# GENERAL MONOTONICITY, INTERPOLATION OF OPERATORS, AND APPLICATIONS

by

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## ABSTRACT

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Assume that $\{\phi_n\}$ is an orthonormal uniformly bounded $(ONB)$ sequence of complex-valued functions defined on a measure space $(\Omega, \Sigma, \mu)$, and $f \in L^1(\Omega, \Sigma, \mu)$. Let $c_n = \int\limits_{\Omega} f \cdot \overline{\phi_n} d\mu$ be the Fourier coefficients of $f$ with respect to $\{\phi_n\}$ .

R.E.A.C. Paley proved a theorem connecting the $L^p$-norm of $f$ with a related norm of the sequence $\{c_n\}$. Hardy and Littlewood subsequently proved that Paley's result is best possible within its context. Their results were generalized by Dikarev, Macaev, Askey, Wainger, Sagher, and later by Tikhonov, Liflyand, Booton and others. The present work continues the generalization of these results.

*To my father, Mikhail, my first math teacher, and Professor Vladimir Zalyapin, my first graduate advisor*

GENERAL MONOTONICITY, INTERPOLATION OF OPERATORS, AND APPLICATIONS

# CHAPTER 1

# INTRODUCTION

## 1.1 ELEMENTS OF THE THEORY OF $L(p,q)$ SPACES AND APPLICATIONS IN FOURIER ANALYSIS.

Recall the definition of the decreasing rearrangement of a function.

**Definition 1.1**. Let $(\Omega, \Sigma, \mu)$ be a measure space, $f$ be $\mu$-measurable $\Omega \to \mathbb{C}$ function. Define, for each $\alpha \geq 0$, the *distribution function* of $f$: $f_*(\alpha) = \mu\left(\{x : |f(x)| > \alpha\}\right)$. Define, for each $x \geq 0$,

$$f^*(x) = \begin{cases} \inf\left(\{\alpha : f_*(\alpha) \leq x\}\right) & \text{if } \exists \alpha : f_*(\alpha) \leq x \\ \infty & \text{otherwise} \end{cases}. \tag{1.1}$$

$f^*$ is called the *decreasing rearrangement* of $f$.

Considering sequences as functions defined on $\Omega \subseteq \mathbb{Z}$, with $\mu = \#$, the counting measure, we use formula (1.1) to define for each $k \in \mathbb{N}$,

$$a_k^* = \left(\{a_n\}^*\right)(k-1). \tag{1.2}$$

**Remark 1.2**. $\left(f^*\right)_*(\alpha) = f_*(\alpha), \forall \alpha \geq 0$.

**Remark 1.3**. If $|f| \leq g$, where $g \searrow$, then $f_* (\alpha) \leq g_* (\alpha)$. Hence for each $x > 0$, $f^* (x) \leq g^* (x) = g (x+) \leq g (x)$. Similarly, if $|f| \geq g$, where $g \searrow$ and right-continuous, then for each $x > 0$, $f^* (x) \geq g^* (x)$.

We consider only functions $f$ such that $f^*(x) < \infty$ for all $x$, and sequences $\{a_n\}$ such that $a_n^* < \infty$ for all $n$.

As usual, $p'$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

**Theorem 1.4** (R.E.A.C. Paley, [17]).

Let $\{\phi_n\}_{n=1}^{\infty}$ be an $ONB$ on $(\Omega, \Sigma, \mu)$ and let $1 < p \leq 2$.

(i) Let $f \in L^p(\Omega, \Sigma, \mu)$, and let $c_n$ be the Fourier coefficients of $f$ with respect to $\{\phi_n\}$. Then $\exists B (p, \{\phi_n\}) < \infty$ so that:

$$\left( \sum_{n=1}^{\infty} (c_n^*)^p \, n^{p-2} \right)^{\frac{1}{p}} \leq B (p, \{\phi_n\}) \, \|f\|_{L^p} . \tag{1.3}$$

(ii) If $\{c_n\}$ satisfies $\sum_{n=1}^{\infty} (c_n^*)^{p'} n^{p'-2} < \infty$, then $\exists B' (p, \{\phi_n\})$ so that $f := \sum_{n=1}^{\infty} c_n \phi_n$ satisfies

$$\|f\|_{L^{p'}} \leq B' (p, \{\phi_n\}) \left( \sum_{n=1}^{\infty} (c_n^*)^{p'} \, n^{p'-2} \right)^{\frac{1}{p'}} . \tag{1.4}$$

**Remark 1.5**. Since a rearrangement of an $ONB$ is an $ONB$, we can permute the $ONB$ sequence in such a way that the magnitudes of the corresponding Fourier coefficients will form a decreasing sequence. Therefore, there is no point in stating Paley's theorem in terms of decreasing rearrangements.

Paley's theorem led to the introduction of $L(p, q)$-spaces, one of the main tools in Real Interpolation theory.

We recall the definition of weighted $L^q$-norms.

**Definition 1.6**. Let $0 < q \leq \infty$, and let $w$ be a measurable function on $(\Omega, \Sigma, \mu)$.

Assume that $f$ is a measurable function on $(\Omega, \Sigma, \mu)$. Define the *weighted $L^q$-norm* of $f$ with the weight $w$:

$$\|f\|_{L^q_w(\Omega,\Sigma,\mu)} = \|f \cdot w\|_{L^q(\Omega,\Sigma,\mu)} .$$

$L^q_w = L^q_w(\Omega, \Sigma, \mu)$ is the space of all measurable functions $f$ for which $\|f\|_{L^q_w(\Omega,\Sigma,\mu)}$ is finite.

In particular, working with $(\mathbb{N}, 2^{\mathbb{N}}, \#)$, we follow the practice of writing

$$\|\{a_n\}\|_{l^q_w} = \|\{a_n \cdot w(n)\}\|_{l^q} .$$

$l^q_w$ is the space of sequences $\{a_n\}$ for which $\|\{a_n\}\|_{l^q_w}$ is finite.

We will make extensive use of the spaces of functions defined on $(0, \infty)$ with the following weight:

**Definition 1.7**. Let $0 < p \leq \infty$, $0 < q \leq \infty$. We denote, $\forall x > 0$:

$$w(p, q)(x) = x^{\frac{1}{p} - \frac{1}{q}}. \tag{1.5}$$

When we work with sequences, we denote, $\forall n \geq 1$:

$$w(p, q)(n) = n^{\frac{1}{p} - \frac{1}{q}}. \tag{1.6}$$

**Definition 1.8**.

(i) Assume that $f$ is a measurable function on $(\Omega, \Sigma, \mu)$ and that $f^*(x) < \infty$, $\forall x > 0$. For $0 < p < \infty$, $0 < q \leq \infty$, and $p = q = \infty$, define the *$L(p, q)$-norm* of $f$:

$$\|f\|_{L(p,q)} = \|f\|_{L(p,q)(\Omega,\Sigma,\mu)} = \|f^*\|_{L^q_{w(p,q)}(\mathbb{R}^+,\mathcal{B},\lambda)} \tag{1.7}$$

$L(p, q) = L(p, q)(\Omega, \Sigma, \mu)$ is the space of all measurable functions, $f$, for which $\|f\|_{L(p,q)}$ is finite.

(ii) Analogously, for sequences, the following norms are equivalent to the ones in (1.7):

$$\|\{a_n\}\|_{l(p,q)} = \|\{a_n^*\}\|_{l^q_{w(p,q)}}. \tag{1.8}$$

$l(p,q)$ is the collection of all sequences $\{a_n\}$ for which $\|\{a_n\}\|_{l(p,q)}$ is finite.

Spaces $L(p,q)$ and $l(p,q)$ are often called *Lorentz spaces.* They generalize $L^p$ spaces:

**Remark 1.9**. Note that, since $f$ and $f^*$ have the same distribution function, $L(p,p) = L^p$ and $l(p,p) = l^p$.

We can now restate Paley's theorem: Let $\{\phi_n\}$ be an $ONB$ on $(\Omega, \Sigma, \mu)$, $1 < p \leq 2$, and $\{c_n\}$ be the sequence of Fourier coefficients of $f$ with the respect to $\{\phi_n\}$.

(i) If $f \in L^p(\Omega, \Sigma, \mu)$, then $\|\{c_n\}\|_{l(p',p)} \leq B\|f\|_{L^p}$.

(ii) If $\|\{c_n\}\|_{l(p,p')} < \infty$ then $\|f\|_{L^{p'}} \leq B'\|\{c_n\}\|_{l(p,p')}$.

We can also state Paley's theorem in an operator form:

Let $T_1(f) = \{c_n\}$ and $T_2(\{c_n\}) = \sum_{n=1}^{\infty} c_n\phi_n$. Then, $\forall p \in (1,2]$ , $T_1 : L^p \to l(p',p)$ and $T_2 : l(p,p') \to L^{p'}$ are bounded linear operators.

With $T_1$, $T_2$ defined as above,

$$T_1 : L^1 \to l^\infty, T_1 : L^2 \to l^2 \text{ and } T_2 : l^1 \to L^\infty, T_2 : l^2 \to L^2.$$

Using Real Interpolation theory, it follows that:

**Theorem 1.10**. For $1 < p < 2$, $0 < q \leq \infty$,

$$\|T_1 f\|_{l(p',q)} \leq C(p,q)\|f\|_{L(p,q)}, \tag{1.9}$$

and

$$\|T_2(\{c_n\})\|_{L(p',q)} \leq C(p,q)\|\{c_n\}\|_{l(p,q)}. \tag{1.10}$$

If we take $q = p$ in (1.9), we get (1.3). If we take $q = p'$ in (1.10), we get (1.4).

Theorem 1.10 was stated in [7]. However, two restrictions imposed there, namely, the measure space $(\Omega, \Sigma, \mu)$ is atom-free and $q \geq 1$, are not needed.

If $0 < q_1 < q_2 < \infty$, then $L(p, q_1) \subseteq L(p, q_2)$. Moreover, $L(p, q_1) = L(p, q_2)$ if and only if $(\Omega, \Sigma, \mu)$ consists of finitely many atoms. When $p \in [1, 2]$, we have $p' \geq 2 \geq p$. Hence, for an arbitrary complex sequence $\{c_n\}$ we have $\|\{c_n\}\|_{l^{p'}} = \|\{c_n\}\|_{l(p',p')} \leq C(p) \|\{c_n\}\|_{l(p',p)}$. If $\{c_n\}$ is the sequence of Fourier coefficients of $f$, then by (1.3):

$$\|\{c_n\}\|_{l^{p'}} \leq B'(p) \|f\|_{L^p([0,2\pi])}. \tag{1.11}$$

With the optimal constant, inequality (1.11) is the Hausdorff inequality for Fourier series, which preceded Paley's theorem.

**Theorem 1.11** (F. Hausdorff, [12]).

Let $p \in [1, 2]$, $f \in L^p([0, 2\pi])$, and $\{c_n\}$ the sequence of Fourier coefficients of $f$ with respect to $\{e^{int}\}_{n \in \mathbb{Z}}$, then

$$\|\{c_n\}\|_{l^{p'}} \leq \|f\|_{L^p([0,2\pi])}. \tag{1.12}$$

**Remark 1.12**. The Hausdorff inequality can be proved with the correct constant, 1, using the Riesz-Thorin Interpolation theorem, rather than the Marcinkiewicz-Calderón-Hunt Interpolation theorem. Taking $f(t) = ce^{i\theta t}$ shows that $B'(p) = 1$ is the optimal constant for (1.11).

We shall frequently use the following notation.

**Definition 1.13**. We say that functions $f$ and $g$ are equivalent and write $f \sim g$ if there exist finite constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 f \leq g \leq C_2 f.$$

The following classical theorem shows that, in the context of $L(p,q)$ spaces, Paley's theorem is best possible.

**Theorem 1.14** (G.H. Hardy and J.E. Littlewood, [9]).

Let $\{c_n\} \searrow 0$, $f(x) = \sum_{n=0}^{\infty} c_n \cos nx$ or $f(x) = \sum_{n=1}^{\infty} c_n \sin nx$, $p > 1$. A necessary and sufficient condition that $f \in L^p(0,\pi)$ , is that $\{c_n\} \in l_{(p',p)}$. Furthermore, $\|f\|_{L^p(0,\pi)} \sim \|\{c_n\}\|_{l(p',p)}$ .

From now on, if $\{c_n\}$ is the sequence of sine or cosine Fourier coefficients of $f$, we shall say $c_n$ are *trigonometric* Fourier coefficients of $f$. Thus, Theorem 1.14 establishes an equivalence between the $L^p$ norm of $f$ and the $l(p',p)$ norm of the sequence of its trigonometric Fourier coefficients when the sequence is monotone.

For a simple proof of Theorem 1.14, using Real Interpolation theory, see [20].

Dikarev and Macaev [7] stated the following generalization of Theorem 1.14 to $L(p,q)(0,\pi)$ spaces as follows:

Let $\{c_n\} \searrow 0$, $1 < p < \infty$, $1 \le q \le \infty$ and let $c_n$ be the trigonometric Fourier coefficients of a function $f$. Then

$$\|f\|_{L(p,q)} \sim \|\{c_n\}\|_{l(p',q)} . \tag{1.13}$$

Hardy and Littlewood also described a norm relation between monotone decreasing functions and their trigonometric Fourier coefficients:

**Theorem 1.15** (G.H. Hardy and J.E. Littlewood, [11]).

Let $f \searrow$ on $[0,\pi]$, $f \ge 0$, $1 < p < \infty$, and let $\{c_n\}$ be the sequence of trigonometric Fourier coefficients of $f$. Then:

$$\|f\|_{L^p(0,\pi)} \sim |c_0| + \|\{c_n\}\|_{l^p_{w(p',p)}}.$$

## 1.2 A CONCEPT OF GENERAL MONOTONICITY

The extension of Theorem 1.14 to broader classes of sequences started with the introduction of *quasi-monotone sequences* by Shah [21]; this concept also applies to functions defined on $(0, \infty)$:

**Definition 1.16**. Let $\beta > 0$.

(a) If $a_n \geq 0$ and $\left\{a_n \cdot n^{-\beta}\right\} \searrow$, the sequence $\{a_n\}$ is called a *$\beta$-quasi-monotone decreasing sequence* (written as $\{a_n\} \in QDS\,(\beta)$).

(b) If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $x^{-\beta} f(x) \searrow$ then $f$ is called a *$\beta$-quasi-monotone decreasing function*, $f \in QD\,(\beta)$.

Furthermore, $QDS := \underset{\beta>0}{\cup} QDS\,(\beta)$; $QD := \underset{\beta>0}{\cup} QD\,(\beta)$. We say that $\{a_n\}$ is a *quasi-monotone decreasing sequence* if $\{a_n\} \in QDS$ and that $f$ is a *quasi-monotone decreasing function* if $f \in QD$.

The interest in quasi-monotone sequences stems from the fact that the class of trigonometric series with quasi-monotone Fourier coefficients is closed under term-by-term differentiation. For an application, see [18].

Askey and Wainger [1] proved that if $\{c_n\}$ is a quasi-monotone sequence of trigonometric Fourier coefficients of $f$ then for $1 < p < \infty$, $1 \leq q < \infty$,

$$\|f\|_{L^q_{w(p,q)(0,\pi)}} \sim \|\{c_n\}\|_{l^q_{w(p',q)}}. \tag{1.14}$$

An extension of Real Interpolation theory made it possible to simplify the proof of (1.14). The extension consists of interpolation of normed *monoids* (see Definition 2.1) rather than normed vector spaces.

**Theorem 1.17** (Y. Sagher, [20]).

Let $p \in (1, \infty)$, $q \in [1, \infty]$. Let $\{c_n\} \in QDS$ be the sequence of trigonometric

Fourier coefficients of $f$ . Then:

$$\|f\|_{L_{(p,q)}} \sim \|f\|_{L^q_{w(p,q)}} \sim \|\{c_n\}\|_{l^q_{w(p',q)}} \sim \|\{c_n\}\|_{l_{(p',q)}} . \tag{1.15}$$

Sagher [19] also showed that every quasi-monotone decreasing $\mathbb{R}^+ \to \mathbb{R}^+$ function $f$ can be written as a difference of two monotone decreasing functions $f_1$ and $f_2$. Furthermore, if, in addition, $f \in L(p,q)$, then $f_1$ and $f_2$ can be chosen so that:

$$\|f_j\|_{L(p,q)} \leq C(p,q) \|f\|_{L(p,q)} , \quad j = 1, 2.$$

The immediate consequence of this fact is that for every $f \in QD$ the $L(p,q)$ and $L^q_{w(p,q)}$ norms are equivalent. This has led to a generalization of Theorem 1.15:

**Theorem 1.18** (Y. Sagher, [19]).

Assume that $f \in QD$, $p \in (1,\infty)$, $q \in [1,\infty]$. Let $\{c_n\}$ be the sequence of trigonometric Fourier coefficients of $f$. Then:

$$\|f\|_{L(p,q)} \sim \|\{c_n\}\|_{l^q_{w(p',q)}} \sim \|\{c_n\}\|_{l_{(p',q)}} . \tag{1.16}$$

Tikhonov [23] defined the following extension of the class of quasi-monotone sequences:

**Definition 1.19**. Let $B \geq 1$. We write $\{a_n\} \in GMS\,(B)$ if for all $n \geq 1$:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq B\,|a_n| . \tag{1.17}$$

Define $GMS = \underset{B\geq 1}{\cup} GMS(B)$. If $\{a_n\} \in GMS$, we say that $\{a_n\}$ is a *general monotone sequence.*

We write $GMS^+ = GMS \cap \{\{a_n\} : a_n \geq 0,\ \text{all } n\}$.

Recall the definition of the variation of a function on an interval.

