Minimal Zero-Dimensional Extensions

by

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Abstract

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The structure of minimal zero-dimensional extensions of rings with Noetherian spectrum in which zero is a primary ideal and with at most one prime ideal of height greater than one is determined. These rings include $K[[X, Y]]$ where K is a field and Dedekind domains, but need not be Noetherian nor integrally closed. We show that for such a ring R there is a one-to-one correspondence between isomorphism classes of minimal zero-dimensional extensions of R and sets $\mathcal{M}$, where the elements of $\mathcal{M}$ are ideals of R primary for distinct prime ideals of height greater than zero. A subsidiary result is the classification of minimal zero-dimensional extensions of general ZPI-rings.

Contents

1 Introduction

Any field has only one prime ideal which is maximal. An integral domain can be embedded in a field, the smallest field which works for this being the field of fractions. An analogy can be made between a field and a zero-dimensional extension of a ring. A zero-dimensional extension of a ring has all its prime ideals maximal, so a zero-dimensional extension can be thought of as a generalization of a field. In our analogy the integral domain can be generalized by a ring. The smallest ring which satisfies the embeddability condition above is the minimal zero-dimensional extension, so a minimal zero-dimensional extension can be thought of as generalization of a field of fractions. While the structure of a field of fractions is well-known, the structure of minimal zero-dimensional extensions has received considerably less attention, despite the fact that the study of zero dimensionality in commutative rings has been a hot topic in commutative algebra community in the past twenty-five years. There are many studies related to zero-dimensional extensions and zero-dimensional subrings of a commutative ring, but none investigates the structure of minimal zero-dimensional extensions of a commutative ring, which is the objective of this work.

Zero-dimensional extensions of a commutative ring was the primary focus of some of Arapović's papers [1]-[3]. He was particularly interested in the question of embeddability of a ring in a zero-dimensional ring and he also was the first one to give a criterion of embeddability of a ring in a zero-dimensional ring ([3, Theorem 7]). Another criterion, less known is provided by Burgess and Menal in reference [5] and it will be presented here in Section 2.2, together with examples of rings which have zero-dimensional extensions.

The existence of a unique minimal zero-dimensional extension of a commutative ring within a zero-dimensional ring was proved by Arapović ([2]) and by Gilmer and Heinzer ([10]). Arapović's proof provides a method for constructing a minimal zero-dimensional extension. We refer to his method as Arapović's construction and we describe it in detail in Section 3.1. We show that for rings R with Noetherian spectrum in which zero is primary and with only finitely many primes of height greater than one, minimal zero-dimensional R-algebras can be constructed using Arapović's construction. Using Arapović's construction we determine the structure of minimal zero-dimensional extensions of rings with Noetherian spectrum in which zero is a primary ideal and with at most one prime ideal of height greater than one in Section 3.2. These rings include $K[[X,Y]]$ where K is a field, Dedekind domains but need not be Noetherian nor integrally closed. We show that for such rings there is a one-to-one correspondence between isomorphism classes of minimal zero-dimensional extensions of R and sets $\mathcal{M}$, where the elements of $\mathcal{M}$ are ideals of R primary for distinct prime ideals of height greater than zero. Using the structure of minimal zero-dimensional extensions of Dedekind domains and primary rings we determine the structure of minimal zero-dimensional extensions of general ZPI-rings in Section 3.3. The structure of minimal zero-dimensional R-algebra with finite spectrum is also determined using Arapović's construction in Section 3.4. In Section 3.5 we describe a few properties of minimal zero-dimensional algebras.

We end the thesis by suggesting some open questions in Section 3.6.

2 Background

2.1 Notation and Useful Results

In this section we list some notation, definitions and results which will be used throughout this dissertation. By a ring we always mean a commutative ring with identity. The **prime spectrum** of a ring R, denoted $\mathrm{Spec}(R)$, is the set of prime ideals of R. The **maximal spectrum** of a ring R, denoted $\mathrm{MaxSpec}(R)$, is the subset of $\mathrm{Spec}(R)$ consisting of all maximal ideals of R. All allusions to the dimension of a ring refer to its Krull dimension.

Definition 1 *Let R be a ring. The **Krull dimension** of R is the supremum of the lengths of all chains of its prime ideals.*

Therefore a ring is zero dimensional if every prime ideal is maximal.

Definition 2 *Let R be a ring and P a prime ideal of R. The **height of** P, denoted $\mathrm{ht}(P)$, is the number of strict inclusions in the longest chain of prime ideals contained in P.*

Definition 3 *Let R be a ring. A ring S is an R-**algebra** if S is an R-module and $(ar)s = a(rs) = r(as)$ for all $a \in R$ and $r, s \in S$.*

Definition 4 *Let R be a ring. The **total quotient ring of** R, denoted $T(R)$, is the ring consisting of all fractions of the form r/s where $r, s \in R$ and s is regular.*

Lemma 5 *Let R_i be a ring for every $i \in I$. Then $\prod_{i\in I} T(R_i) = T(\prod_{i\in I} R_i)$.*

Proof. Let $x = (x_i) \in \prod_{i \in I} T(R_i)$. Then, for every x_i there exist $a_i, b_i \in R_i$ with b_i regular in R_i such that $b_i x_i = a_i$. Let $a = (a_i) \in \prod_{i \in I} R_i$ and $b = (b_i) \in \prod_{i \in I} R_i$. Since b_i is regular in R_i for every $i \in I$, then b is regular in $\prod_{i=1}^{n} R_i$. Therefore, $bx = a$ with $a, b \in \prod_{i \in I} R_i$ and b regular in $\prod_{i \in I} R_i$. Hence, $x \in T(\prod_{i \in I} R_i)$.

Conversely, let $x \in T(\prod_{i \in I} R_i)$. Then, there exist $a, b \in \prod_{i \in I} R_i$ with b regular in $\prod_{i \in I} R_i$ such that $bx = a$. Let $a = (a_i) \in \prod_{i \in I} R_i$ and $b = (b_i) \in \prod_{i \in I} R_i$. Since b is regular in $\prod_{i \in I} R_i$, then b_i is regular in R_i for every $i \in I$. Then $b_i x_i = a_i$ where $a_i, b_i \in R_i$ and b_i is regular in R_i for every $i \in I$. Hence, $x \in \prod_{i \in I} T(R_i)$. ■

Definition 6 *Let R be a ring, S an R-module and I a subset of M. The* ***annihilator of*** *I, denoted $\mathrm{ann}_R(I)$, is the set of elements $r \in R$ such that $rI = 0$.*

Definition 7 *Let R be a ring. The* ***radical of an ideal*** *I of R, denoted $\mathrm{rad}(I)$, is the ideal of elements $r \in R$ such that $r^n \in I$ for some positive integer n.*

Definition 8 *Let R be a ring and I an ideal of R. We say that I is a* ***primary ideal*** *if $I \neq R$ and if $ab \in I$, then either $a \in I$ or $b^n \in I$ for some positive integer n.*

Definition 9 *Let R be a ring. The prime spectrum of R is said to be Noetherian if R has the ascending chain condition on radical ideals.*

Definition 10 *A nonzero element b of a Boolean algebra is said to be an* ***atom*** *if for all elements x of the algebra $x \wedge b = b$ or $x \wedge b = 0$.*

Theorem 11 ([6]) *A finitely generated Boolean algebra is finite.*

Proof. Let A be a Boolean algebra generated by n elements $e_1, e_2, \ldots, e_n$. Then A contains elements of the form

$$e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \cdots \wedge e_n^{\epsilon_n} \tag{1}$$

where each $e_i^{\epsilon_i}$ is either e_i or the complement e_i'. There are 2^n meets like (1). Now consider joins of zero and these 2^n elements. Let B be the set of all these joins. Claim that B is a subalgebra of A. Since any element of B is a join of elements like (1), B is closed under join. Using the commutativity property the meet of any two elements like (1) is zero, so B is closed under meet. Each

$$e_i = (e_1 \vee e_1') \wedge \cdots \wedge (e_{i-1} \vee e_{i-1}') \wedge e_i \wedge (e_{i+1} \vee e_{i+1}') \wedge \cdots \wedge (e_n \vee e_n').$$

If we expand e_i by distributive law, we get e_i in B. Similarly we get e_i' in B. Any element of B is a join of meets of elements e_i and e_i'. By DeMorgan's laws any complement of an element of B is a meet of joins of elements e_i and e_i' in B. Since B is closed under meets and joins, then it is closed under complement. We have $1 = (e_1 \vee e_1') \wedge \cdots \wedge (e_n \vee e_n')$. Using the distributivity property, we get 1 as a meet of these 2^n elements like (1). Then, B is a subalgebra containing $e_1, e_2, \ldots, e_n$. Hence B is the algebra generated by the elements $e_1, e_2, \ldots, e_n$. It has at most 2^{2^n} different elements, corresponding to the 2^{2^n} subsets of the set of 2^n elements like (1). ∎

The non-zero elements (1) are the atoms of B.

Definition 12 *A ring is said to be* ***primary*** *if it contains at most one prime ideal.*

2.2 Zero-Dimensional Extensions

This section offers examples of rings which have zero-dimensional extensions and presents criteria of embeddability of a ring in a zero-dimensional ring.

Rings which have zero-dimensional extensions are reduced rings and rings whose zero ideal is a finite intersection of primary ideals (integral domains, Noetherian rings and Laskerian rings). Zero-dimensional extensions of these rings are obtained using the following well-known result.

Theorem 13 *Let R be a ring and $\{I_\alpha\}_{\alpha\in A}$ a nonempty family of ideals such that $0 = \bigcap_{\alpha\in A} I_\alpha$. Then R is embedded in $\prod_{\alpha\in A} R/I_\alpha$, which is embedded in $\prod_{\alpha\in A} T(R/I_\alpha)$.*

Applying Theorem 13 to a reduced ring R and a family $\{P_\alpha\}_{\alpha\in A}$ of prime ideals of R, leads to R being embedded in $\prod_{\alpha\in A} T(R/P_\alpha)$, which is a product of fields, and so it is von Neumann regular. Since a von Neumann regular ring is zero dimensional, it follows that R is embedded in a zero-dimensional ring.

For a ring R and a finite family $\{I_\alpha\}_{\alpha\in A}$ of P_α-primary ideals with intersection zero, Theorem 13 implies that R is embedded in $\prod_{\alpha\in A} T(R/I_\alpha)$. Since $\mathrm{rad}(I_\alpha) = P_\alpha$, we have that $T(R/I_\alpha) = (R/I_\alpha)_{(P_\alpha/I_\alpha)}$ is zero dimensional, therefore $\prod_{\alpha\in A} T(R/I_\alpha)$ is zero dimensional, since a finite product of zero-dimensional rings is zero dimensional. Hence R has a zero-dimensional extension.

Determining conditions for a ring to be embeddable in a zero-dimensional ring attracted the interest of many mathematicians in the past twenty-five years. This issue was addressed first by Arapović in references [1]-[3], who gave a criterion for embeddability. Independent of Arapović's work, Burgess and Menal in reference [5] gave two more criteria. The work on embeddability was continued by Gilmer and Heinzer in references [9]-[12].

Next we present the most representative criteria, the first one given by Arapović, which we refer to as Arapović's Embedding Theorem (AET).

Theorem 14 *A ring R is embeddable in a zero-dimensional ring if and only if R has a family of primary ideals $\{Q_\lambda\}_{\lambda\in\Lambda}$, such that:*

A1. $\bigcap_{\lambda\in\Lambda} Q_\lambda = 0$ *and*

A2. For each $x \in R$, there is $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$, if $x \in \mathrm{rad}(Q_\lambda)$, then $x^n \in Q_\lambda$.

According to AET, one of the necessary conditions for a ring R to be embeddable in a zero-dimensional ring is that the zero ideal of R is an intersection of primary ideals of

R. But this condition alone is not enough to imply embeddability. An example is given by Gilmer and Heinzer ([12]), who showed that $R = \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$ satisfies condition A1, but R is not embeddable in a zero-dimensional ring. Since A2 is not easy to verify in general, it makes AET hard to apply in practice. Nevertheless, A1 can be used to obtain examples of rings which are not embeddable in zero-dimensional rings. Gilmer proved in reference [9] that A1 is a necessary and sufficient condition for a chained ring to have a zero dimensional extension.

