Weakly Integrally Closed Domains and Forbidden Patterns

by

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Abstract

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An integral domain D is weakly integrally closed if whenever there is an element x in the quotient field of D and a nonzero finitely generated ideal J of D such that $xJ \subseteq J^2$, then x is in D. We define weakly integrally closed numerical monoids similarly. If a monoid algebra is weakly integrally closed, then so is the monoid. A pattern F of finitely many 0's and 1's is forbidden if whenever the characteristic binary string of a numerical monoid M contains F, then M is not weakly integrally closed. Any stretch of the pattern 11011 is forbidden. A numerical monoid M is weakly integrally closed if and only if it has a forbidden pattern. For every finite set S of forbidden patterns, there exists a monoid that is not weakly integrally closed and that contains no stretch of a pattern in S. It is shown that particular monoid algebras are weakly integrally closed.

Contents

1 Introduction

Noether introduced the concept of integrality in the 1920s. Let $D \subseteq D'$ be integral domains. An element $x \in D'$ is said to be **integral** over D if $x^n + d_{n-1}x^{n-1} + \cdots + d_1x + d_0 = 0$ for some natural number n and $d_i \in D$. Kaplansky cites an equivalent definition in [6]: $x \in D'$ is integral over D if there is a finitely generated D-submodule J of D' such that $xJ \subseteq J$. An element $x \in D'$ is **almost integral** over D if there is a nonzero $d \in D$ such that $dx^n \in D$ for every natural number n. Any element $x \in D'$ that is integral over D is also almost integral over D. We say that the domain D is **(completely) integrally closed** in D' if whenever an element $x \in D'$ is (almost) integral over D, then x is in D.

In [3], Brown gives an example of a 3-by-3 matrix over the ring $D = K[[t^3, t^4]]$, where K is a field and t an indeterminate, whose minimum polynomial, $X^2 - t^5X$, has coefficients which do not lie in D. Brewer and Richman, intrigued by Brown's example, ask when must the coefficients of the minimum polynomial of a n-by-n matrix over a domain lie in that domain. In [2], this question led to the notion of a weakly integrally closed domain. An element $x \in D'$ is said to be **strongly integral** over D if there is a nonzero finitely generated ideal J of D such that $xJ \subseteq J^2$. If we use Kaplansky's definition of an integral element, then it is easy to see that any element $x \in D'$ that is strongly integral over D is also integral over D. We say that D is **weakly integrally closed** in D' if every element $x \in D'$ that is strongly integral over D is in D. If D is weakly integrally closed in it's quotient field, then we simply say that D is weakly integrally closed.

Let M be a numerical monoid (i.e. an additive submonoid of the natural numbers) and let $K[M] = \{k_0 + k_1 t^{m_1} + k_2 t^{m_2} + \ldots : k_i \in K \text{ and } m_i \in M\}$, where K is a field. We represent M by its infinite characteristic binary string. For example, the string 10111... represents the monoid generated by 2 and 3. Brewer and Richman show the following important results:

Theorem 1 ([2], Theorems 1 **and** 3**)** *If the binary string of a monoid M contains a stretch of the pattern* 11011*, then $K[M]$ is not weakly integrally closed.*

For example, the monoids $M = \{0, 3, 4, 6, 7, 8 \ldots\}$ and $M' = \{0, 6, 8, 12, 14, 15, 16, \ldots\}$ have the following infinite binary string representations:

$$M \quad : \quad 100\mathbf{11011}111\ldots \text{ and} \tag{1}$$

$$M' \quad : \quad 100000\mathbf{1}0\mathbf{1}0\mathbf{0}0\mathbf{1}0\mathbf{1}111\ldots. \tag{2}$$

Since the binary strings of M and M' contain the patterns 11011 and $1 \cdot 1 \cdot 0 \cdot 1 \cdot 1$, respectively, then the rings $K[M]$ and $K[M']$ are not weakly integrally closed. Thus, Brown's ring $K[t^3, t^4]$ is not weakly integrally closed.

Theorem 2 ([2], Theorem 5**)** *Let i and m be a pair of natural numbers such that $m \leq 2i$ and let $M = \{0, i, m, m+1, \ldots\}$. Then $K[M]$ is weakly integrally closed.*

For example, the ring $K[t^4, t^7, t^8, \ldots]$ is weakly integrally closed. The motivation for this dissertation is to answer the following two questions asked by Brewer and Richman in [2].

Question 1 Are weakly integrally closed monoid rings $K[M]$ characterized by the property that the binary string of the monoid M contains no stretch of the pattern 11011?

Question 2 Is the ring $K[t^4, t^5, t^{11}]$ weakly integrally closed?

When investigating a particular property of monoid rings, it is common for mathematicians to study the analogous property in the monoid itself. In Gilmer's book, *Commutative Semigroup Rings*, he extends the concept of integrality to monoids and studies the relationship between integrally closed/completely integrally closed monoids and their corresponding monoid rings. Similarly, Geroldinger has extensively researched completely integrally closed monoids and their rings in papers such as *The Complete Integral Closure of Monoids and Domains* [4]. We will do the same by defining strongly integral elements for monoids. In doing so, we will negatively answer Question 1 and affirmatively answer Question 2.

In Chapter 2, the concept of being weakly integrally closed is extended to monoids and it is shown that if a monoid ring is weakly integrally closed, then so is the monoid. In Chapter 3, we give examples of patterns of finitely many 0's and 1's such that when found in the characteristic binary string of a numerical monoid, then the monoid ring is not weakly integrally closed; we call such patterns **forbidden**. Question 1 is answered in Example 31. In Chapter 4, we discuss a JavaScript program that we wrote which determines if a pattern is forbidden. We prove that this program determines whether or not a numerical monoid is weakly integrally closed. Chapter 5 is devoted to showing that there is no finite set S of forbidden patterns such that whenever a monoid M contains no stretch of a pattern in S, then M is weakly integrally closed. In Chapter 6, we show that particular monoid rings are weakly integrally closed. We answer Question 2 and show that the ring $K\left[t^l, t^{l+1}, t^{l+s}, t^{l+s+1}, \ldots\right]$ is weakly integrally closed for all $3 < s \leq l$.

2 Background and Preliminary Results

We list some relevant notation, definitions and results used throughout the rest of this dissertation. Throughout this dissertation, all rings are commutative and contain 1. Let $D \subseteq D'$ be domains.

Definition 3 ([2]) *An element x in D' is said to be **strongly integral over** D if there exists a nonzero finitely generated ideal J of D such that $xJ \subseteq J^2$.*

Definition 4 ([2]) *The domain D is **weakly integrally closed** in D' if every element $x \in D'$ that is strongly integral over D is an element of D.*

Theorem 5 ([2], Theorem 2)

1. *(Intersections) If each member in a family of subrings D_i of D is weakly integrally closed in D, then $\cap_i D_i$ is weakly integrally closed in D.*

2. *(Transitivity) If D is weakly integrally closed in D', and D' is weakly integrally closed in D'', then D is weakly integrally closed in D''.*

3. *(Weakness) If D is integrally closed in D', then D is weakly integrally closed in D'.*

Definition 6 *A **monoid** is a nonempty set closed under an associative binary operation that has an identity element.*

For the rest of this dissertation, we assume that all monoids are commutative and we denote the monoid operation by $+$. Also, 0 is contained in the natural numbers. Then the natural numbers $\mathbb{N}$ form a monoid under addition.

Definition 7 *A monoid is* ***numerical*** *if it is an additive submonoid of the natural numbers.*

We extend the concept of a strongly integral element to monoids. Let U be an arbitrary monoid and let S be a submonoid of U.

Definition 8 ([5]) *An* ***ideal*** I *of* S *is a nonempty subset of* S *such that* $I = I + S = \{i + s : i \in I \text{ and } s \in S\}$.

Definition 9 ([5]) *An ideal* I *of* S *is* ***finitely generated*** *if there is a finite subset* B *of* S *such that* $I = B + S$.

Definition 10 *An element* $u \in U$ *is* ***strongly integral over S*** *if there exists a finitely generated ideal* J *of* S *such that* $u + J \subseteq J + J$.

Definition 11 ([5]) *An element* $u \in U$ *is* ***integral*** *over* S *if* $nu \in S$ *for some* $n > 0$.

Definition 12 *A monoid* M *is* ***cancellative*** *if whenever there are* $m, a, b \in M$ *such that* $m + a = m + b$, *then* $a = b$.

Clearly, any numerical monoid is cancellative.

Theorem 13 *Let* U *be a cancellative monoid and let* S *be a submonoid of* U. *If* $u \in U$ *is strongly integral over* S, *then* u *is integral over* S.

Proof. Let $u \in U$ be strongly integral over S. Then there is a finitely generated ideal J of S such that $u + J \subseteq J + J \subseteq J$. We will proceed by induction on the number of generators of any finitely generated ideal J' of S such that $u + J' \subseteq J'$.

Let $F = \{f_1, \ldots, f_n\}$ be a set of generators of J and let $N = \{1, \ldots, n\}$. Then there is a map $\sigma : N \longrightarrow N$ such that for each $i \in N$,

$$u + f_i = f_{\sigma(i)} + s_i$$

for some $s_i \in S$. Let $F' = \{f_{\sigma(i)} : i \in N\}$. If $F' = F$, then $\sum_{i \in N} f_i = \sum_{i \in N} f_{\sigma(i)}$ so $nu = \sum_{i \in N} s_i$ since U is cancellative. Thus, $nu \in S$.

