UNIQUE DECOMPOSITION OF DIRECT SUMS OF IDEALS

by

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I would like to dedicate this dissertation to my family.

ABSTRACT

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We say that a commutative ring R has the unique decomposition into ideals (UDI) property if, for any R-module which decomposes into a finite direct sum of indecomposable ideals, this decomposition is unique up to the order and isomorphism class of the ideals. In a 2001 paper, Goeters and Olberding characterize the UDI property for Noetherian integral domains. In Chapters 1-3 the UDI property for reduced Noetherian rings is characterized.

In Chapter 4 it is shown that overrings of one-dimensional reduced commutative Noetherian rings with the UDI property have the UDI property, also.

In Chapter 5 we show that the UDI property implies the Krull-Schmidt property for direct sums of torsion-free rank one modules for a reduced local commutative Noetherian one-dimensional ring R.

In Chapter 6 we give some examples of rings which satisfy the UDI property and an example of a ring which does not.

Contents

Chapter 1

Introduction

1.1 BACKGROUND

A class *C* is said to have the *Krull-Schmidt property* if every module in *C* is a direct sum of indecomposable members of *C*, and such a direct decomposition is unique up to isomorphism. The Krull-Schmidt property was first formulated for finite groups by W. Krull and O. Yu. Schmidt. Later on, G. Azumaya showed that the direct sums of countably many indecomposable modules with local endomorphism rings satisfy the Krull-Schmidt property.

In this dissertation we are interested in finite direct sums. We impose two more finiteness conditions: rings are Noetherian and modules are finitely generated. For a commutative Noetherian ring R, the Krull-Schmidt property may not hold for all finitely generated R-modules. For example, if R is a Dedekind domain but not a PID, then for ideals I, J of R, $I \oplus J \cong R \oplus I \cdot J$, so that R fails to have the Krull-Schmidt property.

We restrict our attention to modules which are finite direct sums of ideals. The ring R is said to have *the unique decomposition into ideals (UDI) property*, if for any R-module that decomposes into a finite direct sum of indecomposable ideals, this decomposition is unique apart form the order and isomorphism class of the ideals. In other words, for any indecomposable ideals $I_1, \ldots, I_n, J_1, \ldots, J_m$ of R, if $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$, then $n = m$ and, after reindexing, $I_i \cong J_i$, for each index i.

1.2 NOTATION AND SOME DEFINITIONS

In [6] Goeters and Olberding characterize the UDI property for Noetherian integral domains. Furthermore, they showed that an overring of a Noetherian domain with Krull dimension one inherits the UDI property. They also consider the unique decomposition into submodules of the field of fractions of a one-dimensional Noetherian Domain. In this dissertation a characterization of the UDI property for reduced commutative Noetherian rings is given and it is shown that an overring of such rings satisfies the UDI property, also. Moreover, we investigate the unique decomposition into submodules of the total quotient ring.

Let R be a reduced (i.e. R has no nonzero nilpotent elements) commutative Noetherian ring; our first goal is to determine necessary and sufficient conditions that R have the UDI property. We remark that, if R is isomorphic to $R_1 \times R_2$, then the rings R_1 and R_2 are also reduced (commutative) Noetherian, and R has the UDI property if and only if both R_1 and R_2 have the UDI property. Therefore, it suffices to characterize the UDI property in case R is also assumed to be indecomposable. Thus, throughout this dissertation, R will always denote an indecomposable, reduced, commutative, Noetherian ring.

We fix the following notation. The set of all prime ideals of R is denoted by $\operatorname{Spec}(R)$, and the Zariski topology on $\operatorname{Spec}(R)$ has closed sets consisting of sets of the form $V(I)$, the set of all prime ideals containing I, as I ranges over all ideals of R. Since R is Noetherian, $\operatorname{Spec}(R)$ contains only finitely many minimal primes, which we denote by $P_1, \ldots, P_t$. If any of these minimal prime ideals, say P_1, is also a maximal ideal, then $V(P_1) = \{P_1\}$

would be both open and closed in $\mathrm{Spec}(R)$. Under the assumption that the ring R is indecomposable, the space $\mathrm{Spec}(R)$ is also indecomposable, so $\{P_1\}$ open and closed would make $\mathrm{Spec}(R) = \{P_1\}$; that is, P_1 would be the only prime ideal of R. Since we also assume that R is reduced, this would force $P_1 = 0$ and R to be a field. A field has the UDI property, so to avoid this trivial case, we shall assume that none of the minimal prime ideals $P_1, \ldots, P_t$ is a maximal ideal. We remark that, since R is assumed to be reduced the set of all of its zero-divisors of R is precisely the set $P_1 \cup \cdots \cup P_t$. Therefore, since the minimal prime ideals are all assumed to be non-maximal, it follows that no maximal ideal of R is contained in the set of zero-divisors. Recall that an ideal is called *regular* if it contains a regular element. Thus, every maximal ideal of R must be regular.

Let us fix $R_i = R/P_i$ for each index i. Because R is reduced, $P_1 \cap \cdots \cap P_t = 0$, so the natural map $\phi : R \to R_1 \times \cdots \times R_t$ is injective. We usually identify R with its isomorphic image $\phi(R)$. Also let $R_{P_i} = Q_i = Q(R_i)$, the field of fractions of R_i, and let $\tilde{R}_i$ be the integral closure of R_i in Q_i, for each index i. Then the total quotient ring of R is the ring $Q = Q_1 \times \cdots \times Q_t$, and the integral closure of R is the ring $\tilde{R} = \tilde{R}_1 \times \cdots \times \tilde{R}_t$.

We say that the R-module G is *torsion-free* if no nonzero element is annihilated by a regular element of R. The *rank* of G is the t-tuple $\mathrm{rank}(G) = (r_1, \ldots, r_t)$, where r_i is the rank of the Q_i-vector-space $G_{P_i} = G \otimes R_{P_i}$, for each index i.

This dissertation is organized as follows. In Chapter 2 we characterize the UDI property by showing that the ring R has the UDI property if and only if R has at most one nonprincipal maximal ideal, and R has the UDI property locally at every maximal ideal. In Chapter 3 we give an explicit description of the local rings R with the UDI property. Here our answer is more complicated than the corresponding result in [6], which is one of three cases in our Theorem 21. In Chapter 4 we show that if R has Krull dimension one, and satisfies the UDI property, then any overring of R has the UDI property also. In Chapter 5 we prove that for R local and one-dimensional, if R satisfies the UDI property, then R

has the unique decomposition into R-submodules of Q. In Chapter 6 we provide some examples of rings which satisfy the UDI property, including a non-local ring of dimension two with the UDI property.

1.3 SOME FACTS

We note the following well-known facts which we will find helpful throughout this dissertation. For completeness, we sketch the proofs.

Recall that, for ideals I and J of R, the colon ideal $[I : J]$ is defined to be $\{q \in Q : qJ \subseteq I\}$.

Lemma 1. *Let I and J be ideals of the ring R, with J regular. Then the natural map from $[I : J]$ to $\mathrm{Hom}_R(J, I)$ is an isomorphism.*

Proof. Let $\theta : [I : J] \to \mathrm{Hom}_R(J, I)$ be defined by $\theta(q) = \mu_q$, multiplication by q. Since J is regular, clearly θ is injective. Let $\gamma \in \mathrm{Hom}_R(J, I)$, and fix a regular element $x \in J$. For any $y \in J$, $y \cdot \gamma(x) = \gamma(yx) = x \cdot \gamma(y)$, so that $\gamma = \theta(x^{-1}\gamma(x))$, and hence θ is surjective. □

Recall that a ring R' is called an *overring* of R if $R \subseteq R' \subseteq Q$.

Lemma 2. *If A and B are torsion-free R'-modules, where R' is an overring of R, then $\mathrm{Hom}_{R'}(A, B) = \mathrm{Hom}_R(A, B)$.*

Proof. Certainly any R'-homomorphism is also an R-homomorphism. Conversely, suppose that $\phi : A \to B$ is an R-module homomorphism. Since $R \subseteq R' \subseteq Q$, for any $s \in R'$, we can write $s = xy^{-1}$ for some $x, y \in R$, with y regular in R, so that $ys = x$. For each $a \in A$, since $sa \in A$, we get

$$ys\phi(a) = x\phi(a) = \phi(xa) = \phi(ysa) = y\phi(sa).$$

Now, since y is regular and B is a torsion-free R-module, $s\phi(a) = \phi(sa)$, making ϕ an R'-module homomorphism. □

Lemma 3. *Let P be a prime ideal of R and $Q(R_P)$ the total quotient ring of R_P. Then $Q_P \subseteq Q(R_P)$.*

Proof. Let S_1 be the set of all the regular elements of R and $S_2 = R - P$. Then $Q_P = S_2^{-1}(S_1^{-1}R) = (S_1S_2)^{-1}R = (\bar{S}_1)^{-1}(S_2^{-1}R) = (\bar{S}_1)^{-1}(R_P)$, where $\bar{S}_1$ denotes the image of S_1 in R_P. Since regular elements of R remain regular in R_P, we get that $\bar{S}_1$ is contained in the set of regular elements of R_P, and hence $(\bar{S}_1)^{-1}(R_P) \subseteq Q(R_P)$. □

Chapter 2

Reduction to the Local Case

We remind the reader that, throughout this chapter, R is assumed to be a reduced commutative Noetherian (indecomposable) ring but not a field.

We begin with the key observation concerning the UDI property.

Proposition 4. *If R has the UDI property, then at most one maximal ideal of R is nonprincipal.*

Proof. Suppose that M_1 and M_2 are two distinct maximal ideals of R. Let $\rho : M_1 \oplus M_2 \to R$ such that $\rho(x, y) = x + y$ for $x \in M_1$, $y \in M_2$. Since M_1 and M_2 are comaximal, ρ is surjective and hence splits because R is projective, so that $M_1 \oplus M_2 \cong R \oplus (M_1 \cap M_2)$. Since R is indecomposable and has the UDI property, R must be isomorphic to a summand of M_1 or M_2, say $M_1 = Rt \oplus J$ for some regular element $t \in R$ and some ideal $J \subseteq R$. Then $Rt \cap J = 0$, implies that $tJ = 0$, and since t is regular, it follows that $J = 0$, so that M_1 is principal. Therefore, R can have no more than one nonprincipal maximal ideal. □

We can already see that the UDI property is not a local property. Indeed, if F is a field, then the polynomial ring $F[X, Y]$ in two indeterminates certainly has more than one nonprincipal maximal ideal, so by Proposition 4, $F[X, Y]$ does not have the UDI property. On the other hand, since $F[X,Y]$ is integrally closed, the localization $F[X,Y]_N$ is integrally closed for each maximal ideal N of $F[X,Y]$. Thus, the endomorphism ring of each nonzero ideal of $F[X,Y]_N$ must be $F[X,Y]_N$ itself, and hence by [2, Theorem I.3.6], $F[X,Y]_N$ has the UDI property for each maximal ideal N.

In this chapter we show that if R has a unique nonprincipal maximal ideal M, and if R_M has the UDI property, then R has the UDI property. Before we prove this fundamental result, we collect several useful facts.

Lemma 5. *The following hold for the ring R.*

(1) For every principal maximal ideal $N \subseteq R$, the localization R_N is a DVR, and N properly contains a unique prime ideal, necessarily a minimal prime of R.

(2) If every maximal ideal of R principal, then R is a PID.

(3) If R has a nonprincipal maximal ideal, then every minimal prime ideal of R is contained in a nonprincipal maximal ideal. In particular, if R has a unique nonprincipal maximal ideal M, then M contains all the minimal prime ideals of R, so that M contains all of the zero-divisors of R.

(4) If R has a unique nonprincipal maximal ideal M, then $\operatorname{rank}(G) = \operatorname{rank}(G_M)$ for every finitely generated torsion-free R-module G.

(5) Let G and H be finitely generated torsion-free R-modules. If R has a nonprincipal maximal ideal, and if G and H are locally isomorphic at each nonprincipal maximal ideal, then $\operatorname{rank}(G) = \operatorname{rank}(H)$.

Proof. (1) Since R is reduced, R_N must be reduced also. Thus, if N is a principal maximal ideal of R, then R_N is a local Noetherian ring with maximal ideal N_N generated by a non-nilpotent element, so that R_N is a DVR [13, Proposition I.2.2]. Therefore, the maximal ideal N properly contains only one prime ideal, which must be a minimal prime of R.

(2) Suppose that all maximal ideals of R are principal. From (1) it follows that each maximal ideal of R contains a unique minimal prime ideal, so that $V(P_1)$ is an open and closed subset of $\mathrm{Spec}(R)$. Because R is assumed to be indecomposable, the space $\mathrm{Spec}(R)$ is connected, and hence P_1 is the only minimal prime of R. Since R is reduced, $P_1 = 0$, and hence R is a domain. Because R is a Noetherian domain and locally a DVR, R is a Dedekind domain. Since all of the maximal ideals of R are assumed to be principal, R is a PID.