Definition 15 *A ring R is said to be a **chained ring** if the ideals of R are linearly ordered with respect to inclusion.*

Theorem 16 ([9]) *In a chained ring R, the following conditions are equivalent:*

1. *R is embeddable in a zero-dimensional ring.*
2. *Zero is an intersection of primary ideals (Condition A1 of AET holds).*
3. *The ideal zero of R is primary.*

By Theorem 16, if R is a chained ring and I is a nonprimary ideal of R then R/I is not embeddable in a zero-dimensional ring.

Significant work on criteria of embeddability of a ring R in a zero-dimensional ring has also been done by Burgess and Menal, who published their results in reference [5]. They actually offer two of these criteria: the first one, which presents topological arguments, is equivalent to AET and the other, more arithmetic in nature, constitutes a viable alternative to AET. The latter one, since it is much less known, I will briefly present it here.

Theorem 17 *Let R be a ring. Then the following are equivalent:*

1. *R can be embedded in a zero-dimensional ring,*
2. *for each $x \in R$ there is $n(x) \geq 1$ such that for any $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in R$, the ideal $I = \sum_{i=1}^{k} x_i^{n(x_i)} R$ has the property that $I \cap \operatorname{ann}_R(I) = 0$.*

Another necessary condition for embeddability in a zero-dimensional ring, presented by Burgess and Menal in reference [5] and also by Gilmer in reference [9], is the following. We refer to this condition as NC.

Theorem 18 *If R is embeddable in a zero-dimensional ring, then for each $x \in R$ there is a positive integer m_x such that* $\operatorname{ann}_R(x^{m_x}) = \operatorname{ann}_R(x^{m_x+1})$.

Burgess and Menal proved in reference [5] that, in general, condition NC is not sufficient.

Theorem 19 *Let R be a commutative ring such that $R/\operatorname{rad}(0)$ is a principal ideal domain. Then R can be embedded in a zero-dimensional ring if and only if for each $x \in R$ there is a positive integer m_x such that $\operatorname{ann}_R(x^{m_x}) = \operatorname{ann}_R(x^{m_x+1})$.*

We note here that Theorem 19 gives an answer to Gilmer's question from reference [9] of whether condition NC is sufficient for embeddability in a zero-dimensional ring.

Condition NC provides more examples of rings not embeddable in a zero-dimensional ring.

Theorem 20 ([9]) *Suppose that R is a chained ring. If the dimension of $T(R)$ is positive, then R does not satisfy condition NC, and hence R is not embeddable in a zero-dimensional ring.*

More examples of rings which do not have zero-dimensional extensions can also be obtained by using the following result from reference [9].

Theorem 21 *Suppose $\{R_\alpha\}_{\alpha \in A}$ is a family of zero-dimensional rings and $R = \prod_{\alpha \in A} R_\alpha$. Either R is zero dimensional or R is infinite dimensional. In the latter case, R is not embeddable in a zero-dimensional ring.*

3 Minimal Zero-Dimensional Extensions

3.1 Arapović's Construction

Let R be a subring of a ring S. This section focuses on the existence of a minimal zero-dimensional extension of R within S. We describe a method of constructing a minimal zero-dimensional extension of R within S and we give examples of rings for which we can apply this method.

Gilmer and Heinzer proved the existence of a minimal zero-dimensional extension of R within S using the following important result regarding zero-dimensional rings.

Theorem 22 ([10]) *Suppose $\{R_\alpha\}_{\alpha\in A}$ is a nonempty family of zero-dimensional subrings of S. Then $\bigcap_{\alpha\in A} R_\alpha$ is zero-dimensional.*

Thus, if R is contained in a zero-dimensional subring of S, then there exists a unique minimal zero-dimensional subring of S containing R.

Arapović also proved the existence of the minimal zero-dimensional extension of R within S for the case when S is zero-dimensional in reference [2]. His proof provides a method of constructing a minimal zero-dimensional extension of a ring within a zero-dimensional ring. His method says that in order to obtain a minimal zero-dimensional extension of R within S, we need to adjoin a set of idempotents of S to R and then take the total quotient ring within S. We refer to this method as Arapović's construction and we present it next. The idempotents we need to adjoin have to satisfy the following conditions.

Lemma 23 *Let S be a ring and $x \in S$. Then the following conditions on an idempotent e_x of S are equivalent.*

1. *$x(1 - e_x)$ is nilpotent and $x + (1 - e_x)$ is invertible,*
2. *$x(1 - e_x)$ is nilpotent and xe_x is invertible in Se_x,*
3. *Some power of Sx is equal to Se_x,*
4. *$x(1 - e_x)$ is nilpotent and $e_x \in Sx$.*

At most one such idempotent exists.

Proof. The equivalence of the first three conditions and the uniqueness of e_x is [4, Lemma 4.1]. Assume that $x(1 - e_x)$ is nilpotent and xe_x is invertible in Se_x. We see that xe_x is invertible in Se_x if and only if there exists $se_x \in Se_x$ such that $xe_xse_x = e_x$. So, $se_xx = e_x$. This is equivalent to $e_x \in Sx$. Hence, the second condition is equivalent to the fourth condition. ■

We note that if x is nilpotent, then $e_x = 0$ and if x is invertible then $e_x = 1$.

We observe that Lemma 23 also holds for an R-algebra S and $x \in R$ if $x + (1 - e_x)$ is interpreted as $x \cdot 1 + (1 - e_x)$. Therefore, Arapović's construction also works to construct a minimal zero-dimensional R-algebra.

Definition 24 *If, for given x and S, the idempotent e_x satisfying the conditions of Lemma 23 exists, then we say that e_x* ***is defined in*** *S. We denote it by e_x^S or omit the superscript when no confusion arises.*

We remark that e_x could be defined in S and be in a subring of S, yet not be defined in that subring. Take for example $\mathbb{Z} \subseteq \mathbb{Q}$. We have e_2 defined in $\mathbb{Q}$ and equal to 1, so it is in $\mathbb{Z}$, but it is not defined in $\mathbb{Z}$.

The following arithmetic characterization of a zero-dimensional ring proved by Arapović is very useful for constructing minimal zero-dimensional extensions. See also [4, Theorem 2.2].

Theorem 25 *A commutative ring S is zero dimensional if and only if e_x is defined in S for every $x \in S$.*

Definition 26 *Let R be a subring of the ring S. By the* ***total quotient ring of*** *R* ***within*** *S we mean the ring of elements x such that there exist $a, b \in R$ with $bx = a$ and b is invertible in S.*

We observe that in general regular elements in R need not be invertible in S even if S is a total quotient ring. Take for example $R = \mathbb{Z}$ and $S = \prod_{p \in P} \mathbb{Z}/p\mathbb{Z}$ where P is the set of all positive prime integers. Since a product of total quotient rings is a total quotient ring, then S is a total quotient ring, by Lemma 5. But, the total quotient ring of $\mathbb{Z}$ within S is $\mathbb{Z}$.

Next we present some results regarding the idempotents satisfying the conditions of Lemma 23, which lead to Arapović's construction.

Let S be a ring. Define on the set of idempotents of S the operations:

$$e \vee f = e + f - ef, \ e \wedge f = ef \text{ and } e' = 1 - e$$

It is easily checked that the set of idempotents with those operations constitutes a Boolean algebra.

Let S be an R-algebra and E a set of idempotents of S. We denote $\overline{E}$ the Boolean algebra generated by E and $R[E]$ the R-subalgebra of S generated by E. We note that $\overline{E} \subseteq R[E]$.

Theorem 27 *Let R be a ring and S and R-algebra. Let E be a set of idempotents of S. Then every element $x \in R[E]$ can be written as $\sum_{i=1}^{n} r_i f_i$ where $r_i \in R$ and $f_i \in \overline{E}$ are orthogonal.*

Proof. Note that $R[E] = R[\overline{E}]$, so we may assume that E is a Boolean algebra. Let $s \in R[E]$. Then $s = \sum_{i=1}^{n} x_i e_i$ where $x_i \in R$ and $e_i \in E$. Let F be the Boolean subalgebra of E generated by $e_1, \ldots, e_n$. Then F contains elements of the form

$$e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \cdots \wedge e_n^{\epsilon_n} \tag{2}$$

where each $e_i^{\epsilon_i}$ is either e_i or the complement e_i'. By Theorem 11, a finitely generated Boolean algebra is finite. We also know that every element of F is a join of finitely many elements like (2). Clearly the elements (2) are orthogonal. Then $s = \sum_{i=1}^{n} r_i f_i$ where $r_i \in R$ and f_i are elements like (2). ■

Theorem 28 *Let S be an R-algebra such that e_x is defined in S for every $x \in R$. Let $E = \{e_x : x \in R\}$. Then:*

1. *e_x is defined in S, and is in $R[E]$, for every $x \in R[E]$,*
2. *Every regular element of $R[E]$ is invertible in S.*

Proof. To prove (1), consider the Boolean algebra $\overline{E}$ generated by the idempotents $\{e_x : x \in R\}$. Every element $x \in R[E]$ can be written as $x = r_1 f_1 + \cdots + r_n f_n$ where $r_i \in R$ and $f_i \in \overline{E}$ are orthogonal, by Theorem 27. Let $e = e_{r_1} f_1 + \cdots + e_{r_n} f_n \in R[E]$. Then

$$x(1-e) = r_1(1-e_{r_1})f_1 + \cdots + r_n(1-e_{r_n})f_n$$

is nilpotent because the elements $r_i(1-e_{r_i})$ are nilpotent. Since $e_{r_i} \in Sr_i$, then $e_{r_i} f_i \in Sr_i f_i$, so $e \in \sum_{i=1}^{n} Sr_i f_i = Sx$.

To prove (2), let x be a regular element of $R[E]$. Claim that x is invertible in S. The e constructed above has the property that $x(1-e)$ is nilpotent and $e \in Sx$. So $x^n(1-e) = 0$ for some positive integer n. Since x is regular, $e = 1$, so $1 \in Sx$. Hence, x is invertible in S. Thus regular elements of $R[E]$ are invertible in S. ■

Lemma 29 *Let S be an R-algebra and $a, b \in R$. If e_a and e_b are defined, then e_{ab} is defined and $e_{ab} = e_a e_b$.*

Proof. If e_a and e_b are defined then $a(1-e_a)$, $b(1-e_b)$ are nilpotent, $e_a \in Sa$ and $e_b \in Sb$. Then $ab(1-e_a)(1-e_b)$, $abe_a(1-e_b)$ and $ab(1-e_a)e_b$ are nilpotent. Since

$$ab(1-e_ae_b) = ab(1-e_a)(1-e_b) + ab[e_a(1-e_b) + (1-e_a)e_b],$$

then $ab(1-e_ae_b)$ is nilpotent. Since $e_a \in Sa$ and $e_b \in Sb$, then $e_ae_b \in Sab$. Hence, e_{ab} is defined and $e_{ab} = e_ae_b$. ■

Now we are ready for Arapović's construction, for which we follow the proof of [4, Theorem 4.2].

Theorem 30 *Let R be a ring and S an R-algebra such that e_x is defined in S for every $x \in R$. Let $E = \{e_x : x \in R\}$. Then the total quotient ring T' of $R[E]$ within S is the minimal zero-dimensional R-algebra within S.*

Proof. By Theorem 28, the regular elements of $R[E]$ are invertible in S.

We observe that T' is contained in any zero-dimensional subring of S containing R since such a ring must be a total quotient ring and must contain $R[E]$ because of the uniqueness of the idempotents e_x. Let $x \in T'$. Then $x = a/b$ where a,$b \in R[E]$ with b regular in $R[E]$. By Theorem 28, e_a is defined in S, and $e_a \in R[E]$. So, $e_a \in T'$. Claim that e_a is defined in T'. Since e_a is defined in S, then $a(1-e_a)$ is nilpotent and $a + (1-e_a)$ is invertible in S. So, $a + (1-e_a)$ is invertible in T'. We observe that for an invertible element $y \in T'$ the idempotent e_y is defined in T' and is equal to 1. By Theorem 28 and Lemma 29, then $e_x = e_{a/b} = e_a$. Hence, e_x is defined in T' for every $x \in T'$, so T' is zero dimensional. ■

We note here that the total quotient ring T' of $R[E]$ within S is the same as the total quotient ring $T(R[E])$ of $R[E]$, by Theorem 28.