**Definition 1.20**. Let $-\infty < c \leq d < \infty$, $f : [c, d] \to \mathbb{C}$. The *variation* of $f$ on $[c, d]$ is defined as

$$V_f([c,d]) = \sup_{\{x_j\} \in \mathcal{P}([c,d])} \left\{ \sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)| \right\}. \tag{1.18}$$

Here, $\mathcal{P}([c, d])$ is the collection of all partitions of $[c, d]$, that is,

$$\{x_j\}_{j=0}^{n} \in \mathcal{P}([c,d]) \Rightarrow c = x_0 \leq x_1 \leq \ldots \leq x_n = d.$$

We say that $f$ has a *bounded variation* on $[c, d]$, or $f \in BV([c, d])$ if $V_f([c, d]) < \infty$.

**Definition 1.21**. Let $f : J \to \mathbb{C}$, where $J$ is a subinterval of $\mathbb{R}$. We say that $f$ has *locally bounded variation* on $J$, or $f \in BV_{loc}(J)$, if for all compact subintervals $[c, d]$ of $J$, $f \in BV([c, d])$.

If the interval $J$ is clear from the context, we write $f \in BV_{loc}$.

Liflyand and Tikhonov [15] defined *general monotone functions* ($GM$). We will work with a special case of their definition.

**Definition 1.22**. Let $B \geq 1$. We write $f \in GM(B)$ if $f \in BV_{loc}((0, \infty))$ and for all $x > 0$:

$$V_f([x, 2x]) \leq B\, |f(x)|\,. \tag{1.19}$$

Define $GM = \bigcup_{B \geq 1} GM(B)$. If $f \in GM$, we say that $f$ is a *general monotone function.*

We write $GM^+ = \{f \in GM : f \geq 0\}$.

**Remark 1.23**. From the definitions of $GM$ and $GMS$ it is clear that if $\{a_n\} \in GM$ then $\{|a_n|\} \in GM^+$ and if $f \in GMS$ then $|f| \in GMS^+$.

We will often use the following classes of functions and sequences:

**Definition 1.24**.

$$GM_1(B) := \{f \in \mathcal{M}_{\mathbb{R}^+} : |f(t)| \leq B|f(x)|, \ \forall x \leq t \leq 2x\}; \tag{1.20}$$

$$GMS_1(B) := \{\{a_k\}_{k=1}^{\infty} : |a_k| \leq B|a_n|, \ \forall n \leq k \leq 2n\}; \tag{1.21}$$

$$GM_2(B) := \left\{ f \in BV_{loc}\left(\mathbb{R}^+\right) : V_f([x, M]) \leq B \left( |f(x)| + \int_x^M |f(t)| \frac{dt}{t} \right), \forall M > 0, \forall 0 < x < M \right\}; \tag{1.22}$$

$$GMS_2(B) := \left\{ \{a_k\}_{k=1}^{\infty} : \sum_{k=n}^{N-1} |a_k - a_{k+1}| \leq B \left( |a_n| + \sum_{k=n+1}^{N} \frac{|a_k|}{k} \right), \forall N > 1, \forall 1 \leq n < N \right\}. \tag{1.23}$$

We write for $j = 1, 2$, $GM_j = \bigcup_{B \geq 1} GM_j(B)$, $GMS_j = \bigcup_{B \geq 1} GMS_j(B)$, $GM_j^+ = \{f \in GM_j : f \geq 0\}$, and $GMS_j^+ = GMS_j \cap \{\{a_n\} : a_n \geq 0, \forall n\}$.

The following is a useful characterization of $GM$:

**Lemma 1.25** (E. Liflyand and S. Tikhonov, [15]).

$$GM(B) \subset GM_1(2B) \cap GM_2\left(2B^2\right); \tag{1.24}$$

$$GM_1(B) \cap GM_2(B) \subset GM\left(2B^2\right). \tag{1.25}$$

A characterization analogous to Lemma 1.25 holds for general monotone sequences:

**Lemma 1.26** (S. Tikhonov, [23]).

$$GMS(B) \subset GMS_1(2B) \cap GMS_2\left(2B^2\right); \tag{1.26}$$

$$GMS_1(B) \cap GMS_2(B) \subset GMS\left(2B^2\right). \tag{1.27}$$

**Theorem 1.27** (S. Tikhonov, [23]).

Let $\{c_n\} \in GMS^+$, $\forall n$, let $1 < p < \infty$, $1 \leq q < \infty$ and let $c_n$ be the trigonometric Fourier coefficients of a function $f$. Then $f \in L^q_{w(p,q)}(0,\pi)$ if and only if $\{c_n\} \in l^q_{w(p',q)}$, and

$$\|f\|_{L^q_{w(p,q)}(0,\pi)} \sim \|\{c_n\}\|_{l^q_{w(p',q)}} . \tag{1.28}$$

The following theorem extends Theorem 1.27. It also generalizes (1.15) to $GMS$.

**Theorem 1.28** (B. Booton, [3]).

Let $\{c_n\} \in GMS^+$, $\forall n$, let $1 < p < \infty$, $1 \leq q \leq \infty$ and let $c_n$ be the trigonometric Fourier coefficients of a function $f$. Then (1.15) holds. If $\{c_n\}$ is the sequence of sine Fourier coefficients of $f$, (1.15) holds also for the case $p = q = \infty$.

In the present work, we use Real Interpolation theory to derive relations similar to (1.15) for functions with general monotone Fourier coefficients. For instance, we extend the results of Tikhonov and Booton to a class of complex-valued general monotone sequences $\{c_n\}$ and all positive $p$ and $q$ (Theorem 7.5).

Because of the central role of Hardy's inequality in Classical Analysis, it may be worthwhile pointing out that we give a short interpolation proof that Hardy's inequality, when applied to general monotone functions, is an equivalence that holds for the full range of parameters (see Theorem 8.2). For a longer proof of a somewhat stronger theorem, see Booton [4].

# CHAPTER 2

# SOME RESULTS FROM REAL INTERPOLATION THEORY

**Definition 2.1**. A *semigroup* in a topological vector space (t.v.s.) is a subset $G$ of the t.v.s. that is closed under addition. A semigroup that contains 0 is a *monoid*. A monoid that is closed under multiplication by any positive scalar is a *cone*.

**Definition 2.2**. A *quasi-norm* on a monoid $G$ is a function $\|\cdot\|_G : G \to [0, \infty)$ that satisfies:

(a) $\|a\|_G = 0 \Leftrightarrow a = 0$.

(b) $\exists k = k(G)$ such that $\forall a_1, a_2 \in G$:

$$\|a_1 + a_2\|_G \leq k \left( \|a_1\|_G + \|a_2\|_G \right).$$

**Definition 2.3**. A quasi-norm on a cone $G$ is a quasi-norm also satisfying:

(c) $\|\lambda a\|_G = \lambda \|a\|_G$, $\forall \lambda > 0, a \in G$.

**Defintion 2.4**. A quasi-norm on a vector space $G$ is a quasi-norm for which the condition (c) holds for all complex $\lambda$:

(c') $\|\lambda a\|_G = |\lambda| \|a\|_G$, $\forall \lambda \in \mathbb{C}, a \in G$.

In the Definitions 2.2-2.4, if $k = 1$ then $\|\cdot\|_G$ is a *norm* on $G$.

A vector space equipped with a quasi-norm is a *quasi-normed vector space*, a complete quasi-normed vector space is called a *quasi-Banach space*. A monoid or a cone which is a subset of a quasi-normed vector space is a *quasi-normed monoid* or *quasi-normed cone* respectively. When the quasi-norm is a norm, we follow the usual convention and say that the corresponding structure is *normed*.

**Definition 2.5**. If $G_0$ and $G_1$ are monoids continuously embedded in a t.v.s., we say that $(G_0, G_1)$ is an *interpolation couple*.

Throughout this section, $G_0$, $G_1$ will be quasi-normed monoids which form an interpolation couple in a t.v.s. of measurable functions, the topology of the t.v.s. is defined by the convergence in measure.

**Definition 2.6**. For $f \in G_0 + G_1$, $t > 0$, define

$$K(t, f) = K(t, f, G_0, G_1)$$

$$= \inf \left\{ \|f_0\|_{G_0} + t\,\|f_1\|_{G_1} \,\middle|\, f_0 + f_1 = f, f_j \in G_j, j = 0, 1 \right\}. \tag{2.1}$$

**Definition 2.7**. ($K$-method of interpolation of monoids).

For $f \in G_0 + G_1$, and for each $0 < \theta < 1$ and $0 < q \leq \infty$ or for $0 \leq \theta \leq 1$ and $q = \infty$, define

$$\|f\|_{(G_0,G_1)_{\theta,q;K}} = \left\|t^{-\theta} K(t, f)\right\|_{L^q\left((0,\infty), \frac{dt}{t}\right)}. \tag{2.2}$$

Define the *Interpolation monoid with respect to* $K$ as

$$(G_0, G_1)_{\theta,q;K} = \left\{ f \in G_0 + G_1 \,\middle|\, \|f\|_{(G_0,G_1)_{\theta,q;K}} < \infty \right\}. \tag{2.3}$$

It is clear that $(G_0, G_1)_{\theta,q;K}$ is a monoid. When $G_0$ and $G_1$ are quasi-normed vector spaces, we call $(G_0, G_1)_{\theta,q;K}$ an *Interpolation space.*

**Definition 2.8**. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $\mathcal{M} = \mathcal{M}(\Omega, \Sigma, \mu; \mathbb{C})$ be the space of $\mu$-measurable complex-valued functions on $\Omega$. A Banach space $X$ of functions in $\mathcal{M}$ is called a *Banach function space* provided:

(1) If $g \in \mathcal{M}$ and $|g| \leq |f|$ $\mu$-a.e. for some $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$;

(2) If $f_n \geq 0$, $f_n \in X$, all $n$ and $f_n \nearrow f \in X$ $\mu$-a.e., then $\lim_{n\to\infty} \|f_n\|_X = \|f\|_X$.

The next definition generalizes Definition 1.6 to all Banach function spaces.

**Definition 2.9**. Assume that $X$ is a Banach function space, $w \in \mathcal{M}^+(\Omega, \Sigma, \mu)$, and $f \in \mathcal{M}(\Omega, \Sigma, \mu)$, are such that $f \cdot w \in X$. Define the *weighted $X$-norm* of $f$ with the weight $w$:

$$\|f\|_{X_w} = \|f \cdot w\|_X .$$

$X_w$ is the space of all measurable functions $f$ for which $\|f\|_{X_w}$ is finite.

**Definition 2.10**. For $g \in G_0 \cap G_1$, $t > 0$, define

$$J(t, g) = J(t, g, G_0, G_1) = \max\left(\|g\|_{G_0}, t\,\|g\|_{G_1}\right). \tag{2.4}$$

**Definition 2.11** ($J$-method of interpolation of Banach spaces).

Assume $G_0$ and $G_1$ are Banach spaces. For $f \in G_0 \cap G_1$, and for each $0 < \theta < 1$ and $0 < q \leq \infty$ or for $\theta = 0, 1$ and $q = 1$, define

$$\|f\|_{\theta,q;J} = \|f\|_{\theta,q,G_0,G_1;J}$$

$$= \inf\left\{ \left\|t^{-\theta} J(t, u(t))\right\|_{L^q\left((0,\infty),\frac{dt}{t}\right)} \middle| u : \mathcal{M}(\mathbb{R}^+) \to G_0 \cap G_1, \int_0^\infty u(t)\frac{dt}{t} = f \right\}. \tag{2.5}$$

Define the *Interpolation space with respect to $J$* as

$$(G_0, G_1)_{\theta,q;J} = \left\{ f \in G_0 + G_1 \middle| \|f\|_{\theta,q;J} < \infty \right\}. \tag{2.6}$$

**Lemma 2.12**. $f \in (G_0, G_1)_{\theta,q;J}$ if and only if there exist $u_\nu \in G_0 + G_1$, $\nu \in \mathbb{Z}$, with

$$f = \sum_\nu u_\nu$$

(convergence in $G_0+G_1$), and such that

$$\sum_{\nu\in\mathbb{Z}}\left(2^{-\nu\theta}J\left(2^{\nu},u_{\nu}\right)^{q}\right)<\infty.$$

Moreover,

$$\|f\|_{\theta,q;J}\sim\inf\left\{\left(\sum_{\nu\in\mathbb{Z}}\left(2^{-\nu\theta}J\left(2^{\nu},u_{\nu}\right)^{q}\right)\right)^{\frac{1}{q}}:f=\sum_{\nu}u_{\nu}\right\}.$$

Furthermore, $G_0$ and $G_1$ are quazi-normed monoids, then $\forall 0<\theta<1$, $\forall 0<q\leq\infty$,

$$\|f\|_{\theta,q;K}\sim\inf\left\{\left(\sum_{\nu\in\mathbb{Z}}\left(2^{-\nu\theta}J\left(2^{\nu},u_{\nu}\right)^{q}\right)\right)^{\frac{1}{q}}:f=\sum_{\nu}u_{\nu}\right\}., \tag{2.7}$$

(see [2], Lemmas 3.2.3 and 3.11.3).

When $G_0$ and $G_1$ are quazi-normed monoids, we therefore can write $(G_0,G_1)_{\theta,q}$ without specifying which interpolation method was used.

**Lemma 2.13**. If $G_0,G_1$ are monoids then $K(t,\cdot)$ is a quasi-norm on $G_0+G_1$. If $G_0$ and $G_1$ are vector spaces then $J(t,\cdot)$ is a quasi-norm on $G_0\cap G_1$, and moreover, $K(\cdot,f)$ and $J(\cdot,f)$ are continuous and increasing functions on $(0,\infty)$.

If $G_0$, $G_1$ are normed monoids, $K(t,\cdot)$ is a norm; if $G_0$ and $G_1$ are normed vector spaces, $J(t,\cdot)$ is a norm.

**Theorem 2.14** (J. Gilbert, [8]).

Assume $X$ is a Banach function space, $w$ is a measurable positive function on $\Omega$, $\sigma$ is positive, piecewise continuous function such that both $\sigma$ and $\sigma_t$, where $\sigma_t(\lambda):=t\lambda\sigma(t\lambda)$, are in $L^{\infty}(0,\infty)$, $\forall t>0$. For any $f\in(X,X_w)_{\theta,q}$, $\theta\in(0,1)$, $q\in(0,\infty]$, there are $C_j = C_j(\sigma, \theta, q)$ $(j = 1, 2)$ such that

$$C_1 \|f\|_{\theta,q;J} \leq \left( \int_0^\infty \left( t^{-\theta} \|f \cdot \sigma_t \circ w\|_X \right)^q \frac{dt}{t} \right)^{1/q} \leq C_2 \|f\|_{\theta,q;K} . \tag{2.8}$$

J.-L. Lions and J.Peetre proved that for $0 < \theta < 1$, $0 < q \leq \infty$, $0 \leq p_0, p_1 \leq \infty$, $p_0 \neq p_1$, $(L^{p_0}, L^{p_1})_{\theta,q} = L(p, q)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (see [16]). Their result was generalized by J. Bergh and J. Löfström for the interpolation of Lorentz spaces:

**Theorem 2.15** (J. Bergh; J. Löfström, [2]).

Suppose that $p_0$, $p_1$, $q_0$, and $q_1$ are positive, possibly infinite, numbers, $p_0 \neq p_1$, and write $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, where $0 < \theta < 1$. Then for $0 < q \leq \infty$:

$$\left( L(p_0, q_0), L(p_1, q_1) \right)_{\theta,q} = L(p, q). \tag{2.9}$$

The formula is also true in the case $p_0 = p_1 = p$, provided that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

**Definition 2.16**. Let $(G_0, G_1)$ and $(Y_0, Y_1)$ be two interpolation couples of quasi-normed monoids. An operator $T : G_0 + G_1 \to Y_0 + Y_1$ is called *quasi-linear* if $\exists K_j$, $j = 0, 1$, so that for every $a_0 \in G_0$, $a_1 \in G_1$ we can find $b_0 \in Y_0$, $b_1 \in Y_1$ for which

$$\begin{cases} T(a_0 + a_1) = b_0 + b_1 \\ \|b_j\|_{Y_j} \leq K_j \|a_j\|_{G_j}, \qquad j = 0, 1 \end{cases}$$

For quasi-linear operators the following interpolation theorem holds:

**Theorem 2.17**. Let $T : G_j \to Y_j$, $j = 0, 1$, $T$ a quasi-linear operator. Then for $0 < \theta < 1$, $0 < q \leq \infty$, $T$ maps $(G_0, G_1)_{\theta,q}$ into $(Y_0, Y_1)_{\theta,q}$ and $\forall a \in G_0 + G_1$.

$$\|Ta\|_{\theta,q} \leq K_0^{1-\theta} K_1^{\theta} \|a\|_{\theta,q} \tag{2.10}$$

(see e.g. [18]).

Theorems 2.15 and 2.17 imply the following important fact about interpolation of operators in Lorentz spaces.

**Theorem 2.18** (Marcinkiewicz-Calderón-Hunt, [6],[13]).

Suppose that $T$ is a quasi-linear operator mapping the vector space $(L(p_0, q_0) + L(p_1, q_1))\,(\Omega_0, \Sigma_0, \mu_0)$ to the vector space of measurable functions on $(\Omega_1, \Sigma_1, \mu_1)$. Assume also that for all simple integrable functions, $f$,

$$\|Tf\|_{L(\tilde{p}_j,\tilde{q}_j)} \leq M_j \|f\|_{L(p_j,q_j)}\,,$$

where $0 < p_0 < p_1 \leq \infty$, $\tilde{p}_0 \neq \tilde{p}_1$, $0 < \tilde{p}_j \leq \infty$, $0 < q_j, \tilde{q}_j \leq \infty$ (but if $p_1 = \infty$ then also $q_1 = \infty$ and if $\tilde{p}_1 = \infty$ then also $\tilde{q}_1 = \infty$).