We might as well assume F' is properly contained in F. Then F' has fewer than n elements. Let $J' = F' + S$. Then $u + J' \subseteq J'$. Thus, u is integral over S by induction. ■

Let M be a numerical monoid. If $d = \gcd(M)$, then M/d has finite complement in the natural numbers and is isomorphic to M [5]. Thus, we may assume that M has finite complement in the natural numbers. Let D a domain and t an indeterminate. Then $D[M] = \{d_0 + d_1 t^{m_1} + \cdots + d_n t^{m_n} : n \in \mathbb{N},\, d_i \in D,\, m_i \in M\}$ and $D[[M]] = \{d_0 + d_1 t^{m_1} + d_2 t^{m_2} \cdots : d_i \in D,\, m_i \in M\}$. In the case where D is a field, then $D[t]$ and $D[[t]]$ are integrally closed, so they are weakly integrally closed by Theorem 5.

Lemma 14 *Let K be a field and M a numerical monoid. Then:*

1. *$K[[M]]$ is WIC if and only if $K[[M]]$ is WIC in $K[[t]]$*
2. *$K[M]$ is WIC if and only if $K[M]$ is WIC in $K[t]$.*

Proof. We will prove (1); the proof of (2) is identical. Let Q be the quotient field of $K[[M]]$ and let F be the quotient field of $K[[t]]$.

Assume that $K[[M]]$ is WIC. Suppose there is $x \in K[[t]]$ and a nonzero finitely generated ideal J of $K[[M]]$ such that $xJ \subseteq J^2$. Let j be a nonzero element of J. Then $xj = j_1$ for some $j_1 \in J$ since $J^2 \subseteq J$. Thus, $x = j_1/j \in Q$ so $x \in K[[M]]$ by assumption. Hence, $K[[M]]$ is WIC in $K[[t]]$.

Assume $K[[M]]$ is WIC in $K[[t]]$. But $K[[t]]$ is integrally closed in F so is WIC in F. Thus, $K[[M]]$ is WIC in F by Theorem 5. Let $x \in Q$ be strongly integral over $K[[M]]$. Then $x \in F$ since $Q \subseteq F$ so $x \in K[[M]]$. Hence, $K[[M]]$ is WIC. ■

Therefore, when deciding whether or not the monoid rings $K[M]$ or $K[[M]]$ are weakly integrally closed, it suffices to consider only the strongly integral elements of

$K[[t]]$.

Definition 15 *A numerical monoid M is* ***weakly integrally closed*** *(or WIC) if every natural number x that is strongly integral over M is an element of M.*

Theorem 16 *Let K be a field and M a numerical monoid. Then the following is true:*

$$K[[M]] \text{ is WIC} \Longrightarrow K[M] \text{ is WIC} \Longrightarrow M \text{ is WIC.}$$

Proof. Assume $K[[M]]$ is WIC. Then $K[[M]]$ is WIC in $K[[t]]$ by Lemma 14. Let $x \in K[t]$ and let J be a nonzero finitely generated ideal of $K[M]$ such that $xJ \subseteq J^2$ for some . Let F be a finite set of generators for J. Then $xf \in J^2$ for every $f \in F$. Let L be the ideal generated by F in $K[[M]]$. Then $J \subseteq L \subseteq K[[M]]$ since $K[M] \subseteq K[[M]]$. Thus, L is a nonzero finitely generated ideal of $K[[M]]$ for which $xf \in L^2$ for every $f \in F$ so $xL \subseteq L^2$. Moreover, $x \in K[[t]]$ since $K[t] \subseteq K[[t]]$. Therefore, $x \in K[[M]]$ since $K[[M]]$ is WIC in $K[[t]]$. Consequently, $x \in K[t] \cap K[[M]] = K[M]$ so $K[M]$ is WIC in $K[t]$.

Assume $K[M]$ is WIC. Then $K[M]$ is WIC in $K[t]$ by Lemma 14. Let $x \in \mathbb{N}$ and let J be a nonempty ideal of M such that $x + J \subseteq J + J$. Let F be a finite set of generators for J. For every $f \in F$ there exist $k_f, l_f \in F$ and $m_f \in M$ such that

$$x + f = k_f + l_f + m_f.$$

Let L be the ideal generated by $\{t^f : f \in F\}$ in $K[M]$. Then $t^x t^f = t^{k_f + l_f + m_f} \in L^2$ for every $f \in F$. Moreover, L is nonzero since F is nonempty. Thus, $t^x \in K[M]$ since $K[M]$ is weakly integrally closed in $K[t]$. Hence, $x \in M$ so M is weakly integrally closed. ■

Thus, the monoid $M = \{0, i, m, m+1, \ldots\}$ is weakly integrally closed by Theorem 2. Neither of the converses to Theorem 16 are known to be true. Unfortunately, we have no counterexamples.

Theorem 17 *Let M be a numerical monoid. If the binary string of M contains a stretch of the pattern* 11011*, then M is not weakly integrally closed.*

Proof. If M contains a stretch of the pattern 11011, then there is $k \geq 0$ and $n > 0$ such that $k, k+n, k+3n$, and $k+4n$ are elements of M and such that $k+2n$ is not an element of M. Let $x = k+2n$ and let $F = \{k, k+n, k+3n, k+4n\}$. Then

$$\begin{aligned} x + k &= (k+n) + (k+n)\,, \\ x + (k+n) &= k + (k+3n)\,, \\ x + (k+3n) &= (k+n) + (k+4n)\,, \text{ and} \\ x + (k+4n) &= (k+3n) + (k+3n)\,. \end{aligned}$$

Thus, $x + F \subseteq F + F$, so $x + J \subseteq J + J$. Hence, M is not weakly integrally closed since $x \notin M$. ∎

Thus, we have another proof of Theorem 1. If the binary string of M contains the pattern 11011, then both $K[M]$ and $K[[M]]$ are not weakly integrally closed by Theorem 16. In the next chapter, we investigate patterns such as 11011 that necessarily lead to monoids that are not weakly integrally closed. We will see that there are infinitely such patterns. Then, by Theorem 16, their corresponding monoid rings are not weakly integrally closed.

3 Forbidden Patterns

3.1 Terminology and Preliminaries

Are there patterns other than the pattern 11011 such that whenever a monoid contains it, then it is necessarily weakly integrally closed? We will see that the answer is yes in the next section. But first, we list definitions and results that are used throughout the rest of this dissertation.

Definition 18 *A* ***pattern*** *F is a pair of disjoint finite subsets, (F_0, F_1), of the natural numbers whose union is either empty or contains 0. Let $U_F = F_0 \cup F_1$. If U_F is nonempty, then $0 = \min(U_F)$.*

We identify a pattern F with a string, where elements of F_0 are represented by zeros, elements of F_1 are represented by ones, and elements $y \in \mathbb{N} \setminus U_F$ such that $\min(U_F) < y < \max(U_F)$ are represented by dots. For example, the string $1 \cdot 010 \cdot 01111$ represents the pair $(\{2,4,6\},\{0,3,7,8,9,10\})$. We will use the term pattern to refer either to the pair or its string.

Definition 19 *The* ***length*** *l_F of a pattern F is defined as follows:*

$$l_F = \begin{cases} \max(U_F) + 1 & ;\ \text{if } U_F \text{ is nonempty} \\ 0 & ;\ \text{if } U_F \text{ is empty.} \end{cases}$$

Definition 20 *Let F and L be patterns. Then L is a* ***stretch*** *(or m-stretch) of F if $L_0 = mF_0$ and $L_1 = mF_1$ for some $m > 0$, in which case, we write $L = mF$.*

If U_F is nonempty, then the length of mF is $m(l_F - 1) + 1$. Let F be the pattern 11011 and L the pattern $1 \cdot \cdot 1 \cdot \cdot 0 \cdot \cdot 1 \cdot \cdot 1$. Then $(F_0, F_1) = (\{2\}, \{0,1,3,4\})$ and $(L_0, L_1) = (\{6\}, \{0,3,9,12\})$ so L is a 3-stretch of F. Since $l_F = 5$, then $l_L = 3 \cdot 4 + 1 = 13$.

Definition 21 *Let F and L be patterns. Then L* ***contains*** *F (by z) if $z + F_0 \subseteq L_0$ and $z + F_1 \subseteq L_1$ for some integer z.*

If F is the pattern $10 \cdot 1$ and L is the pattern 1101011, then L contains F by $z = 3$.

Definition 22 *Let M be a numerical monoid. Then we say that M* ***contains*** *F (by z) if $(z+ F_0) \cap M$ is empty and $z + F_1 \subseteq M$ for some integer z.*

Let $M = \{0,4,6,8,11,12,13,\ldots\}$ and let F be the pattern $1 \cdot 100111$. Then $(F_0, F_1) = (\{3,4\}, \{0,4,5,6,7\})$ so M contains F by $z = 6$.

Definition 23 *Let F be a pattern such that F_1 is nonempty. Let $F' = \{x \in F_0 : \min(F_1) < x < \max(F_1)\}$ and let $\alpha = \min(F' \cup F_1)$. Define the sets $F_0^T = F' - \alpha$ and $F_1^T = F_1 - \alpha$. Then the* ***trimmed pattern*** *F^T of F is the pair $\left(F_0^T, F_1^T\right)$.*

Suppose that $(F_0, F_1) = (\{0,4,10\}, \{2,3,5,6,8\})$. Then $F' = \{4\}$ and $\alpha = 2$ so $\left(F_0^T, F_1^T\right) = (\{2\}, \{0,1,3,4,6\})$.