Thus, in order to obtain a minimal zero-dimensional R-algebra using Arapović's construction we need to find an R-algebra S such that e_x is defined for every $x \in R$. If S

is zero-dimensional, according to the arithmetic characterization of a zero-dimensional ring, e_x is defined for every $x \in S$. But S need not be zero dimensional, we see next that we can consider $S = \prod_{\lambda \in \Lambda} T(R/Q_\lambda)$, where $\{Q_\lambda\}_{\lambda \in \Lambda}$ is a family of primary ideals of R satisfying A2 of Arapović's Embedding Theorem.

Let R be a ring, $\{Q_\lambda\}_{\lambda \in \Lambda}$ a family of primary ideals of R and $x \in R$. We use A2x when $\{Q_\lambda\}_{\lambda \in \Lambda}$ satisfies Arapović's condition A2 for x.

Lemma 31 *Let R be a ring, $\{Q_\lambda\}_{\lambda \in \Lambda}$ a family of primary ideals of R. For $x \in R$ and an idempotent $e \in S = \prod_{\lambda \in \Lambda} T(R/Q_\lambda)$ the following are equivalent:*

1. *$x(1-e)$ is nilpotent and $x + (1-e)$ is invertible in S;*

2. *The λ-th coordinate of e is 1 if $x \notin \operatorname{rad}(Q_\lambda)$ and 0 if $x \in \operatorname{rad}(Q_\lambda)$, and $\{Q_\lambda\}_{\lambda \in \Lambda}$ satisfies A2x.*

Proof. (1) $\Rightarrow$ (2) Since S is a product of local rings, the λ-th coordinate of e is either 0 or 1. Since $x(1-e)$ is nilpotent, there exists $n \in \mathbb{N}$ such that $x^n(1-e) = 0$. If the λ-th coordinate of e is 0, then $x^n \in Q_\lambda$. So $x \in \operatorname{rad}(Q_\lambda)$. If the λ-th coordinate of e is 1, then x must be invertible in $T(R/Q_\lambda)$. Since $T(R/Q_\lambda)$ is a primary ring with the prime ideal $\operatorname{rad}(Q_\lambda)T(R/Q_\lambda)$, then $x \notin \operatorname{rad}(Q_\lambda)$.

We notice that the λ-th coordinate of $x(1-e)$ is x if $x \in \operatorname{rad}(Q_\lambda)$ and 0 if $x \notin \operatorname{rad}(Q_\lambda)$. Since $x(1-e)$ is nilpotent, for all $\lambda \in \Lambda$ if $x \in \operatorname{rad}(Q_\lambda)$ then $x^n \in Q_\lambda$. Hence, $\{Q_\lambda\}_{\lambda \in \Lambda}$ satisfies A2x.

(2) $\Rightarrow$ (1) First we prove that $x+(1-e)$ is invertible in S. We observe that the λ-th coordinate of $x + (1-e)$ is $x + 1$ if $x \in \operatorname{rad}(Q_\lambda)$ and x if $x \notin \operatorname{rad}(Q_\lambda)$. Since $T(R/Q_\lambda)$ is a primary ring with the prime ideal $\operatorname{rad}(Q_\lambda)T(R/Q_\lambda)$ for every $\lambda \in \Lambda$, $x + (1-e)$ is invertible in S. Since $\{Q_\lambda\}_{\lambda \in \Lambda}$ satisfies A2x, there exists $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$ if $x \in \operatorname{rad}(Q_\lambda)$, then $x^n \in Q_\lambda$. Then, $x^n(1-e) = 0$. So, $x(1-e)$ is nilpotent. ■

Theorem 32 *Let R be a ring, $\{Q_\lambda\}_{\lambda\in\Lambda}$ a family of primary ideals of R. Then $\prod_\lambda T(R/Q_\lambda)$ contains a zero-dimensional R-subalgebra if and only if $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2.*

Proof. Let $S = \prod_\lambda T(R/Q_\lambda)$. Assume first that $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2. In order to prove that S contains a zero-dimensional R-subalgebra, it is enough to prove that for every $x \in R$ there exists the idempotent $e_x \in S$ such that $x(1-e_x)$ is nilpotent and $x + (1-e_x)$ is invertible in S. Then, by Arapović's construction, the total quotient ring of the R-algebra generated by the idempotents $\{e_x : x \in R\}$ is a zero-dimensional R-subalgebra of S. Let x be an element of R and the idempotent $e_x \in S$ with λ-th coordinate 0 if $x \in \mathrm{rad}(Q_\lambda)$ and 1 if $x \notin \mathrm{rad}(Q_\lambda)$. By Lemma 31, since $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2 we have $x(1-e_x)$ is nilpotent and $x + (1-e_x)$ is invertible in S.

Conversely, assume that S contains a zero-dimensional R-subalgebra A. Since A is zero dimensional, for every $x \in R$ there exists an idempotent $e_x \in A$ such that $x(1-e_x)$ is nilpotent and $x + (1-e_x)$ is invertible in A. Since A is a subring of S, $x(1-e_x)$ is nilpotent and $x + (1-e_x)$ is invertible in S. Then by Lemma 31 for all $\lambda \in \Lambda$ if $x \in \mathrm{rad}(Q_\lambda)$ then $x^n \in Q_\lambda$. Hence, the family $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2. ∎

Theorem 32 not only says that e_x is defined in $S = \prod_{\lambda\in\Lambda} T(R/Q_\lambda)$ for every $x \in R$ if $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2 but also it gives a criterion of embeddability of a zero-dimensional R-algebra into the product $\prod_{\lambda\in\Lambda} T(R/Q_\lambda)$.

Clearly, if Λ is finite, for any ring R a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals of R satisfies A2. We show next that, for a ring R with Noetherian spectrum, only one minimal prime ideal and only finitely many primes of height greater than one, a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of ideals primary for distinct prime ideals of R satisfies A2.

Theorem 33 *Let R be a ring with Noetherian spectrum, only one minimal prime and only finitely many primes of height greater than one. Let $\{Q_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals primary for distinct prime ideals of R. Then $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2.*

Proof. We need to prove that for each $x \in R$, there is $n \in \mathbb{N}$ such that for all $\lambda \in \Lambda$, if $x \in \operatorname{rad}(Q_\lambda)$, then $x^n \in Q_\lambda$. Since R has only one minimal prime, then the nilradical of R is prime. If x is nilpotent, then x is contained in $\operatorname{rad}(Q_\lambda)$ for every $\lambda \in \Lambda$. So, $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2x for every nilpotent element $x \in R$. Let x be a nonnilpotent element of R. Since R has Noetherian spectrum, then (x) has finitely many minimal prime ideals, by Theorem 34. It follows that x is contained in only finitely many primes of height greater than zero, so x is contained in finitely many $\operatorname{rad}(Q_\lambda)$ where $\lambda \in \Lambda$. Hence $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2x for every nonnilpotent element $x \in R$. ■

Therefore, for a ring R with Noetherian spectrum, only one minimal prime ideal and only finitely many primes of height greater than one, using Arapović's construction, we obtain a minimal zero-dimensional R-algebra within $S = \prod_{\lambda\in\Lambda} T(R/Q_\lambda)$ where $\{Q_\lambda\}_{\lambda\in\Lambda}$ is a family of ideals primary for distinct prime ideals of R.

Next we present some examples of Arapović's construction for $R = \mathbb{Z}$.

- Let S be a field of characteristic zero. Let $x \in \mathbb{Z}$. If $x \neq 0$, then $e_x = 1$, and $e_0 = 0$. It follows that the ring $R[E]$ is $\mathbb{Z}$, and its total quotient ring is $\mathbb{Q}$.

- Let

$$S = \mathbb{Q} \times \prod_{i=1}^{k} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

  where $p_1, p_2, \ldots, p_k$ are distinct positive prime integers and n_i are positive integers. Let $x \in \mathbb{Z}$. If $x \neq 0$, then e_x has the first coordinate 1 and the i-th coordinate 0 if p_i divides x and 1 if p_i does not divide x. Then $R[E] = \mathbb{Z} \times \prod_{i=1}^{k} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$ and its total quotient ring is S.

- Let

$$S = \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

  where $p_1, p_2, \ldots$ are distinct positive prime integers and n_i are positive integers. Let $x \in \mathbb{Z}$. The idempotent e_x has the i-th coordinate 0 if p_i divides x and 1 if

p_i does not divide x. Then the minimal zero-dimensional extension of $\mathbb{Z}$ within S is

$$\{s \in S : \text{there exist } n, m \in \mathbb{Z} \text{ such that } ns = m \text{ and } n \neq 0\}.$$

We remark that S is a product of zero-dimensional rings but S is not zero dimensional unless the n_i are bounded. In fact, Gilmer and Heinzer proved that an infinite product of zero-dimensional rings is either zero-dimensional or infinite-dimensional ([10]).

3.2 PS-Rings

Throughout this section by a PS-ring we mean a ring with Noetherian spectrum in which zero is primary. In section 3.1 we proved that for a ring R with Noetherian spectrum with only one minimal prime and only finitely many primes of height greater than one, using Arapović's construction, we obtain a minimal zero-dimensional R-algebra within $\prod_{\lambda \in \Lambda} T(R/Q_\lambda)$ where $\{Q_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals primary for distinct prime ideals of R. In this section we determine the structure of the minimal zero-dimensional extensions of PS-rings with at most one prime ideal of height greater than one and we show that all the minimal zero-dimensional extensions of such ring R can be obtained using Arapović's construction and are subrings of $\prod_{\lambda \in \Lambda} T(R/Q_\lambda)$ where $\{Q_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals primary for distinct prime ideals of R.

For our specific purposes more characterizations of Spec(R) being Noetherian are needed.

Theorem 34 ([15, Proposition 2.1]) *For a commutative ring R the following are equivalent:*

a. Spec(R) *is Noetherian,*

b. Every prime ideal is the radical of a finitely generated ideal, *c. The ascending chain condition holds for prime ideals and each ideal of R has only finitely many minimal prime ideals.*

Definition 35 *An ideal of a ring is said to be* ***nil*** *if it consists only of nilpotent elements.*

Generalizing Lemma 23 we introduce the idempotent corresponding to an ideal.

Theorem 36 *Let R be a ring, S an R-algebra and I an ideal of R. Then there is at most one idempotent $e \in S$ such that $I(1-e)$ is nil and $e \in SI$.*

Proof. To see that the idempotent $e \in S$ is unique, assume e and f are idempotents of S such that $e \in SI$ and $I(1-f)$ is nil. Since $e \in SI$, then $e = \sum_{i=1}^{k} s_i a_i$ where $s_i \in S$ and $a_i \in I$. We have that $e = (\sum_{i=1}^{k} s_i a_i)e = (\sum_{i=1}^{k} s_i a_i)(1-f)e + (\sum_{i=1}^{k} s_i a_i)fe$. Since $I(1-f)$ is nil, then $x(1-f)$ is nilpotent for every $x \in I$. Therefore, $(\sum_{i=1}^{k} s_i a_i)(1-f)e$ is nilpotent. It follows that $e = (\sum_{i=1}^{k} s_i a_i)^n fe$ for some positive integer n. Hence, $e \leq f$. By symmetry, the idempotent e is unique. ∎

Whenever there is an idempotent e as in Theorem 36 we denote it by e_I.

We note that if I is nil, then $e_I = 0$. Next we give some properties of e_I.