Then for $0 < \theta < 1$, $p$ and $\tilde{p}$, given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1};\ \frac{1}{\tilde{p}} = \frac{1-\theta}{\tilde{p}_0} + \frac{\theta}{\tilde{p}_1} \tag{2.11}$$

and $f \in L(p, q)$, any $0 < q \leq \infty$,

$$\|Tf\|_{L(\tilde{p},q)} \leq C(p_j, \tilde{p}_j, q_j, \tilde{q}_j, k, M_j, \theta)\,\|f\|_{L(p,q)} \tag{2.12}$$

Note that, although $q_j$ may be different from $\tilde{q}_j$, $q$ does not depend on $q_0$, $q_1$, $\tilde{q}_0$, or $\tilde{q}_1$.

# CHAPTER 3

# SELECTED RESULTS FROM THE THEORY OF $L(p, q)$ SPACES

## 3.1 REARRANGEMENT INEQUALITIES FOR SEQUENCES

We will use following form of the summation by parts formula. For $\{a_k\}$, $\{b_k\}$, $1 \le k \le n$:

$$\sum_{k=1}^{n} a_k b_k = b_n \sum_{k=1}^{n} a_k + \sum_{k=1}^{n-1} \left( \sum_{l=1}^{k} a_l \right) (b_k - b_{k+1}) . \tag{3.1}$$

**Theorem 3.1**. Assume that for all $k \ge 1$, $c_k \ge 0$.

(a) Suppose $\{c_k\} \searrow$, and assume that for all $1 \le m \le n$, $\sum_{k=1}^{m} a_k \le \sum_{k=1}^{m} b_k$. Then:

$$\sum_{k=1}^{n} a_k c_k \le \sum_{k=1}^{n} b_k c_k. \tag{3.2}$$

(b) Suppose $\{c_k\} \nearrow$, and assume that for all $1 \le m \le n$, $\sum_{k=m}^{n} a_k \le \sum_{k=m}^{n} b_k$. Then (3.2) holds.

**Proof**.

(a) Since $c_k \ge c_{k+1}$ and $\sum_{l=1}^{k} b_l \ge \sum_{l=1}^{k} a_l \ge 0$ for all $k$, it follows by (3.1) that

$$\sum_{k=1}^{n} a_k c_k = c_n \sum_{k=1}^{n} a_k + \sum_{k=1}^{n-1} \left( \sum_{l=1}^{k} a_l \right) (c_k - c_{k+1})$$

$$\le c_n \sum_{k=1}^{n} b_k + \sum_{k=1}^{n-1} \left( \sum_{l=1}^{k} b_l \right) (c_k - c_{k+1}) = \sum_{k=1}^{n} b_k c_k.$$

(b) Observe that

$$\sum_{l=1}^{n-m+1} a_{n-l+1} = \sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} b_k = \sum_{l=1}^{n-m+1} b_{n-l+1},$$

and that $\{c_{n-l+1}\} \searrow$ as a function of $l$, hence we can apply the result of part (a):

$$\sum_{k=1}^{n} a_k c_k = \sum_{l=1}^{n} a_{n-l+1} c_{n-l+1} \leq \sum_{l=1}^{n} b_{n-l+1} c_{n-l+1} = \sum_{k=1}^{n} b_k c_k . \square$$

The next lemma can be derived from (3.1). We give a direct proof below:

**Lemma 3.2**. If $\sum_{k=1}^{\infty} a_k$ is a convergent series, then for all $\{c_k\}$,

$$\sum_{k=1}^{n} a_k c_k = \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l \right) (c_k - c_{k-1}) - c_n \sum_{l=n+1}^{\infty} a_l. \tag{3.3}$$

**Proof**.

Let $c_0 = 0$. Then:

$$\sum_{k=1}^{n} a_k c_k = \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l - \sum_{l=k+1}^{\infty} a_l \right) c_k$$

$$= \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l \right) c_k - \sum_{k=1}^{n+1} \left( \sum_{l=k}^{\infty} a_l \right) c_{k-1}$$

$$= \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l \right) (c_k - c_{k-1}) - c_n \sum_{l=n+1}^{\infty} a_l . \square$$

**Lemma 3.3**. Assume $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, and $\{c_k\} \nearrow$. Assume that $\forall m \geq 1$,

$$\sum_{k=m}^{\infty} a_k \leq \sum_{k=m}^{\infty} b_k. \tag{3.4}$$

Assume also that $\exists N$ such that for all $n > N$, $b_n \geq 0$, $c_{n-1} \cdot \sum_{k=n}^{\infty} a_k \geq 0$. Then for all $n \geq N$:

$$\sum_{k=1}^{n} a_k c_k \leq \sum_{k=1}^{\infty} b_k c_k. \tag{3.5}$$

**Proof**.

Since $c_k \nearrow$, it follows that for each $m > n$, $c_n \sum_{l=n+1}^{m} b_l \leq \sum_{k=n+1}^{m} b_k c_k$, and that the products $b_k c_k$ have a constant sign from some $k$ on. Hence, $\lim_{m\to\infty} \sum_{k=n+1}^{m} b_k c_k$ exists, possibly infinite, implying $c_n \sum_{l=n+1}^{\infty} b_l \leq \sum_{k=n+1}^{\infty} b_k c_k$.

Therefore, by Lemma 3.2, for all $n \geq N$:

$$\sum_{k=1}^{n} a_k c_k = \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l \right) (c_k - c_{k-1}) - c_n \sum_{l=n+1}^{\infty} a_l$$

$$\leq \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} a_l \right) (c_k - c_{k-1}) \leq \sum_{k=1}^{n} \left( \sum_{l=k}^{\infty} b_l \right) (c_k - c_{k-1})$$

$$= \sum_{k=1}^{n} b_k c_k + c_n \sum_{l=n+1}^{\infty} b_l \leq \sum_{k=1}^{\infty} b_k c_k. \Box$$

**Theorem 3.4**. Let $a_k, b_k, c_k \geq 0$, all $k \geq 1$.

(a) Suppose $\{c_k\} \searrow$, and assume that for $\forall m \geq 1$,

$$\sum_{k=1}^{m} a_k \leq \sum_{k=1}^{m} b_k. \tag{3.6}$$

Then:

$$\sum_{k=1}^{\infty} a_k c_k \leq \sum_{k=1}^{\infty} b_k c_k. \tag{3.7}$$

(b) Suppose $\{c_k\} \nearrow$, and assume (3.4) holds for all $m$. Then (3.7) holds.

**Proof**.

(a) Follows trivially from part (a) of Theorem 3.1.

(b) We may assume $c_k > 0$ for some $k$. If $\sum_{k=1}^{\infty} b_k = \infty$ then also $\sum_{k=1}^{\infty} b_k c_k = \infty$ and the result is trivial. Therefore, we may assume that $\sum_{k=1}^{\infty} b_k < \infty$. Since $a_k \geq 0$ and due to (3.4), it follows that $\sum_{k=1}^{\infty} a_k < \infty$. Hence, conditions of Lemma 3.3 are met, and so, for each $n \geq 1$, (3.5) holds; (3.7) follows by taking $n \to \infty$.$\Box$

**Corollary 3.4** ([2]). Suppose that $I$ is a subinterval of $\mathbb{Z}$ and that $\{a_k\}_{k\in I}$, $\{c_k\}_{k\in I}$ are nonnegative sequences. If $\{c_k\} \searrow$ then $\sum\limits_{k\in I} a_k c_k \le \sum\limits_{k\in I} a_k^* c_k$. If $\{c_k\} \nearrow$ then $\sum\limits_{k\in I} a_k c_k \ge \sum\limits_{k\in I} a_k^* c_k$.

## 3.2 REARRANGEMENT INEQUALITIES FOR FUNCTIONS

The part (a) of the following Theorem is due to G. H. Hardy, and its proof appears in [14].

**Theorem 3.5** (G.H. Hardy).

Let $f_1, f_2, g$ be nonnegative, measurable functions on $\mathbb{R}^+$.

(a) Assume that $g \searrow$ and that for all $a > 0$, $\int\limits_0^a f_1 dt \le \int\limits_0^a f_2 dt$. Then

$$\int\limits_0^\infty f_1 g dt \le \int\limits_0^\infty f_2 g dt. \tag{3.8}$$

(b) Assume that $g \nearrow$ and that for all $a > 0$, $\int\limits_a^\infty f_1 dt \le \int\limits_a^\infty f_2 dt$. Then (3.8) holds.

**Proof**.

(a) Consider first the case when $g$ is a left-continuous simple function. Then $g$ can be written as

$$g(x) = \sum_{j=1}^N c_j I_{(0,a_j]}(x), \tag{3.9}$$

where $c_j \ge 0$ and $\{a_j\} \nearrow$ $(a_N = \infty)$. Hence:

$$\int\limits_0^\infty f_1 g dt = \sum_{j=1}^N c_j \int\limits_0^{a_j} f_1 dt \le \sum_{j=1}^N c_j \int\limits_0^{a_j} f_2 dt = \int\limits_0^\infty f_2 g dt.$$

Then for an arbitrary decreasing $g$:

$$\int\limits_0^\infty f_1 g dt = \sup\left\{ \int\limits_0^\infty h f_1 dt : h \le g,\ h \text{ is of form (3.9)} \right\}$$

$$\leq \sup \left\{ \int_0^\infty h f_2 dt : h \leq g,\ h \text{ is of form } (3.9) \right\} = \int_0^\infty f_2 g dt.$$

(b) Consider first the case when $g$ is a left-continuous simple function. Then $g$ can be written as

$$g(x) = \sum_{j=1}^{N} c_j I_{(a_j,\infty)}(x), \tag{3.10}$$

where $c_j \geq 0$ and $\{a_j\} \nearrow$. Hence:

$$\int_0^\infty f_1 g dt = \sum_{j=1}^{N} c_j \int_{a_j}^\infty f_1 dt \leq \sum_{j=1}^{N} c_j \int_{a_j}^\infty f_2 dt = \int_0^\infty f_2 g dt.$$

Then for an arbitrary increasing $g$:

$$\int_0^\infty f_1 g dt = \sup \left\{ \int_0^\infty h f_1 dt : h \leq g,\ h \text{ is of form } (3.10) \right\}$$

$$\leq \sup \left\{ \int_0^\infty h f_2 dt : h \leq g,\ h \text{ is of form } (3.10) \right\} = \int_0^\infty f_2 g dt. \square$$

**Corollary 3.5**. Suppose that $f, g$ are nonnegative, measurable on $\mathbb{R}^+$ functions. If $g \searrow$ then $\int_0^\infty f g dt \leq \int_0^\infty f^* g dt$. If $g \nearrow$ then $\int_0^\infty f g dt \geq \int_0^\infty f^* g dt$.
For more related results on decreasing rearrangements, see [14].

## 3.3 FURTHER RESULTS ON $L(p, q)$ SPACES

**Lemma 3.6**. Assume that $\{a_n\}^*$ exists. For $0 < p, q < \infty$,

$$\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} (a_k^*)^q \sim \int_0^\infty x^{\frac{q}{p}} \left(\{a_n\}^*(x)\right)^q \frac{dx}{x}. \tag{3.11}$$

Also, for $0 < p \leq \infty$,

$$\sup_{k \in \mathbb{N}} \left( k^{\frac{1}{p}} a_k^* \right) = \sup_{x \geq 0} \left( x^{\frac{1}{p}} \{a_n\}^*(x) \right). \tag{3.12}$$

**Proof.**

Observe first that $\int\limits_0^1 x^{\frac{q}{p}-1}dx = \frac{p}{q}$. Let $k \geq 2$. Then $k-1 \geq \frac{k}{2}$. If $p \leq q$ then

$$2^{1-\frac{q}{p}}k^{\frac{q}{p}-1} = \left(\frac{k}{2}\right)^{\frac{q}{p}-1} \leq \int\limits_{k-1}^{k} x^{\frac{q}{p}-1}dx \leq k^{\frac{q}{p}-1}.$$

If $p > q$ then

$$k^{\frac{q}{p}-1} \leq \int\limits_{k-1}^{k} x^{\frac{q}{p}-1}dx \leq \left(\frac{k}{2}\right)^{\frac{q}{p}-1} = 2^{1-\frac{q}{p}}k^{\frac{q}{p}-1}.$$

Therefore, for $k \in \mathbb{N}$,

$$\min\left(1, \frac{p}{q}, 2^{1-\frac{q}{p}}\right) k^{\frac{q}{p}-1} \leq \int\limits_{k-1}^{k} x^{\frac{q}{p}-1}dx \leq \max\left(1, \frac{p}{q}, 2^{1-\frac{q}{p}}\right) k^{\frac{q}{p}-1}.$$

It follows that

$$\int\limits_0^\infty x^{\frac{q}{p}} \left(\{a_n\}^*(x)\right)^q \frac{dx}{x} = \sum_{k=1}^{\infty} \int\limits_{k-1}^{k} x^{\frac{q}{p}-1} \left(\{a_n\}^*(x)\right)^q dx$$

$$= \sum_{k=1}^{\infty} (a_k^*)^q \int\limits_{k-1}^{k} x^{\frac{q}{p}-1}dx \sim \sum_{k=1}^{\infty} (a_k^*)^q \, k^{\frac{q}{p}-1}.$$

This proves (3.11). For the proof of (3.12), notice that for $\forall x \in [k-1, k)$, $\{a_n\}^*(x) = \{a_n\}^*(k-1) = a_k^*$. Therefore,

$$\sup_{k-1 \leq x < k} \left(x^{\frac{1}{p}} \{a_n\}^*(x)\right) = \sup_{k-1 \leq x < k} \left(x^{\frac{1}{p}} a_k^*\right) = k^{\frac{1}{p}} a_k^*,$$

and so

$$\sup_{x \geq 0} \left(x^{\frac{1}{p}} \{a_n\}^*(x)\right) = \sup_{k \in \mathbb{N}} \left(k^{\frac{1}{p}} a_k^*\right). \square$$

**Theorem 3.7**. Let $f$ be measurable on $\mathbb{R}^+$. For $0 < p \leq q \leq \infty$,

$$\|f\|_{L(p,q)(0,\infty)} \leq \|f\|_{L^q_{w(p,q)(0,\infty)}}. \tag{3.13}$$

**Proof.**

It suffices to consider $f \in L^q_{w(p,q)}(0,\infty)$.

If $q<\infty$ then $x^{\frac{q}{p}-1} \nearrow$, and therefore, by Corollary 3.5:

$$\|f\|^q_{L^q_{w(p,q)}} = \int_0^\infty x^{\frac{q}{p}-1} |f(x)|^q \, dx \geq \int_0^\infty x^{\frac{q}{p}-1} \left((f(x))^q\right)^* dx$$

$$= \int_0^\infty x^{\frac{q}{p}-1} \left(f^*(x)\right)^q dx = \|f\|^q_{L(p,q)} .$$

If $q=\infty$ then $|f(x)| \leq x^{-\frac{1}{p}} \|f\|_{L^\infty_{w(p,\infty)}}$, for almost every $x>0$, and $x^{-\frac{1}{p}}$ is nonincreasing on $(0,\infty)$. Therefore, taking Remark 1.3 into account, $f^*(x) \leq x^{-\frac{1}{p}} \|f\|_{L^\infty_{w(p,\infty)}}$, $\forall x>0$, and so,

$$\|f\|_{L(p,\infty)} = \sup\left\{x^{\frac{1}{p}} f^*(x)\right\} \leq \|f\|_{L^\infty_{w(p,\infty)}} . \square$$

The following theorem is proved for sine Fourier coefficients in [20].

