Computing Topological Dynamics from Time Series

by

Mark Wess

A Dissertation Submitted to the Faculty of

The Charles E. Schmidt College of Science

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Florida Atlantic University

Boca Raton, Florida

November 2008

Computing Topological Dynamics from Time Series

by

Mark Wess

This dissertation was prepared under the direction of the candidate's dissertation advisor, Dr. William D. Kalies, Department of Mathematical Sciences, and has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

SUPERVISORY COMMITTEE:

William D. Kalies, Ph.D.
Dissertation Advisor

Yuandan Lin, Ph.D.

Tomas Schonbek, Ph.D.

Yuan Wang, Ph.D.

Spyros Magliveras Ph.D.
Chair, Department of Mathematics

Gary W. Perry, Ph.D.
Dean, The Charles E. Schmidt College of Science

Barry T. Rosson, Ph.D.
Dean, Graduate College

11-7-08
Date

ACKNOWLEDGEMENTS

The author wishes to express his sincere thanks to the guidance, time, and energy given by the dissertation advisor Dr. William Kalies for which this manuscript could not be possible without him. The author is grateful for the input and corrections made by the dissertation committee. Also the author is grateful to the Faculty of the Department of Mathematical Sciences for the wonderful instuction received as a graduate student.

ABSTRACT

Author : Mark Wess

Title : Computing Topological Dynamics from Time Series

Institution : Florida Atlantic University

Dissertation Advisor : Dr. William Kalies

Degree : Doctor of Philosophy

Year : 2008

The topological entropy of a continuous map quantifies the amount of chaos observed in the map. In this dissertation we present computational methods which enable us to compute topological entropy for given time series data generated from a continuous map with a transitive attractor. A triangulation is constructed in order to approximate the attractor and to construct a multivalued map that approximates the dynamics of the linear interpolant on the triangulation. The methods utilize simplicial homology and in particular the Lefschetz Fixed Point Theorem to establish the existence of periodic orbits for the linear interpolant. A semiconjugacy is formed with a subshift of finite type for which the entropy can be calculated and provides a lower bound for the entropy of the linear interpolant. The dissertation concludes with a discussion of possible applications of this analysis to experimental time series.

DEDICATIONS

This manuscript is dedicated to my family.

CONTENTS

LIST OF FIGURES

1 Introduction

In modeling a deterministic physical system, one often encounters difficulty in establishing the validity of the model. One way this difficulty can arise is from the inability of the model to completely account for all physical behavior of the system. For example Guckenheimer and Holmes [14] describe an experiment where a cantilever beam is suspended between two permanent magnets and an external force is applied to the apparatus suspending the beam. This external force causes the beam to oscillate between the two magnets. An analysis of the forces acting on the beam gives a model for the motion of the beam given by the Duffing equation $x'' + \delta x - x + x^3 = \gamma \cos(\omega t)$. In Figure 1.1 data from experimental measures are compared to numerical data from the simulation of the Duffing equation. The experimental data seems to have spikier oscillations than the data from the Duffing equation. This visible difference in the data could be due to the fact that the mathematical model does not have the correct number of degrees of freedom.

A second complication can arise when studying numerical data from a mathematical model when the physical system is chaotic. The experimental data often has some distinguishable noise. Therefore even if the numerical simulation is started with the same initial conditions as the experimental data, the numerical orbit and the experimental orbit will drift far apart in a short time due to noise in the experiment and/or numerical errors. In this case, using numerical data from the mathematical model leads to gross inaccuracies when approximating individual orbits of the physical system. However, this does not mean that characteristics of the dynamics cannot be compared. Indeed in this dissertation we present a computational framework in which it may be possible to measure and compare chaotic dynamics in a robust manner using topological methods.

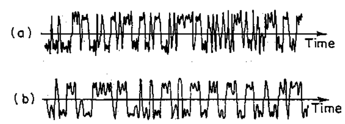


Figure 1.1: Top: Experimental data from motion of beam. Bottom: Numerical data from solution of Duffing equation.

The previous discussion leads us to the idea that perhaps closed-form mathematical models are not the only way to study the dynamics of a deterministic physical system, but rather experimental measures of the system could be beneficial . A theorem that lays the theoretical foundation for an alternative analysis by reconstructing the dynamics from measurement data is the Takens Embedding Theorem.