Lemma 37 *Let R be a ring, S and R-algebra and $I = (x_1, x_2, ..., x_k)$ a finitely generated ideal of R. If e_{x_i} is defined for $i = 1, \ldots, k$, then e_I is defined and $e_I = e_{x_1} \vee e_{x_2} \vee \cdots \vee e_{x_k}$.*

Proof. Let $e = e_{x_1} \vee e_{x_2} \vee \cdots \vee e_{x_k}$. Claim that $I(1-e)$ is nil. Since $x_i(1-e_{x_i})$ is nilpotent for $i = 1, ..., k$, then $x_i(1-e) = x_i(1-e_{x_1})(1-e_{x_2})\cdots(1-e_{x_k})$ is nilpotent for $i = 1, ..., k$. Hence, $x(1-e)$ is nilpotent for every $x \in I$, so $I(1-e)$ is nil. Claim that $e \in SI$. Since $e_{x_i} \in Sx_i$ for $i = 1, ..., k$, then $e \in SI$. Hence, e_I is defined and $e_I = e$. ∎

Lemma 38 *Let R be a ring, S an R-algebra and I, J ideals of R. If e_I and e_J are defined, then e_{IJ} is defined and $e_{IJ} = e_I e_J$.*

Proof. Since e_I and e_J are defined, then $I(1-e_I)$, $J(1-e_J)$ are nil, $e_I \in SI$ and $e_J \in SJ$. Let $x = ab$ where $a \in I$ and $b \in J$. Then, $ab(1-e_I)(1-e_J)$, $abe_I(1-e_J)$ and $ab(1-e_I)e_J$ are nilpotent. But,

$$ab(1 - e_I e_J) = ab(1 - e_I)(1 - e_J) + ab[e_I(1 - e_J) + (1 - e_I)e_J].$$

Hence, $ab(1-e_Ie_J)$ is nilpotent. Since any element of IJ is a finite sum of elements of the form ab, where $a \in I$ and $b \in J$ and a finite sum of nilpotents is a nilpotent, then every element of $IJ(1-e_Ie_J)$ is nilpotent. Hence, $IJ(1-e_Ie_J)$ is nil. Since $e_I \in SI$ and $e_J \in SJ$, then $e_Ie_J \in SIJ$. Hence, e_{IJ} is defined and $e_{IJ} = e_Ie_J$. ■

Lemma 39 *Let R be a ring, S an R-algebra and I an ideal of R. Then e_I is defined if and only if $e_{\mathrm{rad}(I)}$ is defined, and $e_I = e_{\mathrm{rad}(I)}$.*

Proof. Assume e_I is defined. Then, $I(1-e_I)$ is nil and $e_I \in SI$. Let $x \in \mathrm{rad}(I)$. Then, $x^n \in I$ for some positive integer n. Then, $x^n(1-e_I)$ is nilpotent so, $x(1-e_I)$ is nilpotent. Hence, $\mathrm{rad}(I)(1-e_I)$ is nil. We have $e_I \in SI \subset S\,\mathrm{rad}(I)$. Then, $e_{\mathrm{rad}(I)}$ is defined and $e_{\mathrm{rad}(I)} = e_I$.

Conversely, assume that $e_{\mathrm{rad}(I)}$ is defined. Then, $\mathrm{rad}(I)(1-e_{\mathrm{rad}(I)})$ is nil and $e_{\mathrm{rad}(I)} \in S\,\mathrm{rad}(I)$. Since $I \subset \mathrm{rad}(I)$, then $I(1-e_{\mathrm{rad}(I)})$ is nil. Since $e_{\mathrm{rad}(I)} \in S\,\mathrm{rad}(I)$ and $e_{\mathrm{rad}(I)}$ is an idempotent, $e_{\mathrm{rad}(I)} \in SI$. Then, e_I is defined and $e_I = e_{\mathrm{rad}(I)}$. ■

Lemma 40 *Let R be a ring, S an R-algebra and I, J ideals of R. If e_I and e_J are defined, then e_{I+J} is defined and $e_{I+J} = e_I \vee e_J$.*

Proof. Since e_I and e_J are defined, then $I(1-e_I)$, $J(1-e_J)$ are nil, $e_I \in SI$ and $e_J \in SJ$. Let $x+y \in I+J$ where $x \in I$ and $y \in J$. Since $I(1-e_I)$, $J(1-e_J)$ are nil, there exist positive integers n and m such that $x^n(1-e_I)=0$ and $y^m(1-e_J)=0$. Therefore $(x+y)^{n+m}(1-e_I \vee e_J) = (x+y)^{n+m}(1-e_I-e_J+e_Ie_J) = (x+y)^{n+m}(1-e_I)(1-e_J) = 0$. Since $e_I \in SI$ and $e_J \in SJ$, then $e_I \vee e_J = e_I + e_J - e_Ie_J \in S(I+J)$. Hence, e_{I+J} is defined and $e_{I+J} = e_I \vee e_J$. ■

Let R be a subring of a ring S. Assume that e_x is defined in S for every $x \in R$. In order to obtain a minimal zero-dimensional extension of R using Arapović's construction, we need to adjoin the idempotents $e_x \in S$ to R and then take the total quotient ring. We prove that for rings with Noetherian spectrum, if e_x is defined for every $x \in R$, we adjoin the idempotents e_P for every $P \in \operatorname{Spec}(R)$, and then take the total quotient ring.

Theorem 41 *Let R be a ring with Noetherian spectrum and S an R-algebra. Then:*

a. If e_x is defined for every $x \in R$, then e_P is defined for every $P \in \operatorname{Spec}(R)$,

b. If e_P is defined for every $P \in \operatorname{Spec}(R)$, then e_x is defined for every $x \in R$,

c. Assume that hypothesis of (a) or (b) hold. Let $E_1 = \{e_x : x \in R\}$ and $E_2 = \{e_P : P \in \operatorname{Spec}(R)\}$. Then $R[E_1] = R[E_2]$.

Proof. To show (a), we observe that since R has Noetherian spectrum, every radical ideal is the radical of a finitely generated ideal, by Theorem 34. Let $P = \operatorname{rad}(I)$ where $P \in \operatorname{Spec}(R)$ and $I = (x_1, x_2, ..., x_k)$ is a finitely generated ideal of R. From Lemmas 37 and 39 it follows that e_P and e_I are defined and $e_P = e_I = e_{x_1} \vee e_{x_2} \vee \cdots \vee e_{x_k}$.

To prove (b), let $x \in R$ and $\mathcal{P}$ the set of minimal prime ideals containing x which, by Theorem 34, is finite. Using Lemmas 38 and 39 we have

$$\prod\nolimits_{P\in\mathcal{P}} e_P = e_{\Pi_{P\in\mathcal{P}} P} = e_{\Pi_{P\in\mathcal{P}} \operatorname{rad}(P)} = e_{\operatorname{rad}(\cap_{P\in\mathcal{P}} P)} = e_{\cap_{P\in\mathcal{P}} P} = e_{\operatorname{rad}(x)} = e_x.$$

Hence, e_x is defined in S and is a finite product of e_P where $P \in \mathcal{P}$.

The proof of (c) is straightforward. ■

Definition 42 *A* ***rng*** *(pronounced rung) is a commutative ring that does not necessarily have an identity.*

The definition of a rng was introduced by Jacobson ([14]).

Definition 43 *Let R be a ring. A rng A is an R-**rng** if A is an R-module and $a(rs) = (ar)s = r(as)$ for all $a \in R$ and $r, s \in A$.*

Every rng can be turned into a ring by adjoining an identity element. A canonical way to do this is by using the following theorem.

Theorem 44 *Let R be a ring and A an R-rng. On $A^* = R \times A$ define $(r_1, a_1) + (r_2, a_2) = (r_1 + r_2, a_1 + a_2)$ and $(r_1, a_1)(r_2, a_2) = (r_1 r_2, r_1 a_2 + a_1 r_2 + a_1 a_2)$. Then A^* is an R-algebra with identity $(1, 0)$.*

Proof. Straightforward. ■

From now on we use A^* to denote the R-algebra obtained from an R-rng A by adjoining an identity as in Theorem 44.

Lemma 45 *Let R be a ring, B an R-algebra with identity and C an R-rng. Then $(B \oplus C)^*$ is isomorphic to $B \times C^*$.*

Proof. By Theorem 44, $(B \oplus C)^* = R \times (B \oplus C)$ is an R- ring with identity $(1, 0, 0)$ and multiplication

$$(r_1, b_1, c_1)(r_2, b_2, c_2) = (r_1 r_2, r_1 b_2 + b_1 b_2 + r_2 b_1, r_1 c_2 + c_1 c_2 + r_2 c_1).$$

Also, $B \times C^* = B \times R \times C$ is an R-ring with identity $(1, 1, 0)$ and multiplication

$$(b_1, r_1, c_1)(b_2, r_2, c_2) = (b_1 b_2, r_1 r_2, r_1 c_2 + c_1 r_2 + c_1 c_2)$$

Define $f : (B \oplus C)^* \to B \times C^*$ by $f(r, b, c) = (r + b, r, c)$ and $g : B \times C^* \to (B \oplus C)^*$ by $g(b, r, c) = (r, b - r, c)$. It is easy to check that f and g are R-ring homomorphisms. We only show that $f((r_1, b_1, c_1)(r_2, b_2, c_2)) = f(r_1, b_1, c_1)f(r_2, b_2, c_2)$ where $(r_1, b_1, c_1), (r_2, b_2, c_2) \in (B \oplus C)^*$. We have

$$f((r_1, b_1, c_1)(r_2, b_2, c_2)) = f(r_1r_2, r_1b_2 + b_1b_2 + r_2b_1, r_1c_2 + c_1c_2 + r_2c_1) =$$

$$(r_1r_2 + r_1b_2 + b_1b_2 + r_2b_1, r_1r_2, r_1c_2 + c_1c_2 + r_2c_1) =$$

$$((r_1 + b_1)(r_2 + b_2), r_1r_2, r_1c_2 + c_1c_2 + r_2c_1) =$$

$$(r_1 + b_1, r_1, c_1)(r_2 + b_2, r_2, c_2) = f(r_1, b_1, c_1)f(r_2, b_2, c_2)$$

Then $f(g(b, r, c)) = f(r, b-r, c) = (b, r, c)$ for any $(b, r, c) \in B \times C^*$ and $g(f(r, b, c)) = g(r + b, r, c) = (r, b, c)$ for any $(r, b, c) \in (B \oplus C)^*$. Hence, $(B \oplus C)^*$ is isomorphic to $B \times C^*$. ■

Definition 46 *Let R be a ring and I and ideal of R. We say that I is a* ***regular ideal*** *if I contains a regular element of R.*

Next we present a chain of lemmas which leads to the main results.

Lemma 47 *Let R be a ring and $\{A_\alpha\}_{\alpha \in I}$ be a finite family of R-algebras. Let $A = \prod_{\alpha \in I} A_\alpha$ and $x \in R$. If $e_x^{A_\alpha}$ is defined for every $\alpha \in I$, then e_x^A is defined and $e_x^A = \prod_{\alpha \in I} e_x^{A_\alpha}$.*

Proof. Since $e_x^{A_\alpha}$ is defined , then $x(1 - e_x^{A_\alpha})$ is nilpotent and $e_x^{A_\alpha} \in A_\alpha x$ for every $\alpha \in I$. Since I is finite, there exists a positive integer n such that $x^n(1 - \prod_{\alpha \in I} e_x^{A_\alpha}) = \prod_{\alpha \in I} x^n(1 - e_x^{A_\alpha}) = 0$. Clearly, $\prod_{\alpha \in I} e_x^{A_\alpha} \in Ax$. Hence, e_x^A is defined and $e_x^A = \prod_{\alpha \in I} e_x^{A_\alpha}$. ■

Lemma 48 *Let R be a ring with Noetherian spectrum and with only finitely many prime ideals of height greater than one, and S an R-algebra such that e_x is defined for every $x \in R$. Let P be a prime ideal of R of height greater than zero such that $e_M = 1$ for every prime ideal M properly containing P. If $e_P \neq 1$, then $\mathrm{ann}_R(1 - e_P)$ is P-primary.*

Proof. Let P be a prime ideal of R of height greater than zero such that $e_M = 1$ for every prime ideal M properly containing P and $e_P \neq 1$. Let $\mathrm{ann}_R(1 - e_P) = Q$. Let $x, y \in R$ such that $xy \in Q$. We assume that $y \notin P$ and we prove that $x \in Q$. Let $\mathcal{P}$ be the set of prime ideals M of R such that $(y) + P \subseteq M$. Since R has finitely many prime ideals of height greater than one, $\mathcal{P}$ is finite. By Lemmas 38 and 39 we have

$$e_{(y)+P} = e_{\mathrm{rad}((y)+P)} = e_{\bigcap_{M\in\mathcal{P}} M} =$$

$$e_{\mathrm{rad}(\bigcap_{M\in\mathcal{P}} M)} = e_{\prod_{M\in\mathcal{P}} \mathrm{rad}(M)} = \prod_{M\in\mathcal{P}} e_M = 1$$