**Theorem 3.8**. Let $f \in L(p',q)$, $1<p<\infty$, $0<q\leq\infty$ and assume that $\{c_n\}$ are Fourier coefficients of $f$ with respect to $\{e^{int}\}_{n=-\infty}^{\infty}$. Define $\sigma_n = \frac{1}{2n+1}\sum_{k=-n}^{n} c_k$, and let $\{m(a_n)\}$ be the least bell-shaped majorant of $\{a_n\}$, that is to say, $m(a_n) := \sup\{|a_k| : |k| \geq |n|\}$. Then $\exists A(p)>0$ so that:

$$\|\{m(\sigma_n)\}\|_{l(p,q)} \leq A(p) \|f\|_{L(p',q)(0,2\pi)} , \tag{3.14}$$

Consequently,

$$\|\{\sigma_n\}\|_{l(p,q)} \leq A(p) \|f\|_{L(p',q)(0,2\pi)} . \tag{3.15}$$

**Proof**.

$$\sigma_n = \frac{1}{2n+1} \int_0^{2\pi} f(t) \left(\sum_{k=-n}^{n} e^{ikt}\right) dt$$

$$= \frac{1}{2n+1} \int_0^{2\pi} f(t) \left(1 + 2\sum_{k=1}^{n} \cos kt\right) dt.$$

Since

$$\left(1+2\sum_{k=1}^{n}\cos kt\right)\cdot\sin\frac{t}{2}$$

$$=\sin\frac{t}{2}+\sum_{k=1}^{n}\left(\sin t\left(k+\frac{1}{2}\right)-\sin t\left(k-\frac{1}{2}\right)\right)=\sin t\left(n+\frac{1}{2}\right),$$

it follows that

$$|\sigma_n|\le\frac{1}{2n+1}\,\|f\|_{L^{p'}(0,2\pi)}\left(\int_0^{2\pi}\left|1+2\sum_{k=1}^{n}\cos kt\right|^p dt\right)^{\frac{1}{p}}$$

$$\le\|f\|_{L^{p'}(0,2\pi)}\left(\int_0^{2\pi}\left|\frac{\sin t\left(n+\frac{1}{2}\right)}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}},$$

but

$$\left(\int_0^{2\pi}\left|\frac{\sin t\left(n+\frac{1}{2}\right)}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}}=\left(\int_0^{2\pi}\left|\frac{\sin nt\cos\frac{t}{2}+\cos nt\sin\frac{t}{2}}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\le\left(\int_0^{2\pi}\left|\frac{\sin nt}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}}+\frac{(2\pi)^{\frac{1}{p}}}{2n}=A(n,p)+c_0n^{-1},$$

where

$$A(n,p)=\left(\int_0^{\pi}\left|\frac{\sin nt}{2n\sin\frac{t}{2}}\right|^p dt+\int_{\pi}^{2\pi}\left|\frac{\sin nt}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}}$$

$$=\left(\int_0^{\pi}\left|\frac{\sin nt}{2n\sin\frac{t}{2}}\right|^p dt+\int_0^{\pi}\left|\frac{\sin n\left(2\pi-u\right)}{2n\sin\frac{2\pi-u}{2}}\right|^p du\right)^{\frac{1}{p}}$$

$$=\left(2\int_0^{\pi}\left|\frac{\sin nt}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}}.$$

Since for $0\le t\le\pi$, $\sin\dfrac{t}{2}\ge\dfrac{t}{\pi}$, it follows that

$$A(n,p)\le C_1(p)\left(\int_0^{\pi}\left|\frac{\sin nt}{nt}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq C_1(p)\left(\int_0^{\infty}\left|\frac{\sin u}{u}\right|^p d\left(\frac{u}{n}\right)\right)^{\frac{1}{p}} = C_2(p)n^{-\frac{1}{p}},$$

and so,

$$\left(\int_0^{2\pi}\left|\frac{\sin t\left(n+\frac{1}{2}\right)}{2n\sin\frac{t}{2}}\right|^p dt\right)^{\frac{1}{p}} \leq C_2(p)n^{-\frac{1}{p}} + c_0 n^{-1} \leq C(p)n^{-\frac{1}{p}},$$

proving that

$$m\left(\sigma_n\right) \leq C(p)n^{-\frac{1}{p}} \left\|f\right\|_{L^{p'}(0,2\pi)}. \tag{3.16}$$

If $Tf = m\left(\sigma_n\right)$, then (3.16) proves that for all $1 < p < \infty$,

$$T : L^{p'}\left(0, 2\pi\right) \to l\left(p, \infty\right).$$

Given $1 < p < \infty$, take $1 < p_0 < p < p_1 < \infty$. Since $T : L^{p_j'}\left(0, 2\pi\right) \to l\left(p_j, \infty\right)$, by Theorem 3.17 it follows that

$$T : L\left(p', q\right)\left(0, 2\pi\right) \to l\left(p, q\right),$$

for all $0 < q \leq \infty$, proving (3.14).□

# CHAPTER 4

# SOME ELEMENTARY PROPERTIES OF $GM$ AND $GMS$

## 4.1 RELATIONS BETWEEN VARIOUS TYPES OF GENERAL MONOTONICITY

**Lemma 4.1**. For $j = 1,2$, if $f \in GM_j$ then $\{f(n)\} \in GMS_j$. Moreover, if $\{a_n\} \in GMS_1$ and $f(x) = a_{\lceil x \rceil}$ then $f \in GM_1$.

**Proof.**

If $f \in GM_1(B)$ then, for all $n \geq 1$ and $k \in [n, 2n]$, $|f(k)| \leq B|f(n)|$, and so $\{f(n)\} \in GMS_1(B)$.

Conversely, assume that $\{a_n\} \in GMS_1(B)$. For all $0 < x \leq t \leq 2x$, $\lceil x \rceil \leq \lceil t \rceil \leq \lceil 2x \rceil \leq 2\lceil x \rceil$ and so:

$$|f(t)| = \left|a_{\lceil t \rceil}\right| \leq B\left|a_{\lceil x \rceil}\right| = B|f(x)|.$$

Let $f \in GM_2(B)$. For $t \in [k, k+1]$:

$$|f(t)| \leq |f(k)| + V_f([k,t]) \leq |f(k)| + V_f([k,k+1])$$
$$\leq |f(k)| + B\left(|f(k)| + \int_k^{k+1} |f(s)| \frac{ds}{s}\right) \leq 2B|f(k)| + B\int_k^{k+1} |f(s)| \frac{ds}{s}.$$

It follows that

$$\int_k^{k+1} |f(t)| \frac{dt}{t} \leq \frac{1}{k}\int_k^{k+1} |f(t)|\, dt \leq 2B\frac{|f(k)|}{k} + \frac{B}{k}\int_k^{k+1} |f(t)| \frac{dt}{t}.$$

Consider the sequence $\{f(n)\}$. Assume first that $n \geq \lfloor 2B \rfloor$. Notice that for $B \geq 1$, $\lfloor 2B \rfloor > \frac{4}{3}B$ and so for $\forall k \geq n$, $1 - \frac{B}{k} > 1 - \frac{3}{4} = \frac{1}{4}$ and therefore:

$$\int_k^{k+1} |f(t)| \frac{dt}{t} \leq \frac{2B}{1-\frac{B}{k}} \frac{|f(k)|}{k} \leq 8B \frac{|f(k)|}{k}.$$

It follows now for $\forall N > 1$, $\forall \lfloor 2B \rfloor \leq n < N$ that

$$\sum_{k=n}^{N-1} |f(k+1) - f(k)| \leq V_f([n,N]) \leq B \left( |f(n)| + \int_n^N |f(t)| \frac{dt}{t} \right)$$

$$= B \left( |f(n)| + \sum_{k=n}^{N-1} \int_k^{k+1} |f(t)| \frac{dt}{t} \right) \leq B|f(n)| + 8B^2 \sum_{k=n}^{N-1} \frac{|f(k)|}{k}$$

$$\leq \left(B + \tfrac{8B^2}{n}\right) |f(n)| + 8B^2 \sum_{k=n+1}^{N-1} \frac{|f(k)|}{k} \leq 8B^2 \left( |f(n)| + \sum_{k=n+1}^{N-1} \frac{|f(k)|}{k} \right). \tag{4.1}$$

Assume now that $n < \lfloor 2B \rfloor$. Observe that for any function $f$, $\forall N > n$:

$$\sum_{k=n}^{N-1} |f(k+1) - f(k)| \leq |f(n)| + |f(N)| + 2 \sum_{k=n+1}^{N-1} |f(k)|$$

$$\leq 2N \left( |f(n)| + \sum_{k=n+1}^{N} \frac{|f(k)|}{k} \right) \leq 4B \left( |f(n)| + \sum_{k=n+1}^{N} \frac{|f(k)|}{k} \right), \tag{4.2}$$

whenever $N \leq 2B$. If $N > 2B$ then, applying (4.2), (4.1):

$$\sum_{k=n}^{N-1} |f(k+1) - f(k)| = \sum_{k=n}^{\lfloor 2B \rfloor - 1} |f(k+1) - f(k)| + \sum_{k=\lfloor 2B \rfloor}^{N-1} |f(k+1) - f(k)|$$

$$\leq 4B \left( |f(n)| + \sum_{k=n+1}^{\lfloor 2B \rfloor} \frac{|f(k)|}{k} \right) + 8B^2 \left( |f(\lfloor 2B \rfloor)| + \sum_{k=\lfloor 2B \rfloor + 1}^{N-1} \frac{|f(k)|}{k} \right)$$

$$\leq 4B \left( |f(n)| + \sum_{k=n+1}^{\lfloor 2B \rfloor - 1} \frac{|f(k)|}{k} \right) + \left(4B + 16B^3\right) \frac{|f(\lfloor 2B \rfloor)|}{\lfloor 2B \rfloor} + 8B^2 \left( \sum_{k=\lfloor 2B \rfloor + 1}^{N-1} \frac{|f(k)|}{k} \right)$$

$$\leq 20B^3 \left( |f(n)| + \sum_{k=n+1}^{N} \frac{|f(k)|}{k} \right). \square$$

However, $\{a_n\} \in GMS_2$ does not imply that $f$, defined as $f(x) = a_{\lceil x \rceil}$, is in $GM_2$, as the following example shows. Let $a_k = 0$ if $k \neq 2$ and $a_2 = 1$. Then $\{a_n\} \in GMS_2(4)$: $|a_n| + \sum_{k=n+1}^{N} \frac{|a_k|}{k} \geq \frac{1}{2}$ if $n \leq 2$, and 0 otherwise. If $n > 2$ then $\sum_{k=n}^{N-1} |a_k - a_{k+1}| = 0$, otherwise $\sum_{k=n}^{N-1} |a_k - a_{k+1}| \leq 2 \leq 4 \left( |a_n| + \sum_{k=n+1}^{N} \frac{|a_k|}{k} \right)$. For $f(x) = a_{\lceil x \rceil} = I_{(1,2]}(x)$, $f \notin GM_2$, since for $\forall 0 < \epsilon < 1$:

$$|f(1)| + \int_1^{1+\epsilon} |f(t)| \frac{dt}{t} < \epsilon = \epsilon \left( V_f \left( [1, 1+\epsilon] \right) \right).$$

**Lemma 4.2**. If $f \in GM$ then $\{f(n)\} \in GMS$. Conversely, for each $\{a_n\} \in GMS$ there is an $f \in GM$ such that $f(n) = a_n$, $\forall n \geq 1$.

**Proof.**

If $f \in GM(B)$ then for all $n \geq 1$:

$$\sum_{k=n}^{2n-1} |f(k+1) - f(k)| \leq V_f([n, 2n]) \leq B |f(n)|.$$

If $\{a_n\} \in GMS(B)$, let $f(x) = \sum_{n=1}^{\infty} a_n I_{(n-1,n]}(x)$, so that $f(x) = a_{\lceil x \rceil}$. For all $x > 0$:

$$V_f([x, 2x]) = \sum_{n=\lceil x \rceil}^{\lceil 2x \rceil - 1} |a_{n+1} - a_n| \leq \sum_{n=\lceil x \rceil}^{2\lceil x \rceil - 1} |a_{n+1} - a_n| \leq B \left| a_{\lceil x \rceil} \right| = B |f(x)|. \square$$

Using Lemma 4.2, results for $GMS$ follow from corresponding results for $GM$, and therefore we shall only give proofs for $GM$ functions.

Tikhonov [23] proved that $QDS \subset GMS$ as a part of a more general claim. We present a direct proof that $QD \subset GM$:

**Lemma 4.3**. $QD(\beta) \subset GM\left(2^{\beta+1} - 1\right)$.

**Proof**.

For $f \in QD(\beta)$, if $x \leq y$ then $\left(\frac{y}{x}\right)^{\beta} f(x) \geq f(y)$, and so:

$$2\left(\frac{y}{x}\right)^{\beta} f(x) - f(x) - f(y) \geq f(y) - f(x).$$

On the other hand, for any $f \geq 0$ and $x \leq y$:

$$2\left(\frac{y}{x}\right)^{\beta} f(x) - f(x) - f(y) \geq -\left(f(y) - f(x)\right).$$

That is to say, if $f \in QD(\beta)$, then for $x \leq y$ we have:

$$|f(y) - f(x)| \leq 2\left(\frac{y}{x}\right)^{\beta} f(x) - f(x) - f(y)$$

$$= 2\frac{f(x)}{x^{\beta}}\left(y^{\beta} - x^{\beta}\right) + f(x) - f(y).$$

Let $x > 0$ and $\{x_k\}_{k=0}^{n}$ be a partition of $[x, 2x]$. Then $\dfrac{f(x)}{x^{\beta}} \geq \dfrac{f\left(x_k\right)}{x_k^{\beta}}$, and therefore:

$$\sum_{k=0}^{n-1} \left|f\left(x_k\right) - f\left(x_{k+1}\right)\right|$$

$$\leq \sum_{k=0}^{n-1} \left(2\frac{f\left(x_k\right)}{x_k^{\beta}}\left(x_{k+1}^{\beta} - x_k^{\beta}\right) + f\left(x_k\right) - f\left(x_{k+1}\right)\right)$$

$$\leq 2\frac{f\left(x\right)}{x^{\beta}} \sum_{k=0}^{n-1} \left(x_{k+1}^{\beta} - x_k^{\beta}\right) + f\left(x\right) - f\left(2x\right)$$

$$\leq 2\frac{f\left(x\right)}{x^{\beta}}\left((2x)^{\beta} - x^{\beta}\right) + f(x) = f(x)\left(2^{\beta+1} - 1\right),$$

proving that

$$V_f([x, 2x]) \leq f(x)\left(2^{\beta+1} - 1\right). \square$$

## 4.2 GENERAL MONOTONICITY IN LORENTZ SPACES

The following lemma first appeared in [3] and relates a $GMS_1$ sequence with its decreasing rearrangement.

**Lemma 4.4** (B. Booton, [3]).

Let $\{c_k\} \in GMS_1(B)$ be such that $\{c_k^*\}$ exists. Then $\forall n \geq 1$,

$$|c_n| \leq B c^*_{\lfloor \frac{n}{2} \rfloor + 1}.$$

The analogous inequality holds for $GM_1$ functions. We give a proof which is similar to the one of Lemma 4.4, although somewhat shorter and more transparent.

**Lemma 4.5**. Assume that $f \in GM_1(B)$ and that $f^*$ exists. Then $\forall x > 0$,

$$|f(x)| \leq B f^*\left(\left(\frac{x}{2}\right)-\right). \tag{4.3}$$

**Proof**.

Let $x > 0$, $0 < \theta < 1$. Since (4.3) trivially holds if $f(x) = 0$, we can assume that $f(x) \neq 0$. For all $t \in \left[\frac{x}{2}, x\right]$, $t \leq x \leq 2t$, and by (1.20),

$$|f(t)| \geq \frac{|f(x)|}{B} > \frac{\theta\,|f(x)|}{B}.$$

Therefore:

$$f_*\left(\frac{\theta\,|f(x)|}{B}\right) = \lambda\left\{t : |f(t)| > \frac{\theta\,|f(x)|}{B}\right\} \geq \frac{x}{2} > \frac{\theta x}{2},$$

and since $f_* \searrow$:

$$f^*\left(\frac{\theta x}{2}\right) = \inf\left\{\alpha : f_*(\alpha) \leq \frac{\theta x}{2}\right\} \geq \frac{\theta\,|f(x)|}{B},$$

or $|f(x)| \leq \frac{B}{\theta} f^*\left(\frac{\theta x}{2}\right)$. Letting $\theta \to 1^-$, (4.3) follows. $\square$

Theorem 3.7 shows that if $p \leq q$ then every $L^q_{w(p,q)}$ function is in $L(p,q)$. However, the function $f(x) = \sum\limits_{n=1}^{\infty} n I_{[2^n, 2^n+1]}(x)$ is in $\bigcap\limits_{p>q} L^q_{w(p,q)}$:

$$\|f\|^q_{L^q_{w(p,q)}} = \int\limits_{[2,\infty)} x^{\frac{q}{p}-1} (f(x))^q \, dx$$

$$= \sum_{n=1}^{\infty} n^q \int\limits_{2^n}^{2^n+1} x^{\frac{q}{p}-1} dx \leq \sum_{n=1}^{\infty} n^q 2^{n\left(\frac{q}{p}-1\right)} < \infty.$$

But $f_*(\alpha) = \infty$ for all $\alpha > 0$, and so $f^* = \infty$.

At the same time, for all positive $p, q$, at least one of which is finite, if $f \in L^q_{w(p,q)} \cap GM_1(B)$ then $f^*$ is finite-valued. More generally:

**Lemma 4.6**. Assume that $f \in GM_1$, $A > 0$, $\alpha \in \mathbb{R}$.
(i) If $\int_A^\infty |f(x)|\, x^\alpha dx < \infty$ then $\lim_{x\to\infty} x^{\alpha+1} f(x) = 0$.
(ii) If $\int_0^A |f(x)|\, x^\alpha dx < \infty$ then $\lim_{x\to 0+} x^{\alpha+1} f(x) = 0$.
**Proof**.

Let $2^k \le x \le 2^{k+1}$. If $\alpha < 0$ then $2^\alpha \cdot 2^{k\alpha} = 2^{(k+1)\alpha} \le x^\alpha \le 2^{k\alpha}$. If $\alpha \ge 0$ then $2^{k\alpha} \le x^\alpha \le 2^{(k+1)\alpha} = 2^\alpha \cdot 2^{k\alpha}$. That is to say, for any $\alpha \in \mathbb{R}$:

$$\min\{2^\alpha, 1\} \cdot 2^{k\alpha} \le x^\alpha \le \max\{2^\alpha, 1\} \cdot 2^{k\alpha}.$$

Furthermore, by (1.20), $\frac{1}{B}\left|f\left(2^{k+1}\right)\right| \le |f(x)| \le B\left|f\left(2^k\right)\right|$. Therefore:

$$\frac{1}{B}\min\{2^\alpha, 1\}\, 2^{k\alpha}\left|f\left(2^{k+1}\right)\right| \le x^\alpha |f(x)| \le B\max\{2^\alpha, 1\}\, 2^{k\alpha}\left|f\left(2^k\right)\right|. \tag{4.4}$$

It follows from (4.4) that for all $-\infty \le n < N \le \infty$,

$$\int_{2^n}^{2^N} |f(x)|\, x^\alpha dx = \sum_{k=n}^{N-1} \int_{2^k}^{2^{k+1}} x^\alpha |f(x)|\, dx$$

$$\ge \frac{1}{B} \cdot \min\{2^\alpha, 1\} \cdot \sum_{k=n}^{N-1} 2^{k\alpha}\left|f\left(2^{k+1}\right)\right| \cdot 2^k = \frac{\min\{2^\alpha, 1\}}{B \cdot 2^{\alpha+1}} \sum_{k=n+1}^{N} 2^{k(\alpha+1)}\left|f\left(2^k\right)\right|.$$

Therefore, if $\int_A^\infty |f(x)|\, x^\alpha dx < \infty$ then $\sum_{k=\lceil \log_2 A\rceil}^{\infty} 2^{k(\alpha+1)}\left|f\left(2^k\right)\right| < \infty$, and so $\lim_{k\to\infty} 2^{k(\alpha+1)}\left|f\left(2^k\right)\right| = 0$. If $\int_0^A |f(x)|\, x^\alpha dx < \infty$ then $\sum_{k=-\infty}^{\lfloor \log_2 A\rfloor} 2^{k(\alpha+1)}\left|f\left(2^k\right)\right| < \infty$, and so $\lim_{k\to-\infty} 2^{k(\alpha+1)}\left|f\left(2^k\right)\right| =$

0. The claim of the Lemma now follows by substituting $\alpha+1$ for $\alpha$ in the right-hand side of (4.4). $\square$

**Corollary 4.6**. If $0<p\le\infty$, $0<q<\infty$ and $f\in L^q_{w(p,q)}\cap GM_1$ then $\lim\limits_{x\to 0+} x^{\frac{1}{p}} f(x)=\lim\limits_{x\to\infty} x^{\frac{1}{p}} f(x)=0$.

