 2004.

[32] Jacod, J., Shiryaev, A. : *Limit Theorems for Stochastic Processes*, Springer-Verlag 1987.

[33] Jeanblanc-Picqué, M. : *Impulse control method and exchange rate*, Mathematical Finance, 3 (161-177), 1993.

[34] Jeanblanc-Picqué, M., Shiryaev, A. N. : *Optimization of the flow of dividends*, Russian Math. Survey, 50 (257-277), 1995.

[35] Karatzas, I., Shreve, S. : *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 2nd edition, 2003.

[36] Korn, R. : *Optimal impulse control when control actions have random consequences*, Math. Oper. Res., 22 (639-667), 1997.

[37] Kou, S. : *A jump-diffusion model for option pricing*, Management Science, 48 (1086-1101), 2002.

[38] Krugman, P. : *Trigger strategies and price dynamics in equity and foreign exchange markets*, NBER Working Paper # 2459, 1987.

[39] Krugman, P. : *Target zones and exchange rate dynamics*, Quarterly Journal of Economics 106 (669-682), 1991.

[40] Krylov, N.V. : *Controlled Diffusion Processes*, Springer-Verlag, 1980.

[41] Kunita, H. : *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, 1990.

[42] Kushner, H.J., Dupuis, P. : *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, 1992.

[43] Lapeyre, B., Sulem, A., Talay, D. : *Simulation of Financial Models: Mathematical Foundations and Applications*, Cambridge University Press (to appear).

[44] Makasu, C. : *On Some Optimal Stopping and Stochastic Control Problems with Jump Diffusions*, Ph.D. Thesis, University of Zimbabwe 2002.

[45] Miller, M., Weller, P. : *Currency bands, target zones and price flexibility*, IMF Staff Papers, 38 (184-215), 1991.

[46] Moreno, J. F. : *Impulse Control Problems under Non-constant Volatility*, PhD Thesis, College of Arts and Sciences, Florida State University, 2007.

[47] Mundaca, G., Øksendal, B. : *Optimal stochastic intervention control with application to the exchange rate*, Journal of Mathematical Economics, 29 (225-243), 1998.

[48] Øksendal, B. : *Stochastic control problems where small intervention costs have big effects*, Applied. Math. Optim, 40 (355-375), 1999.

[49] Øksendal, B. : *Stochastic Differential Equations*, Springer-Verlag, New York, 6th edition, 2003.

[50] Øksendal, B. : *Optimal stopping with delayed information*, Stoch. Dyn., 5 (271-280), 2005.

[51] Øksendal, B., Sulem, S. : *Optimal consumption and portfolio with both fixed and proportional transaction costs*, SIAM J. Control and Optim, 40 (1765-1790), 2002.

[52] Øksendal, B., Sulem, S. : *Applied Stochastic Control of Jump Diffusions*, Springer-Verlag, Berlin, Germany, 2004.

[53] Øksendal, B., Ubøe, J., Zhang, T. : *Nonrobustness of some impulse control problems with respect to intervention costs*' Stoch. Anal. and Appl., 20 (999-1026), 2002.

[54] Peskir, G., Shiryaev, A. : *Optimal Stopping and Free-Boundary Problems*, Birkhauser, Basel, Switzerland, 2006.

[55] Protter, P. : *Stochastic Integration and Differential Equations*, Springer Verlag, 2004.

[56] Sato, K. : *Lévy Processes and Infinite Divisibility*, Cambridge University Press, 1999.

[57] Schoutens, W. : *Lévy Processes in Finance*, John Wiley and Sons, 2005.

[58] Sethi, S. P., Thompson G. L. : *Optimal Control Theory, Applications to Management Science and Economics, Second Edition*, Springer, 2000.

[59] Sulem, A. : *Explicit solution of a two-dimensional deterministic inventory problem*, Math. Oper. Res., 11 (134-146), 1986.

[60] Svensson, L. : *The term structure of interest rate differentials in a target zone*, J. Monet. Econ., 28 (87-116), 1991.

[61] Varner, H. : *Some Impulse Control Problems with Applications to Economics*, Cand. Scient. Thesis, University of Oslo, 1997.