By Lemma 40, $e_{(y)+P} = e_y \vee e_P = 1$. Then $(1 - e_y)(1 - e_P) = 1 - e_y - e_P + e_y e_P = 0$. Since $e_y \in Sy$, then $e_y = sy$ for some $s \in S$. It follows that $0 = sxy(1 - e_P) = xe_y(1 - e_P) = x(1-(1-e_y))(1-e_P) = x(1-e_P)$. Hence $x \in Q$, and so Q is P-primary. ■

Lemma 49 *Let R be a ring with Noetherian spectrum and with only one minimal prime ideal. Let $\mathcal{Q}$ be a set of ideals that are primary for distinct prime ideals of height at most one and $S = \prod_{Q\in\mathcal{Q}} T(R/Q)$. Then e_P is defined in S for every $P \in \mathrm{Spec}(R)$ and e_P has the Q-th coordinate 0 if $P \subseteq \mathrm{rad}(Q)$ and 1 if $P \not\subseteq \mathrm{rad}(Q)$ for every $P \in \mathrm{Spec}(R)$.*

Proof. Let $P_0 = \mathrm{rad}(0)$. Then $e_{P_0} = e_{\mathrm{rad}(0)} = e_0 = 0$. Since R has only one minimal prime ideal P_0 and $e_{P_0} = 0$, we only have to prove that e_P is defined in S for every prime ideal P of height greater than zero. We prove that $e_x \in S$ is defined for every $x \in R$. If $x \in R$ is nilpotent, it is clearly that $e_x = 0$ is defined in S. Let $x \in R$ be nonnilpotent. Claim that $e_x \in S$ has the Q-th coordinate 0 if $x \in \operatorname{rad}(Q)$ and 1 if $x \notin \operatorname{rad}(Q)$ for every $Q \in \mathcal{Q}$. Since R has Noetherian spectrum, then (x) has finitely many minimal prime ideals by Theorem 34. It follows that x is contained in only finitely many $\operatorname{rad}(Q)$ where $Q \in \mathcal{Q}$. Therefore there exists a positive integer n such that $x^n(1-e_x) = 0$. Clearly, $e_x \in Sx$. Hence, $e_x \in S$ is defined for every $x \in R$. Let P be a prime ideal of height greater than zero. Since R has Noetherian spectrum, then $P = \operatorname{rad}(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k \in R$. Then e_P is defined in S and $e_P = e_{\operatorname{rad}(x_1,\ldots,x_k)} = e_{(x_1,\ldots,x_k)} = e_{x_1} \vee e_{x_2} \vee \cdots \vee e_{x_k}$, by Lemmas 37 and 39. It follows that $1 - e_P = 1 - e_{x_1} \vee e_{x_2} \vee \cdots \vee e_{x_k} = (1-e_{x_1})(1-e_{x_2})\cdots(1-e_{x_k})$ has the Q-th coordinate 1 if $(x_1, \ldots, x_k) \subseteq \operatorname{rad}(Q)$ and 0 if $(x_1, \ldots, x_k) \nsubseteq \operatorname{rad}(Q)$. Therefore e_P has the Q-th coordinate 0 if $P \subseteq \operatorname{rad}(Q)$ and 1 if $P \nsubseteq \operatorname{rad}(Q)$. ■

Lemma 50 *Let R be a PS-ring with only finitely many prime ideals of height greater than one and S an R-algebra such that e_P is defined for every $P \in \operatorname{Spec}(R)$ and $e_M = 1$ for every prime ideal M of R of height greater than one. Let P, Q be two distinct prime ideals of R of height one. Then $f_P f_Q = 0$.*

Proof. Let $\mathcal{P}$ be the set of prime ideals M of R such that $P + Q \subseteq M$. We have

$$e_{P+Q} = e_{\operatorname{rad}(P+Q)} = e_{\bigcap_{M\in\mathcal{P}} M} =$$

$$e_{\operatorname{rad}(\bigcap_{M\in\mathcal{P}} M)} = e_{\prod_{M\in\mathcal{P}} \operatorname{rad}(M)} = \textstyle\prod_{M\in\mathcal{P}} e_M = 1$$

Then $f_P f_Q = (1-e_P)(1-e_Q) = 1 - e_P - e_Q + e_P e_Q = 1 - e_P \vee e_Q = 1 - e_{P+Q} = 0$. ■

Lemma 51 *Let R be a ring, $\mathcal{M}$ a set of regular ideals of R, and S an extension ring of R. For every $Q \in \mathcal{M}$, let f_Q be an idempotent of S such that $\operatorname{ann}_R(f_Q) = Q$. Assume* *that $f_P f_Q = 0$ for any two distinct ideals P and Q of $\mathcal{M}$. Let $R[E]$ be the subring of S generated by R and the idempotents $E = \{f_Q \in S : Q \in \mathcal{M}\}$. Then $R[E]$ is isomorphic to $(\bigoplus_{Q\in\mathcal{M}} R/Q)^*$.*

Proof. Claim that every $x \in R[E]$ can be written uniquely as

$$x = r + \textstyle\sum_{Q\in\mathcal{M}} r_Q f_Q$$

where $r \in R, r_Q \in R/Q$ and only finitely many r_Q are nonzero. Since $\mathrm{ann}_R(f_Q) = Q$ for every $Q \in \mathcal{M}$ and $f_P f_Q = 0$ for $P \neq Q$ and $P, Q \in \mathcal{M}$, every $x \in R[E]$ can be written as

$$x = r + \textstyle\sum_{Q\in\mathcal{M}} r_Q f_Q.$$

Assume that

$$r + \textstyle\sum_{Q\in\mathcal{M}} r_Q f_Q = s + \sum_{Q\in\mathcal{M}} s_Q f_Q. \tag{3}$$

Let $\mathcal{M}_1$ be a (finite) set of ideals $Q \in \mathcal{M}$ such that $r_Q \neq 0$ or $s_Q \neq 0$. Let t be a product of regular elements of R, one from each $Q \in \mathcal{M}_1$. Multiplying (3) by t, we obtain $rt = st$. Since t is a regular element in R, then $r = s$, so we have

$$\textstyle\sum_{Q\in\mathcal{M}} r_Q f_Q = \sum_{Q\in\mathcal{M}} s_Q f_Q. \tag{4}$$

Let $Q \in \mathcal{M}_1$. Multiply (4) by f_Q. Since $f_P f_Q = 0$ for $P \neq Q$ and f_Q is an idempotent, then $(r_Q - s_Q)f_Q = 0$. So, $(r_Q - s_Q)f_Q = 0$ for all $Q \in \mathcal{M}_1$. Therefore, $r_Q = s_Q$ in R/Q for all $Q \in \mathcal{M}_1$, because $\mathrm{ann}_R(f_Q) = Q$.

Define $f : R[E] \to (\bigoplus_{Q\in\mathcal{M}} R/Q)^* = R \times (\bigoplus_{Q\in\mathcal{M}} R/Q)$ as follows: if $x = r + \sum_{Q\in\mathcal{M}} r_Q f_Q$, then $f(x) = (r, \sum_{Q\in\mathcal{M}} r_Q)$ where $r_Q \in R/Q \subseteq \bigoplus_{Q\in\mathcal{M}} R/Q$. Since x has unique representation, f is well defined. Clearly f is a ring isomorphism. ■

Now we are ready for the main results.

Let R be a ring and S an R-algebra. We denote by S' the R-subalgebra of S consisting of all those $a \in S$ such that $ta = u \cdot 1$ for some $t, u \in R$ with t regular in R.

Theorem 52 *Let R be a PS-ring with only finitely many prime ideals of height greater than one. Let $\mathcal{Q}$ be a set of ideals of R primary for distinct prime ideals of height at most one such that $\bigcap_{Q \in \mathcal{Q}} Q = 0$ and $S = \prod_{Q \in \mathcal{Q}} T(R/Q)$. Let $\mathcal{M}$ be the subset of $\mathcal{Q}$ consisting of ideals that are primary for prime ideals of height one. Then $A = T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*)$ is a minimal zero-dimensional extension of R, $e_M^A = 1$ for every prime ideal M of R of height greater than one and*

$$\mathcal{M} = \{\mathrm{ann}_R(1 - e_P^A) : e_P^A \neq 1 \text{ and } P \in \mathrm{Spec}(R) \text{ with } \mathrm{ht}(P) = 1\}.$$

Moreover, $A \cong S'$.

Proof. Since $\bigcap_{Q \in \mathcal{Q}} Q = 0$, the natural ring homomorphism from R into S is one-to-one, so S is an extension of R. By Lemma 49, e_P is defined in S for every $P \in \mathrm{Spec}(R)$. For every prime ideal P of height greater than zero we have that $e_P = e_Q$ if $P = \mathrm{rad}(Q)$ for some $Q \in \mathcal{Q}$ or $e_P = 1$ if $P \neq \mathrm{rad}(Q)$ for every $Q \in \mathcal{M}$. Therefore $e_M = 1$ for every prime ideal M of R of height greater than one. Let $f_Q = 1 - e_Q$ for $Q \in \mathcal{M}$. Then, f_Q has the Q-th coordinate 1 and others 0, so $\mathrm{ann}_R(f_Q) = Q$. Let $E = \{f_Q \in S : Q \in \mathcal{M}\}$. Let $B = \bigoplus_{Q \in \mathcal{M}} R/Q$. Since the zero ideal of R is primary, then all the prime ideals of R of height greater than zero are regular. So $\mathcal{M}$ is a set of regular ideals. By Lemmas 51 and 50, $R[E] \cong B^*$. Then by Theorems 30 and 41(c), $T(B^*)$ is a minimal zero-dimensional extension of R. Since $e_P^{T(R[E])}$ is defined for every $P \in \mathrm{Spec}(R)$ and $T(R[E]) \cong T(B^*)$, then $e_P^{T(B^*)}$ is defined for every $P \in \mathrm{Spec}(R)$. So $1 = e_M^{T(B^*)}$ for every prime ideal M of R of height greater than one and $\mathrm{ann}_R(f_Q) = Q$ for every $Q \in \mathcal{M}$.

Let C be the ring of elements $x \in S$ such that $x_Q = r + Q$ for some $r \in R$ and all $Q \in \mathcal{Q} \setminus \mathcal{F}$ where $\mathcal{F} \subseteq \mathcal{M}$ is finite. Clearly $R[E] = C$. By Lemmas 51 and 50, $R[E] \cong B^*$, so $C \cong B^*$. By Theorem 28, every regular element of C is invertible in S, so $T(C) \subseteq S$. Claim that $T(C) = S'$. Let $x \in T(C)$. Then there exist $a, b \in C$ such that $bx = a$ and b regular in C. So, b is invertible in S. We also have $a_Q = r + Q$ for some $r \in R$ and all $Q \in \mathcal{Q} \setminus \mathcal{F}$ where $\mathcal{F} \subseteq \mathcal{M}$ is finite and $b_Q = s + Q$ for some $s \in R$ and all $Q \in \mathcal{Q} \setminus \mathcal{G}$ where $\mathcal{G} \subseteq \mathcal{M}$ is finite.

We prove that s is regular in R. Claim that $\mathcal{Q} \setminus \mathcal{G}$ is nonempty. Assume that $\mathcal{Q} \setminus \mathcal{G}$ is empty. Then $\bigcap_{Q \in \mathcal{G}} Q = 0$, so minimal prime ideals of R have height one, which is a contradiction with the fact that zero is primary. Hence, there exists an $N \in \mathcal{Q} \setminus \mathcal{G}$. Since b is invertible in S, then s is regular in $T(R/Q)$ for every $Q \in \mathcal{Q} \setminus \mathcal{G}$. So, $s \notin \mathrm{rad}(N)$. It follows that $s \notin \mathrm{rad}(0)$. Since the zero ideal of R is primary, $\mathrm{rad}(0)$ is the set of all zero-divisors of R. Hence, s is regular in R.