By Lemma 4.1:

**Lemma 4.7**. Assume $\{a_n\}\in GMS_1$. If $\alpha\in\mathbb{R}$ is such that $\sum\limits_{n=1}^{\infty} n^\alpha |a_n|<\infty$ then $\lim\limits_{n\to\infty} n^{\alpha+1} a_n=0$.

**Corollary 4.7**. If $0<p\le\infty$, $0<q<\infty$ and $\{a_n\}\in l^q_{w(p,q)}\cap GMS_1$ then $\lim\limits_{n\to\infty} n^{\frac{1}{p}} a_n=0$.

If $\alpha=-1$ then the claim of Lemma 4.7 is also true for $GMS_2$:

**Lemma 4.8**. Assume $\{a_n\}\in GMS_2$. If $\sum\limits_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$ then $\lim\limits_{n\to\infty} a_n=0$.

**Proof.**

By the assumption of the Lemma,

$$\sum_{j=1}^{\infty}|a_j-a_{j+1}|\le B\left(|a_1|+\sum_{k=2}^{\infty}\frac{|a_k|}{k}\right)<\infty. \tag{4.5}$$

Assume that $\{a_n\}$ does not converge to 0. Then $\exists\epsilon>0, \{n_k\}_{k=1}^{\infty}$ such that for all $k$, $n_{k+1}>n_k$ and $|a_{n_k}|>\epsilon$. By (4.5),

$$\sum_{k=1}^{\infty}\sum_{j=n_k}^{n_{k+1}-1}|a_j-a_{j+1}|<\infty,$$

implying that $\lim\limits_{k\to\infty}\sum\limits_{j=n_k}^{n_{k+1}-1}|a_j-a_{j+1}|=0$, and so, $\exists k_0$ such that $\sum\limits_{j=n_k}^{n_{k+1}-1}|a_j-a_{j+1}|<\frac{\epsilon}{2}$, $\forall k\ge k_0$. For each $j>n_{k_0}$ $\exists k$ so that $n_k<j\le n_{k+1}$, and so,

$$|a_j|\ge|a_{n_k}|-|a_{n_k}-a_j|\ge|a_{n_k}|-\sum_{l=n_k}^{j-1}|a_l-a_{l+1}|>\epsilon-\frac{\epsilon}{2}=\frac{\epsilon}{2},$$

contracting to $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.□

**Lemma 4.9**. Let $B \geq 1$, $f \in GM_1(B)$. Then for $0 < p < \infty$, $0 < q < \infty$:

$$\frac{C_1(p,q)}{B} \|f\|_{L^q_{w(p,q)}} \leq \left( \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left| f\left(2^k\right) \right|^q \right)^{\frac{1}{q}} \leq BC_2(p,q) \|f\|_{L(p,q)}, \tag{4.6}$$

with $0 < C_j(p,q) < \infty$. For $0 < q < \infty$,

$$\|f\|_{L^q_{w(\infty,q)}} = \left( \int_0^{\infty} |f(x)|^q \frac{dx}{x} \right)^{\frac{1}{q}} \leq B (\ln 2)^{\frac{1}{q}} \left( \sum_{k=-\infty}^{\infty} \left| f\left(2^k\right) \right|^q \right)^{\frac{1}{q}}. \tag{4.7}$$

Furthermore, for $0 < p \leq \infty$:

$$\frac{1}{2^{\frac{1}{p}} B} \|f\|_{L(p,\infty)} \leq \frac{1}{2^{\frac{1}{p}} B} \|f\|_{L^\infty_{w(p,\infty)}} \leq \sup_{k \in \mathbb{Z}} \left\{ 2^{\frac{k}{p}} \left| f\left(2^k\right) \right| \right\}$$

$$\leq 2^{\frac{1}{p}} B \|f\|_{L^\infty_{w(p,\infty)}} \leq 2^{\frac{2}{p}} B^2 \|f\|_{L(p,\infty)}. \tag{4.8}$$

**Proof**.

For $f \in GM_1(B)$ and $2^k \leq x \leq 2^{k+1}$, $|f(x)| \leq B \left| f\left(2^k\right) \right|$, and so:

$$\|f\|^q_{L^q_{w(p,q)}} = \int_0^{\infty} x^{\frac{q}{p}-1} |f(x)|^q \, dx = 2^{\frac{q}{p}} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \left(\frac{x}{2}\right)^{\frac{q}{p}} |f(x)|^q \frac{dx}{x}$$

$$\leq 2^{\frac{q}{p}} B^q \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left| f\left(2^k\right) \right|^q \int_{2^k}^{2^{k+1}} \frac{dx}{x} = 2^{\frac{q}{p}} B^q \ln 2 \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left| f\left(2^k\right) \right|^q,$$

proving the first inequality in (4.6) and (4.7). Assume $2^{k-2} \leq x \leq 2^{k-1}$, or equivalently, $2x \leq 2^k \leq 4x$. If $q > p$ then $2^{k\left(\frac{q}{p}-1\right)} \leq (4x)^{\frac{q}{p}-1}$, if $q \leq p$ then $2^{k\left(\frac{q}{p}-1\right)} \leq (2x)^{\frac{q}{p}-1}$ and so, for all $0 < p, q < \infty$:

$$2^{k\left(\frac{q}{p}-1\right)} \leq A(p,q) x^{\frac{q}{p}-1},$$

$A(p,q) = \max\left\{2^{\frac{q}{p}-1}, 4^{\frac{q}{p}-1}\right\}$. By Lemma 4.5, $\left| f\left(2^k\right) \right| \leq B f^*\left(2^{k-1}-\right)$, and so:

$$\sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left| f\left(2^k\right) \right|^q \leq B^q \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left( f^*\left(2^{k-1}-\right) \right)^q$$

$$= 4B^q \sum_{k=-\infty}^{\infty} \int_{2^{k-2}}^{2^{k-1}} 2^{k\left(\frac{q}{p}-1\right)} \left(f^*\left(2^{k-1}-\right)\right)^q dx \leq 4A(p,q)B^q \int_0^\infty x^{\frac{q}{p}-1} \left(f^*(x)\right)^q dx$$

$$= B^q A(p,q) \int_0^\infty x^{\frac{q}{p}-1} \left(f^*(x)\right)^q dx = B^q A(p,q) \|f\|_{L(p,q)}^q ,$$

proving the second inequality in (4.6).

Let us consider (4.8). Observe that $|f(x)| \leq x^{-\frac{1}{p}} \sup_{x>0} \left\{ x^{\frac{1}{p}} |f(x)| \right\}$, and so, $|f^*(x)| \leq x^{-\frac{1}{p}} \sup_{x>0} \left\{ x^{\frac{1}{p}} |f(x)| \right\}$, that is to say, $\|f\|_{L(p,\infty)} \leq \|f\|_{L^\infty_{w(p,\infty)}}$, proving the first inequality. For $2^k \leq x \leq 2^{k+1}$ and $\alpha = \dfrac{1}{p}$, (4.4) shows that:

$$\frac{1}{B} \cdot 2^{\frac{k}{p}} \left| f\left(2^{k+1}\right) \right| \leq x^{\frac{1}{p}} |f(x)| \leq 2^{\frac{1}{p}} B \cdot 2^{\frac{k}{p}} \left| f\left(2^k\right) \right| ,$$

or equivalenty, using $2^{k-1} \leq \dfrac{x}{2} \leq 2^k$:

$$\frac{1}{2^{\frac{1}{p}} B} x^{\frac{1}{p}} |f(x)| \leq 2^{\frac{k}{p}} \left| f\left(2^k\right) \right| \leq 2^{\frac{1}{p}} B \cdot \left(\frac{x}{2}\right)^{\frac{1}{p}} \left| f\left(\frac{x}{2}\right) \right| .$$

Taking the supremum over all $k \in \mathbb{Z}$ we obtain:

$$\frac{1}{2^{\frac{1}{p}} B} \|f\|_{L^\infty_{w(p,\infty)}} \leq \sup_{k \in \mathbb{Z}} \left\{ 2^{\frac{k}{p}} \left| f\left(2^k\right) \right| \right\} \leq 2^{\frac{1}{p}} B \|f\|_{L^\infty_{w(p,\infty)}} ,$$

proving the second and the third inequality. Finally, by Lemma 4.5, for all $x > 0$, $|f(x)| \leq B f^*\left(\left(\frac{x}{2}\right) -\right)$, therefore, for all $0 < \theta < 1$:

$$\sup_{x>0} \left\{ x^{\frac{1}{p}} |f(x)| \right\} \leq B \sup \left\{ x^{\frac{1}{p}} f^*\left(\left(\frac{x}{2}\right) -\right) \right\} = 2^{\frac{1}{p}} B \sup \left\{ x^{\frac{1}{p}} f^*(x-) \right\}$$

$$\leq 2^{\frac{1}{p}} B \sup_{x>0} \left\{ x^{\frac{1}{p}} f^*(\theta x) \right\} = \frac{2^{\frac{1}{p}} B}{\theta^{\frac{1}{p}}} \sup_{x>0} \left\{ (\theta x)^{\frac{1}{p}} f^*(\theta x) \right\} = \frac{2^{\frac{1}{p}} B}{\theta^{\frac{1}{p}}} \sup_{x>0} \left\{ x^{\frac{1}{p}} f^*(x) \right\} .$$

Letting $\theta \to 1-$, it follows that $\|f\|_{L^\infty_{w(p,\infty)}} \leq 2^{\frac{1}{p}} B \|f\|_{L(p,\infty)}$, and the last inequality in (4.8) follows.□

The next lemma shows that the statement of Theorem 3.7 holds for $q < p$ if $f \in GM_1$. The proof is similar to the proof by B. Booton [3] of the analogous statement for $GMS_1$.

**Lemma 4.10**. Let $f \in GM_1(B)$, $0 < q < p$. Then

$$\|f\|_{L(p,q)} \le \left(\frac{2p}{q}\right)^{\frac{1}{q}} B^2 \|f\|_{L^q_{w(p,q)}}.$$

**Proof**.

Let $g(x) = \sum_{k=-\infty}^{\infty} \left|f\left(2^k\right)\right| I_{\left[2^k,2^{k+1}\right)}(x)$. Then by (1.20), $|f(x)| \le Bg(x)$. Furthermore, $g^*$ is decreasing, right-continuous and for each $x > 0$, $g^*(x) = \left|f\left(2^k\right)\right|$ for some $k \in \mathbb{Z}$. Therefore, $g^*$ can be written as

$$g^*(x) = \sum_{k \in J \cap \mathbb{Z}} \alpha_k I_{[x_k, x_{k+1})}(x),$$

where $J$ is an interval, $\alpha_k > a_{k+1}$, and $[x_k, x_{k+1}) = \{g^* = \alpha_k\}$, $\forall k$. And so:

$$\|f\|^q_{L(p,q)} \le B^q \|g\|^q_{L(p,q)} = B^q \int_0^\infty x^{\frac{q}{p}} (g^*(x))^q \frac{dx}{x}$$

$$= B^q \sum_{k=-\infty}^{\infty} \int_{\{g^* = \alpha_k\}} x^{\frac{q}{p}} \alpha_k^q \frac{dx}{x} = B^q \sum_{k=-\infty}^{\infty} \alpha_k^q \int_{x_k}^{x_{k+1}} x^{\frac{q}{p}} \frac{dx}{x}$$

$$= B^q \frac{p}{q} \sum_{k=-\infty}^{\infty} \alpha_k^q \left(x_{k+1}^{\frac{q}{p}} - x_k^{\frac{q}{p}}\right) \le B^q \frac{p}{q} \sum_{k=-\infty}^{\infty} \alpha_k^q (x_{k+1} - x_k)^{\frac{q}{p}}$$

$$= B^q \frac{p}{q} \sum_{k=-\infty}^{\infty} \alpha_k^q \left(\sum_{\{m : |f(2^m)| = \alpha_k\}} 2^m\right)^{\frac{q}{p}} \le B^q \frac{p}{q} \sum_{k=-\infty}^{\infty} \sum_{\{m : |f(2^m)| = \alpha_k\}} |f(2^m)|^q 2^{\frac{mq}{p}}$$

$$= B^q \frac{p}{q} \sum_{m \in \mathbb{Z}} |f(2^m)|^q 2^{\frac{mq}{p}} = 2B^q \frac{p}{q} \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} |f(2^m)|^{\frac{q}{p}} 2^{m\left(\frac{q}{p}-1\right)} dx$$

$$\le 2B^{2q} \frac{p}{q} \int_0^\infty |f(x)|^q x^{\frac{q}{p}-1} dx. \square$$

**Theorem 4.11**. Let $B > 1$, $f \in GM_1(B)$ . If $0 < p < \infty$, $0 < q < \infty$ then:

$$\|f\|_{L(p,q)} \sim \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left(f^*\left(2^k\right)\right)^q \sim \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left|f\left(2^k\right)\right|^q \sim \|f\|_{L^q_{w(p,q)}}. \qquad (4.9)$$

If $0 < p \leq \infty$ then

$$\|f\|_{L(p,\infty)} \sim \sup_{k\in\mathbb{Z}} \left\{ 2^{\frac{k}{p}} \left| f(2^k) \right| \right\} \sim \sup_{k\in\mathbb{Z}} \left\{ 2^{\frac{k}{p}} \left| f^*(2^k) \right| \right\} \sim \|f\|_{L^\infty_{w(p,\infty)}}. \tag{4.10}$$

**Proof**.

First, show that $\|f\|_{L(p,q)} \sim \|f\|_{L^q_{w(p,q)}}$ for all $0 < q \leq \infty$.

The inequality $\|f\|_{L^q_{w(p,q)}} \leq B^2 C(p,q) \|f\|_{L(p,q)}$ follows from Lemma 4.9, more specifically, for $0 < p < \infty$, $0 < q < \infty$ it is a consequence of (4.6), and for $0 < p \leq \infty$, $q = \infty$ it is a consequence of (4.8).

The inequality $\|f\|_{L(p,q)} \leq B^2 C(p,q) \|f\|_{L^q_{w(p,q)}}$ follows from Theorem 3.7 for all $0 < p \leq q \leq \infty$ and it is a consequence of Lemma 4.9 for all $0 < q < p$.

$f^* \in GMS_1(1)$ as a decreasing function, and so by Lemma 4.9, if $0 < p < \infty$ then $\left( \sum_{k=-\infty}^{\infty} 2^{\frac{kq}{p}} \left| f^*\left(2^k\right) \right|^q \right)^{\frac{1}{q}} \sim \|f\|_{L(p,q)}$, and $\sup_{k\in\mathbb{Z}} \left\{ 2^{\frac{k}{p}} \left| f^*\left(2^k\right) \right| \right\} \sim \|f\|_{L(p,\infty)}$, that is, the remaining equivalences in (4.9) and (4.10) follow.□

# CHAPTER 5

# CONES IN $GM$ AND $GMS$

**Definition 5.1**. For each $0 \leq \varphi < \dfrac{\pi}{2}$ and $\alpha \in \mathbb{R}$, define

$$S_{\alpha,\varphi} = \left\{ z \in \mathbb{C} : \left| \arg \left( e^{-i\alpha} z \right) \right| \leq \varphi \right\} \cup \{0\} .$$

**Lemma 5.2**. Let $(\Omega, \Sigma, \mu)$ be a measure space and assume that $f(\omega) \in S_{\alpha,\varphi}$, $\mu$-a.e. on $\Omega$. Then $\int\limits_{\Omega} f d\mu \in S_{\alpha,\varphi}$.