Definition 24 *We say that F is* ***forbidden*** *if whenever a monoid M contains F, then M is not weakly integrally closed.*

The pattern 11011 is forbidden by Theorem 17. It is easy to see that any pattern that contains a forbidden pattern is forbidden.

Definition 25 *If $x \in F_0$ and C is a nonempty subset of F_1 such that $x + C \subseteq C + C$, then we say that x is a* ***bad zero*** *of F.*

Observe that we used the fact that the pattern 11011 has a bad zero to show that it is forbidden in Theorem 17.

Theorem 26 *Any pattern that has a bad zero is forbidden.*

Proof. Let F be a pattern that has a bad zero and let M be a monoid that contains F. Then there exist an $x \in \mathbb{N} \setminus M$ and a finite subset $F' \subseteq M$ such that $x + F' \subseteq F' + F'$. Let J be the ideal of M generated by F'. Then clearly $x + J \subseteq J + J$ so M is not weakly integrally closed. Hence, F is forbidden. ■

The next result follows immediately from the definition of a forbidden pattern and Theorem 16.

Theorem 27 *Let K be a field. If the binary string of a monoid M contains a forbidden pattern, then $K[M]$ is not weakly integrally closed, so neither is $K[[M]]$.*

Definition 28 *A pattern F is a* ***block*** *if U_F is nonempty and if $y \in U_F$ for every $0 < y < \max(U_F)$ (i.e. its string representation has no dots).*

For example, the patterns 11011 and 1101011 are blocks.

Definition 29 *The* ***block set*** *of F is the set of all blocks H such that H contains F and $l_H = l_F$. We refer to an element of the block set of a pattern F as* ***a block of*** *F. The block set of the pattern $011 \cdot 01$ is $\{011001, 011101\}$.*

3.2 Examples of Forbidden Patterns

When convenient, we will use exponents $l \geq 0$ in the string representation of a pattern. For example, $10^2 1 \cdot^2 01^4$ is shorthand for the pattern $1001 \cdot \cdot 01111$.

The next theorem generalizes the forbidden pattern 11011 to an infinite family of forbidden patterns.

Theorem 30 *Let $j \geq 1$. The pattern $11\,(01)^j\,1$ has a bad zero.*

Proof. Let F be the pattern $11\,(01)^j\,1$. Then

$$\begin{aligned} F_0 &= \{2i : 1 \leq i \leq j\} \text{ and} \\ F_1 &= \{0\} \cup \{(2i+1) : 0 \leq i \leq j\} \cup \{2\,(j+1)\}. \end{aligned}$$

Let $x = 2i$ for some $1 \leq i \leq j$. Then

$$\begin{aligned} x + 0 &= 1 + (2i - 1) \text{ and} \\ x + 2\,(j+1) &= (2i+1) + (2j+1)\,. \end{aligned}$$

Let $f = (2k+1)$ for some $0 \leq k \leq j$. Then

$$x + f = \begin{cases} 0 + [2(i+k)+1] & ;\text{ if } j \geq i+k \\ 2\,(j+1) + [2\,(i+k-j) - 1] & ;\text{ if } j < i+k. \end{cases}$$

Thus, $x + f \in F_1 + F_1$ for all $f \in F_1$, so every element of F_0 is a bad zero. Hence, F has a bad zero. ■

So, if a monoid M contains the pattern $11\,(01)^j\,1$, then M is not weakly integrally closed and neither is its monoid algebra $K\,[M]$ by Theorem 16.

This next example negatively answers Question 1. We will show that the exclusion of the pattern 11011 and all of its stretches from a monoid is not enough to guarantee that its monoid algebra will be weakly integrally closed.

Example 31 *Let $M = \{0, 5, 6, 8, 10, 11, 12, \ldots\}$. Then the monoid M contains no stretch of the pattern 11011 but its monoid algebra, $K\,[t^5, t^6, t^8, t^{14}]$, is not weakly integrally closed.*

Proof. The binary string of M,

$$10000\mathbf{1101011}1\ldots,$$

contains the pattern 1101011. Thus, M is not weakly integrally closed by Theorem 30 so $K[M]$ is not weakly integrally closed by Theorem 16. Clearly, M does not contain the pattern 11011. Suppose M contains an n-stretch of the pattern 11011 for some $n \geq 2$. Then $k, k+n, k+3n, k+4n \in M$ and $k+2n \notin M$ for some $k \geq 0$. If $k = 0$, then $k+n \geq 5$ so $k+2n \geq 10$. So we might as well assume that $k \geq 5$. Thus, $k+n \geq 8$ since $k+n \in M$ so $k+2n \geq 10$. Therefore, M contains no stretch of the pattern 11011. Hence, $K[t^5, t^6, t^8, t^{14}]$ is a monoid algebra that is not weakly integrally closed but whose corresponding monoid contains no stretch of the pattern 11011. ■

The next theorem is another generalization of the forbidden pattern 11011.

Theorem 32 *Let $l \geq 0$. The pattern $1 \cdot^{l} 1 \cdot^{l} 01^{l+2}$ has a bad zero.*

Proof. Let F be the pattern $1 \cdot^{l} 1 \cdot^{l} 01^{l+2}$. Then

$$\begin{aligned} F_0 &= \{2(l+1)\} \text{ and} \\ F_1 &= \{0\} \cup \{l+1\} \cup \{2l+i : 3 \leq i \leq l+4\}. \end{aligned}$$

Let $x = 2(l+1)$. Then

$$\begin{aligned} x+0 &= (l+1)+(l+1), \\ x+(l+1)n &= 0+3(l+1), \text{ and} \\ x+(2l+3)\,n &= (l+1)+(3l+4). \end{aligned}$$

Let $f = 2l+i$ for some $4 \leq i \leq l+4$. Then $x+f = (2l+3)+(2l+i-1)$. Hence,

$x + f \in F_1 + F_1$ for every $f \in F_1$, so F has a bad zero. ■

For example, the pattern 10100111 has a bad zero. Therefore, the monoid $M = \{5, 7, 10, 11, 12, \ldots\}$, as well as its monoid algebra $K[t^5, t^7, t^{11}]$, are not weakly integrally closed.

We will see that a forbidden pattern need not have a bad zero. But first we list an important lemma that will be used repeatedly throughout this dissertation.

Lemma 33 *If C is a nonempty finite subset in $\mathbb{N}$ and x is an element of $\mathbb{N} \diagdown C$ such that $x + C \subseteq C + C$, then there are $a, b, c, d \in C$ such that $a < b < x < c < d$.*

Proof. Let $a = \min(C)$ and $d = \max(C)$. Then $x + a = c_1 + c_2$ and $x + d = c_3 + c_4$ for some $c_1, c_2, c_3, c_4 \in C$ such that $a \leq c_1, c_2$ and $c_3, c_4 \leq d$.

Let $i, j \in \{1, 2\}$ such that $i \neq j$. If $c_i = a$, then $c_j = x$ which contradicts $x \notin C$. Thus, $a < c_1, c_2$. If $x < c_i$, then $c_j = x + a - c_i < x + a - x = a$ which contradicts the minimality of a. Thus, $c_1, c_2 < x$. Therefore, there exists some $b \in C$ such that $a < b < x$.

Let $k, l \in \{3, 4\}$ such that $k \neq l$. If $c_k = d$, then $c_l = x$ which contradicts $x \notin C$. Thus, $c_3, c_4 < d$. If $c_k < x$, then $c_l = x + d - c_k > x + d - x = d$ which contradicts the maximality of d. Thus, $x < c_3, c_4$. Therefore, there exists some $c \in C$ such that $x < c < d$.

Hence, there exist a, b, c, d in C such that $a < b < x < c < d$. ■

So, for example, the pattern $100 \ldots 00111 \ldots 1$ contains no bad zeros.

Theorem 34 *The pattern $1101 \cdot\cdot 111$ has no bad zeros, but each of its blocks has a bad zero, so $1101 \cdot\cdot 111$ is forbidden.*

Proof. Let F be the pattern $1101 \cdot\cdot 111$. Then $F_0 = \{2\}$ and $F_1 = \{0, 1, 3, 6, 7, 8\}$. Let C be a nonempty subset of F_1 such that $2 + C \subseteq C + C$. Then $1 \in C$ by Lemma

33. Thus, $2+1 \in C+C$ so $0,3 \in C$. Then $2+3 \in C+C$. But $5 \notin F_1+F_1$ so $5 \notin C+C$. Hence, F has no bad zeros.

We will show that every block of F has a bad zero. The following is a list of all the blocks of F:

$$H^1 = 110100111,\ \ H^2 = 110101111,\ \ H^3 = 110110111,\ \ H^4 = 110111111$$

It is easy to see that H^1 contains $1 \cdot 1 \cdot 0111$, H^2 contains 1101011, H^3 contains 11011, and H^4 contains 11011. Let $1 \leq i \leq 4$. Then H^i contains a pattern with a bad zero by Theorems 30 and 32. Thus, H^i itself has a bad zero. Let M be a monoid that contains the pattern F. Then M contains some block H of F, so contains a pattern that has a bad zero. Hence, M is not weakly integrally closed, so F is forbidden. ■

3.3 A Characterization of Forbidden Patterns

In Lemma 26, we saw that any pattern with a bad zero is forbidden. The converse need not be true, as we saw with the forbidden pattern $1101 \cdot\cdot 111$. We generalize the results of Lemma 26 to characterized an arbitrary forbidden pattern. But first we make a brief remark.