Then $sx_Q = r + Q$ for every $Q \in \mathcal{Q} \setminus (\mathcal{F} \cup \mathcal{G})$. Since every $Q \in \mathcal{M}$ is primary for a primary ideal of height one and the zero ideal of R is primary, then every $Q \in \mathcal{M}$ is regular. Let q be a product of regular elements, one in each $Q \in \mathcal{F} \cup \mathcal{G}$. Let $u = rq$ and $t = sq$. Then $tx = u \cdot 1$ in S and t is regular in R, so $x \in S'$.

Conversely, let $x \in S'$. Then, there exist $t, u \in R$ such that $tx = u \cdot 1$ in S and t is regular in R. Since t is regular, t is contained in only finitely many prime ideals of height one. We can consider $b \in C$ with $b_Q = 1 + Q$ if $t \in \mathrm{rad}(Q)$ and $b_Q = t + Q$ if $t \notin \mathrm{rad}(Q)$. Then b_Q is regular in $T(R/Q)$ for every $Q \in \mathcal{Q}$, so b is regular in S. Let $a \in C$ such that $a_Q = x_Q$ if $t \in \mathrm{rad}(Q)$ and $a_Q = u + Q$ if $t \notin \mathrm{rad}(Q)$. Then $b_Q x_Q = x_Q = a_Q$ if $t \in \mathrm{rad}(Q)$ and $b_Q x_Q = t x_Q = u + Q = a_Q$ if $t \notin \mathrm{rad}(Q)$. So, $bx = a$ and $x \in T(C)$. ■

Theorem 53 *Let R be a PS-ring with only finitely many prime ideals of height greater than one. Let A be a minimal zero-dimensional extension of R such that $e_M = 1$ for* *every prime ideal M of R of height greater than one. Then the set*

$$\mathcal{M} = \{\mathrm{ann}_R(1 - e_P) : 1 \neq e_P \text{ and } P \in \mathrm{Spec}(R) \text{ with } \mathrm{ht}(P) = 1\}$$

consists of ideals of R primary for distinct prime ideals of height one, and A is isomorphic to $T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*)$.

Proof. Since A is zero dimensional, e_x is defined for every $x \in A$. By Theorem 41(a), every e_P is defined for every $P \in \mathrm{Spec}(R)$. Let P be a prime ideal of R of height one. If $e_P \neq 1$, $\mathrm{ann}_R(1 - e_P)$ is P-primary by Lemma 48.

Let $\mathcal{M}$ be the set of primary ideals of R that are annihilators of nonzero $1 - e_P$. Let E be the set of idempotents $1 - e_P$ such that $\mathrm{ann}_R(1 - e_P) \in \mathcal{M}$. Since R is a ring in which zero is primary, then every prime ideal of R of height greater than zero is regular. Therefore, every element of $\mathcal{M}$ is regular. By Lemmas 51 and 50, $R[E] \cong (\bigoplus_{Q \in \mathcal{M}} R/Q)^*$. Then by Theorems 30 and 41(c), $A = T(R[E]) \cong T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*)$.
■

We note that if $\mathcal{M}$ is empty, then $\bigoplus_{Q \in \mathcal{M}} R/Q = 0, (\bigoplus_{Q \in \mathcal{M}} R/Q)^* = R$ and $A = T(R)$.

Let R be a one-dimensional PS-ring and let M be a maximal ideal of R. Theorem 53 implies that there is a minimal zero-dimensional extension of R associated to each M-primary ideal Q. Since there exists infinitely many distinct M-primary ideals, there exists infinitely many non-isomorphic minimal zero-dimensional extensions of R.

In summary, for a one-dimensional PS-ring R, Theorems 52 and 53 state the following.

Corollary 54 *Let R be a one-dimensional PS-ring. There is a one-to-one correspondence between isomorphism classes of minimal zero-dimensional extensions of R and sets $\mathcal{M}$, where the elements of $\mathcal{M}$ are primary ideals for distinct maximal ideals of R.*

The sets of ideals primary for distinct maximal ideals are complete sets of invariants of the minimal zero-dimensional extensions of R.

Proof. Let $\mathcal{A}$ be the class of minimal zero-dimensional extensions of R and $\mathcal{I}$ be the set of sets of ideals of R that are primary for distinct maximal ideals. Let $\mathcal{M} \in \mathcal{I}$. If we take $\mathcal{Q} = \mathcal{M} \cup \{0\}$ in Theorem 52, then $T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)$ is a minimal zero-dimensional extension of R. Define $F : \mathcal{I} \to \mathcal{A}$ by

$$F(\mathcal{M}) = T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)$$

where $\mathcal{M} \in \mathcal{I}$.

Now, let $A \in \mathcal{A}$. By Theorem 53, the set

$$G(A) = \{\mathrm{ann}_R(1 - e_M) : 1 \neq e_M \in A \text{ and } M \in \mathrm{MaxSpec}(R)\}$$

is in $\mathcal{I}$, and $T((\bigoplus_{Q\in G(A)} R/Q)^*) \cong A$. Then, for $A \in \mathcal{A}$ we have

$$FG(A) = T((\bigoplus_{Q\in G(A)} R/Q)^*) \cong A.$$

For $\mathcal{M} \in \mathcal{I}$, we have

$$GF(\mathcal{M}) = G(T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)) =$$

$$\{\mathrm{ann}_R(1 - e_M) : 1 \neq e_M \in T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*) \text{ and } M \in \mathrm{MaxSpec}(R)\} = \mathcal{M},$$

by Theorem 52.

The fact that the sets of ideals primary for distinct maximal ideals are complete sets of invariants of the minimal zero-dimensional extensions of R follows immediately.

■

Examples of one-dimensional PS-rings include $\mathbb{Z}[X]/(X^2)$, $K[X,Y]/(X^2)$ and

$K[X,Y]/(Y^2 - X^2(X+1))$ where K is a field. They also include Dedekind domains but need not be Noetherian nor integrally closed. (A **Dedekind domain** is an integrally closed, Noetherian one-dimensional domain.) Take for example $D = \mathbb{Q} + X\mathbb{A}[X]$ where $\mathbb{A}$ is the field of algebraic numbers. Then D is a one-dimensional domain with Noetherian spectrum but is neither Noetherian nor integrally closed.

Using Theorems 52 and 53 we determine the structure of minimal zero-dimensional extensions of PS-rings with only one prime ideal of height greater than one. Examples of those rings include $K[[X,Y]]$ and $K[X,Y]_{(X,Y)}$ where K is a field.

Let R be a ring and A be an R-algebra. For the following results we use E_A for the set of idempotents $\{e_x^A \in A : x \in R\}$.

Theorem 55 *Let R be a ring and A, B two minimal zero-dimensional R-algebras. Assume that $(1,0) \in E_{A\times B}$. Then $A \times B$ is a minimal zero-dimensional R-algebra.*

Proof. Since A and B are zero dimensional, then e_x^A and e_x^B are defined for every $x \in R$. By Lemma 47, $e_x^{A\times B}$ is defined and $e_x^{A\times B} = (e_x^A, e_x^B)$ for every $x \in R$. Then $E_{A\times B} \subseteq E_A \times E_B$. Since $(1,0) \in E_{A\times B}$, then $(e_x^A, 0) = (1,0)e_x^{A\times B} \in E_{A\times B}$ and $(0, e_x^B) = (1,0)e_x^{A\times B} \in E_{A\times B}$. Therefore, $E_A \subseteq E_{A\times B}$ and $E_B \subseteq E_{A\times B}$. Hence, $E_{A\times B} = E_A \times E_B$. It follows that $T(R[E_{A\times B}]) = T(R[E_A] \times R[E_B]) = T(R[E_A]) \times T(R[E_B])$. By Arapović's construction $T(R[E_A])$ is the minimal zero-dimensional R-algebra within A, $T(R[E_B])$ is the minimal zero-dimensional R-algebra within B and $T(R[E_{A\times B}])$ is the minimal zero-dimensional within $A \times B$. Thus, $T(R[E_A]) = A$ and $T(R[E_B]) = B$, and $A \times B = T(R[E_A]) \times T(R[E_B]) = T(R[E_{A\times B}])$ is a minimal zero-dimensional R-algebra.

■

Theorem 56 *Let R be a PS-ring with only one prime ideal of height greater than one. Let $\mathcal{Q}$ be a set of ideals of R primary for distinct prime ideals of height greater than zero. Then $T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$ is a minimal zero-dimensional extension of R.*

Proof. If $\mathcal{Q}$ contains only ideals of R primary for distinct prime ideals of height one, then by Theorem 52, $T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$ is a minimal zero-dimensional extension of R.

Let M be the ideal of R of height greater than one. Assume that $\mathcal{Q}$ contains an M-primary ideal I. Then R/I is a minimal zero-dimensional R-algebra. Let $\mathcal{M}$ be the subset of $\mathcal{Q}$ consisting of ideals primary for primes of height one. By Theorem 52, $T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)$ is a minimal zero-dimensional extension of R. Then $R/I \times T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)$ is a zero-dimensional extension of R. Let $A = R/I$, $B = T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*)$ and $C = A \times B$. Then e_x^C is defined for every $x \in R$, by Lemma 47. Since R has Noetherian spectrum, e_P^C is defined for every $P \in \mathrm{Spec}(R)$, by Theorem 41(a). Using Lemma 47 and Theorem 52, we obtain $e_M^C = (e_M^A, e_M^B) = (0,1) \in E_C$. By Theorem 55, it follows that C is a minimal zero-dimensional extension of R. By Lemma 45, $R/I \times (\bigoplus_{Q\in\mathcal{M}} R/Q)^* \cong (\bigoplus_{Q\in\mathcal{Q}} R/Q)^*$, so $R/I \times T((\bigoplus_{Q\in\mathcal{M}} R/Q)^*) = T(R/I \times (\bigoplus_{Q\in\mathcal{M}} R/Q)^*) \cong T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$. Hence $T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$ is a minimal zero-dimensional extension of R. ■

Theorem 57 *Let R be a PS-ring with only one prime ideal of height greater than one. Let A be a minimal zero-dimensional extension of R. Then A is isomorphic to $T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$ where $\mathcal{Q}$ is a set of ideals of R primary for distinct prime ideals of height greater than zero.*

Proof. Since A is zero dimensional, e_x is defined for every $x \in A$. By Theorem 41(a), every e_P is defined for every $P \in \mathrm{Spec}(R)$. Let M be the prime ideal of R of height greater than one.

If $e_M^A = 1$, then by Theorem 53, A is isomorphic to $T((\bigoplus_{Q\in\mathcal{Q}} R/Q)^*)$ where $\mathcal{Q}$ is a set of ideals of R primary for distinct prime ideals of height one.

Assume that $e_M^A \neq 1$. Then $A = A(1-e_M^A) \times Ae_M^A$. Let $B = A(1-e_M^A)$ and $C = Ae_M^A$. Since A is a minimal zero-dimensional extension of R, then B and C are minimal zero-dimensional R-algebras. Since $e_P^A = 1$ for every prime ideal P properly containing M, then $\mathrm{ann}_R(1 - e_M^A)$ is M-primary by Lemma 48. Let $I = \mathrm{ann}_R(1 - e_M^A)$. Then R/I is zero dimensional and B is an R/I-algebra. Therefore, $B \cong R/I$. Since A is an extension of R, then $I \bigcap \mathrm{ann}_R(e_M^A) = 0$. Since zero is primary, then M is regular, so I is regular. Therefore, $\mathrm{ann}_R(e_M^A) = 0$ and C is an extension of R. Claim that $e_M^C = 1_C (= e_M^A)$. Clearly $M(1 - e_M^C)$ is nil. We need to show that $e_M^A \in MC$. Since $e_M^A \in MA$, then $e_M^A \in MAe_M^A = MC$. Since $e_M^C = 1_C$, by Theorem 53, $C \cong T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*)$ where $\mathcal{M}$ is a set of ideals of R primary for distinct prime ideals of height one. Therefore $A \cong R/I \times T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*)$. Let $\mathcal{Q} = \mathcal{M} \cup \{I\}$. Then by Lemma 45, $R/I \times (\bigoplus_{Q \in \mathcal{M}} R/Q)^* \cong (\bigoplus_{Q \in \mathcal{Q}} R/Q)^*$, so $R/I \times T((\bigoplus_{Q \in \mathcal{M}} R/Q)^*) = T(R/I \times (\bigoplus_{Q \in \mathcal{M}} R/Q)^*) \cong T((\bigoplus_{Q \in \mathcal{Q}} R/Q)^*) \cong A$. ■

Theorems 56 and 57 imply the following.