Theorem 1.1 (Takens [31])**.** *Let M be a compact manifold of dimension m. For a diffeomorphism $\phi \in C^2(M)$ and a real valued function $y \in C^2(M, \mathbb{R})$, it is a generic property that the map $\Phi_T : M \to \mathbb{R}^{2m+1}$ with $T > 0$ defined by*

$$\Phi(x)_T := (y(x), y(\phi^T(x)), \dots, y(\phi^{2mT}(x)))$$

is an embedding.

The function $y : M \to \mathbb{R}$ is called a **measurement function**, and it should be noted that the theorem does not imply that $2m+1$ is the lowest dimension for such an embedding. This theorem gives rise to what is called a time-delay reconstruction of the dynamics of the map ϕ. In the case that real-valued experimental measures $\{y_1, \ldots, y_n\}$ from an unknown physical system ϕ are known, then given an embedding dimension k and delay time T, we can construct a series of data points $\{d_1, \ldots, d_{n-Tk+1}\}$ in $\mathbb{R}^k$ where $d_i = (y_i, \ldots, y_{i+T(k-1)})$. If we choose the correct embedding dimension, then the data points generically lie on an embedded manifold, and we can treat the sequence d_i as an orbit segment, then we may be able to analyze the dynamics of the physical system. The sequence of data points d_i is an example of what we call a **time series**, which is a finite sequence of data points in $\mathbb{R}^n$. A time series can be constructed by a time-delay reconstruction of experimental data, or it can be constructed simply by taking a finite segment of a forward orbit of some continuous map, or finally by projecting a forward orbit segment of some continuous map into lower dimensions. Our goal in this thesis is to characterize dynamics and in particular chaos from a given sequence of time series data.

One of the main goals of dynamical systems theory is to locate certain structures of interest and to describe the behavior of the dynamics locally around and within these structures, and if possible, to determine the global behavior of the dynamics. These structures include, but are not limited to, equilibrium points, periodic points, stable and unstable manifolds, chaotic sets, and other invariant sets. Besides the many analytic tools for describing dynamics, there exist certain techniques from algebraic topology that can be utilized. Indeed, Poincaré founded this discipline of mathematics in order to study dynamical systems.

One can use homology theory combined with fixed-point theorems to provide some measurement of how chaotic a system can be. Methods using cubical homology along with the Lefschetz Fixed Point Theorem to compute the presence of periodic orbits and chaos for a continuous map were first implemented by Szymczak [29, 28]. For an introductory description of these methods see [17]. In Day [11], these methods are extended to provide algorithms to prove the existence of a semiconjugacy between the Hénon map and a shift map on a symbol sequence. A rigorous lower bound on the entropy of the Hénon map is computed from the entropy of the shift map. These cubical homology techniques are shown to be extendable to time-series data in principle by Szymczak [29]. A complete description of an application of the techniques to a specific set of time-series data derived from a real experiment is given in [20, 21]. These cubical homologoy techniques have the advantage that they are easily implemented on a computer, but they have some disadvantages that we discuss later. These disadvantages and the desire to try to approximate a covering of manifolds and attractors led us to use simplicial homology. Simplicial homology has the disadvantage that it is not as computationally friendly as cubical homology. Therefore we introduce new theory, algorithms, and implementation for simplicial homology maps. We now give some basic theorems before returning to the discussion of time series.

1.1 Dynamical Systems Preliminaries

We begin by stating some basic definitions in dynamical systems theory. Most of these definitions are standard and can be found in any textbook, see Robinson [25].

Let X be a metric space and denote time by $\mathbb{T}$ where $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T}^+$ is either $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ or $\mathbb{R}^+ = [0, \infty)$.

Definition 1.2. *A **dynamical system** is a continuous map $\varphi : \mathbb{T}^+ \times X \to X$ such that*

(1) $\varphi(0, x) = x$ *and*

(2) $\varphi(s + t, x) = \varphi(s, \varphi(t, x))$ *for all* $x \in X$ *and* $s, t \in \mathbb{T}^+$.

When $\mathbb{T}$ is $\mathbb{R}$ we call the dynamical system a ***flow*** and when $\mathbb{T}$ is $\mathbb{Z}$ we say the dynamical system is ***discrete***. In this thesis we only consider discrete dynamical systems which are generated by iterating the map $f(x) = \varphi(1, x)$. Thus $\varphi(n, x)$ will be denoted by $f^n(x)$ where f^n is the composition of the map f n-times.