Remark 35 *Let M be a numerical monoid, let e be the smallest positive integer in M and let g be the largest integer not in M. We know that g exists since M has finite complement in $\mathbb{N}$. Let $x \in \mathbb{N}$ and let J be an ideal of M such that $x+J \subseteq J+J$. Let $n = \min(J)$. Then $x+n = j_1+j_2$ for some $j_1, j_2 \in J$. We will show that when trying to prove that M is weakly integrally closed, then one might as well assume that*

$$e \leq n < j_1, j_2 < x \leq g.$$

Proof. If $n = 0$, then $x = x + 0 = j_1 + j_2 \in J + J \subseteq M$. If $n = j_1$ or $n = j_2$, then $x \in J \subseteq M$. Thus, we might as well assume that $0 < n < j_1, j_2$ or else $x \in M$. If $j_1 \geq x$ or $j_2 \geq x$, then either $x \in J \subseteq M$ (and we are done) or $x + n = j_1 + j_2 > x + n$, a contradiction. Thus, we might as well assume that $j_1, j_2 < x$. If $x > g$, then $x \in M$. Also, $n \geq e$ since $n > 0$. Hence, we might as well assume that $e \leq n < j_1, j_2 < x \leq g$. ■

The results from this remark are used in the proofs of Lemma 36 and Theorem 47.

Lemma 36 *Let H be a block of length d and let*

$$M = \{0\} \cup \left(d^2 + H_1\right) \cup \{2d^2, 2d^2 + 1, 2d^2 + 2, \ldots\}.$$

If H has no bad zeros, then M is weakly integrally closed.

Proof. It is clear that M is a monoid that contains H. Assume that H has no bad zeros. Let $x \in \mathbb{N}$ and let J be an ideal of M such that $x + J \subseteq J + J$. We will show that $x \in M$.

Let $n = \min(J)$. Then $x + n = j_1 + j_2$ for some $j_1, j_2 \in J$. By Remark 35, we might as well assume that $d^2 \leq n < j_1, j_2 < x < 2d^2$ or else $x \in M$. Then $n, j_1, j_2 \in d^2 + H_1$, so $d > 1$ and $d^2 \leq n, j_1, j_2 \leq d^2 + d - 1$. Thus, $x + n = j_1 + j_2 \leq 2(d^2 + d - 1)$ which implies that $x \leq d^2 + 2d - 2$.

Let $I = J \cap (d^2 + H_1)$. Suppose that there is $i \in I$ such that $x + i \notin I + I$. But $i \in J$, so $x + i \in J + J$. Thus, $x + i = j_3 + j_4$ for some $j_3, j_4 \in J$. Without loss of generality, we may assume that $j_3 \notin I$. Then $j_3 \geq 2d^2$ since $0 \notin J$. Thus, $x + i = j_3 + j_4 \geq 3d^2$, so $x \geq 2d^2 - d + 1$. Then $2d^2 - d + 1 \leq x \leq d^2 + 2d - 2$, so $d^2 - 3d + 3 \leq 0$. But $d^2 - 3d + 3 \geq 3/4$ for all real numbers d, so we have a contradiction. Thus, $x + I \subseteq I + I$. Then $\min(I) \leq x \leq \max(I)$, so $x \in d^2 + (H_0 \cup H_1)$ since H is a block. But H has no bad zeros, so $x \in d^2 + H_1 \subseteq M$. Hence, M is weakly integrally closed. ■

Therefore, a block is forbidden if and only if it has a bad zero. Now, we are ready for our characterization of forbidden patterns.

Theorem 37 *A pattern is forbidden if and only if each of its blocks has a bad zero.*

Proof. Let F be a forbidden pattern and let H be a block of F. Let M be the monoid of Lemma 36, so M contains H. Then M contains F, a forbidden pattern, since H contains F. Thus, M is not weakly integrally closed. Hence, H has a bad zero by Lemma 36.

Let F be an arbitrary pattern and assume that each of its blocks has a bad zero. Let M be a monoid containing F. Then M contains a block of F. But each block of F has a bad zero, so is forbidden. Thus, M is not weakly integrally closed. ■

The next theorem implies that we can generalize Lemma 36.

Theorem 38 *Let N be a finite set of blocks that contain no bad zeros. Then there is a block that contains every block in N and that contains no bad zeros.*

Proof. We will first show that the theorem is true when N contains exactly two elements. Let $S = \{G, H\}$. Then G and H are blocks that contain no bad zeros. Let $n = 2l_G + l_H$ and let L be the pair (L_0, L_1), where

$$\begin{aligned} L_0 &= G_0 \cup \{l_G, l_G + 1, \ldots, n - 1\} \cup (n + H_0) \text{ and} \\ L_1 &= G_1 \cup (n + H_1)\,. \end{aligned}$$

It is clear that L contains the blocks G and H. The pattern L is a block since G and H are blocks. Let x be an element of $L_0 \cup L_1$ and S a nonempty subset of L_1 such that $x + S \subseteq S + S$. We will show that L has no bad zeros by showing $x \in L_1$.

Let $G' = S \cap G_1$ and let $H' = S \cap (n + H_1)$. Suppose that $x + G' \subseteq G' + G'$. Then $\min(G_1) \le x \le \max(G_1)$ by Lemma 33. Thus, x is contained in $G_0 \cup G_1$ since G is a block. So $x \in G_1 \subseteq L_1$ since G has no bad zeros. The same argument shows that if $x + H' \subseteq H' + H'$, then $x \in (n + H_1) \subseteq L_1$. Thus, we might as well assume that there is $g \in G'$ and $h \in H'$ such that $x + g \notin G' + G'$ and $x + h \notin H' + H'$. Since $g \le l_G - 1$, then $x + g \le (n + l_H - 1) + (l_G - 1) < 2n$ and since $h \ge n$, then $x + h \ge n$.

Note the following observations. If f is an element of $G'+G'$, then $f \le 2(l_G - 1) < n$. If f is an element of $H' + H'$, then $f \ge 2n$. If f is an element of $G' + H'$, then $n \le f \le (l_G - 1) + (n + l_H - 1) = n + l_G + l_H - 2$.

By the above observations, $x + g \notin H' + H'$ since $x + g < 2n$ and $x + h \notin G' + G'$ since $x + h \ge n$. Then $x + g$ and $x + h$ are elements of $G' + H'$. Therefore, $x + g \ge n$ and $x + h \le n + l_G + l_H - 2$, so $x \ge n - g \ge n - (l_G - 1) = l_G + l_H + 1$ and $x \le n + l_G + l_H - 2 - h \le n + l_G + l_H - 2 - n = l_G + l_H - 2$. Thus, $l_G + l_H < x < l_G + l_H$, a contradiction. Therefore, x is an element of L_1, so L has no bad zeros. Hence, the theorem is true by induction. ■

Definition 39 *A pattern F is said to be* ***safe*** *if it is not forbidden.*

Then a pattern F is safe if and only if a block of F has no bad zeros by Theorem 37.

Theorem 40 *For every finite set S of safe patterns, there is a weakly integrally closed monoid that contains every pattern in S.*

Proof. Let S be a finite set of safe patterns. Each pattern in S has a block that has no bad zeros. For each pattern F of S, fix H^F to be a block of F that has no bad zeros. Let $N = \{H^F : F \text{ is an element of } S\}$. Then, by Theorem 38, there is some block H such that H contains every block in N and such that H contains no bad zeros. Let M be the monoid from Lemma 36. Then M is weakly integrally closed since H has no bad zeros. Moreover, M contains every block in N, so M contains every element of S. ■

4 An easy test to determine if a monoid is weakly integrally closed

4.1 Algorithm to determine if a pattern has a bad zero

We have developed a JavaScript program that finds any bad zeros of a pattern. Before discussing the algorithm that the program is based on, we list some relevant definitions and lemmas.

Definition 41 *If a pattern F has a bad zero $x \in F_0$ by $C \subsetneq F_1$, then we say that the pattern $(\{x\}, C)$, denoted C^x, is a **closed pattern** of F (by x).*

Definition 42 *Let F be a pattern and let U^x be a closed pattern of F. Then U^x is said to be the **maximum** pattern of F by x if $L \subseteq U$ for every closed pattern L^x of F.*

For example, if F is the pattern 1011011, then it is not difficult to see that $x = 4$ is the only bad zero of F and that $(4, F_1)$ and $(4, \{2, 3, 5, 6\})$ are the only closed patterns of F. Then $(4, F_1)$ is the maximum pattern of F by $x = 4$.