(3) If a minimal prime ideal P_j is contained in no nonprincipal maximal ideals, then $V(P_j)$ is open and closed in $\mathrm{Spec}(R)$, so R would be a PID by the same argument as in (2). Since R is assumed to have at least one nonprincipal maximal ideal, it follows that each minimal prime ideal of R is contained in at least one nonprincipal maximal ideal. In particular, if R has a unique nonprincipal maximal ideal M, then M contains all the minimal prime ideals of R, so that, since $P_1 \cup \cdots \cup P_t$ is the set of all zero-divisors of R, it follows that M contains all the zero-divisors of R.

(4) By part (3), all minimal prime ideals are contained in M, so it follows that $\mathrm{rank}((G_M)_P) = \mathrm{rank}(G_P)$ for each minimal prime P of R, and hence $\mathrm{rank}(G_M) = \mathrm{rank}(G)$.

(5) Let P be a minimal prime ideal of R; by part (3), P is contained in a nonprincipal maximal ideal M. By assumption $G_M \cong H_M$, so that $\mathrm{rank}(G_P) = \mathrm{rank}((G_M)_P) = \mathrm{rank}((H_M)_P) = \mathrm{rank}(H_P)$. Since this holds for every minimal prime ideal P, $\mathrm{rank}(G) = \mathrm{rank}(H)$. □

Recall that $P_1, ..., P_t$ are the minimal primes of R.

Lemma 6. *Let I be the intersection of any minimal prime ideals of R and $\rho : R \to R/I$ be the natural map. Set $\bar{R} = R/I$. If R has the UDI property, then $\bar{R}$ has the UDI property.*

Proof. Suppose that $I = P_1 \cap \cdots \cap P_s$ for $1 \leqslant s \leqslant t$. Let $\bar{J}$ be an ideal of $\bar{R}$, so that $\bar{J} = J/I$ for some ideal J of R. We claim that $\bar{J}$ is isomorphic to an ideal of R. Let

$z \in (P_{s+1} \cup \cdots \cup P_t) - (P_1 \cap \cdots \cap P_s)$, so that $\bar{z}$ is a regular element of $\bar{R}$. Now, ρ maps zJ onto $\bar{z}\bar{J}$. Since $z \cdot I = 0$, it follows that $(zJ \cap I)^2 = 0$, so, since R is reduced, $zJ \cap I = 0$, and hence $zJ \cap \mathrm{Ker}(\rho) = 0$. Thus, $\rho|_{zJ}$ is an isomorphism onto $\bar{z}\bar{J}$, and therefore $\bar{z}\bar{J} \cong zJ$ as R-modules. Since $\bar{z}$ is regular in $\bar{R}$, $\bar{z}\bar{J} \cong \bar{J}$ as $\bar{R}$-modules, so as R-modules.

Finally, let $\bar{I}_1 \oplus \cdots \oplus \bar{I}_n \cong \bar{J}_1 \oplus \cdots \oplus \bar{J}_m$ for some (indecomposable) ideals $\bar{I}_1, \ldots, \bar{I}_n$, $\bar{J}_1, \ldots, \bar{J}_m$ of $\bar{R}$. Each $\bar{I}_i$ and each $\bar{J}_j$ is isomorphic to an ideal of R, and they remain indecomposable as R-modules. Since R has the UDI property, it follows that $n = m$ and after reindexing, $\bar{I}_j \cong \bar{J}_j$ for each index j, as R-modules, so also as $\bar{R}$-modules.

□

We say that a ring R is of *finite character* if every regular element of R is contained in only finitely many maximal ideals of R. If R is an integral domain of finite character, and if every nonzero prime ideal of R is contained in a unique maximal ideal, then R is called an *h-local domain*.

Proposition 7. *If R has only finitely many nonprincipal maximal ideals, then R is of finite character.*

Proof. Let a be a regular element in R, and suppose that $a \in N$, where N is a principal maximal ideal of R. By Lemma 5(1), N contains exactly one prime ideal properly, which is a minimal prime ideal, say P. But since a is regular, $a \notin P$, and hence N/Ra is both a maximal ideal and a minimal prime ideal of R/Ra. Since R/Ra has only finitely many minimal prime ideals, a can be contained in only finitely many principal maximal ideals and hence in only finitely many maximal ideals of R. □

Corollary 8. *If R has a unique nonprincipal maximal ideal, then for every prime ideal $P \subseteq R$, R/P is an h-local domain. In particular, if R has the UDI property, then each R_i is an h-local domain.*

Proof. Suppose that R has a unique nonprincipal maximal ideal, and let P be a prime ideal of R. Then R/P has at most one nonprincipal maximal ideal, so, by Proposition 7, R/P is of finite character. Each nonzero prime ideal of R/P is an image of a non-minimal prime ideal of R. Hence, a nonzero prime ideal of R/P is either a principal maximal ideal or it is contained in the unique nonprincipal maximal ideal. Therefore, R/P is an h-local domain.

Now suppose that R has the UDI property. By Lemma 4, R has at most one nonprincipal maximal ideal. If such a maximal ideal exists, then each R_i is an h-local domain, by the first part of this corollary. If every maximal ideal of R is principal, then R is a PID by Lemma 5(2), and hence $R_i = R$ is an h-local domain. □

Two torsion-free R-modules G and H are said to be *nearly isomorphic* if, for each regular ideal I, there exists an embedding $f : G \to H$ such that the ideal $\mathrm{Ann}_R(\mathrm{Coker}(f))$ is comaximal with I. The R-modules G and H are called *locally isomorphic* if $G_M \cong H_M$ for all maximal ideals M of R. It is easy to see that, if G and H are torsion-free nearly isomorphic R-modules, then G and H are locally isomorphic. We show that the converse holds for finitely generated torsion-free R-modules in case R has only finitely many nonprincipal maximal ideals.

Proposition 9. *Suppose that R has at least one but only finitely many nonprincipal maximal ideals and that G and H are finitely generated torsion-free R-modules. If, for each nonprincipal maximal ideal $M \subseteq R$, $G_M \cong H_M$, then G and H are nearly isomorphic.*

Proof. Let I be a regular ideal, and $N_1, \ldots, N_v$ be the nonprincipal maximal ideals and $N_{v+1}, \ldots, N_{v+k}$ the principal maximal ideals of R that contain I. Since $N_1, \ldots, N_{v+k}$ are distinct maximal ideals of R, we can write $R = N_j + \prod_{i \neq j} N_i$ for each index j, so that $1 = a_j + b_j$ with $a_j \in N_j$ and $b_j \in \prod_{i \neq j} N_i$. By assumption, $\mathrm{rank}(G_M) = \mathrm{rank}(H_M)$ for each nonprincipal maximal ideal M, so by Lemma 5(5), $\mathrm{rank}(G) = \mathrm{rank}(H)$. By Lemma 5(1), R_{N_i} is a DVR for each index i, $v + 1 \leq i \leq v + k$, so each G_{N_j} is a free R_{N_j}-module implying that $G_{N_i} \cong H_{N_i}$. Since G is finitely presentable by [7, Proposition 1.10, p 95], for each index j, $1 \leq j \leq v+k$, there exists a map $f_j : G \to H$ such that $(f_j)_{N_j}$ is an isomorphism. Then we can write $f_j = h_j + g_j$, where $h_j = f_j \cdot a_j$ and $g_j = f_j \cdot b_j$. Let $g = g_1 + \cdots + g_{v+k}$. Since $\mathrm{Im}((h_j)_{N_j}) \subseteq N_j H_{N_j}$, by Nakayama's Lemma the map $(g_j)_{N_j}$ is surjective for each index j. For $i \neq j$, $\mathrm{Im}((g_i)_{N_j}) \subseteq N_j H_{N_j}$ by choice of b_i, so by Nakayama's Lemma again, g_{N_j} must be surjective, for each index j. Since G is a Noetherian R-module and $G_{N_j} \cong H_{N_j}$, it follows that g_{N_j} is an isomorphism, for each index j.

We claim that g is injective. For each index j, $1 \leq j \leq v+k$, the map g_{N_j} is an isomorphism from G_{N_j} onto H_{N_j}, so that $\mathrm{Ker}(g_{N_j}) = 0$. Thus, $(\mathrm{Ker}(g))_{N_j} = 0$ for each index j, which implies that $(\mathrm{Ker}(g))_P = 0$ for every minimal prime ideal P of R, by Lemma 5(3). Therefore, $\mathrm{Ann}_R(\mathrm{Ker}(g))$ is not contained in any minimal prime ideal of R, so that $\mathrm{Ann}_R(\mathrm{Ker}(g))$ contains a regular element, and hence $\mathrm{Ker}(g) = 0$, since G is torsion-free. That is, $g : G \to H$ is an injection such that g_{N_j} is an isomorphism for each index j. Thus, $(\mathrm{Coker}(g))_{N_j} = 0$, where $\mathrm{Coker}(g)$ is a finitely generated R-module, and hence $\mathrm{Ann}_R(\mathrm{Coker}(g))_{N_j} = R_{N_j}$ for each index j. That is, $\mathrm{Ann}_R(\mathrm{Coker}(g))$ is not contained in any of the maximal ideals $N_1, \ldots, N_{v+k}$, which include all the maximal ideals that contain I. Therefore, I and $\mathrm{Ann}_R(\mathrm{Coker}(g))$ are comaximal. □

We show that local isomorphism reflects decompositions for finitely generated torsion-free R-modules in case R has only finitely many nonprincipal maximal ideals.

Proposition 10. *Suppose that R has only finitely many nonprincipal maximal ideals, and A, B, and C are finitely generated torsion-free R-modules. If A and $B \oplus C$ are locally isomorphic, then $A = B' \oplus C'$ for some submodules B' and C' locally isomorphic to B and C, respectively.*

Proof. If R has no nonprincipal maximal ideals, then by Lemma 5(2) R is a PID, so that

A, B, and C are free R-modules, and the result is clear. Therefore, we can assume that R has at least one nonprincipal maximal ideal.

Let $M_1, \ldots, M_k$ be the nonprincipal maximal ideals of R. Then by Proposition 9, there is an embedding $f : A \to B \oplus C$ such that $M_1 \cdots M_k$ and $\mathrm{Ann}_R(\mathrm{Coker}(f))$ are relatively prime. Let $g = \pi \circ f$ where π is the projection map $\pi : B \oplus C \to B$, and $I = \mathrm{Im}(g)$ and $K = \mathrm{Ker}(g)$. We note that f_{M_i} is surjective for each index i, because $M_1 \cdots M_k$ and $\mathrm{Ann}_R(\mathrm{Coker}(f))$ are relatively prime. Then $I_{M_i} = B_{M_i}$ for each i, since $I_{M_i} = g(A)_{M_i} = g_{M_i}(A_{M_i}) = (\pi \circ f)_{M_i}(A_{M_i}) = \pi_{M_i}(B_{M_i} \oplus C_{M_i}) = B_{M_i}$. Thus, $\mathrm{rank}(I_{M_i}) = \mathrm{rank}(B_{M_i})$ for each index i, and hence $\mathrm{rank}(I) = \mathrm{rank}(B)$, by Lemma 5(5). Since, for all principal maximal ideals N of R, the ring R_N is a DVR (Lemma 5(1)), it follows that I_N and B_N are free modules of the same rank and hence isomorphic; therefore I is locally isomorphic to B. Moreover, for each principal maximal ideal N of R, since R_N is a DVR and I_N is a finitely generated torsion-free R_N-module, I_N is projective, so the map g_N splits. On the other hand, for each nonprincipal maximal ideal M of R, $g_M = \pi_M \circ f_M$, where f_M is an isomorphism, while π_M is a split surjection, so that the map g_M also splits. Therefore, the map g splits, because $I = \mathrm{Im}(g)$ is finitely presented. Thus, $A \cong I \oplus K$, where I is locally isomorphic to B. By [4, Theorem 2], locally at each maximal ideal of R we can cancel those summands from both sides of the isomorphism, so that K is locally isomorphic to C. □

Lemma 11. *Suppose that R has a unique nonprincipal maximal ideal M. Let G and H be finitely generated torsion-free modules, with H isomorphic to a direct sum of indecomposable ideals of R. If $G_M \cong H_M$, then $G \cong H$.*

Proof. Let us first assume that H is an indecomposable ideal of R. Given that $G_M \cong H_M$, there exists a map $f : G \to H$ such that f_M is an isomorphism. Then $\mathrm{Ker}(f_M) = (\mathrm{Ker}(f))_M = 0$ implies that $s \cdot \mathrm{Ker}(f) = 0$ for some $s \in R - M$. Since s is regular by Lemma 5(3), and $\mathrm{Ker}(f)$ is torsion-free, it follows that $\mathrm{Ker}(f) = 0$, and hence f is injective. Set $I = \mathrm{Ann}_R(H/\mathrm{Im}(f))$, and note that $(H/\mathrm{Im}(f))_M = 0$, so that $I \nsubseteq M$. By Lemma 13, $I = tR$ for some $t \in I - M$, so that t is regular by Lemma 5(3).