Definition 1.3. *The **forward orbit** of a point $x \in X$ is the set $\{f^n(x) : n \geq 0\}$.*

We can denote the forward orbit of a point x as a sequence x_n where $x_0 = x$ and $x_n = f^n(x)$ with $n \in \mathbb{Z}^+$. Similarly we can define a backward orbit.

Definition 1.4. *A **backward orbit** of a point $x \in X$ is a sequence x_{-n} with $n \in \mathbb{Z}^+$ such that $f(x_{-n-1}) = x_{-n}$ and $f(x_{-1}) = x_0$. An **orbit** through a point $x \in X$ is a sequence x_n with $n \in \mathbb{Z}$ such that $f(x_{n-1}) = x_n$ for all $n \in \mathbb{Z}$.*

Note that if f is invertible, then the backward orbit is uniquely determined by composing the inverse i.e. $x_{-n} = f^{-n}(x_0)$.

Definition 1.5. *A point x is called a **periodic point of least period** n if $f^n(x) = x$ and $f^j(x) \neq x$ for all $0 \leq j < n$. If x has period 1, then we call x a **fixed point** or an **equilibrium point**. If x is a periodic point of least period n, then we call the forward orbit of x a **periodic orbit**.*

In dynamical systems one often wishes to decompose the space X into fundamental pieces so that the dynamics can be studied on each of the pieces separately, and then the global description of dynamics can be inferred from the interactions of those pieces. This decomposition is done by the invariance of subsets of X which we define now.

Definition 1.6. *A subset $S \subset X$ is said to be **invariant** if $f(S) = S$. A subset $S \subset X$ is said to be **forward invariant** if $f(S) \subset S$, and finally a subset $S \subset X$ is said to be **backward invariant** if $f^{-1}(S) \subset S$.*

The next definition describes some important invariant sets in a dynamical system that characterize asymptotic behavior in time.

Definition 1.7. *A point y is an ω-**limit point of** x if there exists a sequence of n_k going to infinity such that*

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

where d is the distance function of X. The set of all ω-limit points of x is called the ω-limit set of x which we denote by $\omega(x)$.

One can easily show that $\omega(x)$ is a closed, forward invariant set. As stated earlier, we would like to decompose the space into invariant sets so now we give the notion of the most fundamental or indecomposable invariant sets.

Definition 1.8. *A set S is a **minimal set** for f provided that it is nonempty, closed, and invariant and has no closed, invariant, proper subset that is nonempty.*

Periodic orbits and fixed points are obvious examples of minimal sets, and later we will see another example when we define chaotic sets. We have a simple proposition that tells us when certain sets are minimal sets.

Proposition 1.9 ([25])**.** *Let X be a metric space, $f : X \to X$ a continuous map, and $S \subset X$ a nonempty, compact subset. Then S is a minimal set if and only if $\omega(x) = S$ for all $x \in S$.*

Now we give a definition of an important topological property of a map.

Definition 1.10. *A map $f : X \to X$ is transitive on an invariant set S if the forward orbit of some point $p \in S$ is dense in S.*

Next we define some important topological properties of sets on which a map is defined.

Definition 1.11. *Let $f : M \to M$ be a map. A nonempty compact region $N \subset M$ is called a* ***trapping region*** *if $f(N) \subset \text{int}(N)$. A set Λ is an* ***attractor*** *if there is a trapping region N such that $\Lambda = \cap_{k \geq 0} f^k(N)$. A set Λ is a* ***transitive attractor*** *if it is an attractor and $f|\Lambda$ is transitive.*

If there is a point p in an invariant set S such that the forward orbit of p is dense, then it is easy to see that $\omega(p) = S$. This leads to a simple proposition.

Proposition 1.12. *If a map $f : X \to X$ is transitive on an invariant set S, then there exist no trapping regions that are strictly contained in S, and either S is a minimal set or the only minimal subsets of S are nowhere dense sets.*

Proof. Let $p \in S$ such that the forward orbit of p is dense in Y. Then $\omega(p) = S$. Let T be a trapping region such that $T \subset S$ and $T \neq S$, then there exists $n \geq 0$ such that $f^n(p) \in T$. This implies that $\omega(p) \subset T$ which is a contradiction.