Lemma 43 *If F is a pattern with a bad zero $x \in F_0$, then there is a maximum pattern U^x of F for every element $x \in F_0$.*

Proof. Let $x \in F_0$ be a bad zero of F. Let $\Sigma_x = \{C \subseteq F_1 : C$ is nonempty, $x + C \subseteq C + C\}$ and let $U = \bigcup_{C \in \Sigma_x} C$. Then $U \subseteq F_1$ since $C \subseteq F_1$ for every $C \in \Sigma_x$. Let $u \in U$. Then $u \in C$ for some $C \in \Sigma_x$. Thus, $x + u \in C + C \subseteq U + U$, so (x, U) is a closed pattern of F by x. Then it is clear that the pattern U^x is maximum pattern of F by x since $C \subseteq U$ for every $C \in \Omega_x$. ■

Let F be a pattern and let x be an element of F_0. Let $\Omega = \{S \subseteq F_1 : S$ is nonempty$\}$. Define the map $\mu_x : \Omega \longrightarrow \Omega$ as

$$\mu_x(S) = \{s \in S : x + s \in S + S\}$$

for each $S \in \Omega$. If μ_x maps S onto itself, then (x, S) is a closed pattern of F by x. Thus, if $\mu_x(F_1) = F_1$, then (x, F_1) is the maximum closed pattern of F by $x \in F_0$

Lemma 44 *Let F be a pattern and x an element of F_0. Let $L_0 = \mu_x(F_1)$ and $L_{i+1} = \mu_x(L_i)$ for $i \geq 0$. If C^x is a closed pattern of F, then $C \subseteq L_i$ for all $i \geq 0$.*

Proof. Let C^x be a closed pattern of F. Then $C \subseteq F_1$ and $x + C \subseteq C + C$, so $x + C \subseteq F_1 + F_1$. Thus, $C \subseteq L_0 = \mu_x(F_1) = \{f \in F_1 : x + f \subseteq F_1 + F_1\}$. If $C \subseteq L_{i-1}$ for some $i > 0$, then $x + C \subseteq C + C \subseteq L_{i-1} + L_{i-1}$. Thus, $C \subseteq \{l \in L_{i-1} : x + l \in L_{i-1} + L_{i-1}\} = \mu_x(L_{i-1}) = L_i$. Therefore, $C \subseteq L_i$ for all $i \geq 0$ by induction. ■

The following algorithm is the basis of our JavaScript program.

Algorithm 45 *Let F be a pattern and x an element of F_0. Let $L_0 = \mu_x(F_1)$ and $L_{i+1} = \mu_x(L_i)$ for $i \geq 0$.*

1. *If L_i is empty for some $i \geq 0$, then x is not a bad zero.*
2. *If L_i is nonempty and $L_{i+1} = L_i$ for some $i \geq 0$, then x is a bad zero and (x, L_i) is the maximum pattern of F by x.*

Proof. It is clear that 1 follows immediately from Lemma 44. To show 2, suppose that $L_k = L_{k+1}$. Then $L_k = \mu_x(L_k) = \{l \in L_k : x + l \in L_k + L_k\}$, so $x + L_k \subseteq L_k + L_k$. Note that

$$F_1 \supseteq L_0 \supseteq L_1 \supseteq \ldots \supseteq L_k = L_{k+1}.$$

Thus, $L_k \subseteq F_1$, so (x, L_k) is a closed pattern of F. Let U^x be the maximum pattern of F by x. Then $L_k \subseteq U$. Since U^x is a closed pattern of F, then $U \subseteq L_k$ by 44. Therefore, $U = L_k$, so (x, L_k) is the maximum pattern of F by x. ■

If the pattern (x, L_i) is the maximum pattern of F, then for each element $l \in L_i$, we will apply the map μ_x recursively to $G_l^i = L_i \diagdown \{l\}$. We give an example to demonstrate how the program works.

Example 46 *Let F be the pattern* 101001111*. Then we use the algorithm to show that F contains one bad zero $x \in F_0$ and three closed patterns.*

Proof. If F is the pattern 101001111, then $F_0 = \{1, 3, 4\}$ and $F_1 = \{0, 2, 5, 6, 7, 8\}$. By Lemma 33, $x = 1$ cannot be a bad zero of F. If $x = 3$, then $L_0 = \{2, 5, 6, 7, 8\}$, $L_1 = \{5, 6, 7, 8\}$, $L_3 = \{7, 8\}$ and L_4 is empty. Thus, $x = 3$ cannot be a bad zero by 1 of Algorithm 45.

Let $x = 4$. Then $L_0 = F_1$. Thus, the pattern $(4, F_1)$, i.e. $1 \cdot 1 \cdot 01111$, is the maximum pattern of F by $x = 4$. Let $G_l^i = L_i \diagdown \{l\}$. By Lemma 33, it suffices to check G_l for $4 < f < 9$. If $L_0 = \mu_x(G_5)$, then $L_1 = \{0, 2, 6, 8\} = L_2$. Thus, the pattern $1 \cdot 1 \cdot 0 \cdot 1 \cdot 1$ is a closed pattern of F by $x = 4$. Note that by Lemma 33, the pattern $1 \cdot 1 \cdot 0 \cdot 1 \cdot 1$ contains no closed patterns other than itself. If $L_0 = \mu_x(G_6)$, then $L_1 = \{0, 5, 8\}$, so (x, L_1) is not a bad zero by Lemma 33. If $L_0 = \mu_x(G_7)$, then $L_1 = \{0, 6, 8\}$, so (x, L_1) is not a bad zero by Lemma 33. If $L_0 = \mu_x(G_8)$, then $L_1 = \{0, 2, 5, 6, 7\} = L_2$, so the pattern $1 \cdot 1 \cdot 0111$ is a closed pattern. We repeat the algorithm on $L_1 \diagdown \{l\}$ for each $l \in L_1$. Let $G_l = L_1 \diagdown \{l\}$. Note that it suffices to check only G_5, G_6 and G_7 by Lemma 33. We leave it to the reader to see that the pattern $1 \cdot 1 \cdot 0111$ contains no closed patterns other than itself. Hence, the pattern

101001111 contains exactly the following closed patterns:

$$1 \cdot 1 \cdot 0 \cdot 1 \cdot 1,$$

$$1 \cdot 1 \cdot 0111, \text{ and}$$

$$1 \cdot 1 \cdot 01111.$$

■

Next, we will characterize weakly integrally closed monoids in a way that allows us to use Algorithm 45 to easily determine if a numerical monoid is weakly integrally closed.

4.2 A characterization of weakly integrally closed monoids

Let M be a monoid, let e be the smallest positive integer in M and let g be the largest integer not in M. Let

$$\begin{aligned} H_0 &= \{x \in \mathbb{N} \diagdown M : e < x \leq g\} \text{ and} \\ H_1 &= \{m \in M : e \leq m \leq 2g + 1 - e\}. \end{aligned}$$

Then the **test block** H of M is the pattern (H_0, H_1). Clearly, M contains its test block H. We will show that it suffices to consider only the test block of a monoid M when determining if M is weakly integrally closed.

Theorem 47 *A monoid is weakly integrally closed if and only if its test block contains no bad zeros.*

Proof. Let M be a monoid and H its test block. If H has a bad zero, then H is forbidden by Theorem 26, so M is not weakly integrally closed.

Assume that H has no bad zeros. Let $x \in \mathbb{N}$ and let J be an ideal of M such that $x + J \subseteq J + J$. Let $n = \min(J)$. Then $x + n = j_1 + j_2$ for some $j_1, j_2 \in J$. By Remark 35, we might as well assume that $e \leq n < j_1, j_2 < x \leq g$ or else $x \in M$ and we are done. Then $x \in H_0 \cup H_1$. Let

$$I = \{j \in J : j < g\} \cup \{g+1, g+2, \ldots, 2g+1-e\}.$$

Then I is not empty since $n \in I$. Let $i \in I$. We will show that $x + i \in I + I$.

Suppose that $i < g$. Then $i \in J$, so $x + i \in J + J$. Thus, there are $j_3, j_4 \in J$ such that $j_3 + j_4 = x + i < 2g$. Then $j_3 < 2g - j_4 \leq 2g - e$ and $j_4 < 2g - j_3 \leq 2g - e$, so $j_3, j_4 \in I$. Thus, if $i < g$, then $x + i \in I + I$.

Suppose that $i > g$. Observe that

$$\begin{aligned} x + i &= n + (i + x - n) \\ &= (g+1) + (i + x - (g+1)). \end{aligned}$$

Clearly, $n, g+1 \in I$. We will show that either $i + x - n \in I$ or $i + x - (g+1) \in I$. Assume that $i + x - n \notin I$. But $i + x - n > g$ since $i > g$ and $x > n$. Thus, $i + x - n > 2g + 1 - e$ since $i + x - n \notin I$. Then $i + x - (g+1) > g$ since $n \geq e$. But also, $i + x - (g+1) < i$ since $x < g+1$. Thus, $i + x - (g+1) \in I$. Then $x + i \in I + I$, so $x + I \subseteq I + I$. But $I \subseteq H_1$ and H has no bad zeros. Hence, $x \in H_1 \subseteq M$, so M is weakly integrally closed. ■

So, if the binary string of a monoid M is $10001001111\ldots$, then it suffices to check the pattern 100111 for any bad zeros to determine whether or not M is weakly integrally closed. It is easy to see that this pattern has no bad zeros by Lemma 33, so M is weakly integrally closed by Theorem 47.

5 Question 1 Generalized

We saw that excluding the pattern 11011 and its stretches from a monoid does not guarantee that the monoid is weakly integrally closed. Does there exist a finite set of forbidden patterns that suffices? Specifically,

> Is there a finite set of patterns such that whenever a monoid M contains no stretch of any of these patterns, then $K[M]$ is necessarily weakly integrally closed?