We claim that $R/tR \cong H/tH$. By Lemma 7, t is contained in only finitely many maximal ideals, all of which are principal. If P is a maximal ideal such that $t \in P$, then R_P is a DVR, and $t \in I$ implies that $H_P \neq 0$, so that $R_P \cong H_P$, and hence $R_P/tR_P \cong H_P/tH_P$. Thus, if we write $\bar{P} = P/tR$, then $(R/tR)_{\bar{P}} \cong (H/tH)_{\bar{P}}$ for each maximal ideal $\bar{P}$ of R/tR, and therefore H/tH is a locally free R/tR-module of rank one. Since R/tR is semi-local, H/tH is free [11, Theorem 4.30], proving the claim.

From the above claim, and because $tH \subseteq \mathrm{Im}(f)$, there is a surjection $\alpha : R/tR \to H/\mathrm{Im}(f)$. Since $tR = \mathrm{Ann}_R(H/\mathrm{Im}(f))$, it follows that $\mathrm{Ker}(\alpha) = tR$, and hence $H/tH \cong R/tR \cong H/\mathrm{Im}(f)$. We note that R/tR is Artinian, so that H/tH and H/$\mathrm{Im}(f)$ have the same finite length. Then $tH \subseteq \mathrm{Im}(f)$ forces $\mathrm{Im}(f) = tH$, so that $G \cong \mathrm{Im}(f) = tH \cong H$.

In the general case, write $H = A \oplus B$, where A is an indecomposable ideal of R. By Lemma 10 we have $G = A' \oplus B'$ for some submodules A' and B' such that $A'_M \cong A_M$ and $B'_M \cong B_M$. By the first part of the proof, $A' \cong A$ and $B' \cong B$, so by induction on the rank the lemma follows.

□

We call an R-module G *completely decomposable* if G is isomorphic to a finite direct sum of indecomposable ideals of R.

Theorem 12. *Suppose that M is the unique nonprincipal maximal ideal of R. Let G and H be finitely generated torsion-free modules, with H completely decomposable. The following are equivalent.*

(1) $G_M \cong H_M$.

(2) G and H are locally isomorphic.

(3) G and H are nearly isomorphic.

(4) G and H are isomorphic.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3) follows immediately from Proposition 9.

(3) $\Rightarrow$ (2) was noted in the remark preceding Proposition 9.

(4) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (4) follows immediately from Lemma 11.

□

The set of all invertible fractional ideals of a ring S forms a group and the set of all principal fractional ideals of S is a subgroup of it. The Picard group of S is defined as the quotient group of the fractional invertible ideals of S by the subgroup of principal fractional ideals, and we denote it by $\mathrm{Pic}(S)$.

We show that UDI property implies a trivial Picard group.

Lemma 13. *Suppose R has only finitely many nonprincipal maximal ideals. If an ideal I of R is not contained in any of the nonprincipal maximal ideals, then I is principal. Moreover, $\mathrm{Pic}(R)$ is trivial.*

Proof. If R has no nonprincipal maximal ideals, then R is a PID by Lemma 5(2), so the current lemma holds. Therefore, we can suppose that R has at least one nonprincipal maximal ideal.

Let $N_1, \ldots, N_k$ be all of the nonprincipal maximal ideals of R, and suppose that the ideal I is not contained in any of $N_1, \ldots, N_k$, so that there exists an element $a \in I - (N_1 \cup \cdots \cup N_k)$. Note that a is regular, by Lemma 5(3). For any maximal ideal P of R such that $a \notin P$ (in particular, for each N_j), we have $R_P = aR_P \subseteq I_P \subseteq R_P$, so that $aR_P = I_P$. Since a is regular, by Proposition 7 there are only finitely many maximal ideals P of R

such that $a \in P$. Hence, there can be only finitely many maximal ideals P of R such that $aR_P \neq I_P$, that is $aR_P \subsetneq I_P$. If there are no such maximal ideals, then because aR and I are locally equal at every maximal ideal, they are equal. Otherwise, fix a (principal) maximal ideal P such that $I_P \nsubseteq aR_P$, and set $P = \pi R$ for some $\pi \in P$. Thus, $a = \pi b$ for some $b \in R$. Since a is regular in R, it follows that b is regular as well, and hence $aR \subsetneq bR$, because otherwise π would be a unit. Now, we claim that $bR \subseteq I$. For all maximal ideals N such that $N \neq P$, π becomes a unit in R_N, so that $bR_N = aR_N \subseteq I_N$. Therefore, it suffices to prove the claim locally at the fixed maximal ideal P. Since R_P is a DVR (Lemma 5(1)), and $PR_P = \pi R_P$ is its maximal ideal, we can write $I_P = \pi^n R_P$ and $aR_P = \pi^m R_P$ for some positive integers m, n. Then $m > n$, because $aR_P \neq I_P$, and hence $bR_P = \pi^{m-1} R_P \subseteq \pi^n R_P \subseteq I_P$. Thus, $bR \subseteq I$, as claimed. Set $a_1 = a$ and $a_2 = b$, and apply the same procedure inductively (so long as $a_i R \nsubseteq I$), to get an ascending chain of ideals $a_1 R \subsetneq a_2 R \subsetneq \ldots \subseteq I$. Since R is Noetherian, the procedure must terminate with $a_n R = I$ for some n.

Finally, we claim that $\mathrm{Pic}(R)$ is trivial. Let J be an invertible ideal of R, so that J is a projective regular ideal. Thus, J is locally free, so that the trace map $\mathrm{Hom}_R(J, R) \otimes_R J \to R$ is locally surjective at all maximal ideals of R. Hence the trace map is surjective, so there exist $f_j \in \mathrm{Hom}_R(J, R)$ such that $f_j(J) \nsubseteq N_j$, for each index j. Since $N_1, \ldots, N_k$ are distinct maximal ideals of R, we can write $R = N_j + \prod_{i \neq j} N_i$, for each index j, so that $1 = a_j + b_j$ with $a_j \in N_j$ and $b_j \in \prod_{l \neq j} N_i - N_j$. Let $f = b_1 f_1 + \cdots + b_k f_k$. Since $\mathrm{Im}(a_j f_j) \subseteq N_j$ and $\mathrm{Im}(b_j f_j) \nsubseteq N_j$, it follows that $\mathrm{Im}(f) \nsubseteq N_j$ for each index j. Therefore, by the part of the lemma already proven, $\mathrm{Im}(f)$ is principal, say $\mathrm{Im}(f) = f(x)R$ for some $x \in J$. Since $\mathrm{Im}(f)$ is not contained in any N_j, $\mathrm{Im}(f)$ is not contained in any minimal prime ideal, so by Lemma 5(3), $f(x)R$ is regular. Thus, $\mathrm{rank}(f(x)R_P) = 1$ for every minimal prime ideal P. Since J is also regular, by comparing the ranks we conclude that f is injective. Therefore, f is an isomorphism, so that $J \cong \mathrm{Im}(f) = f(x)R$ is principal. □

Theorem 14. *If R has the UDI property, then $\mathrm{Pic}(R) = 0$, and R_N has the UDI property for each maximal ideal N of R.*

Proof. By Lemma 4, R has at most one nonprincipal maximal ideal, so either R has a unique nonprincipal maximal ideal, or R is a PID. In each case, by Lemma 13, $\mathrm{Pic}(R) = 0$. Moreover, if $N \subseteq R$ is a principal maximal ideal, then by Lemma 5(1), R_N is a DVR, so R_N has the UDI property. Therefore, we can suppose that M is the unique nonprincipal maximal ideal of R, and we show that R_M has the UDI property.

Let $A'_1, \ldots, A'_t, B'_1, \ldots, B'_s$ be indecomposable ideals of R_M such that $A'_1 \oplus \cdots \oplus A'_t \cong B'_1 \oplus \cdots \oplus B'_s$. Let $A_j = A'_j \cap R$ and $B_i = B'_i \cap R$ for each index j and each index i. Since every ideal of R_M is an extended ideal, we get that $A'_j = A_j R_M$ and $B'_i = B_i R_M$ for each index j and each index i. Moreover, A'_j and B'_i indecomposable implies that each A_j and B_i is also indecomposable. Then $G = A_1 \oplus \cdots \oplus A_t$ and $H = B_1 \oplus \cdots \oplus B_s$ are finitely generated torsion-free R-modules such that $G_M \cong H_M$. Then $G \cong H$ by Lemma 11, and hence $t = s$, and $A_j \cong B_j$ for each index j (reordering if necessary). Therefore, $A'_j \cong B'_j$ for each index j. □

In general, the converse of Theorem 14 need not hold. For example, consider the polynomial ring $F[X, Y]$, where F is a field. Since $F[X, Y]$ is integrally closed, $F[X, Y]_N$ is integrally closed for each maximal ideal N of $F[X, Y]$. Thus, the endomorphism ring of each nonzero ideal of $F[X, Y]_N$ must be $F[X, Y]_N$ itself, and hence by [2, Theorem I.3.6], $F[X, Y]_N$ has the UDI property for each maximal ideal N. Moreover, as is well-known, every finitely generated projective $F[X, Y]$-module is free (for example, see [12, Theorem 4.59], so that $\mathrm{Pic}(F[X, Y])$ is trivial. On the other hand, $F[X, Y]$ certainly has has more than one nonprincipal maximal ideal, so by Lemma 4, $F[X, Y]$ does not have the UDI property.

The UDI property is not quite a local property. Our main theorem of this section shows that the UDI property can be characterized locally, provided that the ring has at most one nonprincipal maximal ideal.

Theorem 15. *R has the UDI property if and only if R has at most one nonprincipal maximal ideal, and R_P has the UDI property for every maximal ideal P of R.*

Proof. If R has the UDI property, then by Lemma 4, it has at most one nonprincipal maximal ideal, and, by Theorem 14, R_N has the UDI property for every maximal ideal N.

Conversely, suppose that R has at most one nonprincipal maximal ideal, and R_P has the UDI property for every maximal ideal P of R. If R has no nonprincipal maximal ideal, then R is a PID, and hence R has the UDI property. Therefore, we can suppose that R has a unique nonprincipal maximal ideal M, and that R_M has the UDI property.

Suppose that $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$ for some indecomposable ideals $I_1, \ldots, I_n$, $J_1, \ldots, J_m$ of R. Note that, if J is an indecomposable ideal of R, then J_M must be an indecomposable ideal of R_M. For if $J_M \cong X' \oplus Y'$ for some ideals X' and Y' of R_M, then setting $X = R \cap X'$ and $Y = R \cap Y'$, as above we obtain $J_M \cong X_M \oplus Y_M \cong (X \oplus Y)_M$. From Lemma 11 it follows that $J \cong X \oplus Y$, so that $X = 0$ or $Y = 0$, and hence $X' = 0$ or $Y' = 0$, forcing J_M to be indecomposable as well. Therefore, since each $(I_i)_M$ and $(J_j)_M$ is indecomposable as an R_M-module, and R_M is assumed to have the UDI property, it follows that $n = m$, and reindexing if necessary, $(I_j)_M \cong (J_j)_M$ for each index j. By Lemma 11 again, $I_j \cong J_j$ for each index j.

□

Chapter 3

Local Rings with the UDI Property

Having reduced characterization of the UDI property for reduced commutative Noetherian rings to the case in which the ring is local, in this chapter we give a precise description of the reduced, commutative, Noetherian, local rings which have the UDI property. Thus, throughout this chapter we assume that R is a reduced, commutative, Noetherian, local ring with maximal ideal M.

We begin by recalling a fundamental result [6, Theorem 3.2].

Theorem 16. *Assume that R is a local Noetherian domain with maximal ideal M, and let $\tilde{R}$ be the integral closure of R in its field of fractions. Then R has the UDI property if and only if there exists a fractional overring R' of R with the same number of maximal ideals as $\tilde{R}$, such that one of the following holds.*

(1) R' is local.

(2) R' has exactly two distinct maximal ideals M_1', M_2' such that M_1' is principal, with $M \nsubseteq (M_1')^2$ and $R'/M_1' \cong R/M$.

(3) $R' = \tilde{R}$ has exactly three distinct maximal ideals M_1', M_2', M_3', such that all three are principal, with $M \nsubseteq (M_j')^2$ and $R'/M_j' \cong R/M$ for each index j.

Recall that $P_1, \ldots, P_t$ are the minimal prime ideals of our ring R, and we set $R_i = R/P_i$ for each index i. We view $R \subseteq R_1 \times \cdots \times R_t$, and $Q = Q_1 \times \cdots \times Q_t$ is the total quotient ring of R, where Q_i is the field of fractions of R_i for each index i. The integral closure of R is written as $\tilde{R} = \tilde{R}_1 \times \cdots \tilde{R}_t$, where $\tilde{R}_i$ is the integral closure of R_i in Q_i for each index i.