Now assume $\emptyset \neq B \subset S$ is closed and invariant. If B has nonempty interior, then there exists some $n \geq 0$ such that $f^n(p) \in B$. This implies that $\omega(p) = \omega(f^n(p)) \subset B$ since B is invariant. However, $\omega(p) = S$, therefore $B = S$ so that S is a minimal set. If B has empty interior, then B is nowhere dense. ∎

Given two maps $f : X \to X$ and $g : Y \to Y$, we would like to know if we can compare the dynamics of the two maps and conclude some sort of equivalence. This enables us to work with a map on a space where we have an advantage and to infer behavior of the dynamics in the other space where we might not. This leads to the definition of conjugacy.

Definition 1.13. *Let $f : X \to X$ and $g : Y \to Y$ be two maps. A map $h : X \to Y$ is called a* ***semiconjugacy*** *from f to g provided that h is continuous, onto, and*

$h \circ f = g \circ h$. The map h is called a ***conjugacy*** *if it is a semiconjugacy and h is a homeomorphism.*

One of the consequences of having a conjugacy is that fixed points and periodic points for one map corresponds to fixed points and periodic points in the other, or more generally the dynamics in one map corresponds to the dynamics in the other. In essence, conjugacy gives us topological equivalence of the two systems.

1.2 Embedology

Theorem 1.1 is stated in terms of manifolds, but a similar theorem holds for attractors of continuous maps [27, 3, 24].

Theorem 1.14 (Sauer, Yorke, Casdagli [27])**.** *Let A be an attractor for $\phi \in C^2(\mathbb{R}^n)$ with box-counting dimension m. If $k > 2m$, then for a real valued function $y \in C^2(\mathbb{R}^n, \mathbb{R})$, it is a generic property that the map $\Phi_T : A \to \mathbb{R}^k$ with $T > 0$ defined by*

$$\Phi(x) := (y(x), y(\phi^T(x)), \ldots, y(\phi^{Tk}(x)))$$

is an embedding.

It is important to note that embeddings can exist for $k \leq 2m$, but the generic property does not necessarily hold. There has been much study of time delay reconstructions from time series data since Theorems 1.1 and 1.14 were introduced, see [18, 24, 3] and the references therein. Typically, one utilizes these theorems by picking a large enough embedding dimension k and constructing a time-delay reconstruction of the phase space. Then this dynamics in dimension k, is analyzed often statistically. In typical applications, chaotic attractors have small box-counting dimension, but the problem of determining an embedding dimension which is large enough to contain the correct dynamics but small enough to compute with is a central one. Our topological approach also requires knowledge of a good embedding dimension in order to be applied to experimental or measured data.

For the purposes of this thesis, we assume that A is embedded in some low-dimensional manifold M in $\mathbb{R}^n$ and that there exists a suitable projection into this lower dimension or collection of projections on charts in an atlas of M that cover A. Indeed, for the computations performed in Chapter 7, we consider only maps $f : \mathbb{R}^2 \to \mathbb{R}^2$. However, in general, these projections and the subsequent phase space reconstruction in lower dimensions introduced in this thesis would provide a method for reducing the dimension in the computational topological analysis of a dynamical system, which would allow for the computation of the dynamics of a high-dimensional system in the case where projections into lower dimensions are known. For example, in previous work by Day, Junge, and Mischaikow [12] rigorous computations of symbolic dynamics have been performed on a system in which the ambient dimension is 6 but the dynamics could be embedded in 3 dimensions. Using cubical structures, topological computations in 6 dimensions are still feasible. However, if the ambient dimension is much higher than 6, the cost for the topological computations becomes large, even if the effective dimension is low. The algorithms in this thesis are an attempt to replace the high-dimensional phase space with a lower-dimensional reconstruction, but at the cost of replacing cubical structures with simplicial ones. The computational cost of using simplicies is higher, but this cost should be offset by the savings in dimension reduction for high-dimensional problems, see also Chapter 8.

2 Chaos and Symbolic Dynamics

We now give a precise definition of chaos used in this thesis. We also describe how to measure chaos by making entropy calculations and utilizing symbolic dynamics.

2.1 Definition of chaos

Usually when one looks for a definition of chaos there is no single all-encompassing definition that everyone agrees defines chaotic behavior. We state the fundamental properties of what we consider as chaotic behavior on a set, and then give the assumptions needed to characterize these properties. Unless stated otherwise, we assume that f is a continuous map on a complete metric space X.