In Example 31, we used the fact that the forbidden pattern 1101011 contains no stretch of the pattern 11011. Similarly, we will show that for every finite set of forbidden patterns S, there is $j > 0$ such that the pattern $11(01)^j1$ contains stretch of an element of S. Then we construct a monoid containing the pattern $11(01)^j1$ but containing no stretch of a pattern in S. First, some technical lemmas:

Lemma 48 *Any forbidden pattern that* $11(01)^j1$ *contains has length equal to* $2j+3$.

Proof. Suppose $11(01)^j1$ contains a forbidden pattern L. Then there is a block H of L such that $11(01)^j1$ contains H. Thus, $l_L = l_H \leq 2j+3$, the length of $11(01)^j1$, and there is $m \geq 0$ such that

$$\begin{aligned} m + H_0 &\subseteq \{2i : 1 \leq i \leq j\} \text{ and} \\ m + H_1 &\subseteq \{0\} \cup \{2i+1 : 0 \leq i \leq j\} \cup \{2(j+1)\}. \end{aligned}$$

Furthermore, H has a bad zero by Theorem 37 since L is forbidden. Thus, there is $x \in m+H_0$ and nonempty $H' \subseteq m+H_1$ such that $x+H' \subseteq H'+H'$. Let $\alpha = \min(H')$

and $\beta = \max(H')$. Then $0 \leq \alpha < \beta \leq 2(j+1)$. We will show that $\beta - \alpha + 1 = 2j + 3$.

Observe that $x + \alpha = h_1 + h_2$ and $x + \beta = h_3 + h_4$ for some $h_1, h_2, h_3, h_4 \in H'_1$ such that $\alpha < h_1, h_2, h_3, h_4 < \beta$. Thus, $h_1, h_2, h_3, h_4 \in B$ so $x + \alpha, x + \beta \in \{2i : i \geq 1\}$. As a result, $\alpha, \beta \in \{2k : k \geq 0\}$ since $x \in A$. Therefore, $\alpha = 0$ and $\beta = 2(j+1)$ so $\beta - \alpha + 1 = 2j + 3$. Then $2j + 3 \leq l_H$ since $H' \subseteq m + H_1$. Hence, $l_L = l_H = 2j + 3$. ∎

For example, the pattern 1101010101011 has length 13 and

$$1101 \cdot 1 \cdot 1 \cdot 1 \cdot 11,$$

$$11 \cdot 101 \cdot 1 \cdot 1 \cdot 11,$$

$$11 \cdots\ 101 \cdots\ 11,$$

$$11 \cdot 1 \cdot 101 \cdot 1 \cdot 11,$$

$$11 \cdot 1 \cdot 1 \cdot 101 \cdot 11,$$

$$11 \cdot 1 \cdot 1 \cdot 1 \cdot 1011 \text{ and}$$

$$1 \cdot \cdot 1 \cdot \cdot 0 \cdot \cdot 1 \cdot \cdot 1$$

lists every closed pattern of F, all of which have length 13.

Lemma 49 *Any stretch of a forbidden pattern is forbidden.*

Proof. Let F be a forbidden pattern and let $m > 0$. Then mF is a stretch of F. Let G be in the block set of mF. Then G contains mH for some block H of F. And H has a bad zero by Theorem 37. Thus, there is $x \in H_0$ such that $x + H' \subseteq H' + H'$ for some nonempty $H' \subseteq H_1$. Let $m > 0$, $y = mx$ and $L = mH'$. Then $y \in mH_0$, $L \subseteq mH_1$ and $y + L \subseteq L + L$, so mH has a bad zero. Thus, G has a bad zero since it contains mH. Hence, mF is forbidden by Theorem 37. ∎

Lemma 50 *For every forbidden pattern F, the set F_1 has at least 4 elements.*

Proof. Let F be a forbidden pattern. Let V be the set of all $y \notin (F_0 \cup F_1)$ such that $0 < y < \max(F_0 \cup F_1)$. Let $G_0 = F_0 \cup V$ and $G_1 = F_1$. Then G is a block of F. Thus, G has a bad zero by Theorem 37 since F is forbidden. Then there exist $a, b, c, d \in G_1 = F_1$ such that $a < b < c < d$ by Lemma 33, so F_1 has at least 4 elements. ■

The pattern 11011, with length 5, has the smallest length of all forbidden patterns.

Lemma 51 *A pattern is forbidden if and only if its trimmed pattern is forbidden.*

Proof. Let F be a pattern. Suppose its trimmed pattern F^T is forbidden. Then F is forbidden since F contains F^T.

Assume that F is forbidden. Let G be a block of F^T. Then $G = H^T$ for some block H of F and H has a bad zero since F is forbidden. Thus, there is $x \in H_0$ and a nonempty finite subset $H' \subseteq H_1$ such that $x + H' \subseteq H' + H'$. Then $\min(H_1) < x < \max(H_1)$ by Lemma 33, so $x \in H_0^T$. Thus, H^T has a bad zero. Hence, G has a bad zero for every block G of F^T, so F^T is forbidden. ■

Theorem 52 *Let S be a finite set of forbidden patterns. Then there exists $j > 0$ such that $2j+3$ is greater than the length of every pattern in S and $11(01)^j1$ contains no stretch of an element in S.*

Proof. Fix $j = \prod_{F \in S} (l_F - 1)$. Let $F \in S$. Then $2j + 3 \geq 2(l_F - 1) + 3 > l_F$. Let $m > 0$ and $K = mF$. Suppose $l_K = 2j + 3$. Then $m(l_F - 1) = 2\prod_{F \in S}(l_F - 1) + 2$ so $l_F - 1$ divides 2. But $l_F - 1 \geq 4$ by Lemma 50, so $l_F - 1$ does not divides 2. Therefore, $l_K \neq 2j + 3$. Hence, $11(01)^j1$ contains no stretch of any element of S by Lemma 48. ■

Lemma 53 *Let L be a pattern. There is a monoid M containing L such that for any $m > 0$ and any trimmed forbidden pattern F for which $l_F < l_L$,*

$$\text{if } M \text{ contains } mF, \text{ then } L \text{ contains } mF.$$

Proof. Let $d = l_L$ and $M = \{0\} \cup (d^2 + L_1) \cup \{2d^2, 2d^2 + 1, 2d^2 + 2, \ldots\}$. Then M is a monoid that contains L. Let $K = mF$. Then K is a trimmed forbidden pattern by Lemma 49. Assume that M contains K. Then $z + K_1 \subseteq M$ and $(z + K_0) \cap M$ is empty for some $z \geq 0$. Let $K_0' = z + K_0$, $K_1' = z + K_1$ and $I = (d^2 + L_1) \cap K_1'$. Let $\beta = z + l_K - 1$. We will show that I contains exactly one element.

Suppose that I is empty. Then $100\ldots01_{2d^2}11\ldots1_\beta$ is a block of K, so it has a bad zero by Theorem 37 since K is forbidden. But this contradicts Lemma 33. Thus, I is not empty. We might as well assume that $z = 0$ or $\beta \geq 2d^2$ or else we are done. Then $l_K = \beta - z + 1 \geq 2d^2 - (d^2 + d - 1) = d^2 - d + 1$. Thus, $m\,(l_F - 1) \geq d^2 - d$, so $m > d$ since $l_F < l_L = d$. Then I contains exactly one element since K is an m-stretch.

If $k = z$, then $100\ldots01_{2d^2-z}11\ldots1_{\beta-z}$ is a block of K, a contradiction to Lemma 33. If $k = \beta$, then K_1 contains only two elements, a contradiction to Lemma 50. Thus, $z = 0$ and $\beta \geq 2d^2$, so $100\ldots01_k00\ldots01_{2d^2}11\ldots1_\beta$ is a block, call H, of K. Then there is $x \in H_0$ and $H' \subseteq H_1$ such that $x + H' \subseteq H' + H'$ since K is forbidden. Also, $0 \in H'$ and $k < x < 2d^2$ by Lemma 33. Thus, $x + 0 = k + k = 2k \geq 2d^2$, a contradiction. Hence, z, $\beta \in (d^2 + L_1)$ so L contains K. ■

Theorem 54 *For every finite set S of forbidden patterns, there is a monoid that is not weakly integrally closed and whose binary string contains no stretch of an element of S.*

Proof. Let S be a finite set of forbidden patterns and let S^T be the set of the trims of elements of S. By Theorem 52, there is $j > 0$ such that $11(01)^j 1$ contains no stretch of an element in S^T. Let M be the monoid from Lemma 53, where $11(01)^j 1$ is the block L. Then M contains the forbidden pattern $11(01)^j 1$, so M is not WIC. By Lemma 53, M contains no stretch of an element of S^T since $11(01)^j 1$ doesn't. Thus, M contains no stretch of an element of S. ■

Thus, there is no finite set S of forbidden patterns such that whenever a monoid contains no stretch of an element in S, then its monoid algebra is necessarily WIC.

6 Weakly integrally closed domains

Let K be a field. Recall that the ring $K[t^i, t^m, t^{m+1}, \ldots]$ is weakly integrally closed by Theorem 2. The proof to show that the power series ring is also weakly integrally closed is identical to Brewer and Richman's proof of Theorem 2. We restate Theorem 2 for formal power series rings.