In order to characterize the UDI property in the case in which the local ring R need not be a domain, we note a few cases in which UDI fails.

Lemma 17. *If $t = 2$ and $\tilde{R}_2$ has at least three distinct maximal ideals, then R does not have the UDI property.*

Proof. Given that $\tilde{R}_2$ has at least three distinct maximal ideals, we note that there exists a finitely generated overring R_2' of R_2 which has at least three distinct maximal ideals M_1', M_2', M_3', and we let M_1 be a maximal ideal of R_1. We observe that R_2' is a fractional ideal of R. Let $A = R + (M_1 \times M_1')$ and $B = R_2 + M_2'M_3'$; then A and B are finitely generated R-submodules of Q and hence fractional ideals of R. Now, let us consider the map

$$f : A \oplus B \to R_2'$$

defined by $f((r,s),s') = s + s'$. The map f is split by the map

$$g : R_2' \to A \oplus B$$

defined by $g(t) = ((0,ta),tb)$, for elements $a \in M_1'$ and $b \in M_2'M_3'$ such that $a + b = 1$. Thus, $\mathrm{Ker}(f) \oplus R_2' \cong A \oplus B$.

Since A and B are multiplicatively closed, they are rings. In particular, $A = \mathrm{End}_A(A)$. Since A is an overring of R, it follows from Lemma 2 that $\mathrm{End}_A(A) = \mathrm{End}_R(A)$, and hence $A = \mathrm{End}_R(A)$. Similarly, B is an overring of $R_2 = R/P_2$, so that $B = \mathrm{End}_B(B) = \mathrm{End}_{R_2}(B) = \mathrm{End}_R(B)$.

The nontrivial idempotent elements of $\tilde{R}$ are $(1,0)$ and $(0,1)$. Let $M_2 = M_1' \cap R_2$. If $(0,1) \in A$, then we could write $(0,1) = (r + m_1, s + m_1')$ for some elements $(r,s) \in R$, $m_1 \in M_1$, and $m_1' \in M_1'$. Then $s + m_1' = 1$ implies that $s = 1 - m_1' \notin M_2$, while $r + m_1 = 0$ implies that $r = -m_1 \in M_1$. On the other hand, R is local (with maximal ideal M) mapping onto the local rings R_1 and R_2 (with maximal ideals M_1 and M_2, respectively), so that either $(r,s) \in M$, in which case $r \in M_1$ and $s \in M_2$, or $(r,s) \notin M$, in which case $r \notin M_1$ and $s \notin M_2$. This contradiction shows that $(0,1) \notin A$. Since $(1,1) \in A$, but $(0,1) \notin A$, it follows that $(1,0) \notin A$, and hence A is an indecomposable fractional ideal of R. Clearly B is also an indecomposable fractional ideal of R.

By considering ranks, each of A, B, R_2', and $\operatorname{Ker}(f)$ is a fractional R-ideal, so if R had the UDI property, comparison of ranks would force $R_2' \cong B$ as R-modules, and hence as R_2-modules. We show that this leads to a contradiction, so that R cannot have the UDI property.

We claim that B is an R_2'-module. Since $R_2 \subseteq Q_2$, for any $s' \in R_2'$ we can write $s' = r_1 s_1^{-1}$ for some $r_1, s_1 \in R_2$. Let $\theta : R_2' \to B$ be an R_2-module isomorphism, and for $b \in B$, write $b = \theta(r')$ for some $r' \in R_2'$. Since R_2 is a domain, it follows that

$$s_1 s' b = r_1 \theta(r') = \theta(r_1 r') = \theta(s_1 s' r') = s_1 \theta(s' r'),$$

so that $s'b = \theta(s'r') \in B$. Thus, B is an R_2'-module, so that R_2' and B are isomorphic as R_2'-modules by Lemma 2, and hence B is an ideal of R_2'. But B is a ring, so that $1 \in B$, and therefore $B = R_2'$. Since $M_2 \subseteq M_2' \cap M_3' = M_2' M_3'$, where $M_2 = M/P_2$ is maximal, it follows that $M_2 = M_2' M_3' \cap R_2$, so that

$$R_2'/M_2'M_3' = (R_2 + M_2'M_3')/M_2'M_3' \cong R_2/M_2$$

contradicting the fact that $M_2'M_3'$ is not a prime ideal in R_2'. Therefore, R cannot have the UDI property. □

Lemma 18. *If $t = 2$ and $\tilde{R}_1$ and $\tilde{R}_2$ each has at least two distinct maximal ideals, then R does not have the UDI property.*

Proof. Given that each of the integral closures $\tilde{R}_1$ and $\tilde{R}_2$ has at least two distinct maximal ideals, as above there exist finitely generated overrings R_1' of R_1 and R_2' of R_2, with distinct maximal ideals M_1', M_2' and N_1', N_2', respectively. Also as above, R_1' and R_2' are fractional ideals of R. Let us consider the map

$$f : (R + M_1' \times N_1') \oplus (R + M_2' \times N_2') \to R_1' \oplus R_2'$$

defined by $f((r, s), (r', s')) = (r + r', s + s')$. The map f is split by the map

$$g : R_1' \oplus R_2' \to (R + M_1' \times N_1') \oplus (R + M_2' \times N_2')$$

defined by $g(t, t') = ((ta, t'b), (ta', t'b'))$, for elements $a \in M_1'$ and $a' \in M_2'$ such that $a + a' = 1$, and $b \in N_1'$ and $b' \in N_2'$ such that $b + b' = 1$.

Therefore, $\mathrm{Ker}(f) \oplus R_1' \oplus R_2' \cong (R + M_1' \times N_1') \oplus (R + M_2' \times N_2')$. Since the rank of $\mathrm{Ker}(f)$ is at most 1 at each minimal prime ideal, it is a fractional ideal of R. By a similar argument as in Lemma 17, $R + M_1' \times N_1'$ and $R + M_2' \times N_2'$ are indecomposable fractional ideals of R. But their ranks are different than the rank of R_1', so it follows that R cannot have the UDI property. □

Lemma 19. *If $t \geq 3$ and $\tilde{R}_1$ has at least two distinct maximal ideals, then R does not have the UDI property.*

Proof. Let $\nu_1 : R \to R_1 \times R_2$ and $\nu_2 : R \to R_1 \times R_3$ be the projection maps, with $R' = \mathrm{Im}(\nu_1)$ and $R'' = \mathrm{Im}(\nu_2)$. There exists a finitely generated overring R_1' of R_1 with at least two distinct maximal ideals M_1' and M_2'; let M_2 and M_3 be the maximal ideals of R_2 and R_3, respectively. As above, R_1' is a fractional ideal of R. Now, let us consider the map

$$f : (R' + M_1' \times M_2) \oplus (R'' + M_2' \times M_3) \to R_1'$$

defined by $f((r,s),(r',s')) = r + r'$. The map f is split by the map

$$g : R_1' \to (R' + M_1' \times M_2) \oplus (R'' + M_2' \times M_3)$$

defined by $g(t) = ((ta, 0), (ta', 0))$, for elements $a \in M_1'$ and $a' \in M_2'$ such that $a + a' = 1$. It follows that $\mathrm{Ker}(f) \oplus R_1' \cong (R' + M_1' \times M_2) \oplus (R'' + M_2' \times M_3)$, and comparing ranks shows that $\mathrm{Ker}(f)$ is a fractional ideal of R. By a similar argument as in Lemma 17, one can show that $R' + M_1' \times M_2$, and $R'' + M_2' \times M_3$ are indecomposable fractional ideals of R, and their ranks are different than that of R_1', so that R fails to have the UDI property. □

The next lemma will allow us to replace the ring R by a homomorphic image in which a given ideal becomes regular.

Lemma 20. *Suppose that R has a unique nonprincipal maximal ideal M, and let J be a nonzero non-regular indecomposable ideal of R. Let $\psi : R \to R/\mathrm{Ann}_R(J)$ be the natural map, and set $\bar{R} = R/\mathrm{Ann}_R(J)$, $\bar{M} = \psi(M) = (M + \mathrm{Ann}_R(J))/\mathrm{Ann}_R(J)$, and $\bar{J} = (J + \mathrm{Ann}_R(J))/\mathrm{Ann}_R(J)$. The following all hold.*

(1) $\mathrm{Ann}_R(J) \subseteq M$, so that $\bar{M} = M/\mathrm{Ann}_R(J)$.

(2) $\bar{R}$ is reduced.

(3) $\bar{J} \cong J$ as R-modules.

(4) $\bar{J}$ is an indecomposable regular ideal of $\bar{R}$.

Proof. (1) Since $\mathrm{Ann}_R(J)$ consists of zero-divisors and M contains all the zero-divisors of R, $\mathrm{Ann}_R(J) \subseteq M$.

(2) If $r^n \in \mathrm{Ann}_R(J)$, for some n, then $r^n J^n = (rJ)^n = 0$, which implies $rJ = 0$, since R is reduced, so $r \in \mathrm{Ann}_R(J)$.

(3) Since R is reduced, $J \cap \mathrm{Ann}_R(J) = 0$, implying that $\bar{J} = \psi(J) = (J + \mathrm{Ann}_R(J))/\mathrm{Ann}_R(J) \cong J/(\mathrm{Ann}_R(J) \cap J) \cong J$ as R-modules.

(4) Let $\bar{x} \in \bar{R}$ such that $\bar{x}\bar{J} = 0$. Then $xJ \subseteq \mathrm{Ann}_R(J)$, so $xJ \cdot J = 0$ implies that $xJ = 0$, since R is reduced. Thus, $x \in \mathrm{Ann}_R(J)$, and hence $\bar{x} = 0$; that is $\bar{J}$ is regular. Since $\bar{J} \cong J$ and J is indecomposable, it follows that $\bar{J}$ is indecomposable also. □

Theorem 21. *Let R be a reduced, commutative, Noetherian, local ring with maximal ideal M. Then R has the UDI property if and only if one of the following holds.*

(1) $t = 1$, and R satisfies one of the conditions of Theorem 16.

(2) $t = 2$, $\tilde{R}_1$ is local with maximal ideal $\tilde{N}$, and $\tilde{R}_2$ is finite over R_2 and a PID with exactly two maximal ideals $\tilde{M}_1$ and $\tilde{M}_2$, such that $M \nsubseteq \tilde{R}_1 \times (\tilde{M}_i)^2$ and $\tilde{R}_2/\tilde{M}_i \cong R/M$ for $i = 1, 2$.

(3) $t \geqslant 2$, and $\tilde{R}_i$ is local for all i, $1 \leqslant i \leqslant t$.

Proof. Suppose first that R has the UDI property. If $t = 1$, then (1) holds by Theorem 16, so we can suppose that $t \geq 2$. If every $\tilde{R}_i$ is local, then (3) holds, so we can suppose also that at least one $\tilde{R}_i$ is non-local. By Lemma 19, it must be the case that $t = 2$. By Lemma 18, one of $\tilde{R}_1$ or $\tilde{R}_2$ must be local, say $\tilde{R}_1$. By Lemma 17, the ring $\tilde{R}_2$ must have exactly two distinct maximal ideals $\tilde{M}_1$ and $\tilde{M}_2$. As noted above, there exists a finite overring R_2' of R_2, such that R_2' has exactly two distinct maximal ideals M_1' and M_2', where $M_i' = R_2' \cap \tilde{M}_i$ for each index i.

Suppose that X is an R_2-submodule of R_2' such that $(M_2')^2 \subseteq X$, so that $m_1 + x = 1$ for some $m_1 \in M_1'$ and $x \in (M_2')^2 \subseteq X$. We observe that X is a fractional ideal of the integral domain R_2, so that X is an indecomposable fractional R-module. Then consider the homomorphism

$$f : (R + M_1') \oplus X \to R_2'$$

defined by $f((s, r), r') = r + r'$. The map f is split by the map

$$g : R_2' \to (R + M_1') \oplus X$$

defined by $g(r) = ((0, m_1 r), xr)$, so that $\mathrm{Ker}(f) \oplus R_2' \cong (R + M_1') \oplus X$. By a similar argument as in Lemma 17, we note that $R + M_1'$ is an overring of R as well as an indecomposable fractional ideal of R. Furthermore, R_2' is a fractional ideal of R, because R_2' is a finitely generated R_2-submodule of Q_2 and hence R-submodule of Q. Moreover, by considering ranks, $\mathrm{Ker}(f)$ must be a fractional ideal of R as well. Since R has the UDI property, by comparing ranks, we see that $X \cong R_2'$ as R-modules and hence as R_2-modules. As shown in Lemma 17, this forces X to be an R_2'-module as well, so that $X = yR_2'$ for some $y \in X$. We shall apply this construction several times.

First, by choosing $X = M_2'$, we see that M_2' must be a principal ideal of R_2'. Reversing the roles of M_1' and M_2', we see that M_1' is also principal, and hence R_2' is a PID. Since R_2' is integrally closed, it follows that $R_2' = \tilde{R}_2$.