The first property that we would like a chaotic set to have is to be a minimal set. It was stated earlier in Proposition 2.2 that if f is transitive on an invariant set Y, then Y is a minimal set or the only minimal subsets of Y are nowhere dense. So having transitivity on an invariant set implies that the set is indecomposable, but transitivity actually gives us even more if we invoke the Birkhoff Transitivity Theorem, which we state now.

Theorem 2.1 (Birkhoff Transitivity Theorem [25]). *Let X be seperable and assume that for every open set U of X, we have $\bigcup_{n\leq 0} f^n(U)$ is dense in X. Then there is a residual set A such that for every $p \in A$ the forward orbit of p is dense in X.*

Now we get the following proposition.

Proposition 2.2. *A map f is transitive on Y if and only if given any two non-empty open sets U and V in Y, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.*

Proof. First suppose that f is transitive, and let p be the point with a dense forward orbit. Let U and V be any two open subsets of Y. There exists a $q \in U$ such that $q = f^k(p)$ for some $k \geq 0$. However $f^m(p) \in V$ for an infinite number of positive integers m so there exists a $m > k$ such that $f^m(p) \in V$. Now let $n = m - k$ and we see that $f^n(q) \in V$.

Now we prove the converse. Let U be open in Y. We must show that $\bigcup_{n\leq 0} f^n(U)$ is dense in Y. Let $y \in Y$. Let V be any open set in Y such that $y \in V$. By the assumption there exists a $n > 0$ such that $f^n(V) \cap U \neq \emptyset$. This implies that $f^{-n}(U) \cap V \neq \emptyset$. Since V was arbitrary this implies that $\bigcup_{n\leq 0} f^n(U)$ is dense. Therefore by the Birkhoff Transitivity Theorem we have that f is transitive. ∎

The property that given any two open sets U and V in Y, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$ is a property that describes a kind of mixing of the points in the forward iteration of the map. So, not only does f being transitive on Y give us indecomposability, but we see that being transitive is equivalent to points being shuffled around all of Y in future time.

Although f being transitive on Y shuffles points around in Y, this is still not enough to be chaotic, since the shuffling can be done in an organized way. For example, a periodic orbit for the graph of f is a compact invariant set where f is transitive but we do not want to consider this set to be chaotic since neighborhoods of points map in a predictable fashion inside this set. This leads to our next property of chaos.

Definition 2.3. *A map f on a metric space X is said to have **sensitive dependence on initial conditions** provided there is an $r > 0$ such that for each point $x \in X$*

and for each $\epsilon > 0$ there is a point $y \in X$ with $d(x, y) < \epsilon$ and a $k \geq 0$ such that $d(f^k(x), f^k(y) \geq r$. We call r a ***sensitivity constant****.*

The property of having sensitive dependence on initial conditions implies that small errors in experimental measurements could lead to large scale divergence of the future data. With these two properties we give the following definition.

Definition 2.4. *A map f is said to be* ***chaotic-like on an invariant set*** *Y if*

(1) *f is transitive on Y, and*

(2) *f has sensitive dependence on initial conditions on Y.*

Now a formal definition of chaos is given that is slightly different from the usual definitions. See [26, 5].

Definition 2.5. *A map f is* ***chaotic on an invariant set*** *Y if f is chaotic-like on Y and if h is a conjugacy between f and any other map g, then g is chaotic-like on $h(Y)$.*

We have the following useful theorem based on a theorem in Banks [5], in the case when Y is a compact, invariant set.

Theorem 2.6. *If f is chaotic-like on a compact invariant set Y, then f is chaotic on Y.*

Proof. Let $g : Z \to Z$ be a map that is conjugate to f by h. We have that h is a homeomorphism from X to Z and $h(Y)$ is compact. Let r be the sensitivity constant for f. Let D_r be the set of ordered pairs in $(y_{,1} y_2) \in Y \times Y$ such that $d(y_1, y_2) \geq r$. Since D_r is a closed subset of the compact space $Y \times Y$, it is compact. Let E_r be the image under the map $(y_1, y_2) \mapsto (h(y_1), h(y_2))$ where $(y_1, y_2) \in D_r$. Since h is a homeomorphism we have that E_r is a compact set in $h(Y) \times h(Y)$. For $(v_1, v_2) \in E_r$

we have that $v_1 \neq v_2$ and since the diagonal of $h(Y) \times h(Y)$ is a compact set we have a minimum distance $\delta > 0$ from E_r to the diagonal. We now show that g has sensitive dependence on initial conditions with sensitivity constant δ.