**Proof**.

Since $0 \leq \varphi < \dfrac{\pi}{2}$, it follows that

$$S_{\alpha,\varphi} = \left\{ z \in \mathbb{C} : \left| \arg \left( e^{-i\alpha} z \right) \right| \leq \varphi \right\} \cup \{0\}$$

$$= \left( \left\{ z : 0 \leq \arg \left( e^{i(\varphi-\alpha)} z \right) \leq 2\varphi \right\} \cap \left\{ z : -2\varphi \leq \arg \left( e^{-i(\varphi+\alpha)} z \right) \leq 0 \right\} \right) \cup \{0\}$$

$$\subseteq \left\{ z : \operatorname{Im} \left( e^{i(\varphi-\alpha)} z \right) \geq 0 \right\} \cap \left\{ z : \operatorname{Im} \left( e^{-i(\varphi+\alpha)} z \right) \leq 0 \right\}$$

$$= \left( \left\{ z : 0 \leq \arg \left( e^{i(\varphi-\alpha)} z \right) \leq \pi \right\} \cap \left\{ z : 0 \leq \arg \left( e^{i(\pi-\varphi-\alpha)} z \right) \leq \pi \right\} \right) \cup \{0\}$$

$$= \left( \left\{ z : -\varphi \leq \arg \left( e^{-i\alpha} z \right) \leq \pi - \varphi \right\} \cap \left\{ z : -\pi + \varphi \leq \arg \left( e^{-i\alpha} z \right) \leq \varphi \right\} \right) \cup \{0\}$$

$$= S_{\alpha,\varphi}.$$

That is to say,

$$S_{\alpha,\varphi} = \left\{ z : \operatorname{Im} \left( e^{i(\varphi-\alpha)} z \right) \geq 0 \right\} \cap \left\{ z : \operatorname{Im} \left( e^{-i(\varphi+\alpha)} z \right) \leq 0 \right\} .$$

If $f(\omega) \in S_{\alpha,\varphi}$ for a.e. $\omega \in \Omega$ then

$$\operatorname{Im}\left(e^{i(\varphi-\alpha)}\int\limits_{\Omega} f d\mu\right) = \int\limits_{\Omega} \operatorname{Im}\left(e^{i(\varphi-\alpha)} f\right) d\mu \geq 0,$$

and

$$\operatorname{Im}\left(e^{-i(\varphi+\alpha)}\int\limits_{\Omega} f d\mu\right) = \int\limits_{\Omega} \operatorname{Im}\left(e^{-i(\varphi+\alpha)} f\right) d\mu \leq 0.$$

Therefore, $\int\limits_{\Omega} f d\mu \in S_{\alpha,\varphi}$.$\square$

**Corollary 5.2**. $S_{\alpha,\varphi}$ is a cone in $\mathbb{C}$.

**Lemma 5.3**. Let $(\Omega, \Sigma, \mu)$ be a measure space and assume that $f(\omega) \in S_{\alpha,\varphi}$ for a.e. $\omega \in \Omega$. Then

$$\int\limits_{\Omega} |f|\, d\mu \leq \frac{1}{\cos\varphi}\left|\int\limits_{\Omega} f d\mu\right|. \tag{5.1}$$

**Proof**.

$$\int\limits_{\Omega} |f| \cos\varphi d\mu \leq \int\limits_{\Omega} \operatorname{Re}\left(f \cdot e^{-i\alpha}\right) d\mu = \operatorname{Re}\int\limits_{\Omega} \left(f \cdot e^{-i\alpha}\right) d\mu$$

$$\leq \left|\int\limits_{\Omega} \left(f \cdot e^{-i\alpha}\right) d\mu\right| = \left|e^{-i\alpha}\int\limits_{\Omega} f d\mu\right| = \left|\int\limits_{\Omega} f d\mu\right|.\square$$

**Corollary 5.3**. If $z_j \in S_{\alpha,\varphi}$, all $1 \leq j \leq n$, then:

$$\sum_{j=1}^{n} |z_j| \leq \frac{1}{\cos\varphi}\left|\sum_{j=1}^{n} z_j\right|. \tag{5.2}$$

**Definition 5.4**. For each $0 \leq \varphi < \frac{\pi}{2}$ and $0 \leq \alpha < 2\pi$, define

$$GM_{\alpha,\varphi} = \left\{f \in GM : f(x) \in S_{\alpha,\varphi},\ \mu\text{-a.e. } x > 0\right\};$$

$$GMS_{\alpha,\varphi} = \left\{\{a_n\} \in GMS : a_n \in S_{\alpha,\varphi}, \forall n \geq 1\right\}.$$

We will also use the notation $GM_{\alpha,\varphi}(B) = GM_{\alpha,\varphi} \cap GM(B)$ and $GMS_{\alpha,\varphi}(B) = GMS_{\alpha,\varphi} \cap GMS(B)$.

**Remark 5.5**. Observe that $GM_{0,0} = GM^+$, $GMS_{0,0} = GMS^+$.

**Lemma 5.6**. Let $(\Omega, \Sigma, \mu)$ be a measure space, $-\infty < c < d < \infty$, and assume that $g : \Omega \times \mathbb{R} \to \mathbb{C}$ is such that for all $c \leq x \leq d$, $g(\cdot, x)$ is a $\mu$-integrable function. Then:

$$V_{\int_\Omega g(\omega,\cdot)d\mu}([c,d]) \leq \int\limits_\Omega V_{g(\omega,\cdot)}([c,d])d\mu. \tag{5.3}$$

**Proof**.

Let $\{x_j\} = \{x_j\}_{j=1}^n \in \mathcal{P}([c,d])$. Then:

$$\sum_{j=1}^n \left| \int\limits_\Omega (g(\omega, x_{j+1}) - g(\omega, x_j))\, d\mu \right| \leq \int\limits_\Omega \left[ \sum_{j=1}^n |g(\omega, x_{j+1}) - g(\omega, x_j)| \right] d\mu$$

$$\leq \int\limits_\Omega V_{g(\omega,\cdot)}([c,d])d\mu,$$

and (5.3) follows by taking the supremum of the left-hand side over all $\{x_j\} \in \mathcal{P}([c,d])$.$\square$

**Lemma 5.7**. Let $(\Omega, \Sigma, \mu)$ be a measure space, and assume that for a.e. $\omega \in \Omega$, $g(\omega, \cdot) \in GM_{\alpha,\varphi}(B)$. Then

$$f(\cdot) = \int\limits_\Omega g(\omega,\cdot)d\mu \in GM_{\alpha,\varphi}\left(\frac{B}{\cos\varphi}\right). \tag{5.4}$$

**Proof**.

For $x > 0$,

$$V_f([x,2x]) \leq \int\limits_\Omega V_{g(\omega,\cdot)}([x,2x])d\mu \leq B \int\limits_\Omega |g(\omega,x)|\, d\mu \leq$$

$$\leq \frac{B}{\cos\varphi}\left|\int_{\Omega} g(\omega,x)d\mu\right| = \frac{B}{\cos\varphi}\left|f(x)\right|.\square$$

**Corollary 5.7**. $GM_{\alpha,\varphi}$ is a cone in $GM$. Moreover, if $f_j \in GM_{\alpha,\varphi}(B)$, $1 \leq j \leq N$ then:

$$\sum_{j=1}^{N} f_j \in GM_{\alpha,\varphi}\left(\frac{B}{\cos\varphi}\right). \tag{5.5}$$

Similarly, $GMS_{\alpha,\varphi}$ is a cone in $GMS$ and for $b_n = \sum_{j=1}^{N} a_{n,j}$, where for each $j$ the sequence $\{a_{n,j}\}_{n=1}^{\infty}$ is in $GMS(B)$, we have that

$$\{b_n\}_{n=1}^{\infty} \in GMS_{\alpha,\varphi}\left(\frac{B}{\cos\varphi}\right).\square \tag{5.6}$$

**Definition 5.8**. For each $f \in L^1_{loc}(\mathbb{R}^+, \mu)$, where $\mu$ is a Borel measure, define for each $x$ such that $\mu\left((0,x]\right) > 0$, the *average of $f$ on $(0,x]$*, as

$$\sigma_x(f,\mu) = \frac{1}{\mu\left((0,x]\right)} \int_{(0,x]} f d\mu. \tag{5.7}$$

Formula (5.7) defines a function of $x$, which we shall write as $\sigma(f,\mu)$. If $f$ or $\mu$ are clear from the context, we may omit them from the notation.

**Remark 5.9**. When $\mu$ is the counting measure, we have the following important case of $\sigma$, defined for sequences:

$$\sigma_n = \sigma_n\left(\{a_k\}\right) = \frac{1}{n}\sum_{k=1}^{n} a_k,\ n \geq 1. \tag{5.8}$$

**Lemma 5.10**. Assume that for a.e. $x > 0$, $f(x) \in S_{\alpha,\varphi}$, and that $\mu$ is a Borel measure on $\mathbb{R}^+$ for which there is $A \geq 0$ such that for all $a > A$, $\mu\left((0,a]\right) > 0$. Then, $\forall x > A$:

$$\sigma_x(|f|) \leq \frac{1}{\cos\varphi}\left|\sigma_x(f)\right|. \tag{5.9}$$

If, moreover, $f \in GM_1(B)$ and $\exists K > 0$ such that $\forall a > 0$,

$$\mu((0,a]) \le K\mu((a,2a]) \tag{5.10}$$

then, $\forall x > A$:

$$|f(x)| \le \frac{(K+1)B}{\cos\varphi} |\sigma_x(f)| \,. \tag{5.11}$$

**Proof**.

(5.9) follows from (5.1):

$$\sigma_x(|f|) = \frac{1}{\mu((0,x])} \int_{(0,x]} |f| \, d\mu$$

$$\le \frac{1}{\mu((0,x]) \cos\varphi} \left| \int_{(0,x]} f d\mu \right| = \frac{1}{\cos\varphi} |\sigma_x(f)| \,.$$

Since $f \in GM_1(B)$, it follows that $|f(x)| \le B|f(t)|$, $\forall t \in \left[\frac{x}{2}, x\right]$. Furthermore, by (5.10) for all $x > 0$:

$$\mu((0,x]) = \mu\left(\left(0, \tfrac{x}{2}\right]\right) + \mu\left(\left(\tfrac{x}{2}, x\right]\right) \le (K+1)\,\mu\left(\left(\tfrac{x}{2}, x\right]\right).$$

Therefore, for $x > A$:

$$|f(x)| \le \frac{B}{\mu\left(\left(\frac{x}{2}, x\right]\right)} \int_{\left(\frac{x}{2}, x\right]} |f(t)| \, d\mu \le \frac{(K+1)B}{\mu((0,x])} \int_{(0,x]} |f(t)| \, d\mu$$

$$= (K+1) B \sigma_x(|f|) \le \frac{(K+1)B}{\cos\varphi} |\sigma_x(f)| \,.\square$$

Analogously, for sequences, since the counting measure satisfies (5.10) with $K = 1$, we have:

**Lemma 5.11**. Let $a_k \in S_{\alpha,\varphi}$, $\forall k \ge 1$. Then, $\forall n \ge 1$:

$$\sigma_n(\{|a_k|\}) \le \frac{1}{\cos\varphi} |\sigma_n(\{a_k\})| \,.$$

If, moreover, $\{a_k\} \in GMS_1(B)$, then, $\forall n \ge 1$:

$$|a_n| \leq \frac{2B}{\cos\varphi} |\sigma_n(\{a_k\})| .$$

**Lemma 5.12.** Let $f \in GM_{\alpha,\varphi}(B)$ and assume that $\mu$ is a Borel measure on $\mathbb{R}^+$, for which there are $A \geq 0$, $K > 0$ such that for all $a > A$,

$$\begin{cases} \mu((0,a]) > 0 \\ \mu((a,2a]) \leq K\mu((0,a]) \leq K^2\mu((0,a]) \end{cases} . \tag{5.12}$$

Then $\forall x > A$, $\sigma_x(f,\mu) \in S_{\alpha,\varphi}$. Moreover, there is $B' = B'(B,K,\varphi) > 0$ such that for all $x > A$, $\sigma(f,\mu)$ satisfies

$$V_{\sigma(f,\mu)}([x,2x]) \leq B' |\sigma_x(f,\mu)| . \tag{5.13}$$

In particular, if $A = 0$ then $\sigma(f,\mu) \in GM_{\alpha,\varphi}(B')$.

**Proof**.

The fact that $\sigma_x(f,\mu) \in S_{\alpha,\varphi}$ follows from Lemma 5.2.

Let $a > A$. Define, for $x \geq a$:

$$g(t,x) = f(t)\frac{I_{(0,x]}(t)}{\mu((0,x])}. \tag{5.14}$$

Since $f \in L^1_{loc}(\mathbb{R}^+,\mu)$, it follows that $g(\cdot,x) \in L^1(\mathbb{R}^+,\mu)$. Furthermore, $\sigma_x(f) = \int\limits_{\mathbb{R}^+} g(t,x)d\mu$. Applying Lemma 5.6, we obtain:

$$V_{\sigma(f)}([a,2a]) = V_{\int\limits_{\mathbb{R}^+} g(t,\cdot)d\mu}([a,2a]) \leq \int\limits_{\mathbb{R}^+} V_{g(t,\cdot)}([a,2a])\, d\mu.$$

From (5.14) it follows that $g(t,x) = f(t)\dfrac{I_{[t,\infty)}(x)}{\mu((0,x])}$ is a decreasing function of $x$ on $[t,\infty)$. Consider the following three cases: $0 \leq t \leq a$, $a < t \leq 2a$, $t > 2a$.

If $0 \leq t \leq a$ then

$$V_{g(t,x)}([a,2a]) = V_{\frac{f(t)}{\mu((0,x])}}([a,2a]) = \frac{|f(t)|}{\mu((0,a])} - \frac{|f(t)|}{\mu((0,2a])}.$$

If $a < t \leq 2a$, observe that for $a \leq x < t$, $g(t,x) = 0$, and $g(t,t) = \dfrac{|f(t)|}{\mu((0,t])}$, therefore, $V_{g(t,x)}([a,t]) = \dfrac{|f(t)|}{\mu((0,t])}$, and so:

$$V_{g(t,x)}([a,2a]) = V_{g(t,x)}([a,t]) + V_{g(t,x)}([t,2a])$$

$$= \frac{|f(t)|}{\mu((0,t])} + V_{\frac{f(t)}{\mu((0,x])}}([t,2a])$$

$$= \frac{|f(t)|}{\mu((0,a])} + \left(\frac{|f(t)|}{\mu((0,t])} - \frac{|f(t)|}{\mu((0,2a])}\right) = \frac{2|f(t)|}{\mu((0,t])} - \frac{|f(t)|}{\mu((0,2a])}.$$

Finally, if $t > 2a$ then $g(t,x) = 0$, $\forall x \in [a,2a]$, and so, $V_{g(t,x)}([a,2a]) = 0$. Therefore:

$$\int\limits_{\mathbb{R}^+} V_{g(t,\cdot)}([a,2a])\,d\mu = \left(\int\limits_{(0,a]} + \int\limits_{(a,2a]}\right) V_{g(t,\cdot)}([a,2a])\,d\mu$$

$$= \int\limits_{(0,a]} \left(\frac{1}{\mu((0,a])} - \frac{1}{\mu((0,2a])}\right) |f(t)|\,d\mu$$

$$+ \int\limits_{(a,2a]} \left(\frac{2}{\mu((0,t])} - \frac{1}{\mu((0,2a])}\right) |f(t)|\,d\mu$$

$$\leq \left(\frac{2}{\mu((0,a])} - \frac{1}{\mu((0,2a])}\right) \int\limits_{(0,2a]} |f(t)|\,d\mu$$

$$= \frac{2\mu((0,2a]) - \mu((0,a])}{\mu((0,a]) \cdot \mu((0,2a])} \int\limits_{(0,2a]} |f(t)|\,d\mu$$

$$= \frac{2\mu((a,2a]) + \mu((0,a])}{\mu((0,a])} \sigma_{2a}(|f|) \leq (2K+1)\,\sigma_{2a}(|f|).$$

Observe that by Lemma 1.25 for $f \in GM(B)$ and $a < t \leq 2a$, $|f(t)| \leq 2B\,|f(a)|$, and so:

$$\sigma_{2a}(|f|) = \frac{1}{\mu((0,2a])} \int\limits_{(0,2a]} |f|\,d\mu = \frac{1}{\mu((0,2a])} \left(\int\limits_{(0,a]} |f|\,d\mu + \int\limits_{(a,2a]} |f|\,d\mu\right)$$

$$\leq \frac{1}{\mu((0,a])} \int\limits_{(0,a]} |f|\,d\mu + \frac{1}{\mu((a,2a])} \int\limits_{(a,2a]} 2B\,|f(a)|\,d\mu = \sigma_a(|f|) + 2B\,|f(a)|.$$

Since $\mu$ satisfies (5.10) and $f \in GM_{\alpha,\varphi}$, it follows from Lemma 5.10 that:

$$\sigma_a(|f|) \leq \frac{1}{\cos\varphi} |\sigma_a(f)| \, ; \; |f(a)| \leq \frac{(K+1)\, B}{\cos\varphi} |\sigma_a(f)| \, ,$$

and so, $\sigma_{2a}(|f|) \leq \dfrac{1 + 2\,(K+1)\, B^2}{\cos\varphi} |\sigma_a(f)|$. Therefore,

$$V_{\sigma(f,\mu)}\left([a, 2a]\right) \leq (2K+1)\, \sigma_{2a}(|f|) \leq \frac{(2K+1)\,(1 + 2\,(K+1)\, B^2)}{\cos\varphi} |\sigma_a(f,\mu)| \, . \square$$

**Remark 5.13**. Lebesgue measure satisfies (5.10) and (5.12) with $A = 0$ and $K = 1$; the counting measure satisfies (5.10) and (5.12) with $A = 1$ and $K = 2$. In the latter case, from $\{a_n\} \in GMS_{\alpha,\varphi}$ and (5.13) it follows that $\sigma_n\left(\{a_k\}\right) \in GMS_{\alpha,\varphi}$.

# CHAPTER 6

# FOURIER SERIES WITH GENERAL MONOTONE COEFFICIENTS

**Theorem 6.1**. Assume that $a_k \in \mathbb{C}$ , $1 \leq m \leq N$. Then, $\forall x \in (0, \pi]$:

$$\left|\sum_{k=m}^{N} a_k e^{ikx}\right| \leq \frac{4\pi}{x}\left(\frac{|a_m|}{2} + \sum_{k=m}^{N-1} |a_{k+1} - a_k|\right). \tag{6.1}$$

**Proof.**

Since for $x \in (0, \pi]$,

$$\left|\sum_{k=0}^{N} e^{ikx}\right| \leq \frac{1}{\sin \frac{x}{2}} \leq \frac{\pi}{x}, \tag{6.2}$$

it follows, using summation by parts,

$$\sum_{k=m}^{N} a_k e^{ikx} = a_N \sum_{k=m}^{N} e^{ikx} - \sum_{k=m}^{N-1} (a_{k+1} - a_k) \sum_{j=m}^{k} e^{ijx},$$

and so

$$\left|\sum_{k=m}^{N} a_k e^{ikx}\right| \leq \frac{2\pi}{x}\left(|a_N| + \sum_{k=m}^{N-1} |a_{k+1} - a_k|\right).$$

Since $a_N = a_m + \sum_{k=m}^{N-1} (a_{k+1} - a_k)$, it follows that

$$\left|\sum_{k=m}^{N} a_k e^{ikx}\right| \leq \frac{2\pi}{x}\left(|a_m| + 2 \cdot \sum_{k=m}^{N-1} |a_{k+1} - a_k|\right). \square \tag{6.3}$$

**Corollary 6.1**. Assume $\{a_k\} \in GMS_2(B)$. Then $\forall x \in (0, \pi]$, $\forall 1 \leq m \leq N$,

$$\left|\sum_{k=m}^{N} a_k e^{ikx}\right| \leq \frac{6\pi B}{x}\left(|a_m| + \sum_{k=m+1}^{N} \frac{|a_k|}{k}\right). \tag{6.4}$$

The next theorem gives an estimate for the $L^1$-norm of a trigonometric Fourier series with coefficients in $GMS_2$.