Corollary 58 *Let R be a PS-ring with only one prime ideal of height greater than one. There is a one-to-one correspondence between isomorphism classes of minimal zero-dimensional extensions of R and sets $\mathcal{M}$, where the elements of $\mathcal{M}$ are primary ideals for distinct prime ideals of R of height greater than zero.*

The sets of ideals primary for distinct prime ideals of height greater than zero are complete sets of invariants of the minimal zero-dimensional extensions of R.

3.3 General ZPI-Rings

Having the structure of minimal zero-dimensional extensions of a primary ring R and a Dedekind domain R we can determine the structure of minimal zero-dimensional extensions of a general ZPI-ring R. For this we need the following result.

Theorem 59 *Let $R_1, \ldots, R_n$ be rings. The minimal zero-dimensional $\prod_{i=1}^n R_i$-algebras are exactly the products of minimal zero-dimensional R_i-algebras for $i = 1, \ldots, n$.*

Proof. Let A_i be a minimal zero-dimensional R_i-algebra for $i = 1, \ldots, n$. Then $\prod_{i=1}^n A_i$ is a zero-dimensional $\prod_{i=1}^n R_i$-algebra. Assume that B is a zero-dimensional $\prod_{i=1}^n R_i$-

subalgebra of $\prod_{i=1}^{n} A_i$. Then $B1_i$ is a zero-dimensional R_i-subalgebra of A_i for every $i = 1, \ldots, n$. Since A_i is a minimal zero-dimensional R_i-algebra, then $B1_i = A_i$ for every $i = 1, \ldots, n$. Hence, $B = \prod_{i=1}^{n} A_i$ is a minimal zero-dimensional $\prod_{i=1}^{n} R_i$-algebra.

Conversely, assume that A is a minimal zero-dimensional $\prod_{i=1}^{n} R_i$-algebra. Then $A1_i$ is a zero-dimensional R_i-algebra for $i = 1, \ldots, n$ and $A = \prod_{i=1}^{n} A1_i$. Since A is a minimal zero dimensional $\prod_{i=1}^{n} R_i$-algebra, then $A1_i$ is a minimal zero dimensional R_i-algebra for $i = 1, \ldots, n$. ■

Definition 60 *A ring R is a* ***general ZPI-ring*** *if each ideal of R can be represented as a finite product of prime ideals.*

Definition 61 *A* ***special primary ring*** *R is a local ring with maximal ideal M such that each proper ideal of R is a power of M.*

Theorem 62 ([7, Theorem 39.2]) *Let R be a ring. The following conditions are equivalent:*

(1) *R is a general ZPI-ring;*

(2) *R is a finite direct product of Dedekind domains and special primary rings.*

Using Theorems 59 and 62 we can determine the structure of minimal zero-dimensional extensions A of a general ZPI-ring R as follows. By Theorem 62 we can write $R = R_1 \times \cdots \times R_m \times R_{m+1} \times \cdots R_n$ where R_i is a Dedekind domain for $1 \leq i \leq m$ and a special primary ring (hence, a zero-dimensional ring) for $m+1 \leq i \leq n$. Then by Theorems 59 and 53 we have $A = A_1 \times \cdots \times A_m \times R_{m+1} \times \cdots \times R_n$ with $A_i \cong T((\bigoplus_{Q \in \mathcal{M}_i} R_i/Q)^*)$ where $\mathcal{M}_i$ is a set of ideals of R_i primary for distinct maximal ideals for every $1 \leq i \leq m$.

3.4 Rings with Finite Spectrum

Let R be a ring. In this section we classify the minimal zero-dimensional R-algebras S with finitely many idempotents $e_x \in S$ for $x \in R$. These minimal zero-dimensional R-algebras include the ones over a ring R with finite prime spectrum.

First we characterize the minimal zero-dimensional R-algebras S with $e_x = 0$ or 1 for all $x \in R$.

Theorem 63 *Let R be a ring. Then S is a minimal zero-dimensional R-algebra with $e_x = 0$ or 1 for all $x \in R$ if and only if $S = T(R/Q)$ where Q is a primary ideal of R.*

Proof. ($\Rightarrow$) Let $\varphi : R \to S$ be the ring homomorphism. Since S is zero dimensional, then for every element $x \in R$ there exists a positive integer n such that $\varphi(x)^n = ue_x$ where u is a unit in S. Since $e_x = 0$ or 1 for all $x \in R$, then every element of $\varphi(R)$ is either a nilpotent or a unit. Then, the zero ideal of $\varphi(R)$ is a primary ideal. Let $Q = \ker(\varphi)$. Then Q is a primary ideal.

Let $E = \{e_x : x \in R\} = \{0, 1\}$. Then $R[E] = R/Q$ and so, $T(R[E]) = T(R/Q)$. By Arapović's construction $T(R[E])$ is a minimal zero-dimensional R-algebra. Hence, $S = T(R/Q)$.

($\Leftarrow$) Since Q is a primary ideal, $T(R/Q)$ is zero dimensional. If $x \in \text{rad}(Q)$, then $e_x = 0$. If $x \notin \text{rad}(Q)$, then $e_x = 1$. Let $E = \{e_x : x \in R\} = \{0, 1\}$. Then $R[E] = R/Q$, so $T(R[E]) = T(R/Q)$. By Arapović's construction, $T(R/Q)$ is a minimal zero-dimensional R-algebra. ■

Notice that $T(R/Q) = (R/Q)_{P/Q}$, where $\text{rad}(Q) = P$, is a primary ring. So actually, all the idempotents of $T(R/Q)$ are either 0 or 1.

Corollary 64 *Let R be a ring. An R-algebra S is a minimal zero dimensional, and indecomposable, if and only if $S = T(R/Q)$ for some primary ideal Q of R.*

Proof. The proof follows from Theorem 63, since any indecomposable ring has only trivial idempotents. ■

Next we classify the minimal zero-dimensional R-algebras S with finitely many idempotents $e_x \in S$ for $x \in R$.

Theorem 65 *Let R be a ring and S be a minimal zero-dimensional R-algebra with finitely many idempotents e_x for $x \in R$. Then $S = \prod_{i=1}^{k} T(R/Q_i)$ where Q_i is a primary ideal of R for $i = 1, ..., k$ and $\text{rad}(Q_i) \neq \text{rad}(Q_j)$ for $i \neq j$.*

Proof. Let E be the Boolean algebra of idempotents generated by the idempotents $\{e_x : x \in R\}$. Then E has finitely many atoms, by Theorem 11. Let $a_1, \ldots, a_k$ be the atoms of E. Then $S = Sa_1 \times \cdots \times Sa_k$ and every Sa_i is a minimal zero-dimensional R-algebra. Since Sa_i is zero dimensional, then for every $x \in R$ there is an idempotent $a_i e_x \in Sa_i$ such that $x(1 - a_i e_x)$ is nilpotent and $x + (1 - a_i e_x)$ is invertible in Sa_i. Since a_i is an atom of E, then $a_i e_x$ is either 0 or a_i for every $x \in R$. Then by Theorem 63, $Sa_i = T(R/Q_i)$ where Q_i is a primary ideal of R. So, $S = \prod_{i=1}^{k} T(R/Q_i)$ where Q_i is a primary ideal of R for $i = 1, \ldots, k$.

To show that $\text{rad}(Q_i) \neq \text{rad}(Q_j)$ for $i \neq j$, suppose that $\text{rad}(Q_i) = \text{rad}(Q_j) = P$ where P is a prime ideal of R. Then, $Q_i \cap Q_j$ is P-primary. We have that $R/(Q_i \cap Q_j)$ is a subring of $R/Q_i \times R/Q_j$. Claim that $R/(Q_i \cap Q_j)$ is a proper subring of $R/Q_i \times R/Q_j$. Let $(0, 1) \in R/Q_i \times R/Q_j$. Since $\text{rad}(Q_i) = \text{rad}(Q_j)$, there is no $x \in R$ such that $x \in Q_i$ and $x \notin \text{rad}(Q_j)$. So, $(0, 1) \notin R/(Q_i \cap Q_j)$. Thus, $R/(Q_i \cap Q_j)$ is a proper subring of $R/Q_i \times R/Q_j$.

Let x be a regular element in $R/(Q_i \cap Q_j)$. Since $Q_i \cap Q_j$ is primary, every zero-divisor in $R/(Q_i \cap Q_j)$ is nilpotent. Then $x \notin P$. Since Q_i and Q_j are P-primary ideals, then x is regular in R/Q_i and R/Q_j. Thus x is regular in $R/Q_i \times R/Q_j$. Then,

$T(R/(Q_i \cap Q_j))$ is a proper subring of $T(R/Q_i) \times T(R/Q_j)$. But then

$$T(R/(Q_i \cap Q_j)) \times \prod_{l \neq i,j} T(R/Q_l)$$

is a proper subring of $S = \prod_{i=1}^{k} T(R/Q_i)$. Since $Q_i \cap Q_j$ is a primary ideal, $T(R/(Q_i \cap Q_j))$ is zero dimensional. Then

$$T(R/(Q_i \cap Q_j)) \times \prod_{l \neq i,j} T(R/Q_l)$$

is a zero dimensional R-algebra, which is a contradiction with the fact that S is a minimal zero-dimensional R-algebra. ■

Theorem 66 *Let R be a ring and $S = \prod_{i=1}^{k} T(R/Q_i)$ where Q_i is a P_i-primary ideal of R for $i = 1, \ldots, k$ and $P_i \neq P_j$ for $i \neq j$. Then S is a minimal zero-dimensional R-algebra and it has finitely many idempotents.*

Proof. Since Q_i is a primary ideal, $T(R/Q_i)$ is zero-dimensional, so S is zero dimensional. Since S is zero dimensional, for every $x \in R$ there is an idempotent e_x such that $x(1-e_x)$ is nilpotent and $x+(1-e_x)$ is invertible in S. Let $E = \{e_x : x \in R\}$ and $c_i \in S$ be the idempotent with i-th coordinate 1 and others 0. Assume that P_i is minimal in $\{P_i, P_{i+1}, \ldots, P_k\}$ for $i = 1, \ldots, k$. Let $x_1 \in (\bigcap_{i=2}^{k} P_i) \setminus P_1$. Then e_{x_1} has first coordinate 1 and other coordinates 0. Let $x_i \in (\bigcap_{i=j+1}^{k} P_i) \setminus P_j$ where $j = 2, \ldots, k-2$. Then e_{x_i} has j-th coordinate 1 and the i-th coordinates 0 for $i = j+1, \ldots, k$. Claim that $R[E] = \prod_{i=1}^{k} R/Q_i$. We have $c_1 = e_{x_1}$ and $c_i = \prod_{j=1}^{i-1}(1-e_{x_j})e_{x_i} \in R'$ for $i = 2, \ldots, k$. Then, $\prod_{i=1}^{k} R/Q_i \subseteq R[E]$. Conversely, $e_{x_i} = \sum_{j=1}^{i-1} c_j r_j + c_i$ where $i = 2, \ldots, k$ and $r_j \in R/Q_j$. So, $R[E] = \prod_{i=1}^{k} R/Q_i$. Then, $T(R[E]) = T(\prod_{i=1}^{k} R/Q_i) = \prod_{i=1}^{k} T(R/Q_i)$. By Arapović's construction $T(R[E])$ is a minimal zero dimensional R-algebra. Since $T(R/Q_i) = (R/Q_i)_{P_i/Q_i}$ is a local ring, $T(R/Q_i)$ has only trivial idempotents. Then,

$S = \prod_{i=1}^{k} T(R/Q_i)$ has finitely many idempotents. ■

We note that Gilmer and Heinzer proved that for a ring R in which $(0) = \bigcap_{i=1}^{k} Q_i$ is a shortest primary decomposition of (0) and Q_i is a P_i-primary ideal of R, then $S = \prod_{i=1}^{k} T(R/Q_i)$ is a minimal zero-dimensional extension of R.([11, Theorem 4.4])

Next we classify the minimal zero-dimensional R-algebras S where R has finite spectrum.