Let $z \in h(Y)$ and let $\epsilon > 0$ and consider $B_\epsilon(z)$. Let $x = h^{-1}(z)$ and let $\epsilon_2 > 0$ be small enough such that $B_{\epsilon_2}(x) \subset h^{-1}(B_\epsilon(z))$. Since f is chaotic-like, there exists an $x_2 \in B_{\epsilon_2}(x)$ such that $d(f^k(x_2), f^k(x)) > r$. Let $z_2 = h(x_2)$ which implies that $z_2 \in B_\epsilon(z)$. By previous arguments we have that $d(h(f^k(x_2)), h(f^k(x))) > \delta$, but $h(f^k(x_2)) = g^k(h(x_2)) = g^k(z_2)$ and $h(f^k(x)) = g^k(z)$. Therefore g has sensitive dependence on initial conditions with sensitivity constant δ.

The proof that g is transitive on $h(Y)$ follows trivially from the conjugacy. Therefore g is chaotic-like on $h(Y)$, hence f is chaotic on Y. ∎

In the case where f is chaotic-like on Y and Y is not a compact set, then there are examples where f is conjugate to a g where g does not have sensitive dependence on initial conditions. However if we verify some other topological properties of f, then we may be able to conclude that f is chaotic. One such possibility is the property of having dense periodic points. This brings us to the following theorem.

Theorem 2.7 (Banks [5]). *If f is transitive and has dense periodic points on an invariant set Y and also contains more than one periodic orbit, then f has sensitive dependence on initial conditions.*

The fact that you must have more than one periodic orbit is crucial since you could take your space to be a periodic orbit for some map f which would make all points periodic which trivially implies dense periodic points and f would be transitive on this space since the orbit of any point is the space itself. However we do not have sensitive dependence on initial conditions. In this case the combination of being transitive and having dense periodic orbits actually implies either trivial dynamics or very complicated dynamics as the following lemma shows.

Lemma 2.8. *If f is transitive and has dense periodic points on an invariant set Y, then Y consists of a single periodic orbit or it contains infinitely many periodic orbits.*

Proof. First suppose that Y only contains one periodic orbit. Since in this case the periodic points are finite and dense, then Y consists of only these periodic points.

Now assume that the number of periodic orbits is finite and greater than one. Since f is transitive, there exists a q with a dense forward orbit which implies that q cannot be a periodic point. However since the number of periodic orbits is finite, there exists only finitely many periodic points which implies there exists a neighborhood around q that does not contain a periodic point, contradicting the density of the periodic points. Therefore if we have more than one periodic orbit, we must have infinitely many. ∎

We can now state the following theorem which provides a condition for chaos without compactness.

Theorem 2.9. *If a map f is chaotic-like on an invariant set Y, f has dense periodic points in Y, and contains more than one periodic orbit, then f is chaotic on Y.*

Proof. Let g be a map that is conjugate to f by h. The topological properties of transitivity and dense periodic points are preserved by h, therefore g is transitive and has dense periodic points on $h(Y)$ and by the Theorem 2.7 we have that g is chaotic-like. Therefore f is chaotic on Y. ∎

2.2 Symbolic Dynamics

We now give definitions of a specific space and map which we call a symbolic dynamical system. Symbolic dynamical systems have the advantage that the dynamics of the system are easily represented and computable. If one is working with a dynamical system where the dynamics are very complicated, then one strategy is to try to build a semiconjugacy to a symbolic dynamical system and work inside the symbolic dynamical system to make inferences about the original system.

Definition 2.10. *The* ***symbol space on*** n ***symbols****, denoted by* Σ_n*, is a set of sequences given by*

$$\Sigma_n = \{a = (\ldots, a_{-1}, a_0, a_1, \ldots) \mid a_j \in \{1, \ldots, n\} \textit{ for all } j \in \mathbb{Z}\}.$$

Proposition 2.11. *The symbol space on* n *symbols is a complete metric space with the following metric*

$$d(a,b) := \sum_{j=-\infty}^{j=\infty} \frac{\delta(a_j, b_j)}{4^{|j|}},$$

where

$$\delta(t,s) = \begin{cases} 0 \textit{ if } t = s \\ 1 \textit{ if } t \neq s \end{cases}$$

Proposition 2.12. Σ_n *is a compact space.*

Proof. First we define a basis for the topology of Σ_n. Given $t \in \Sigma_n$ we define $B_k(t)$ to be the set

$$B_k(t) := \{a \in \Sigma_n \mid a_j = t_j \text{ for } -k \leq j \leq k\}.$$

Clearly the sets $B_k(t)$ for $t \in \Sigma_n$ are open and satisfy the properties of a topological basis.