In Theorems 6.2, 6.3 and 7.4, we state a norm inequality for an infinite Fourier series. It suffices, however, to prove each Theorem for a finite Fourier series, since the norm inequality for the finite Fourier series implies that the sequence of partial sums of the infinite Fourier series is Cauchy and, therefore, convergent in the underlying complete space.

**Theorem 6.2**. Assume $\{c_k\} \in GMS_2(B)$. If $f(x) = \sum_{k=1}^{\infty} c_k e^{ikx}$, then:

$$\|f\|_{L^1(0,\pi)} \leq 2\pi |c_1| + 27\pi B \|\{c_n\}\|_{l^1_{\frac{\ln k}{k}}} . \tag{6.5}$$

**Proof.**

For each $N \geq 2$, denote $f_N(x) = \sum_{k=1}^{N} c_k e^{ikx}$. Then:

$$\|f_N\|_{L^1(0,\pi)} = \int_0^{\pi} \left| \sum_{k=1}^{N} c_k e^{ikx} \right| dx = \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left| \sum_{k=1}^{N} c_k e^{ikx} \right| dx$$

$$\leq \sum_{n=1}^{N-1} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left| \sum_{k=1}^{N} c_k e^{ikx} \right| dx + \sum_{n=N}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left( \sum_{k=1}^{N} |c_k| \right) dx$$

$$\leq \sum_{n=1}^{N-1} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left( \sum_{k=1}^{n} |c_k| + \left| \sum_{k=n+1}^{N} c_k e^{ikx} \right| \right) dx + \frac{\pi}{N} \sum_{k=1}^{N} |c_k| .$$

Applying (6.4) for $n < N - 1$:

$$\left| \sum_{k=n+1}^{N} c_k e^{ikx} \right| \leq \frac{6\pi B}{x} \left( |c_{n+1}| + \sum_{k=n+2}^{N} \frac{|c_k|}{k} \right) .$$

If $n = N - 1$ then $\left| \sum_{k=n+1}^{N} c_k e^{ikx} \right| = |c_N|$, and so, for $n \leq N - 1$,

$$\left| \sum_{k=n+1}^{N} c_k e^{ikx} \right| \leq \frac{6\pi B}{x} \left( |c_{n+1}| + \sum_{k=n+1}^{N} \frac{|c_k|}{k} \right) .$$

Therefore:

$$\|f_N\|_{L^1(0,\pi)} \leq \sum_{n=1}^{N-1} \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left[\sum_{k=1}^{n} |c_k| + \frac{6\pi B}{x}\left(|c_{n+1}| + \sum_{k=n+1}^{N} \frac{|c_k|}{k}\right)\right] dx + \frac{\pi}{N}\sum_{k=1}^{N} |c_k|$$

$$= I_1 + I_2 + I_3 + \frac{\pi}{N}\sum_{k=1}^{N} |c_k|\,,$$

where, using (3.1):

$$I_1 = \sum_{n=1}^{N-1} \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left(\sum_{k=1}^{n} |c_k|\right) dx = \sum_{n=1}^{N-1} \left(\frac{\pi}{n} - \frac{\pi}{n+1}\right) \sum_{k=1}^{n} |c_k|$$

$$= -\frac{\pi}{N}\sum_{n=1}^{N} |c_n| + \pi \sum_{n=1}^{N} \frac{|c_n|}{n} \leq \pi \sum_{n=1}^{N} \frac{|c_n|}{n},$$

it follows that

$$I_1 \leq \pi \left(|c_1| + \sum_{n=2}^{N} \frac{|c_n|}{n}\right) \leq \pi\,|c_1| + \frac{\pi}{\ln 2}\sum_{n=2}^{N} |c_n|\,\frac{\ln n}{n};$$

$$I_2 = \sum_{n=1}^{N-1} \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{6\pi B}{x}\,|c_{n+1}|\,dx = 6\pi B \sum_{n=1}^{N-1} |c_{n+1}| \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{dx}{x}$$

$$= 6\pi B \sum_{n=1}^{N-1} |c_{n+1}| \ln\frac{n+1}{n} \leq 6\pi B \sum_{n=1}^{N-1} \frac{|c_{n+1}|}{n} = 6\pi B \sum_{n=2}^{N} \frac{|c_n|}{n-1}$$

$$\leq 12\pi B \sum_{n=2}^{N} \frac{|c_n|}{n} \leq \frac{12\pi B}{\ln 2}\sum_{n=2}^{N} |c_n|\,\frac{\ln n}{n};$$

$$I_3 = \sum_{n=1}^{N-1} \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{6\pi B}{x}\left(\sum_{k=n+1}^{N} \frac{|c_k|}{k}\right) dx = 6\pi B \sum_{n=1}^{N-1}\left(\sum_{k=n+1}^{N} \frac{|c_k|}{k}\right) \int\limits_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{dx}{x}$$

$$= 6\pi B \sum_{n=1}^{N-1}\sum_{k=n}^{N-1} \frac{|c_{k+1}|}{k+1} \ln\frac{n+1}{n} = 6\pi B \sum_{k=1}^{N-1} \frac{|c_{k+1}|}{k+1} \underbrace{\sum_{n=1}^{k} \ln\frac{n+1}{n}}_{\ln(k+1)}$$

$$= 6\pi B \sum_{k=1}^{N-1} \frac{|c_{k+1}|}{k+1} \ln(k+1) = 6\pi B \sum_{k=2}^{N} |c_k|\,\frac{\ln k}{k}.$$

Combining the estimates for $I_j$ and taking into account that

$$\frac{\pi}{N}\sum_{k=1}^{N}|c_k| \leq \pi|c_1| + \frac{\pi}{N\ln 2}\sum_{k=2}^{N}|c_k|\ln k \leq \pi|c_1| + \frac{\pi B}{\ln 2}\sum_{k=2}^{N}|c_k|\frac{\ln k}{k},$$

we have:

$$\|f_N\|_{L^1(0,\pi)} \leq 2\pi|c_1| + 27\pi B\sum_{k=2}^{N}|c_k|\frac{\ln k}{k}.$$

The inequality (6.5) is therefore proved.□

**Theorem 6.3**. Assume $\{c_k\} \in GMS_2(B)$. If $f(x) = \sum_{k=1}^{\infty} c_k e^{ikx}$, then

$$\|f\|_{L(1,\infty)(0,\pi)} \leq 6\pi B\,\|\{c_n\}\|_{l^1_{1/k}}. \tag{6.6}$$

**Proof**.

Let $N \geq 1$. Denote $f_N(x) = \sum_{k=1}^{N} c_k e^{ikx}$. Then, applying (6.4) with $m = 1$:

$$|f_N(x)| = \left|\sum_{k=1}^{N} c_k e^{ikx}\right| \leq \frac{6\pi B}{x}\left(|c_1| + \sum_{k=2}^{N}\frac{|c_k|}{k}\right) = \frac{6\pi B}{x}\sum_{k=1}^{N}\frac{|c_k|}{k}.$$

Therefore, for $\forall\alpha > 0$:

$$\lambda(\{x : |f_N(x)| > \alpha\}) \leq \lambda\left(\left\{x : \frac{6\pi B}{x}\sum_{k=1}^{N}\frac{|c_k|}{k} > \alpha\right\}\right)$$

$$\leq \lambda\left(\left\{x : \frac{6\pi B}{x}\|\{c_k\}\|_{l^1_{\frac{1}{k}}} > \alpha\right\}\right) = \frac{6\pi B}{\alpha}\|\{c_k\}\|_{l^1_{\frac{1}{k}}},$$

and (6.6) follows. □

The result of Theorem 6.3 can be interpreted as follows. Let $T$ be the operator mapping $GMS_2$ to functions over $\mathbb{R}$, defined as

$$T(\{c_n\})(x) = \sum_{k=1}^{\infty} c_k e^{ikx}. \tag{6.7}$$

Then

$$T : l^1_{\frac{1}{k}} \cap GMS_2 \to L(1,\infty). \tag{6.8}$$

**Theorem 6.4**. Let $1 < p < \infty$, $0 < q \leq \infty$. If $c_k$ are trigonometric Fourier coefficients of $f$ and $\{c_k\} \in GMS_{\alpha,\varphi}(B)$ then $\exists C = C(p,q)$ such that

$$\|\{c_n\}\|_{l(p',q)} \leq C \|f\|_{L(p,q)(0,2\pi)} . \tag{6.9}$$

**Proof.**

By Lemma 5.10, $|c_n| \leq \dfrac{2B}{\cos\varphi} |\sigma_n|$, and so $\|\{c_n\}\|_{l(p',q)} \leq \dfrac{2B}{\cos\varphi} \|\sigma_n\|_{l(p',q)}$. By (3.14),

$$\|\{c_n\}\|_{l(p',q)} \leq \frac{C(p,q)}{\cos\varphi} \|f\|_{L(p,q)(0,2\pi)} .\square$$

# CHAPTER 7

# INTERPOLATION IN $GMS_{\alpha,\varphi}$

## 7.1 CALCULATION OF THE $K$-FUNCTIONAL BETWEEN $l^1_{\frac{1}{k}} \cap GMS_{\alpha,\varphi}$ AND $l^1$

The following Lemma is a special case of a result by Yu. Brudnui and N. Krugljak ([4], Corollary 3.1.26).

**Lemma 7.1**.

$$K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right) = t \sum_{n \leq \frac{1}{t}} |c_n| + \sum_{n > \frac{1}{t}} \frac{|c_n|}{n} \tag{7.1}$$

**Proof**.

For $x \geq 0$, $a \in \mathbb{C}$:

$$|a| + x\,|1-a| \geq |\mathrm{Re}\,(a)| + x\,|\mathrm{Re}\,(1-a)| = |\mathrm{Re}\,(a)| + x\,|1 - \mathrm{Re}\,(a)|,$$

and so,

$$\inf_{a\in\mathbb{C}} \left(|a| + x\,|1-a|\right) = \inf_{a\in\mathbb{R}} \left(|a| + x\,|1-a|\right).$$

If $a < 0$,

$$|a| + x\,|1-a| = -a + x\,(1 + |a|) \geq |-a| + x\,|1-(-a)|,$$

and so,

$$\inf_{a\in\mathbb{R}} \left(|a| + x\,|1-a|\right) = \inf_{a\in\mathbb{R}^+} \left(|a| + x\,|1-a|\right).$$

If $a > 1$,

$$|a| + x\,|1-a| = a + x(a-1) \geq 1 + x(1-1),$$

and so,

$$\inf_{a\in\mathbb{R}^+} \left(|a| + x\,|1-a|\right) = \inf_{a\in[0,1]} \left(|a| + x\,|1-a|\right).$$

Also, for all $x \geq 0$ and $0 \leq a \leq 1$, $|a| + x\,|1-a| = a + x(1-a)$ is a convex combination of 1 and $x$. We have shown that

$$\inf_{a\in\mathbb{C}} \left(|a| + x\,|1-a|\right) = \min\left(1, x\right) = \begin{cases} 1, & \text{if } a = 1 \\ x, & \text{if } a = 0 \end{cases}. \tag{7.2}$$

Let $\{c_n\} \in l^1_{\frac{1}{k}} + l^1 = l^1_{\frac{1}{k}}$. Then $K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right) \leq \|\{c_n\}\|_{l^1_{\frac{1}{k}}} < \infty$. Therefore, by (7.2):

$$\begin{aligned}
K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right) &= \inf\left\{\|\{a_n\}\|_{l^1_{\frac{1}{k}}} + t\,\|\{b_n\}\|_{l^1} : a_n + b_n = c_n\right\} \\
&= \inf\left\{\sum_{n=1}^{\infty}\left(\frac{|a_n|}{n} + t\,|b_n|\right) : a_n + b_n = c_n\right\} \\
&\geq \sum_{n=1}^{\infty} \inf\left\{\left(\frac{|a_n|}{n} + t\,|b_n|\right) : a_n + b_n = c_n\right\} \\
&= \sum_{\{n:c_n\neq 0\}} \inf\left\{\frac{|a_n|}{n} + t\,|b_n| : a_n + b_n = c_n\right\} \\
&\quad + \sum_{\{n:c_n=0\}} \underbrace{\inf\left\{\frac{|a_n|}{n} + t\,|b_n| : a_n + b_n = 0\right\}}_{=0} \\
&= t \sum_{\{n:c_n\neq 0\}} \inf\left\{|b_n| + \frac{1}{tn}\,|a_n| : a_n + b_n = c_n\right\} \\
&= t \sum_{\{n:c_n\neq 0\}} |c_n| \inf\left\{\left|\frac{b_n}{c_n}\right| + \frac{1}{tn}\left|\frac{a_n}{c_n}\right| : \frac{a_n}{c_n} + \frac{b_n}{c_n} = 1\right\} \\
&= t \sum_{n:c_n\neq 0} |c_n| \min\left(1, \frac{1}{tn}\right) = t\sum_{n=1}^{\infty} |c_n| \min\left(1, \frac{1}{tn}\right) \\
&= t \sum_{n\leq\frac{1}{t}} |c_n| + \sum_{n>\frac{1}{t}} \frac{|c_n|}{n}.
\end{aligned}$$

Let $a_n = \begin{cases} c_n, & \text{if } n > \frac{1}{t} \\ 0, & \text{if } n \leq \frac{1}{t} \end{cases}$, $b_n = \begin{cases} c_n, & \text{if } n \leq \frac{1}{t} \\ 0, & \text{if } n > \frac{1}{t} \end{cases}$. Then $c_n = a_n + b_n$. Since $\{c_n\} \in l^1_{\frac{1}{k}}$, it follows that $\{a_n\} \in l^1_{\frac{1}{k}}$, $\{b_n\} \in l^1$, and so, $K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right) = t\sum_{n \leq \frac{1}{t}} |c_n| + \sum_{n > \frac{1}{t}} \frac{|c_n|}{n}$.□

**Lemma 7.2**. Let $\{a_n\}, \{c_n\} \in GMS(B)$. Assume that $N \geq 1$ and let $\gamma > 0$ satisfy $|c_N| \leq \gamma |a_N|$. Define

$$b_n = \begin{cases} a_n & \text{if } n \leq N \\ c_n & \text{if } n > N \end{cases}.$$

Then $\{b_n\} \in GMS(3B + 6B^2\gamma)$.

**Proof**.

If $n = N = 1$ then

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = |a_1 - c_2| = |a_N - c_{N+1}|.$$

If $n = N > 1$ then

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = |a_N - c_{N+1}| + \sum_{k=N+1}^{2N-1} |c_{k+1} - c_k|.$$

If $n = \frac{N+1}{2} > 1$ then

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = \sum_{k=n}^{N-1} |a_{k+1} - a_k| + |a_N - c_{N+1}|.$$

If $\frac{N+1}{2} < n < N$ then

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = \sum_{k=n}^{N-1} |a_{k+1} - a_k| + |a_N - c_{N+1}| + \sum_{k=N+1}^{2n-1} |c_{k+1} - c_k|.$$

In all four cases above:

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| \leq \sum_{k=n}^{2n-1} |a_{k+1} - a_k| + |a_N - c_{N+1}| + \sum_{k=N}^{2N-1} |c_{k+1} - c_k|.$$

Furthermore, $\frac{N}{2} < n \leq N$ and so, by (1.26), $|a_N| \leq 2B|a_n|$. Also, $|c_{N+1}| \leq 2B|c_N|$, and so:

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| \leq \sum_{k=n}^{2n-1} |a_{k+1} - a_k| + |a_N| + |c_{N+1}| + \sum_{k=N}^{2N-1} |c_{k+1} - c_k|$$

$$\leq B|a_n| + 2B|a_n| + (2B + B)|c_N| \leq 3B|a_n| + 3B\gamma|a_N|$$

$$\leq 3B|a_n| + 6B^2\gamma|a_n| = \left(3B + 6B^2\gamma\right)|b_n|.$$

Two cases remain. If $n < \frac{N+1}{2}$ then $2n - 1 \leq N - 1$, and so:

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = \sum_{k=n}^{2n-1} |a_{k+1} - a_k| \leq B|a_n| = B|b_n|.$$

If $n > N$ then

$$\sum_{k=n}^{2n-1} |b_{k+1} - b_k| = \sum_{k=n}^{2n-1} |c_{k+1} - c_k| \leq B|c_n| = B|b_n|. \square$$

Lemmas 7.1 and 7.2 imply:

**Theorem 7.3**.

$$K\left(t, \{c_n\}, l^1_{\frac{1}{k}} \cap GMS, l^1\right) \leq \frac{9}{2} K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right). \tag{7.3}$$

**Proof**.

Let $\{c_n\} \in GMS(B) \cap \left(l^1_{\frac{1}{k}} + l^1\right) = GMS(B) \cap l^1_{\frac{1}{k}}$.

1) If $t > 1$ then, by (7.1), $K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right) = \sum_{n=1}^{\infty} \frac{|c_n|}{n} = \|\{c_n\}\|_{l^1_{\frac{1}{k}}}$, and (7.3) follows.

2) Assume that $t \leq 1$ .