Next, for any R_2-module X such that $M_2' \subseteq X \subsetneq R_2'$, we see that X must be an ideal of R_2', so that, by the maximality of M_2', $X = M_2'$. Therefore, R_2'/M_2' is a simple R_2-module, so it is a one-dimensional vector space over R_2/M_2, where $M_2 = M_2' \cap R_2$. Thus, the inclusion $R_2 \hookrightarrow R_2'$ induces an isomorphism $R_2/M_2 \cong R_2'/M_2'$, and hence $R_2'/M_2' \cong R_2/M_2 \cong R/M$. Similarly, $R_2'/M_1' \cong R/M$.

Finally, we can choose $X = R_2 + (M_2')^2$, an R_2-submodule of R_2'. Then since $1 \in X$, and X is an ideal of R_2', it follows that $X = R_2'$. Let us write $M_2' = \pi R_2'$ for some $\pi \in M_2'$. Then $\pi \in X$ implies that $\pi = r + s\pi^2$ for some $r \in R_2$ and $s \in R_2'$, so that $r = \pi - s\pi^2 \in R_2 \cap M_2' = M_2$. Since $\pi \notin R_2'\pi^2$, it follows that $r \notin (M_2')^2$, and hence $M \nsubseteq \tilde{R}_1 \times (M_2')^2$. Similarly, $M \nsubseteq \tilde{R}_1 \times (M_1')^2$.

Conversely, let us suppose that one of (1), (2) or (3) holds. We consider each case in turn.

If (1) holds, then by Theorem 16, R has the UDI property.

Suppose instead that (2) holds, and that $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$ for some indecomposable ideals $I_1, \ldots, I_n, J_1, \ldots, J_m$ of R. If some $\mathrm{End}_R(I_k)$ is local, then by

[2, Lemma I.3.4], $I_k \cong J_l$ for some index l, so by local cancellation [4, Theorem 2], we can cancel I_k and J_l from the isomorphism. Thus, by induction we can assume that each endomorphism ring $\mathrm{End}_R(I_k)$ and $\mathrm{End}_R(J_l)$ is non-local.

Note that, if I is a nonzero ideal of R, then $\mathrm{rank}(I)$ must be $(1,0), (0,1)$, or $(1,1)$. If $\mathrm{rank}(I) = (1,0)$, then $\mathrm{Ann}_R(I) = P_1$, and $I \cong \bar{I} = (I + P_1)/P_1$ is an ideal of $\bar{R} = R/P_1 = R_1$. Hence $\mathrm{End}_R(I) \cong \mathrm{End}_{\bar{R}}(\bar{I})$ is a finite overring of R_1, which makes it local, since $\tilde{R}_1$ is local. Thus, we assume that none of the remaining summands I_k and J_l have rank $(1,0)$.

On the other hand, if $\mathrm{rank}(I) = (0,1)$, then $\mathrm{Ann}_R(I) = P_2$, and as above, $I \cong \bar{I} = (I + P_2)/P_2$ is an ideal of $\bar{R} = R/P_2 = R_2$. Let $R' = \mathrm{End}_{\bar{R}}(\bar{I}) \cong \mathrm{End}_R(I)$, so that $R_2 \subseteq R' \subseteq \tilde{R}_2$; we claim that $R' = \tilde{R}_2$. Let $M_i' = R' \cap \tilde{M}_i$ for $i = 1, 2$, so that $M_1' \neq M_2'$, since R' is assumed to be non-local. For each index i, $R/M = R_2/M_2 \subseteq R'/M_i' \subseteq \tilde{R}_2/\tilde{M}_i \cong R/M$ implies that $\tilde{R}_2 = R' + \tilde{M}_i$. Moreover, since $M \not\subseteq (\tilde{R}_1 \times \tilde{M}_i)^2$, it follows that $M_i' \not\subseteq \tilde{M}_i^2$. We note that since $\tilde{R}_2$ is a PID with exactly two maximal ideals, each $(\tilde{R}_2)_{M_i'}$ is a DVR, and $M_i'(\tilde{R}_2)_{M_i'} \not\subseteq (\tilde{M}_i)^2_{M_i'}$. Thus, $(\tilde{M}_i)_{M_i'} = M_i'(\tilde{R}_2)_{M_i'}$, and hence $(\tilde{R}_2)_{M_i'} = R'_{M_i'} + M_i'(\tilde{R}_2)_{M_i'}$. Therefore, by Nakayama's Lemma, since $\tilde{R}_2$ is finitely generated as an R_2-module, $R'_{M_i'} = (\tilde{R}_2)_{M_i'}$ for each index i, from which it follows that $R' = \tilde{R}_2$, as claimed. Thus $I \cong \bar{I}$, an ideal of the PID $\tilde{R}_2$, so that $I \cong \tilde{R}_2$ as $\tilde{R}_2$-modules, and hence as R-modules.

In the direct sums $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$, we assume that each indecomposable summand has rank $(0,1)$ or $(1,1)$ and has a non-local endomorphism ring. Since the sums of the ranks are equal on both sides, there are the same number of ideals of rank $(0,1)$ on both sides. We just showed that each such ideal is isomorphic to $\tilde{R}_2$ as an R-module, so by local cancellation [4, Theorem 2], we can cancel those summands from both sides of the isomorphism. Therefore, we can further assume that each I_k and each J_l has rank $(1,1)$ and hence is regular.

Let us fix the indecomposable regular ideal $I = I_1$ with non-local endomorphism ring $R' = \mathrm{End}_R(I)$. The ring $\tilde{R} = \tilde{R}_1 \times \tilde{R}_2$ has exactly three maximal ideals, $\tilde{N} \times \tilde{R}_2, \tilde{R}_1 \times \tilde{M}_1$, and $\tilde{R}_1 \times \tilde{M}_2$. Let $N' = R' \cap (\tilde{N} \times \tilde{R}_2)$, $M'_i = R' \cap (\tilde{R}_1 \times \tilde{M}_i)$, for $i = 1, 2$. We claim that $N' = M'_1$ or $N' = M'_2$ (but, since R' is non-local, not both). Let us assume, by the way of contradiction, that $N' \neq M'_1$ and $N' \neq M'_2$. The ring $\tilde{R}$ has only two minimal primes, $\tilde{P}_1 = \{0\} \times \tilde{R}_2$ and $\tilde{P}_2 = \tilde{R}_1 \times \{0\}$, so that $\tilde{R}_i = \tilde{R}/\tilde{P}_i$ for each index i. Let $P'_i = R' \cap \tilde{P}_i$ for each index i, so that P'_1 and P'_2 are the (distinct) minimal primes of R'. Now $\tilde{P}_1$ is contained in only one maximal ideal, $\tilde{N} \times \tilde{R}_2$, so $P'_1 \subseteq N'$. Similarly, $\tilde{P}_2$ is contained in two distinct maximal ideals $\tilde{R}_1 \times \tilde{M}_1$ and $\tilde{R}_1 \times \tilde{M}_2$, so that $P'_2 \subseteq M'_1, M'_2$. Since $\tilde{R}$ is integral over R', it follows by "going-up" that N' is the only maximal ideal of R' containing P'_1, and M'_1 and M'_2 are the only maximal ideals of R' containing P'_2. Thus, $\mathrm{Spec}(R')$ decomposes as the disjoint union of $V(P'_1)$ and $V(P'_2)$, which makes R' decomposable, and hence I would be decomposable, contrary to assumption. This proves the claim, so we can assume that $N' = M'_1 \neq M'_2$. As argued above, one can show that $(\tilde{M}_2)_{M'_2} = (M'_2\tilde{R}_2)_{M'_2}$. Moreover, $(\tilde{R}_2)_{M'_2}$ is a finitely generated $R'_{M'_2}$-module, since $\tilde{R}_2$ is finite over R_2, so by Nakayama's Lemma, $(\tilde{R}_2)_{M'_2} = R'_{M'_2}$. Therefore, $R'_{M'_2}$ is a DVR, so that $M'_2 R'_{M'_2}$ is principal.

Without loss of generality, we can assume that $R' = \mathrm{End}_R(I_1)$ is minimal with respect to inclusion among all of the endomorphism rings $\mathrm{End}_R(I_k)$ and $\mathrm{End}_R(J_l)$. Let us regard $L = I_1 \oplus \cdots \oplus I_n = J_1 \oplus \cdots \oplus J_n$ as two internal decompositions of L into direct sums of ideals. Now using the inclusion maps $\iota_j : I_j \to L$ and $\iota'_j : J_j \to L$ and the projection maps $\pi_j : L \to I_j$ and $\pi'_j : L \to J_j$, the identity map on I_1 is $\pi_1\iota_1 = \sum_j \pi_1\iota'_j\pi'_j\iota_1$, where each composition $\pi_1\iota'_j\pi'_j\iota_1$ is in R'. Since $R'_{M'_1}$ is a local ring, one of these compositions must be a unit in $R'_{M'_1}$; we can assume that it is $\pi_1\iota'_1\pi'_1\iota_1$. Let $f = \pi_1\iota'_1$, an R-homomorphism from J_1 to I_1; by assumption $f(J_1)R'_{M'_1} = (I_1)_{M'_1}$. Using the fact that $M'_1 \neq M'_2$, we can choose $a \in M'_2 - (M'_1 \cup P'_2 \cup (M'_2)^2)$, which makes a regular. Since $R'_{M'_2}$ is a DVR, there exists $n \geqslant 0$ such that $f(J_1)R'_{M'_2} = a^n(I_1)_{M'_2}$. Now a is regular, so we can set $h = a^{-n}f$, which is a

Q-endomorphism of Q. Since a is a unit in $R'_{M'_1}$, $h(J_1)R'_{M'_1} = f(J_1)R'_{M'_1} = (I_1)_{M'_1}$, while $h(J_1)R'_{M'_2} = (I_1)_{M'_2}$ by choice of n. Then $h(J_1)R'$ and I_1 are locally equal at both maximal ideals of R', so that $h(J_1)R' = I_1$. We can view $\mathrm{Hom}_R(J_1, I_1)$ as an R-submodule of Q, since I_1 and J_1 are regular (Lemma 1), so let $U = \mathrm{Hom}_R(J_1, I_1)J_1$, an R'-submodule of I_1. Since $h \in \mathrm{Hom}_R(J_1, I_1)$ and $h(J_1) \subseteq U$, it follows that $U = I_1$. If we set $R'' = \mathrm{End}_R(J_1)$, then U is an R''-module, which makes I_1 an R''-module, so that $R'' \subseteq R'$. By minimality of R', it follows that $R' = R''$. Thus, by Lemma 2, $I_1 = h(J_1)R' = h(J_1R') = h(J_1)$, which makes h surjective as an R'-map, so as an R-map, by again Lemma 2. The ranks are equal, so h must also be injective, and hence $h : J_1 \to I_1$ is an R-isomorphism. Therefore, by local cancellation [4, Theorem 2] and induction, it follows that R has the UDI property.

Finally, let us suppose that (3) holds. We claim that every indecomposable ideal of R has local endomorphism ring. Let I be an indecomposable ideal of R, and set $\bar{R} = R/\mathrm{Ann}_R(I)$, and $\bar{I} = (I + \mathrm{Ann}_R(I))/\mathrm{Ann}_R(I)$, a regular ideal of $\bar{R}$. If I is non-regular, then by Lemma 20(3), $I \cong \bar{I}$ as R-modules, so that $\mathrm{End}_R(I) \cong \mathrm{End}_{\bar{R}}(\bar{I})$. Moreover, we can assume that $\mathrm{Ann}_R(I) = P_1 \cap \cdots \cap P_s$ for some $s \leqslant t$, from which it follows that the minimal primes of $\bar{R}$ are $\bar{P}_1, \ldots, \bar{P}_s$, where $\bar{P}_i = P_i/\mathrm{Ann}_R(I)$. Then $\bar{R}/\bar{P}_i \cong R/P_i = R_i$ has local integral closure $\bar{R}_i$ in its field of fractions Q_i for each index i, $1 \leqslant i \leqslant s$. Thus, replacing I by $\bar{I}$ and R by $\bar{R}$, we can assume that I is an indecomposable regular ideal of R.