Theorem 55 *Let $i, m \geq 0$. If $M_{i,m} = \{0, i, m, m+1, \ldots\}$ is a monoid, then $K[[M_{i,m}]]$ is weakly integrally closed.*

Let S be a ring, t an indeterminate and let $R = S[t]$. Let $a = a_0 + a_1 t^1 + \ldots + a_u t^u$ be an element of R. If a is nonzero, then define the value $v_t(a)$ to be the least j such that $a_j \neq 0$. If $a = 0$, then define $v_t(a) = 0$. Let J be any nonzero ideal of R and define the value of J as $v_t(J) = \min\{v_t(e) : 0 \neq e \in J\}$. Let $n = v_t(J)$. If $xJ \subseteq J^2$, then $v_t(xe) \geq 2n$, so $v_t(x) \geq n$. Note that if $b = b_0 + b_1 t^1 + \ldots + b_v t^v$ is an element of R, then $(ab)_i = \sum_{0 \leq k \leq i} a_k b_{i-k}$.

Recall, in Question 2, that Brewer and Richman ask if the ring $K[t^4, t^5, t^{11}]$ is weakly integrally closed. Consider the monoid $M = \{0, i, i+1, m, m+1, \ldots\}$, where $0 \leq m \leq 2i$. If $i = 4$ and $m = 8$, then $K[M]$ is the ring $K[t^4, t^5, t^{11}]$. Suppose that $m = i + 3$. Then the characteristic binary string of M, $10\ldots0\mathbf{1}_i\mathbf{1}_{i+1}\mathbf{0}\mathbf{1}_m\mathbf{1}11\ldots$, contains the pattern 11011. Thus, M is not weakly integrally closed by Theorem 17, so $K[M]$ and $K[[M]]$ are not weakly integrally closed by Theorem 16. Therefore, we will require that $2 < i + 3 < m \leq 2i$ and prove that the formal power series ring $K[[t^i, t^{i+1}, t^m, t^{m+1}, \ldots]]$ is weakly integrally closed.

Lemma 56 *Let $i \geq 0$. If $M = \{0, i, i+1, i+4, i+5, i+6, \ldots\}$, then $K[[M]]$ is weakly integrally closed.*

Proof. Let $R = k[[M]]$ and $R' = k[[t^i, t^{i+1}, t^{i+2}, \ldots]]$. Then R' is weakly integrally closed by Theorem 55. we will show that R is weakly integrally closed in R'.

Let x be an element of R' and J a nonzero finitely generated ideal of R such that $xJ \subseteq J^2$. Let $n = v_t(J)$. Then $v_t(x) \geq n$. If $n \geq i+4$, then $x_{i+2} = x_{i+3} = 0$ so $x \in R$ and we are done. Thus, we might as well assume that $n < i+4$, so $n = 0$, $n = i$ or $n = i+1$.

Before continuing, we insert a remark. Let $a \in R$ and $b \in R'$ such that $v_t(a), v_t(b) \geq n$. Then

$$\begin{aligned}(ab)_{n+i+2} &= \sum\nolimits_{0 \leq k \leq i+2-n} a_{n+k} b_{i+2-k} \text{ and} \\ (ab)_{n+i+3} &= \sum\nolimits_{0 \leq k \leq i+3-n} a_{n+k} b_{i+3-k}.\end{aligned}$$

Thus, if $n = 0$ or $n = i+1$, then

$$\begin{aligned}(ab)_{n+i+2} &= a_n b_{i+2} \text{ and} \\ (ab)_{n+i+3} &= a_n b_{i+3},\end{aligned}$$

and if $n = i$, then

$$(ab)_{n+i+3} = a_i b_{i+3} + a_{i+1} b_{i+2}$$

since $a_k = 0$ for all $0 < k < i$ and all $i+1 < k < i+4$, and $b_k = 0$ for all $0 < k < i$.

Suppose that $e, e' \in J$. If $n = 0$ or $n = i+1$, then $(ee')_{n+i+2} = e_n e'_{i+2} = 0$ and $(ee')_{n+i+3} = e_n e'_{i+3} = 0$. If $n = i$, then $(ee')_{n+i+3} = e_i e'_{i+3} + e_{i+1} e'_{i+2} = 0$. So for any element f of J^2, if $n = 0, i+1$, then $f_{n+i+2} = f_{n+i+3} = 0$ and if $n = i$, then

$f_{n+i+3} = 0$. Therefore, the following equations are true for any element e of J:

$$\begin{aligned} 0 &= e_n x_{i+2} = e_n x_{i+3} && ; \text{if } n = 0, i+1 \\ 0 &= e_n x_{i+3} + e_{n+1} x_{i+2} && ; \text{if } n = i \end{aligned}$$

Let c be an element of J such that $v_t(c) = n$. Then $c_n \neq 0$. If $n = 0$ or $n = i + 1$, then $0 = c_n x_{i+2} = c_n x_{i+3}$ by the above remarks. Thus, $x_{i+2} = x_{i+3} = 0$, so $x \in R$.

Suppose that $n = i$. Then $0 = c_i x_{i+3} + c_{i+1} x_{i+2}$ by the above remarks. If $x_{i+2} = 0$, then $0 = c_i x_{i+3}$. Thus, $x_{i+3} = 0$ since $c_i \neq 0$, so $x \in R$. Therefore, assume that $x_{i+2} \neq 0$. Let $x' = x - (x_n/c_n) \cdot c$. Then $v_t(x') \geq i + 1$, so $v_t(x'c) \geq 2i + 1$.

We need to pause for another remark. If $e \in J$, then $e_{i+1} = e_i c_{i+1}/c_i$ since $x_{i+2} \neq 0$ and $c_i \neq 0$. Thus, $e = \alpha c + b$ for some $\alpha \in K$ and $b \in R$ such that $v_t(b) \geq i + 4$. Therefore, if f is an element of J^2, then

$$f = \gamma c^2 + d$$

for some $\gamma \in k$ and $d \in R'$ such that $v_t(d) \geq 2i + 4$.

Since $xc \in J^2$, then $x'c \in J^2$. Thus,by the above remark, $x'c = \gamma c^2 + d$ for some $\gamma \in k$ and $d \in R$ such that $v_t(d) \geq 2i+4$. If $\gamma \neq 0$, then $v_t(x'c) = 2i$, a contradiction. Thus, $\gamma = 0$, so $v_t(d) \geq 2i+4$. Therefore, $v_t(x'c) \geq 2i+4$, implying that $v_t(x') \geq i+4$. Then $x' \in R$, so $x \in R$. Thus, R is weakly integrally closed in R'. Since R' is weakly integrally closed, then so is R by Theorem 5. ■

Theorem 57 *Let $2 < i + 3 < m \leq 2i$. If $M_{i,m} = \{0, i, i+1, m, m+1, \ldots\}$, then $R_{i,m} = K[[M_{i,m}]]$ is weakly integrally closed.*

Proof. The theorem is true for $m = i + 4$ by Lemma 56. Fix $i \geq 0$ and let $M_{i,m} = M_m$ and $R_{i,m} = R_m$. We will show that R_m is weakly integrally closed in R_{m-1}, for $m > i + 4$.

Let x be an element of R_{m-1} and J a nonzero finitely generated ideal of R_m such that $xJ \subseteq J^2$. Let $n = v_t(J)$. Then $v_t(x) \geq n$. We might as well assume that $n < m$ or else $v_t(x) \geq m$ so $x \in R_m$. Thus, $n = 0$, $n = i$ or $n = i + 1$.

If $a \in R_m$ and $b \in R_{m-1}$ such that $v_t(a), v_t(b) \geq n$, then

$$(ab)_{n+m-1} = a_n b_{m-1} + \sum\nolimits_{1 \leq k \leq m-1-n} a_{n+k} b_{m-1-k}.$$

Suppose that there is $1 \leq k \leq m - 1 - n$ such that $a_{n+k} \neq 0$. We will show that $b_{m-1-k} = 0$. Since $a_{n+k} \neq 0$, then $n + k < m$, so $i \leq n + k \leq i + 1$. Thus, $n \leq i$, so $n = 0$ or $n = i$. Moreover, $-i-1 \leq -n-k \leq -i$ implies that $m-i-2 \leq m-n-k-1 \leq m - i - 1$. Thus, $n + 2 < m - k - 1 \leq i + n - 1$, since $i + 4 < m \leq 2i$. If $n = 0$, then $0 < m - k - 1 < i$, so $b_{m-k-1} = 0$. Suppose that $n = i$. Then $i + 2 < m - k - 1$. Also, $m - k - 1 < m - 1$ since $k \geq 1$. Therefore, $i + 2 < m - k - 1 < m - 1$, so $b_{m-k-1} = 0$. Thus,

$$(ab)_{n+m-1} = a_n b_{m-1}.$$

If $e, e' \in J$, then $(ee')_{n+m-1} = e_n e'_{m-1} = 0$. Thus, $f_{n+m-1} = 0$ for every element f of J^2.

Let e be an element of J such that $v(e) = n$. Then

$$0 = (xe)_{n+m-1} = e_n x_{m-1}$$

since $xe \in J^2$. Thus, $x_{m-1} = 0$ since $e_n \neq 0$. Therefore, $x \in R_m$ so R_m is weakly integrally closed in R_{m-1}. If R_{m-1} is weakly integrally closed, then R_m is weakly integrally closed by Theorem 5. Hence, R_m is weakly integrally closed by induction. ■

Therefore the answer to Question 2 is yes, so Brewer and Richman's ring $K[t^4, t^5, t^{11}]$ is weakly integrally closed..