Theorem 67 *Let R be a ring with finite spectrum. Then S is a minimal zero-dimensional R-algebra if and only if $S = \prod_{i=1}^{k} T(R/Q_i)$ where Q_i is a P_i primary ideal of R for $i = 1, \dots, k$ and $P_i \neq P_j$ for $i \neq j$.*

Proof. Assume that S is a minimal zero-dimensional R-algebra. Then e_x is defined for every $x \in R$. By Lemma 41, e_P is defined for every $P \in \text{Spec}(R)$ and e_x is a finite product of e_P where $P \in \text{Spec}(R)$ contains x. Since $\text{Spec}(R)$ is finite, then S contains finitely many idempotents e_x for $x \in R$. Then, $S = \prod_{i=1}^{k} T(R/Q_i)$ where Q_i is a P_i primary ideal of R for $i = 1, \dots, k$ and $P_i \neq P_j$ for $i \neq j$, by Theorem 65.

The converse of the theorem follows from Theorem 66. ■

3.5 Minimal Zero-Dimensional Algebras

Let R be a ring. In this section we present a few properties of minimal zero-dimensional R-algebras. The set of minimal zero-dimensional R-algebras is closed under taking homomorphic images, it is a partially ordered set and any two elements of the set has an infimum. Another interesting property is that any minimal zero-dimensional R-algebra is an R-epimorph.

Lemma 68 *Let R, S be rings and $f : R \to S$ a ring homomorphism. If e_x^R is defined for some $x \in R$, then $e_{f(x)}^S$ is defined and $e_{f(x)}^S = f(e_x^R)$.*

Proof. Since e_x^R is defined, then $x(1-e_x^R)$ is nilpotent and $e_x^R \in Rx$. We have $f(x(1-e_x^R)) = f(x)(f(1)-f(e_x^R)) = f(x)(1-f(e_x^R))$. Since $x(1-e_x^R)$ is nilpotent, then $f(x(1-e_x^R))$ is nilpotent. So, $f(x)(1-f(e_x^R))$ is nilpotent. We also have $f(e_x^R) \in Sf(x)$. Hence, $e_{f(x)}^S$ is defined and $e_{f(x)}^S = f(e_x^R)$. ■

Theorem 69 *Let R be a ring and S be a minimal zero-dimensional R-algebra. Then a homomorphic image of S is a minimal zero-dimensional R-algebra.*

Proof. Let $f : S \to S'$ be an R-algebra epimorphism. Since S is zero dimensional, then S' is zero dimensional. By Arapović's construction S is the total quotient ring of the R-algebra generated by the idempotents $E_S = \{e_x^S : x \in R\}$. By Lemma 68, $e_{f(x)}^{S'} = f(e_x^S)$ for every $x \in S$, so $e_x^{S'} = f(e_x^S)$ for every $x \in R$. Then $f(R[E_S]) = R[E_{S'}]$. Claim that S' is the total quotient ring T' of $R[E_{S'}]$ within S'. Let $x \in S' = f(S) = f(T(R[E_S]))$. Then $f(b)x = f(a)$ where $a, b \in R[E_S]$ with b regular in $R[E_S]$. By Lemma 28, b is invertible in S. Therefore $f(b)$ is regular in $f(S) = S'$. Then $x \in T'$. By Arapović's construction, T' is the minimal zero-dimensional R-algebra within S'. Hence, $S' = T(R[E_{S'}])$. ■

In order to show that a minimal zero-dimensional algebra over a ring R is an R-epimorph we need to introduce some definitions and results.

Definition 70 *A ring homomorphism $R \to S$ is an* ***epimorphism*** *if is right cancellative in the category of rings. In this case we say that S is an R-**epimorph**.*

For example, $\mathbb{Q}$ is a $\mathbb{Z}$-epimorph, since any two ring homomorphisms from $\mathbb{Q}$ to any ring that agree on $\mathbb{Z}$ are equal.

Theorem 71 ([17]) *S is an R-epimorph if and only if the multiplication map $m : S \bigotimes_R S \to S$, $m(a \bigotimes b) = ab$ is an isomorphism.*

Definition 72 *Let $R \to S$ be a ring homomorphism. The R-**dominion of** S, $d_R(S)$, is the largest subring of S with the property that two homomorphisms from S to any other ring agree on $d_R(S)$ whenever they agree on the image of R.*

Silver proved the following arithmetic characterization of an R-dominion of S in reference [16].

Theorem 73 *Let $R \to S$ be a ring homomorphism. The R-dominion of S is the set of elements $d \in S$ such that $d \bigotimes 1 = 1 \bigotimes d$ in $S \bigotimes_R S$.*

For a ring homomorphism $R \to S$, a relation between the R-dominion of S and an R-epimorph is given by the following result proved by Isbell in reference [13].

Theorem 74 *Let $R \to S$ be a ring homomorphism. Then S is an R-epimorph if and only if $d_R(S) = S$.*

In order to show that a minimal zero-dimensional R-algebra is an R-epimorph we need one more result proved by Burgess and Menal.

Theorem 75 ([5]) *Let $\varphi : R \to S$ be a ring homomorphism where S is zero dimensional. Then $d_R(S)$ is zero dimensional.*

Theorem 76 *If S is a minimal zero-dimensional R-algebra, then S is an R-epimorph.*

Proof. Since S is zero-dimensional, $d_R(S)$ is zero-dimensional, by Theorem 75. Then $d_R(S) = S$, since S is a minimal zero-dimensional R-algebra. Hence, S is an R-epimorph, by Theorem 74. ■

An interesting property of the minimal zero-dimensional R-algebras is that they form a partially ordered set. Let $\mathcal{A}$ be the set of minimal zero-dimensional R-algebras. We define the relation " $\leq$ " on $\mathcal{A}$ by $A \leq B$ if there is a homomorphism from A to B where $A, B \in \mathcal{A}$. We prove that the homomorphism from A to B, if it exists, it is unique and onto.

Proposition 77 *Let A, B be two minimal zero-dimensional algebras over a ring R. If f is a homomorphism from A to B, then f is unique and onto.*

Proof. Let $f : A \to B$ be a homomorphism. By Lemma 69, $f(A)$ is a minimal zero-dimensional R-algebra. Since B is a minimal zero-dimensional R-algebra and $f(A) \subseteq B$, then $f(A) = B$. Hence, f is onto.

For uniqueness, we assume that there are two homomorphisms, f and g, from A to B. Since A is an R-epimorph, then $f \circ h = g \circ h$ implies $f = g$, where h is the homomorphism from R to A. ■

Theorem 78 *The relation "$\leq$" on $\mathcal{A}$ is a partial order. Moreover, the infimum of any two elements exists in this partial order.*

Proof. Clearly "$\leq$" is a partial order.

Let $A, B \in \mathcal{A}$. Since A and B are zero-dimensional R-algebras, then $A \times B$ is a zero-dimensional R-algebra. By Arapović's construction there exists a minimal zero-dimensional R-algebra S within $A \times B$. Claim that S is the infimum of A and B. Clearly there exist homomorphisms $A \times B \to A$ and $A \times B \to B$. Since $S \subseteq A \times B$, then there exist homomorphisms $S \to A$ and $S \to B$. So, $S \leq A$ and $S \leq B$. Next, let $C \in \mathcal{A}$ such that $C \leq A$ and $C \leq B$. Then there exist homomorphisms $f : C \to A$ and $g : C \to B$. Therefore we can define the homomorphism $h : C \to A \times B$ by $h(c) = (f(c), g(c))$ where $c \in C$. By Theorem 69, $h(C)$ is the minimal zero-dimensional R-algebra within $A \times B$. Since the minimal zero-dimensional R-algebra within S is unique, $h(C) = S$. Then h is a homomorphism from C onto S, so $C \leq S$. Hence, S is the infimum of A and B. ■

3.6 Open Questions

We conclude this chapter by suggesting open questions for further research.

Question 1 Can we use Arapović's construction to determine the structure of minimal zero-dimensional extensions of PS-rings with only finitely many primes of height greater than one?

Can we generalize the structure theorem for minimal zero-dimensional extensions of one-dimensional PS-rings?

Theorem 79 *Let R be a ring. The zero ideal of R is primary if and only if $T(R)$ is zero dimensional and the nilradical of R is prime.*

Proof. If the zero ideal of R is primary, clearly $T(R)$ is zero dimensional and the nilradical is prime.

Conversely, let $\text{rad}(0) = P$, where $P \in \text{Spec}(R)$. We have $T(R) = R_S$, where S is the set of all regular elements of R. Since the nilradical of R_S is the prime ideal PR_S and R_S is zero dimensional, then PR_S is a maximal ideal. Therefore, the nilradical of R_S is a maximal ideal. Then, the zero ideal of R_S is primary. Hence, the zero ideal of R is primary. ■

In light of Theorem 79, a natural way to generalize the main result is to weaken the hypothesis that zero is a primary ideal to the nilradical is prime or to $T(R)$ is zero dimensional.

Question 2 What is the structure of minimal zero-dimensional extensions of one-dimensional rings with Noetherian spectrum and the nilradical prime? An example of such ring is $K[X,Y]/(X^2, XY)$, where K is a field.

Question 3 What is the structure of minimal zero-dimensional extensions of one-dimensional rings with Noetherian spetrum and the total quotient ring is zero dimensional? An example of such ring is $K[X,Y]/(XY)$, where K is a field.

References

[1] M. ARAPOVIĆ, Characterization of the 0-dimensional rings, *Glasnik Mat.* **18** (1983) 39-46.

[2] M. ARAPOVIĆ, The minimal 0-dimensional overrings of commutative rings, *Glasnik Mat.***18** (1983) 47-52.

[3] M. ARAPOVIĆ, On the embedding of a commutative ring into a 0-dimensional ring, *Glasnik Mat.***18** (1983) 53-59.

[4] J. W. BREWER and F. RICHMAN, Subrings of zero-dimensional rings, in *Multiplicative Ideal Theory in Commutative Algebra* (Springer, 2006) pp. 73-88.

[5] W.D. BURGESS and P. MENAL, On strongly π-regular rings and homomorphisms into them, *Comm. Algebra,* **16**, (1988), 1701-1725.

[6] P. M. COHN, *Universal algebra,* Mathematics and its Applications 6, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981.

[7] R. W. GILMER, *Multiplicative Ideal Theory,* Marcel Dekker, New York and Basel 1972.

[8] R. W. GILMER, Zero-dimensional extensions rings and subrings, in *Zero-dimensional commutative rings, Lecture notes in pure and applied mathematics* **171,** Marcel Dekker 1995, 27-39.

[9] R. W. GILMER, A new criterion for embeddability in a zero-dimensional commutative ring, *Lecture notes in pure and applied mathematics* **220**, Marcel Dekker 2001, 223-229.

[10] R. W. GILMER and W. J. HEINZER, Zero-dimensionality in commutative rings, *Proc. Amer. Math. Soc.*, **115** (1992) 881-893.

[11] R. W. GILMER and W. J. HEINZER, Products of Commutative Rings and Zero-Dimensionality, *Trans. Amer. Math. Soc.* **331** (1992), no. 2, 663-680.

[12] R. W. GILMER and W. J. HEINZER, On the imbedding of a direct product into a zero-dimensional commutative ring, *Proc. Amer. Math. Soc.*,**106** (1989) 631-637.

[13] J. R. ISBELL, Epimorphisms and dominions, *Proceedings of the Conference on Categorical Algebra*. La Jolla, California 1965. Berlin, Heidelberg, New York: Springer 1966.

[14] N. JACOBSON, *Basic Algebra I,* Second Edition, W. H. Freeman and Company, New York, 1985.

[15] J. OHM and R. PENDLETON, Rings with Noetherian spectrum, *Duke Math. J.* **35** (1968) 631-640.

[16] L. SILVER, Non-commutative localization and applications, *J. Algebra* **7** (1976), 44-76.

[17] H. H. STORRER, Epimorphic extensions of non-commutative rings, *Comment. Math. Helv.* **48** (1973), 72-86.