Let $R' = \mathrm{End}_R(I)$, an overring of R. For each index i, let $\tilde{P}_i$ be the kernel of the natural map from $\tilde{R}$ onto $\tilde{R}_i$, so that $\tilde{P}_1, \ldots, \tilde{P}_t$ are the distinct minimal primes of $\tilde{R}$. Moreover, their contractions to R yield the distinct minimal primes of R, namely, $P_i = R \cap \tilde{P}_i$ for each index i. Let $P'_i = R' \cap \tilde{P}_i$ for each index i; then the ideals $P'_1, \ldots, P'_t$ must be the distinct minimal primes of R'. Now each $\tilde{P}_i$ is contained in only one maximal ideal, since $\tilde{R}/\tilde{P}_i = \tilde{R}_i$ is assumed to be local. Since $\tilde{R}$ is integral over R', by "going-up" each minimal prime ideal of R' is contained in a unique maximal ideal as well. If M' is an arbitrary maximal ideal of R', we can number the minimal primes of R' so that $P'_1, \ldots, P'_r \subseteq M'$ while $P'_{r+1}, \ldots, P'_t \nsubseteq M'$, for some integer r, $1 \leqslant r \leqslant t$. Then $V(P'_1) \cup \cdots \cup V(P'_r)$ is both open and closed in $\operatorname{Spec}(R')$, because each minimal prime of R' is contained in a unique maximal ideal. Since R' must be an indecomposable ring, because I is an indecomposable module, it follows that $r = t$, and every minimal prime of R' is contained in M'. Thus, M' is the unique maximal ideal of R', so that R' is local, as claimed.

Since each indecomposable ideal of R has a local endomorphism ring over R, by [2, Proposition V.3.4], R has the UDI property. □

Chapter 4

UDI for Overrings

Recall that $P_1, \ldots, P_t$ are the minimal prime ideals of R, and set $R_i = R/P_i$ for each index i. We view $R \subseteq R_1 \times \cdots \times R_t$. The integral closure of R is $\tilde{R} = \tilde{R}_1 \times \cdots \tilde{R}_t$, where $\tilde{R}_i$ is the integral closure of R_i, and $Q = Q_1 \times \cdots \times Q_t$ is the total quotient ring of R, where Q_i is the field of fractions of $\tilde{R}_i$ for each index i. We say that D is an overring of R if $R \subseteq D \subseteq Q$. In this chapter we show that D inherits the UDI property from R if R is one-dimensional. Before we restrict our attention to one-dimensional rings, we note that if D is finitely generated as an R-module, then the dimension does not matter.

Proposition 22. *Let R' be an overring of R, and suppose that R' is finitely generated as an R-module. If R has the UDI property, then so does R'.*

Proof. Suppose that $A_1 \oplus A_2 \cdots \oplus A_n \cong B_1 \oplus B_2 \cdots \oplus B_m$, for indecomposable ideals $A_1, \ldots, A_n, B_1, \ldots, B_m$ of R'. Since R' is a finitely generated overring of R, every ideal of R' becomes a finitely generated fractional ideal of R. Since A_j is indecomposable as an R'-module, $\operatorname{End}_{R'}(A_j)$ contains no non-trivial idempotents. By Lemma 2, $\operatorname{End}_{R'}(A_j) = \operatorname{End}_R(A_j)$ does not contain any non-trivial idempotents either, so each A_j remains indecomposable as an R-module. Similarly, each B_j remains indecomposable as an

R-module. Since R has the UDI property, $m = n$, and renumbering if necessary, $A_j \cong B_j$ as R-modules for all indices j. But these isomorphisms are also R'-homomorphisms, again by Lemma 2, and hence each $A_j \cong B_j$ as R'-modules. □

Before we proceed with the proof of our main result, we collect some useful results. The following lemmas most maximal ideals in overrings must be principal.

Lemma 23. *Let R' be an overring of R and N a maximal ideal of R' which contracts to a principal maximal ideal of R. Then N is principal.*

Proof. Let $P = N \cap R$ be the principal maximal ideal, so that $P = xR$ for some $x \in P$, and let $Q(R_P)$ be the ring of fractions of R_P. By Lemma 3, the natural map from Q_P into $Q(R_P)$ is injective, which makes R'_P into an overring of R_P. Since P is principal and R is a reduced Noetherian ring, R_P is a DVR, by Lemma 5(1), so that $Q(R_P)$ is a field. Therefore, $R'_P = R_P$ or $R'_P = Q(R_P)$. Now, N_P is a proper ideal of R'_P, since N is a prime ideal of R' and disjoint from $R - P$. Also, $PR_P \subseteq N_P$ and $PR_P \neq 0$, so that R'_P cannot be a field, and hence $R'_P = R_P$.

We claim that $N = xR'$. If $M \neq P$ is a maximal ideal of R, then since $x \notin M$, it follows that x is a unit in R'_M, while $x \in P \subseteq N$ implies that $R'_M = xR'_M \subseteq N_M \subseteq R'_M$, and hence $N_M = xR'_M$. On the other hand, $P_P = (xR)_P = xR_P = xR'_P \subseteq N_P \subsetneq R'_P$, where P_P is the maximal ideal of $R_P = R'_P$, so that $P_P = N_P$, and therefore $N_P = xR'_P$. Since N and xR' are locally equal at every maximal ideal of R, they are equal, and hence N is principal. □

Lemma 24. *Suppose that R has a unique nonprincipal maximal ideal, M. Let R' be an overring of R and N be a maximal ideal of R' lying over M. If N_M is principal, then N is principal.*

Proof. Let $N_M = aR'_M$, where we can suppose that $a \in N$. Since N is an R-module, $a = ts^{-1}$ for some $s, t \in R$, with s regular in R, and $t \in M$. We claim that a is regular in

R'. Since $M = R \cap N$ is regular in R, and $R \subseteq R' \subseteq Q$, it follows that N is regular in R'. Thus, N_M is regular in R'_M, and hence a is regular in R'_M. Suppose that $ar' = 0$ for some $r' \in R'$. Then $ar' = 0$ in R'_M, so that $r' = 0$ in R'_M, and therefore, $r's' = 0$ in R' for some $s' \in R - M$. But s' is regular by Lemma 5(3), so that $r' = 0$ in R', and hence a is a regular element in R', as claimed. Since a and s are regular, t must be regular also. Thus, by Proposition 7, there are only finitely many maximal ideals P of R such that $s \in P$ or $t \in P$. We note that, if $s, t \notin P$, then s and t both become units in R'_P, so that, at such maximal P, we get $aR'_P = R'_P = N_P$.

Fix a principal maximal ideal $P = \pi R$, for some $\pi \in P$, such that $s \in P$ or $t \in P$. If π divides both s and t, then since π is regular in R and R is Noetherian, we can continue cancelling until at least one of s or t is not divisible by π. It remains to consider the following cases.

Case 1. If $\pi | s$ and $\pi \nmid t$, then t becomes a unit in R_P, which makes s a unit in R'_P, since $s^{-1} = ts^{-1} \cdot t^{-1} \in R'_P$. So $aR'_P = R'_P = N_P$ in this case.

Case 2. Suppose instead that $\pi \nmid s$ and $\pi | t$. Since R is Noetherian, we can suppose that n is the largest positive integer such that $\pi^n | t$, so that, for some $t' \in R - P$, $\pi^n t' = t$. Since $t \in M$ and $\pi \notin M$, it follows that $t' \in M$. Moreover, $N_M = aR'_M = \pi^n t' s^{-1} R'_M = t' s^{-1} R'_M$, so we can replace a by $t' s^{-1}$, and we still have $a \in M$ and $N_M = aR'_M$, but now $aR'_P = R'_P = N_P$.

Repeating this procedure for the finitely many principal maximal ideals containing s or t, we obtain $aR'_P = R'_P = N_P$ for every principal maximal ideal P of R. Also, since $aR'_M = N_M$, it follows that aR' and N are locally equal at every maximal ideal of R. Therefore, $N = aR'$ is a principal ideal. □

If D is a finitely generated R-module, then as noted in the proof of Proposition 22 every ideal of D is isomorphic to an ideal of R. In case D is not necessarily finitely generated, but R is a one-dimensional Noetherian domain, D inherits the UDI property from R by [6, Proposition 4.2].

We say that a ring is a one-dimensional Cohen-Macaulay ring if it is a Noetherian ring of Krull dimensional one, and if every maximal ideal of R contains a regular element. As shown in Chapter 1, every maximal ideal of a reduced commutative ring is regular. Therefore, if R is a reduced commutative Noetherian ring with Krull dimension one, then R is a one-dimensional Cohen-Macaulay ring. This fact allows us to apply the results of [8].

Theorem 25. *Suppose that R is a one-dimensional reduced Noetherian ring. If R has the UDI property, then so does any overring D of R.*

Proof. Let D be any overring of R. Then D is Noetherian and one-dimensional as well [14, Theorem 4.9.2].

If D is decomposable, say $D \cong D' \times D''$, then D' and D'' are overrings of two (one-dimensional, reduced, commutative, Noetherian) homomorphic images R' and R'', respectively of R. Since R' and R'' have the UDI property by Lemma 6, it suffices to prove the theorem in case D itself is indecomposable, which we now assume.

We claim that D has at most one nonprincipal maximal ideal and has the UDI property locally at every maximal ideal, so that by Theorem 14 it follows that D has the UDI property.

Since R has the UDI property, R has at most one nonprincipal maximal ideal. If all the maximal ideals of R are principal, then R is a PID, so that D is a PID by [3, Corollary 5.3]. Therefore we can suppose that R has a unique nonprincipal maximal ideal M.

Let N be a maximal ideal of D, and note that $N \cap R$ is a prime ideal of R. Since R is one-dimensional, $N \cap R$ is a minimal prime or a maximal ideal of R. Suppose that $N \cap R = P$ is a minimal prime ideal in R. Since the injection from Q_P to $Q(R_P) = R_P$

makes D_P into an overring of R_P, $D_P = R_P = Q_P$ is a field. Since $ND_P = 0$, no prime ideal of D is properly contained in N. Thus, N is a minimal prime ideal of D. Since D is indecomposable and reduced, this implies D is a field, as argued in Chapter 1, so D has the UDI property. So, we can assume that $N \cap R$ is maximal in R. If N is itself a principal, D_N is a DVR and has the UDI property. So, we can assume that N is a nonprincipal maximal ideal of D. By Lemmas 23 and 24 for N to be a nonprincipal maximal ideal, $N \cap R = M$ and N_M must be nonprincipal. Thus, we only need to worry about whether D_M has at most one nonprincipal maximal ideal, and whether or not D_M has the UDI property locally at this maximal ideal. So, we replace R by R_M and D by D_M, and assume that R is local.

Let us denote the maximal divisible R-submodule of D/R by $h(D/R)$. Then $h(D/R) = D'/R$, where D' is an R-module such that $R \subseteq D' \subseteq D$. By [8, Corollary 3.5] D' is an overring of R.

We note that Q/R is a torsion divisible module. By [8, Theorem 5.5] Q/R is an Artinian R-module. Since D' is an overring of R, then D'/R is an R-submodule of Q/R, which makes D'/R an Artinian R-module. So, applying [8, Corollary 5.2], we get that $(D/R)/h(D/R)$ is finitely generated as an R-module. Thus, D/D' is a finitely generated R-module, and hence D is finitely generated as D'-module. If we can show that D' has the UDI property, then it follows that D has the UDI property by Proposition 22. Therefore, without loss of generality, we suppose that D/R is divisible, so that $D = D'$. By [8, Theorem 3.4], for an arbitrary maximal ideal N of D, $N = D(N \cap R) = DM$, so that N is the unique maximal ideal of D. If we can show that the overring D_N of R_M has the UDI property, then by Theorem [1, Theorem 2.5] so is D.

The minimal prime ideals of $Q = Q_1 \times \cdots \times Q_t$ are in the form $P_i' = \prod_{i \neq j} Q_j$. Since Q is reduced, the intersection of its minimal prime ideals is 0. Let $P_i = P_i' \cap R$, for each index i. Since the intersection of these prime ideals is 0, there are no containment relations between them. Therefore, P_i are the minimal prime ideals of R. Let $P_i'' = P_i' \cap D$, for each index i. Since the intersection of their contractions in R is 0, $\cap_{i=1}^{t} P_i'' = 0$. Thus, there are no containment relations among P_i'', for each i. So, $P_1'', \ldots, P_t''$ are the minimal prime ideals of D. Therefore, $D \hookrightarrow D_1 \times \cdots \times D_t$ where $D_i = D/P_i''$, for each index i. Therefore, $\tilde{D} = \tilde{D}_1 \times \cdots \times \tilde{D}_t$ where $\tilde{D}_i$ is the integral closure of D_i for each i. Now we break the argument into the three cases of Theorem 21.

If $t = 1$ holds, then by Proposition [6, Proposition 4.2], D has the UDI property.