Theorem 58 *Let $i, m, p \geq 0$ and let*

$$M_{i,p,m} = \{0\} \cup \{i + \alpha p : \alpha \geq 0\} \cup \{m, m+1, \ldots\}$$

be a numerical monoid. Then $K[[M_{i,p,m}]]$ is weakly integrally closed.

Proof. If $m < 2$, then $M_{i,p,m}$ is the set of all natural numbers, so $K[[M_{i,p,m}]] = K[[t]]$ and we are done. If $m \leq i$, then $R_m = K[[t^m, t^{m+1}, \ldots]]$ is weakly integrally closed by Theorem 55. Thus, we might as well assume that $m > i$. Fix $i, p \geq 0$. Let $M_m = M_{l,p,m}$ and $R_m = K[[M_{i,p,m}]]$ for $m > 1$. We will show that R_m is weakly integrally closed in R_{m-1}. Note that there is $j \geq 0$ such that $i + jp < m \leq i + (j+1)p$. Fix j. If $m = i + jp + 1$, then $R_m = R_{m-1}$ and we are done. Thus, we might as well assume $i + jp + 1 < m < i + (j+1)p + 1$.

Let x be an element of R_{m-1} and J a nonzero finitely generated ideal of R_m such that $xJ \subseteq J^2$. We will show that $x_{l+m-1} = 0$. Let $n = v_t(J)$. Then $v_t(x) \geq n$. We may assume that $n < m$ or else we are done. Thus, $n = 0$ or $n = i + \alpha p$ for some $0 \leq \alpha \leq j$.

We will insert a remark before continuing. Let $a \in R_m$ and $b \in R_{m-1}$ such that $v_t(a), v_t(b) \geq n$. Then

$$(ab)_{n+m-1} = a_n b_{m-1} + \sum_{1 \leq k \leq m-1-n} a_{n+k} b_{m-1-k}.$$

Let $1 \leq k \leq m-1-n$. Suppose that $a_{n+k} \neq 0$. We will show that $b_{m-1-k} = 0$. Since $m-1-k < m-1$, then it suffices to show that

$$i + \delta p < m - 1 - k < i + (\delta+1)p \text{ for some } \delta \geq 0 \qquad (3)$$

Since $i+jp+1 < m < i+(j+1)\,p+1$, then $i+jp-k < m-1-k < i+(j+1)\,p-k$. Since $0 < n+k < m$, then $n+k = i+\beta p$ for some $0 \leq \beta \leq j$, so $k = i+\beta p - n$. Thus,

$$(j-\beta)\,p+n < m-1-k < (j-\beta+1)\,p+n$$

Therefore, if $n = i+\alpha p$ for some $0 \leq \alpha \leq j$, then (3) is true and we are done. Therefore, we might as well assume that $n = 0$. Suppose that p does not divide i. Then $2i \geq m$ since M_m is a monoid, so $i \geq m$. Therefore, $R_m = K[[t^m, t^{m+1}, \ldots]]$, so R_m is weakly integrally closed by Theorem 55. Thus, we assume that p divides i. Since $n = 0$, then $(j-\beta)\,p < m-1-k < (j-\beta+1)\,p$. Therefore, (3) is satisfied, so $b_{l+m-1-k} = 0$. Thus, if $a_{n+k} \neq 0$, then $b_{l+m-1-k} = 0$, so $a_{n+k}b_{l+m-1-k} = 0$ for all $1 \leq k \leq m-1-n$. Consequently, $(ab)_{n+m-1} = a_n b_{m-1}$.

Let $e, e' \in J$. Then $(ee')_{n+m-1} = e_n e'_{m-1} = 0$ by the above observation. Thus, if f is any element of J^2, then $f_{n+m-1} = 0$. Assume $e \in J$ has value n. Recall that $v_t(x) \geq n$. Then $0 = (xe)_{n+m-1} = e_n x_{m-1}$, so $x_{m-1} = 0$ since $e_n \neq 0$. Therefore, $x \in R_m$ implying that R_m is weakly integrally closed in R_{m-1}. Hence, R_m is weakly integrally closed by induction. ■

For example, the ring $K[[t^{10}, t^{13}, t^{16}, t^{19}, t^{20}, \ldots]]$ is weakly integrally closed. The next theorem requires the following lemma.

Lemma 59 *Let $A \subseteq B$ be integral domains and let Q be the quotient field of A. If $Q \cap B = A$ and B is weakly integrally closed, then so is A.*

Proof. Let $Q(A)$ and $Q(B)$ be the quotient fields of A and B. We will show that $Q(A)$ is weakly integrally closed in $Q(B)$. Let x be an element of $Q(A)$ and J a nonzero finitely generated ideal of $Q(B)$ such that $xJ \subseteq J^2$. Since $Q(B)$ is a field, then $J = Q(B)$, so $1 \in J$. Thus, $x = x \cdot 1 \in J^2 = Q(A)$, so $Q(A)$ is weakly integrally closed in $Q(B)$. Therefore, if $Q(A) \cap B = A$ and B is weakly integrally closed in

$Q(B)$, then so is A by Theorem 5. ■

The above lemma is also true for B, an integrally closed domain.

Definition 60 *A monoid M is said to be* ***torsion-free*** *if whenever $na = nb$ for some natural number $n > 0$ and $a, b \in M$, then $a = b$.*

Theorem 61 *Let D be an integral domain and M a torsion-free and cancellative monoid. If $D[M]$ is weakly integrally closed, then so is D.*

Proof. The ring $D[M]$ is an integral domain since M is torsion-free and cancellative [5]. Let Q be the quotient field of D. We will show that $Q \cap D[M] = D$. It is clear that $D \subseteq Q \cap D[M]$. Let y be an element of $Q \cap D[M]$. Then $y = u_0/v_0 = d_0 + d_1 t^{m_1} + \ldots + d_n t^{m_n}$, where $u_o, v_0, d_k \in D$, $v_0 \neq 0$, and $m_k \in M \diagdown \{0\}$. Thus, $v_0 (d_1 t^{m_1} + \ldots + d_n t^{m_n}) = 0$, so $d_k = 0$ for all $0 < k \leq n$ since v_0 is nonzero. Therefore, $y = d_0 \in D$, so $Q \cap D[M] = D$. Hence, if $D[M]$ is weakly integrally closed, then so is D. ■

It is not difficult to see that Theorem 61 is also true for the power series ring $D[[M]]$.

7 Open Problems

Let K be a field and $0 \leq 3a < m-i \leq i$. Let $M_{a,i,m} = \{0, i, i+1, \ldots, i+a, m, m+1, \ldots\}$ and $R_{a,i,m} = K[[M_{a,i,m}]]$. Then $R_{0,i,m}$ and $R_{1,i,m}$ are weakly integrally closed by Theorems 55 and 57. Thus, the monoids $M_{0,i,m}$ and $M_{1,i,m}$ are weakly integrally closed by Theorem 16. We will show that the monoid $M_{a,i,m}$ is weakly integrally closed for every $a \geq 0$. Observe that $M_{a,i,m}$ has the following characteristic binary string:

$$M_{a,i,m} : 100\ldots01_i1\ldots1_{i+a}0\ldots01_m1111\ldots.$$

Theorem 62 *Let $0 \leq 3a < m-i \leq i$. Then $M_{a,i,m} = \{0, i, i+1, \ldots, i+a, m, m+1, \ldots\}$ is a weakly integrally closed monoid.*

Proof. Let $M = M_{a,i,m}$. Since $m - i \leq i$, then $m \leq 2i$, so M is a monoid. Let $H_0 = \{x \in \mathbb{N} : i + a < x < m\}$ and let $H_1 = \{l \in M : i \leq l \leq 2m - i - 1\}$. Then (H_0, H_1) is the test block H of M. It suffices to show that H has no bad zeros by Theorem 47. Let x be an element of H_0 and $F \subseteq H_1$. Then we will show that $x + F \not\subseteq F + F$.

Let $n = \min(F)$. Assume that $x + F \subseteq F + F$. Then $x + n = f_1 + f_2$ for some $f_1, f_2 \in F$. If say $f_2 \geq m$, then $x + n = f_1 + f_2 \geq n + m > n + x$ since $x < m$. Thus, $f_1, f_2 < m$, so $f_1, f_2 \leq i + a$, so $x + n = f_1 + f_2 \leq 2(i + a)$. Therefore, $x \leq 2(i+a) - n \leq 2(i+a) - i = i + 2a$, so $x \leq i + 2a$.

Let $J = \{i, i+1, \ldots, i+a\}$ and let $L = J \cap F$. Since $n \in J$, then J is not empty. Since $x > i + a$, then $x + J \not\subseteq J + J$ by Lemma 33. Let j be an element of J such that $x + j \notin J + J$. But $x + j \in F + F$ since $x + F \subseteq F + F$ and $j \in F$.

Thus, $x + j = f_3 + f_4$ for some $f_3, f_4 \in F$. We may assume that $f_4 \notin J$. Therefore, $f_4 \geq m$, so $f_3 + f_4 \geq i + m$. But $x + j \leq (i + 2a) + (i + a) = 2i + 3a < i + m$. Thus, $i + m < i + m$, a contradiction. Hence, $x + F \not\subseteq F + F$, so M is weakly integrally closed. ■

Question 1 Let $0 \leq 3a < m - i \leq i$. Is the ring $K[[M_{a,i,m}]]$ weakly integrally closed?

Question 2 If a monoid M is weakly integrally closed, then must $K[M]$ be weakly integrally closed?

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