Let $N = 1 + \left\lfloor \frac{1}{t} \right\rfloor$, $\sigma_N = \frac{1}{N}\sum_{k=1}^{N} |c_k|$, and define $a_n = \frac{n}{N}\sigma_N$ for all $n \geq 1$. Then $\sum_{k=n}^{2n-1} |a_k - a_{k+1}| = a_{2n} - a_n = a_n$, and so $\{a_n\} \in GMS(1)$.

If $2 \leq N \leq 4$ then $\#\left\{k \in \mathbb{Z} : \frac{N}{4} < k \leq \frac{N}{2}\right\} = 1 \geq \dfrac{N}{5}$.

Assume $N > 4$ and let $\alpha = \frac{N}{4} - \left\lfloor \frac{N}{4} \right\rfloor$.

If $\alpha < \frac{1}{2}$ then $\alpha \leq \frac{1}{4}$, and so:

$$\#\left\{k \in \mathbb{Z} : \frac{N}{4} < k \leq \frac{N}{2}\right\} = \left\lfloor \frac{N}{4} \right\rfloor \geq \frac{N}{4} - \frac{1}{4} \geq \frac{N}{4} - \frac{N/5}{4} = \frac{N}{5}.$$

If $\alpha \geq \frac{1}{2}$ then

$$\#\left\{k \in \mathbb{Z} : \frac{N}{4} < k \leq \frac{N}{2}\right\} \geq \left\lfloor \frac{N}{4} \right\rfloor + 1 > \frac{N}{4} > \frac{N}{5}.$$

Applying (1.20) to $\{c_n\}$, for all $n \in \left[\dfrac{N}{2}, N\right]$ we have:

$$a_n = \frac{n}{N}\sigma_N \geq \frac{1}{2}\sigma_N \geq \frac{1}{2} \cdot \frac{1}{N} \sum_{k \in \left(\frac{N}{4}, \frac{N}{2}\right]} |c_k| \geq \frac{1}{2} \cdot \frac{1}{N} \sum_{k \in \left(\frac{N}{4}, \frac{N}{2}\right]} \frac{1}{B} \left|c_{\lceil \frac{N}{2} \rceil}\right|$$

$$\geq \frac{1}{2N} \cdot \frac{N}{5} \cdot \frac{1}{B} \left|c_{\lceil \frac{N}{2} \rceil}\right| = \frac{1}{10B} \left|c_{\lceil \frac{N}{2} \rceil}\right| \geq \frac{1}{10B^2} |c_n|,$$

in particular, $|c_N| \leq 10B^2 |a_N|$.

Define

$$b_n = \begin{cases} a_n & \text{if } n \leq N \\ c_n & \text{if } n > N \end{cases} \quad ; \qquad d_n = \begin{cases} c_n - a_n & \text{if } n \leq N \\ 0 & \text{if } n > N \end{cases} .$$

By Lemma 7.2, $\{b_n\} \in GMS\left(3B + 6B^2 \cdot 10B^2\right) \subset GMS\left(63B^4\right)$ and $b_n + d_n = c_n$, all $n$. Furthermore:

$$\|\{b_n\}\|_{l^1_{\frac{1}{k}}} = \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \frac{|b_n|}{n}$$

$$= \sum_{n=1}^{N} \frac{a_n}{n} + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n} = \sigma_N + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n};$$

$$\|\{d_n\}\|_{l^1} = \sum_{n=1}^{N} |c_n - a_n|$$

$$\leq \sum_{n=1}^{N} |a_n| + \sum_{n=1}^{N} |c_n| = \frac{\sigma_N}{N} \sum_{n=1}^{N} n + \sum_{n=1}^{N} |c_n|$$

$$= \sigma_N \frac{N+1}{2} + N\sigma_N = \sigma_N \frac{3N+1}{2}.$$

Therefore, observing that $t \le \frac{1}{N-1} \le \frac{2}{N}$, we obtain:

$$\|\{b_n\}\|_{l^1_{\frac{1}{k}}} + t\,\|\{d_n\}\|_{l^1}$$

$$= \sigma_N \left(1 + \frac{3N+1}{2} t\right) + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n} \le \sigma_N \left(1 + \frac{3N+1}{N}\right) + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n}$$

$$\le \frac{9}{2}\sigma_N + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n} = \frac{9}{2N}\sum_{n=1}^{N} |c_n| + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n}$$

$$= \frac{9}{2N} \left( \sum_{n=1}^{\lfloor \frac{1}{t} \rfloor} |c_n| + |c_N| \right) + \sum_{n=N+1}^{\infty} \frac{|c_n|}{n}$$

$$\le \frac{9}{2} \left( t\sum_{n=1}^{\lfloor \frac{1}{t} \rfloor} |c_n| + \sum_{n=N}^{\infty} \frac{|c_n|}{n} \right) = \frac{9}{2} K\left(t, \{c_n\}, l^1_{\frac{1}{k}}, l^1\right),$$

in the last line we applied $N = \lfloor \frac{1}{t} \rfloor + 1 \in \left(\frac{1}{t}, \frac{1}{t} + 1\right]$.□

## 7.2 INTERPOLATION THEOREM FOR A FOURIER OPERATOR ON *GMS*

B. Booton [3] proved the following theorem for $q \ge 1$ and $\{c_k\} \in GMS^+$.

**Theorem 7.4.** Let $1 < p < \infty$, $0 < q \le \infty$. Assume $\{c_k\} \in GMS_{\alpha,\varphi}$. Let $f(x) = \sum_{k=1}^{\infty} c_k e^{ikx}$. Then $\exists C = C(p,q)$ such that

$$\|f\|_{L(p,q)(0,\pi)} \le C\,\|\{c_n\}\|_{l(p',q)}. \tag{7.4}$$

**Proof.**

Let $X = l^1_{\frac{1}{k}}$, that is, $\|\{c_k\}\|_X = \sum_{k=1}^{\infty} \frac{|c_k|}{k}$ and let $w = w(k) = k$. Then $X_w = l^1$, and so $\|\{c_k\}\|_{X_w} = \sum_{k=1}^{\infty} |c_k|$. Both $X$ and $X_w$ are Banach spaces, and so, formula (2.7) applies. Therefore, by Gilbert's theorem (2.14),

$$\|\{c_k\}\|_{(X,X_w)_{\theta,q;K}} \sim \left(\int_0^\infty \left(t^{-\theta}\|\{c_k\}\cdot\sigma_t(k)\|_X\right)^q \frac{dt}{t}\right)^{1/q}$$

$$= \left(\int_0^\infty \left(t^{-\theta}\sum_{k=1}^\infty \sigma_t(k)\frac{|c_k|}{k}\right)^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty \left(t^{-\theta}\sum_{k=1}^\infty tk\sigma(tk)\frac{|c_k|}{k}\right)^q \frac{dt}{t}\right)^{1/q}$$

$$= \left(\int_0^\infty \left(t^{1-\theta}\sum_{k=1}^\infty \sigma(tk)|c_k|\right)^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty \left(t^{\theta-1}\sum_{k=1}^\infty \sigma\left(\frac{k}{t}\right)|c_k|\right)^q \frac{dt}{t}\right)^{1/q}.$$

Take $\sigma = I_{[1,2)}$. $\sigma$ satisfies the conditions of Gilbert's theorem and $\sigma\left(\frac{k}{t}\right) \neq 0$ is equivalent to $t \leq k < 2t$, and so:

$$\left(\int_0^\infty \left(t^{\theta-1}\sum_{k=1}^\infty \sigma\left(\frac{k}{t}\right)|c_k|\right)^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty \left(t^{\theta-1}\sum_{t\leq k<2t} |c_k|\right)^q \frac{dt}{t}\right)^{1/q}.$$

Since $\{c_k\} \in GMS$, it follows that (1.21) holds, therefore $\sum_{t\leq k<2t}|c_k| \leq \#\{k : t \leq k < 2t\}\cdot B\left|c_{\lceil t\rceil}\right| \leq 2Bt\left|c_{\lceil t\rceil}\right|$.

Therefore, for $q < \infty$:

$$\left(\int_0^\infty \left(t^{\theta-1}\sum_{t\leq k<2t} |c_k|\right)^q \frac{dt}{t}\right)^{1/q} \leq \left(\int_0^\infty \left(2Bt^\theta\left|c_{\lceil t\rceil}\right|\right)^q \frac{dt}{t}\right)^{1/q}$$

$$= 2B\left(\sum_{m=1}^\infty \int_{m-1}^m \left(t^\theta\left|c_{\lceil t\rceil}\right|\right)^q \frac{dt}{t}\right)^{1/q} = 2B\left(\sum_{m=1}^\infty |c_m|^q \int_{m-1}^m t^{\theta q}\frac{dt}{t}\right)^{1/q}$$

$$\leq BC(\theta,q)\left(\sum_{m=1}^\infty |c_m|^q m^{\theta q-1}\right)^{1/q} = BC(\theta,q)\|\{c_k\}\|_{l^q_{w\left(\frac{1}{\theta},q\right)}},$$

where $C(\theta,q) = \max\left\{1, \frac{1}{\theta q}, 2^{1-\theta q}\right\}$. Therefore:

$$\|\{c_k\}\|_{(X,X_w)_{\theta,q;K}} \sim \|\{c_k\}\|_{l^q_{w\left(\frac{1}{\theta},q\right)}}. \tag{7.5}$$

On the other hand, Theorem 7.3 shows that

$$\|\{c_k\}\|_{(X,X_w)_{\theta,q;K}} = \|\{c_k\}\|_{\left(l^1_{\frac{1}{k}},l^1\right)_{\theta,q;K}} \sim \|\{c_k\}\|_{\left(l^1_{\frac{1}{k}}\cap GMS_{\alpha,\varphi},l^1\right)_{\theta,q;K}}.$$

$l^1_{\frac{1}{k}} \cap GMS_{\alpha,\varphi}$ is a quasi-normed monoid in the vector space of sequences (see Corollary 5.7), with the norm $\|\cdot\|_{l^1_{\frac{1}{k}}}$. Therefore, Theorem 2.17 applies with $G_0 = l^1_{\frac{1}{k}} \cap GMS_{\alpha,\varphi}$, $G_1 = l^1$, and $T$ defined as in (6.7). By the result of Theorem 6.3, $T$ maps $G_0$ into $L(1,\infty)$, and $\|T\{c_n\}\|_{L(1,\infty)} \leq 6\pi B \|\{c_n\}\|_{l^1_{\frac{1}{k}}}$. Also, since a function with an absolutely convergent Fourier series is bounded, it follows that $T : G_1 \to L(\infty,\infty)$. Therefore, by Theorem 2.17,

$$\|T\{c_n\}\|_{(L(1,\infty),L(\infty,\infty))_{\theta,q;K}} \leq C(B) \|\{c_n\}\|_{(G_0,G_1)_{\theta,q;K}} \sim \|\{c_n\}\|_{(X,X_w)_{\theta,q;K}}.$$

Finally, by the interpolation Theorem 2.15 we have, $\forall\theta \in (0,1), \forall q \in (0,\infty)$:

$$(L(1,\infty), L(\infty,\infty))_{\theta,q} = L(p,q),$$

where $p = \dfrac{1}{1-\theta}$. That is to say, $\theta = \dfrac{1-p}{p} = \dfrac{1}{p'}$.

Since $\{c_k\} \in GMS$ the conditions of Lemma 4.4 hold and, therefore, $|c_n| \leq Bc^*_{\lfloor \frac{n}{2} \rfloor}$. This implies that $\|\{c_k\}\|_{l^q_{w(p',q)}} \leq B'(p,q) \|\{c^*_k\}\|_{l^q_{w(p',q)}} = B'(p,q) \|\{c_k\}\|_{l(p,q)}$, thus obtaining (7.4).□

Combining the results of Theorems 7.4 and 6.4, we obtain the main result of this section.

**Theorem 7.5**. Let $1 < p < \infty$, $0 < q \leq \infty$. Assume $\{c_k\} \in GMS_{\alpha,\varphi}$. Define $f(x) = \sum_{k=1}^{\infty} c_k e^{ikx}$. Then:

$$\|\{c_n\}\|_{l(p',q)} \sim \|f\|_{L(p,q)(0,\pi)}. \tag{7.6}$$

# CHAPTER 8
# ON HARDY'S INEQUALITY FOR GENERAL MONOTONE FUNCTIONS

We use interpolation to give a very short proof of Hardy's inequality for general monotone functions, for $\forall \alpha > 0$, $\forall q > 0$.

## 8.1 *GM* IS CLOSED UNDER MULTIPLICATION

**Lemma 8.1**. If $f, g \in GM$ then $f \cdot g \in GM$.

**Proof**.

$$V_{fg}([x,2x]) = \sup_{\{x_j\}\in\mathcal{P}([x,2x])} \left\{ \sum_{j=0}^{n-1} \left| f(x_{j+1})\, g(x_{j+1}) - f(x_j)\, g(x_j) \right| \right\}$$

$$\leq \sup_{\{x_j\}\in\mathcal{P}([x,2x])} \left\{ \sum_{j=0}^{n-1} \left| f(x_{j+1}) \right| \left| g(x_{j+1}) - g(x_j) \right| \right\}$$

$$+ \sup_{\{x_j\}\in\mathcal{P}([x,2x])} \left\{ \sum_{j=0}^{n-1} \left| g(x_j) \right| \left| f(x_{j+1}) - f(x_j) \right| \right\}$$

$$\leq \sup_{[x,2x]} |f(t)| \cdot V_g([x,2x]) + \sup_{[x,2x]} |g(t)| \cdot V_f([x,2x]) .$$

Since $f, g \in GM$ then $\exists B_1 > 0$, $B_2 > 0$ such that $\forall x > 0$:

$$\begin{cases} V_f([x,2x]) \leq B_1 |f(x)| \\ V_g([x,2x]) \leq B_2 |g(x)| \end{cases} .$$

Also, by (1.24), for all $x > 0$ and $t \in [x, 2x]$:

$$\begin{cases} |f(t)| \leq 2B_1 |f(x)| \\ |g(t)| \leq 2B_2 |g(x)| \end{cases}.$$

Therefore, $\forall x > 0$:

$$V_{fg}([x, 2x]) \leq 4B_1B_2 |f(x)g(x)|,$$

that is, $fg \in GM(4B_1B_2)$.□

## 8.2 HARDY'S INEQUALITY FOR *GM*

**Theorem 8.2** (Hardy's inequality for general monotone functions).

Let $\alpha > 0$, $0 < q \leq \infty$, and $f \in GM^+$. Then

$$\left(\int_0^\infty \left(x^{-\alpha}\int_0^x f(t)\frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \leq C(\alpha, q)\left(\int_0^\infty \left(x^{-\alpha}f(x)\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}. \tag{8.1}$$

**Proof.**

1) Consider $\alpha \leq 1$. Take $g(t) = \dfrac{f(t)}{t}$. By Lemma 8.1, since $\dfrac{1}{t} \in GM^+$, $g \in GM^+$.

Let $T$ be defined for integrable on $[0, a]$ functions, for all $a > 0$, as

$$(Th)(x) = \frac{1}{x}\int_0^x h(t)dt.$$

Then $T : L^\infty \to L^\infty$ and $T : L(1, 1) \to L(1, \infty)$. Applying Theorem 2.15, we obtain

$$T : L(p, q) \to L(p, q),$$

where $1 < p < \infty$, $0 < q \leq \infty$.

This implies

$$\|Tg\|_{L(p,q)} = \left(\int_0^\infty x^{\frac{q}{p}}\left(\left(\frac{1}{x}\int_0^x g(t)dt\right)^*\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\leq C(p,q)\left(\int_0^\infty x^{\frac{q}{p}}\left(g^*(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}} = C(p,q)\left\|g\right\|_{L(p,q)}. \quad (8.2)$$

By Theorem 4.11, $\|g\|_{L(p,q)} \sim \|g\|_{L^q_{w(p,q)}}$. By Lemma 5.12 and Remark 5.5, $Tg \in GM^+$, hence $\|Tg\|_{L(p,q)} \sim \|Tg\|_{L^q_{w(p,q)}}$. Therefore, (8.2) can be written as

$$\left(\int_0^\infty x^{\frac{q}{p}}\left(\frac{1}{x}\int_0^x g(t)dt\right)^q\frac{dx}{x}\right)^{\frac{1}{q}} \sim \|Tg\|_{L(p,q)}$$

$$\leq C(p,q)\left\|g\right\|_{L(p,q)} \sim C(p,q)\left(\int_0^\infty x^{\frac{q}{p}}\left(g(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}}.$$

Since $g(t) = \dfrac{f(t)}{t}$, it follows that

$$\left(\int_0^\infty x^{\frac{q}{p}-q}\left(\int_0^x f(t)\frac{dt}{t}\right)^q\frac{dx}{x}\right)^{\frac{1}{q}} \leq C(p,q)\left(\int_0^\infty x^{\frac{q}{p}-q}\left(f(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}}. \quad (8.3)$$

Inequality (8.1) for $\alpha \leq 1$ follows from (8.3) if we take $\alpha = 1 - \frac{1}{p} = \frac{1}{p'}$.

2) Consider $\alpha > 1$. Take $g(t) = t^{\epsilon-\alpha}f(t)$, where $0 < \epsilon < 1$. Clearly, $g \in GM$. Also:

$$x^{-\alpha}\int_0^x f(t)\frac{dt}{t} = x^{-\epsilon}\cdot\int_0^x x^{\epsilon-\alpha}f(t)\frac{dt}{t} \leq x^{-\epsilon}\cdot\int_0^x g(t)\frac{dt}{t}.$$

By (8.1), for $\epsilon < 1$ and $g \in GM$,

$$\left(\int_0^\infty\left(x^{-\epsilon}\int_0^x g(t)\frac{dt}{t}\right)^q\frac{dx}{x}\right)^{\frac{1}{q}} \leq C(\alpha,q)\left(\int_0^\infty\left(x^{-\epsilon}g(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$= C(\alpha,q)\left(\int_0^\infty\left(x^{-\alpha}f(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}},$$

and (8.1) for $\alpha > 1$ follows.□

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