Suppose instead that Theorem 21(2) holds. Since $\tilde{R}_1$ is local and R_1 is one-dimensional Noetherian domain, $\tilde{R}_1$ is a DVR. Thus, $\tilde{D}_1 = \tilde{R}_1$ or $\tilde{D}_1 = Q_1$, so that $\tilde{D}_1$ is local. Now let us consider $\tilde{D}_2$. Since $\tilde{R}_2$ is a PID with two maximal ideals, $\tilde{D}_2$ is one of $\tilde{R}_2$, $(\tilde{R}_2)_{\tilde{M}_1}$, $(\tilde{R}_2)_{\tilde{M}_2}$, or Q_2. If $\tilde{D}_2$ is local, then by 21(1) D has the UDI property. Suppose that $\tilde{D}_2 = \tilde{R}_2$. Let $\tilde{N}_i$ be a maximal ideal of $\tilde{D}_2$ for each $i = 1, 2$. Since D is an overring of R, $R/M \subseteq D/N$. By Theorem 21(2), $R/M \cong \tilde{R}_2/\tilde{M}_i = \tilde{D}_2/\tilde{N}_i$, so that $D/N \cong \tilde{D}_2/\tilde{N}_i$, for each index i. Since $M \not\subseteq \tilde{R}_1 \times (\tilde{M}_i)^2$ and $M \subseteq N$, $N \not\subseteq \tilde{D}_1 \times (\tilde{N}_i)^2$, for each index i. Therefore, by Theorem 21(2), D has the UDI property.

Finally, suppose that $t \geq 2$ and $\tilde{R}_i$ is local for each i. Since $\tilde{R}_i$ is local for each i and R_i is one-dimensional Noetherian domain, $\tilde{R}_i$ is a DVR. Then $\tilde{D}_i$ is a DVR or $\tilde{D}_i = Q_i$. Thus, $\tilde{D}_i$ is local, for each index i. By Theorem 21 (3), D has the UDI property. $\square$

Chapter 5

Unique Decomposition For Direct Sums of Rank One Modules

Recall that for an R-module G, the ***rank*** of G is the t-tuple $\mathrm{rank}(G) = (r_1, \ldots, r_t)$, where r_i is the rank of the Q_i-vector-space $G_{P_i} = G \otimes R_{P_i}$, for each index i.

For an integral domain R, a rank one module is isomorphic to a submodule of its field of fractions Q. For abelian groups (i.e. $\mathbb{Z}$-modules), such modules have been well-studied, e.g. $\mathbb{Q}$.

In [6, Theorem 4.3], Goeters and Olberding show that, for a one-dimensional Noetherian domain, the UDI property implies the Krull-Schmidt property for direct sums of torsion-free rank one modules. In this chapter we prove the same theorem for reduced commutative Noetherian one-dimensional ring R, under the additional assumption that R is local, with a suitable definition of rank one module.

We shall say that a torsion-free R-module has rank one if its rank at each minimal prime of R is zero or one (and at least one local rank is nonzero), equivalently, if it is isomorphic to a nonzero R-submodule of the total quotient ring Q of R.

Theorem 26. *Let R be reduced, local, Noetherian ring of Krull dimension one. If R*

has the UDI property, then for any indecomposable torsion-free rank one R-modules, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, such that $X_1 \oplus \cdots \oplus X_m \cong Y_1 \oplus \cdots \oplus Y_n$, then $m = n$, and, reindexing if necessary, $X_j \cong Y_j$ for all j.

Proof. We divide the proof into the three cases of Theorem 21.

If Theorem 21(1) holds, then the theorem follows immediately from [6, Theorem 4.3].

Suppose instead that Theorem 21(2) holds. If $\mathrm{End}_R(X_k)$ is local for some k, then by [2, Lemma I.3.4], $X_k \cong Y_l$ for some index l, so by semi-local cancellation [4, Theorem 2], we can cancel X_k and Y_l from the isomorphism. Thus, by induction we can assume that each endomorphism ring $\mathrm{End}_R(X_k)$ and $\mathrm{End}_R(Y_l)$ is non-local.

Note that, if X is an R-submodule of Q, then $\mathrm{rank}(X)$ must be $(1,0), (0,1)$, or $(1,1)$. If $\mathrm{rank}(X) = (1,0)$, then, $X_{P_2} = 0$ implies that $\mathrm{Ann}_R(X) \nsubseteq P_2$, so that, $\mathrm{Ann}_R(X) = P_1$, and $X \cong \bar{X} = (X + \mathrm{Ann}_R(X))/\mathrm{Ann}_R(X) = (X + P_1)/P_1$ is an $\bar{R} = R/P_1 = R_1$ -module. Hence $D = \mathrm{End}_R(X) \cong \mathrm{End}_{\bar{R}}(\bar{X}) = [X : X]$ is an overring of R_1. Let $\tilde{D}$ denote the integral closure of D in Q. Since $\tilde{R}_1$ is one-dimensional, local, integrally closed Noetherian domain, $\tilde{D} = Q_1$, or $\tilde{D} = \tilde{R}_1$, so that $\tilde{D}$ is local, which implies that D is local. Thus, we can assume that none of the remaining summands X_k and Y_l have rank $(1,0)$.

On the other hand, if $\mathrm{rank}(X) = (0,1)$, then $\mathrm{Ann}_R(X) = P_2$, and $X \cong \bar{X} = (X + P_2)/P_2$ is an ideal of $\bar{R} = R/\mathrm{Ann}_R(X) = R/P_2 = R_2$. Let $D = \mathrm{End}_{\bar{R}}(\bar{X}) \cong \mathrm{End}_R(X)$, so that D is an overring of $\tilde{R}_2$. So $\tilde{D}$ is an overring of $\tilde{R}_2$, which implies that $\tilde{D}$ is equal to $Q_2, \tilde{R}_2$, or $(R_2)_{\tilde{M}_i}$, for $i = 1, 2$.

If $\tilde{D} = (\tilde{R}_2)_{\tilde{M}_i}$ for $i = 1, 2$, then $\tilde{D}$ is local. Similarly, if $\tilde{D} = Q_2$, then $\tilde{D}$ is local. Since $\tilde{D}$ integral over D, its unique maximal ideal contracts to the unique maximal ideal of D, contrary to assumption that D be non-local.

Therefore we can suppose that $\tilde{D} = \tilde{R}_2$, so that $R_2 \subseteq D \subseteq \tilde{R}_2$, and hence D is finite over R. We claim that $D = \tilde{R}_2$. Write $M_i' = D \cap \tilde{M}_i$ for $i = 1, 2$, so that $M_1' \neq M_2'$. As in the proof of Theorem 21 one can show that $\tilde{D} = D + \tilde{M}_i$, and since $(\tilde{M}_i)_{M_i'} = M_i'(\tilde{D})_{M_i'}$,

$(\tilde{D})_{M_i'} = D_{M_i'} + M_i'(\tilde{D})_{M_i'}$. Therefore, so that $D_{M_i'} = (\tilde{D})_{M_i'}$ for each index i, from which it follows that $D = \tilde{D} = \tilde{R}_2$.

In the direct sums $X_1 \oplus \cdots \oplus X_n \cong Y_1 \oplus \cdots \oplus Y_n$, we assume that each indecomposable summand has rank $(0,1)$ or $(1,1)$ and has a non-local endomorphism ring. Since the sums of the ranks are equal on both sides, there are the same number of indecomposables of rank $(0,1)$ on both sides. We just showed that each indecomposable X with rank $(1,1)$ has endomorphism ring $\tilde{R}_2$, which implies that X is a rank one $\tilde{R}_2$-module. Then $X \cong \tilde{R}_2$ as $\tilde{R}_2$-modules, so as R-modules. Since $\tilde{R}_2$ is semi-local, by semi-local cancellation [4, Theorem 2], we can cancel those summands from both sides of the isomorphism. Therefore, we can further assume that each X_k and each Y_l has rank $(1,1)$. Let $G = X_1 \oplus \cdots \oplus X_n \cong H = Y_1 \oplus \cdots \oplus Y_n$.

Let us fix the indecomposable faithful R-module $X = X_1$ with non-local endomorphism ring $D = \mathrm{End}_R(X)$. By the above argument, we note that its integral closure $\tilde{D} = \tilde{D}_1 \times \tilde{D}_2$, $\tilde{D}_1$ is local, and $\tilde{D}_2$ is either local, or $\tilde{D}_2 = \tilde{R}_2$.

Suppose that $\tilde{D}_2$ is local, so that $\tilde{D}$ would have two maximal ideals. We claim this would force D to be local. Let $\tilde{N} \times \tilde{D}_2, \tilde{D}_1 \times \tilde{M}$ be the maximal ideals of $\tilde{D}$. Let $N' = D \cap (\tilde{N} \times \tilde{D}_2)$, $M' = D \cap (\tilde{D}_1 \times \tilde{M})$. We show that $N' = M'$, contrary to assumption. The ring $\tilde{D}$ has only two minimal primes, $\tilde{P}_1 = \{0\} \times \tilde{D}_2$ and $\tilde{P}_2 = \tilde{D}_1 \times \{0\}$, so that $\tilde{D}_i = \tilde{D}/\tilde{P}_i$ for each index i. Let $P_i' = D \cap \tilde{P}_i$ for each index i, so that P_1' and P_2' are the (distinct) minimal primes of D. Now $\tilde{P}_1$ is contained in only one maximal ideal, $\tilde{N} \times \tilde{D}_2$, so $P_1' \subseteq N'$. Similarly, $\tilde{P}_2$ is contained in $\tilde{D}_1 \times \tilde{M}$, so that $P_2' \subseteq M'$. Since $\tilde{D}$ is integral over D, it follows by "going-up" that N' is the only maximal ideal of D containing P_1', and M' is the only maximal ideal of D containing P_2'. If $N' \neq M'$, then $\mathrm{Spec}(D)$ decomposes as the disjoint union of $V(P_1')$ and $V(P_2')$, which makes X decomposable, contrary to assumption. But then $N' = M'$, which makes D local, also contrary to assumption.

Therefore, we can assume that $\tilde{D}_2 = \tilde{R}_2$, so that $\tilde{D}$ has three maximal ideals. As above the maximal ideals of D are the contractions of the maximal ideals of $\tilde{D}$. By a similar argument to above, the three maximal ideals of $\tilde{D}$ cannot have distinct contractions to D. But D is assumed to be non-local, so it follows that D has two maximal ideals. Let $N' = D \cap (\tilde{N} \times \tilde{D}_2)$, $M_1' = D \cap (\tilde{D}_1 \times \tilde{M}_1)$ and $M_2' = D \cap (\tilde{D}_1 \times \tilde{M}_2)$, where $\tilde{N}$ is the maximal ideal of $\tilde{D}_1$, and $\tilde{M}_i$ is the maximal ideal of $\tilde{D}_2$, for $i = 1, 2$. If $M_1' = M_2'$, then D becomes decomposable. Thus, $N = M_1'$ or $N = M_2'$. Without loss of generality, suppose $N = M_1'$. As argued above, one can show that $(\tilde{M}_2)_{M_2'} = (M_2'\tilde{D}_2)_{M_2'}$. Moreover, $(\tilde{D}_2)_{M_2'}$ is a finitely generated $D_{M_2'}$-module, since $\tilde{D}_2$ is finite over R_2, so by Nakayama's Lemma, $(\tilde{D}_2)_{M_2'} = D_{M_2'}$. Therefore, $D_{M_2'}$ is a DVR, so that $M_2' D_{M_2'}$ is principal.

Without loss of generality, we can assume that $D = \mathrm{End}_R(X_1)$ is minimal with respect to inclusion among all of the endomorphism rings $\mathrm{End}_R(X_k)$ and $\mathrm{End}_R(Y_l)$. Let us regard $L = X_1 \oplus \cdots \oplus X_n = Y_1 \oplus \cdots \oplus Y_n$ as two internal decompositions of L into direct sums of indecomposable R-modules with rank $(1, 1)$. Now using the inclusion maps $\iota_j : X_j \to L$ and $\iota_j' : Y_j \to L$ and the projection maps $\pi_j : L \to X_j$ and $\pi_j' : L \to Y_j$, the identity map on X_1 is $\pi_1\iota_1 = \sum_j \pi_1\iota_j'\pi_j'\iota_1$, where each composition $\pi_1\iota_j'\pi_j'\iota_1$ is in D. Since $D_{M_1'}$ is a local ring, one of these compositions must be a unit in $D_{M_1'}$; we can assume that it is $\pi_1\iota_1'\pi_1'\iota_1$. Let $f = \pi_1\iota_1'$, an R-homomorphism from Y_1 to X_1; by assumption $f(Y_1)R'_{M_1'} = (X_1)_{M_1'}$. Using the fact that $M_1' \neq M_2'$, we can choose $a \in M_2' - (M_1' \cup P_2' \cup (M_2')^2)$, which makes a regular. Since $D_{M_2'}$ is a DVR, there exists $n \geqslant 0$ such that $f(Y_1)D_{M_2'} = a^n(X_1)_{M_2'}$. Now a is regular, so we can set $h = a^{-n}f$, which is a Q-endomorphism of Q. Since X_1 and Y_1 are faithful, as in the proof of Theorem 21 one can show that $h : Y_1 \to X_1$ is an R-isomorphism. Therefore, by local cancellation [4, Theorem 2] and induction, the theorem follows in this case.

Finally, let us suppose that Theorem 21(3) holds. By [2, Lemma I.3.4] and [4, Theorem 2] it suffices t prove that each X_j has local endomorphism ring. Let $X = X_j$ for some index j, and set $\bar{R} = R/\mathrm{Ann}_R(X)$, and $\bar{X} = (X + \mathrm{Ann}_R(X))/\mathrm{Ann}_R(X)$, which is faithful. We note that $X \cong \bar{X}$ as R-modules, so that $\mathrm{End}_R(X) \cong \mathrm{End}_{\bar{R}}(\bar{X})$. Furthermore, we can assume that $\mathrm{Ann}_R(X) = P_1 \cap \cdots \cap P_s$ for some $s \leqslant t$, from which it follows that the minimal primes of $\bar{R}$ are $\bar{P}_1, \ldots, \bar{P}_s$, where $\bar{P}_i = P_i/\mathrm{Ann}_R(X)$. Then $\bar{R}/\bar{P}_i \cong R/P_i = R_i$ has local integral closure $\bar{R}_i$ in its field of fractions Q_i for each index i, $1 \leqslant i \leqslant s$. Thus, replacing X by $\bar{X}$ and R by $\bar{R}$, we can assume that X is an indecomposable faithful R-module.

Let $D = \mathrm{End}_R(X)$, an overring of R. Since each $\tilde{R}_i$ is one-dimensional, integrally closed, local Noetherian domain, it is a DVR. Then $\tilde{D}_i$ is either a DVR, or $\tilde{D}_i = Q_i$, so that $\tilde{D}_i$ is local. For each index i, let $\tilde{P}_i$ be the kernel of the natural map from $\tilde{D}$ onto $\tilde{D}_i$, so that $\tilde{P}_1, \ldots, \tilde{P}_t$ are the distinct minimal primes of $\tilde{D}$. Moreover, their contractions to R yield the distinct minimal primes of R, namely, $P_i = R \cap \tilde{P}_i$ for each index i. Let $P_i' = D \cap \tilde{P}_i$ for each index i; then the ideals $P_1', \ldots, P_t'$ must be the distinct minimal primes of D. Now each $\tilde{P}_i$ is contained in only one maximal ideal, since $\tilde{D}/\tilde{P}_i = \tilde{D}_i$ is assumed to be local. Since $\tilde{D}$ is integral over D, by "going-up" each minimal prime ideal of D is contained in a unique maximal ideal as well. If M' is an arbitrary maximal ideal of D, we can number the minimal primes of D so that $P_1', \ldots, P_r' \subseteq M'$ while $P_{r+1}', \ldots, P_t' \nsubseteq M'$, for some integer r, $1 \leqslant r \leqslant t$. Then $V(P_1') \cup \cdots \cup V(P_r')$ is both open and closed in $\mathrm{Spec}(D)$, because each minimal prime of D is contained in a unique maximal ideal. Since D must be an indecomposable ring, because X is an indecomposable module, it follows that $r = t$, and every minimal prime of D is contained in M'. Thus, M' is the unique maximal ideal of D, so that D is local.

□

It is an open question whether one can drop the extra assumption that R be local in Theorem 26. One might try an approach as in Theorem 15, but that theorem relies on the fact that, if R has a unique nonprincipal maximal ideal M, then for finitely generated torsion-free completely decomposable modules G and H, $G \cong H$ if and only if $G_M \cong H_M$ (Theorem 11). This fact no longer holds for R-submodules of Q, as for example R and R_M become locally isomorphic at M. Since we have no counterexamples to a global version of Theorem 26, however, we make the following conjecture.

Conjecture 27. *Let R be reduced, local, Noetherian ring of Krull dimension one, but not necessarily local. If R has the UDI property, then for any indecomposable torsion-free rank one R-modules, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, such that $X_1 \oplus \cdots \oplus X_m \cong Y_1 \oplus \cdots \oplus Y_n$, then $m = n$, and, reindexing if necessary, $X_j \cong Y_j$ for all j.*

Chapter 6

Examples

In this chapter, we construct several examples of reduced commutative Noetherian rings which satisfy the UDI property, illustrating both Theorem 15 and Theorem 21. Moreover, we give an example of a Noetherian domain with Krull dimension three which satisfies the UDI property, but it has an overring which does not inherit the UDI property.

The examples of non-local integral domains with the UDI property in [6] all have Krull dimension one; our first example shows that this need not always hold. (Our thanks to Tim Ford for this example.)

Example 1. Let F be an algebraically closed field, and consider the ring $F[X, Y]$ of polynomials over F in two indeterminates X and Y. Let S be the multiplicatively closed set in $F[X, Y]$ generated by polynomials of the form $X - \alpha$ and $Y - \beta$ for all nonzero elements $\alpha, \beta \in F$, and set $R = S^{-1}F[X, Y]$. Since F is algebraically closed, if M is a maximal ideal of $F[X, Y]$, then $(X - a, Y - b)$ for some pair $a, b \in F \times F$. We observe that the only height two maximal ideal of R is $M = S^{-1}(X, Y)$, so that R is a Noetherian domain of Krull dimension two. Moreover, M is the only nonprincipal maximal ideal of R by Krull's Ideal Theorem [10, 12.I]. Since the prime ideals PR of R, where $P = (X + Y + a)$ such that $a \neq 0$, are maximal ideals, there are infinitely many height one (principal) maximal ideals of R. Since $R_M = F[X,Y]_{(X,Y)}$ is integrally closed and local, R has the UDI property by Theorems 15 and 16.

Non-local non-domains with the UDI property and Krull dimension greater than one are easy to construct. In fact, as the following example demonstrates, there is no bound to the possible Krull dimension of a non-local ring with the UDI property.

Example 2. Consider the conductor square

$$\begin{array}{ccc} R & \longrightarrow & F[X] \times F[[X_1, \ldots, X_n]] \\ \downarrow & & \downarrow \beta \\ F & \xrightarrow{\alpha} & F \times F \end{array}$$

where F is a field, $F[X]$ is a polynomial ring in one indeterminate, and $F[[X_1, \ldots, X_n]]$ is a power series ring in n indeterminates for some positive integer n. Here α is the diagonal inclusion map $\alpha(r) = (r, r)$ for each $r \in F$, and β is the map sending polynomials and power series to their constant terms $\beta(f, g) = (f_0, g_0)$. Thus,

$$R = \{(f, g) \in F[X] \times F[[X_1, \ldots, X_n]] : f_0 = g_0\} = \beta^{-1}(\mathrm{Im}(\alpha)),$$

and the conductor $M = (X) \times (X_1, \ldots, X_n)$ is the largest common ideal of R and $F[X] \times F[[X_1, \ldots, X_n]]$.

Note that $F[X] \times F[[X_1, \ldots, X_n]]$ is both integrally closed and finitely generated as a module over R (by the idempotents $(1, 0)$ and $(0, 1)$), so that $F[X] \times F[[X_1, \ldots, X_n]]$ is the integral closure of R in their common total ring of quotients $F(X) \times F((X_1, \ldots, X_n))$. In the notation of the introduction, $\tilde{R} = F[X] \times F[[X_1, \ldots, X_n]]$ and $Q = F(X) \times F((X_1, \ldots, X_n))$. Thus, R is Noetherian [9, Theorem 3.7(i)], because $\tilde{R}$ is Noetherian and finitely generated as a module over R. Moreover, R has dimension n, the dimension of its integral closure $\tilde{R}$. We observe also that R has no non-trivial idempotents, so that R is indecomposable.

We claim that M is the unique nonprincipal maximal ideal of R. Being the kernel of β mapping R onto F, M is a maximal ideal of R. If $(X, 0) \in (f, g)R$ for some $(f, g) \in M$, then $g = 0$ because $(1, 0) \notin R$, and hence $(f, g)R \neq M$; that is, M is nonprincipal. On the other hand, the maximal ideals of $\tilde{R}$ which do not lie over M are of the form $N_p = pF[X] \times F[[X_1, \ldots, X_n]]$, where $p \in F[X]$ is a monic irreducible polynomial other than X, and one easily checks that $N_p \cap R$ is the principal maximal ideal $(p, p(0))R$. Since every maximal ideal of R is the contraction of a maximal ideal of $\tilde{R}$, the claim follows.

Finally, we claim that R has the UDI property. By Theorem 15, it suffices to check that R_M has the UDI property. Since R_M is a flat R-module, by [5, Lemma 1.1.6] localizing the above conductor square at M gives a conductor square

$$\begin{array}{ccc} R_M & \longrightarrow & F[X]_{(X)} \times F[[X_1, \ldots, X_n]] \\ \downarrow & & \downarrow \\ F & \longrightarrow & F \times F \end{array}$$

R_M remains indecomposable, and $\tilde{R}_M = F[X]_{(X)} \times F[[X_1, \ldots, X_n]]$ is a product of local rings, so that R_M has the UDI property by Theorem 21(3).

We provide examples illustrating cases (2) and (3) of Theorem 21. We close this chapter with an example which shows that the claim of Theorem 25 may not hold for rings with Krull dimension greater than one.

Example 3. Consider the conductor square

$$\begin{array}{ccc} R & \longrightarrow & F[[X]] \times S^{-1}F[X] \\ \downarrow & & \downarrow \beta \\ F & \xrightarrow{\alpha} & F \times F \times F \end{array}$$

where F is a field, $F[[X]]$ is a power series ring in one indeterminate, $F[X]$ is a polynomial ring in one variable, and $S = F[X] - ((X) \cup (X - 1))$ is a multiplicatively closed subset of $F[X]$, so that $S^{-1}F[X]$ has exactly two maximal ideals (X) and $(X - 1)$. Again α is the diagonal inclusion, while $\beta(f,g) = (f(0), g(0), g(1))$ for $f \in F[[X]]$ and $g \in S^{-1}F[X]$. Thus,

$$R = \{(f,g) \in F[[X]] \times S^{-1}F[X] : f(0) = g(0) = g(1))\},$$

and the conductor is $M = (X) \times ((X) \cap (X-1))$.

As in Example 2, note that R is an indecomposable, reduced, commutative, Noetherian ring, with integral closure $\tilde{R} = F[[X]] \times S^{-1}F[X]$. The only maximal ideals of $\tilde{R}$ are $F[[X]] \times (X)$, $F[[X]] \times (X-1)$, and $(X) \times S^{-1}F[X]$, and these maximal ideals all contract to M in R, so that R is local. Moreover, $\tilde{R}_1 = F[[X]]$ is local, while $\tilde{R}_2 = S^{-1}F[X]$ has maximal ideals (X) and $(X-1)$, where $M \nsubseteq F[[X]] \times (X)^2$ and $M \nsubseteq F[[X]] \times (X-1)^2$, and $R/M = F = S^{-1}F[X]/(X) = S^{-1}F[X]/(X-1)$. Thus, R has the UDI property by Theorem 21(2).

Example 4. Consider the conductor square

$$\begin{array}{ccc} R & \longrightarrow & F[[X_1]] \times \cdots \times F[[X_n]] \\ \big\downarrow & & \big\downarrow \beta \\ F & \xrightarrow{\alpha} & F \times \cdots \times F \end{array}$$

where again F is a field, each $F[[X_i]]$ is a power series ring in one indeterminate, α is the diagonal inclusion, and β maps each n-tuple of power series to the n-tuple of constant terms. In this case

$$R = \{(f_1, \ldots, f_n) \in F[[X_1]] \times \cdots \times F[[X_n]] : f_1(0) = \cdots = f_n(0)\},$$

and the conductor is $M = (X_1) \times \cdots \times (X_n)$.

Also as in Example 2, we observe that R is an indecomposable, reduced, commutative, Noetherian ring. Moreover, M is the only maximal ideal of R, so that R is local. The integral closure of R is the ring $\tilde{R} = F[[X_1]] \times \cdots \times F[[X_n]]$, a product of local rings, so by Theorem 21(3), R has the UDI property. Note that $\tilde{R}$ has exactly n distinct maximal ideals, where the choice of $n \geq 1$ is arbitrary in this example, so there is no bound to the number of maximal ideals possible for the integral closure of a local ring with the UDI property in Theorem 21, unlike the situation for integral domains in [6, Theorem 3.2].

Example 5. Let F be an algebraically closed field, and consider the ring $F[X, Y, Z]$ of polynomials over F in three indeterminates X, Y, and Z. Let $R = F[X, Y, Z]_{(X,Y,Z)}$, a Noetherian domain of Krull dimension three. Since R is integrally closed and local, R has the UDI property by Theorem 16. Let S be the multiplicatively closed set in $F[X, Y, Z]$ generated by polynomials of the form $X + Y + Z$, and set $R' = S^{-1}R$. We observe that (X, Z) and (Y, Z), which are contained in (X, Y, Z), are height two prime ideals of $F[X, Y, Z]$. Moreover, $(X, Z)R$ and $(Y, Z)R$ become distinct prime ideals with height two in R, also. In R', $(X, Z)R'$ and $(Y, Z)R'$ are maximal ideals with height two. By Krull's Ideal Theorem [10, 12.I] they cannot be principal. Therefore, by Lemma 4 R' does not satisfy the UDI property.

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