Next let S_n be the topological space consisting of the discrete topology on $\{1 \dots n\}$. Now take the infinite product of S_n with itself and give it the product topology and label it S,

$$S = \dots S_n \times S_n \times S_n \dots$$

Since S_n, is a compact space we have by the Tychonoff Theorem that S is compact. Now we form a topological basis for S. For a point $s \in S$ we define $\beta_k(s)$ to be,

$$\beta_k(s) := \dots S_n \times s_{-k} \times s_{-k+1} \times \dots \times s_{k-1} \times s_k \times S_n \dots$$

Since S_n is finite, the set $\beta_k(s)$ for $s \in S$ clearly forms a topological basis.

Finally we define a bijection $h : \Sigma_n \to S$ by

$$h(t) = \dots t_{-k} \times t_{-k+1} \times \dots \times t_0 \times \dots \times t_{k-1} \times t_k \times \dots$$

with the inverse of h given by

$$h^{-1}(s) = (\dots, s_{-k}, \dots, s_0, \dots, s_k \dots)$$

It is easy to see that h and h^{-1} map basis elements to basis elements, therefore h is a homeomorphism. Since S is compact, we have that Σ_n is compact. ■

Definition 2.13. *The* ***shift map on*** n ***symbols****, denoted by* $\sigma : \Sigma_n \to \Sigma_n$*, is defined by*

$$(\sigma(a))_k = a_{k+1}.$$

Proposition 2.14 ([19])**.** *The shift map on* n *symbols is a uniformly continuous* *function.*

Since Σ_n is a complete metric space, and σ_n is uniformly continuous, we can say that (Σ_n, σ_n) is a dynamical sytem which we call a symbolic dynamical system. One can now see the ease in representing the dynamics of a symbolic dynamical system. For example, in Σ_3 the points $(\ldots, 1, 1, 1, \ldots), (\ldots, 2, 2, 2, \ldots), (\ldots, 3, 3, 3, \ldots)$ would be the only fixed points of the system and the point $(\ldots, 1, 2, 2, 1, 2, 2, \ldots)$ would be an example of periodic point with least period 3.

Proposition 2.15. *σ_n is chaotic on Σ_n.*

Proof. We first show that σ_n is transitive. We enumerate all possible sequences of length one, and we let $b_{1,1} = 1, b_{1,2} = 2, \ldots, b_{1,n} = n$ and then we enumerate all sequences of length two with $b_{2,1} = 1, 1, b_{2,2} = 1, 2, \ldots, b_{2,2^n} = n, n$. We continue doing this for all possible finite sequences and we let $b_{k,j}$ denote the jth sequence of length k. Define $q \in \Sigma_n$ by

$$q := (\ldots, b_{k,j}, \ldots, b_{1,2}, b_{1,1}, b_{1,2}, \ldots, b_{k,j}, \ldots)$$

where $q_0 = b_{1,1}$ and q is symmetrically built by adding to the front and back all possible finite sequences.

We now show the forward orbit of q is dense. Let $s \in \Sigma_n$ and let $\epsilon > 0$. Let $B_\epsilon(s)$ denote the open ball of radius ϵ centered at s. There exists a smallest nonnegative integer M such that if $t_i = s_i$ for $-M \leq i \leq M$ then $t \in B_\epsilon(s)$. Let b be the finite sequence $b = s_{-M}, s_{-M+1}, \ldots, s_{M-1}, s_M$. Now there exists an $k \geq 0$ such that

$$\sigma_n^k(q) = (\ldots, b, \ldots)$$

and $\left(\sigma_n^k(q)\right)_i = s_i$ for $-M \leq i \leq M$ which implies that $\sigma_n^k(q) \in B_\epsilon(s)$, and hence the forward orbit of q is dense.

We now show that σ_n has sensitive dependence on initial conditions. Let $r = 1$, $\epsilon > 0$, and let $s \in \Sigma_n$ with $B_\epsilon(s)$ denoting the open ball of radius ϵ centered at s. Let $t \in B_\epsilon(s)$ and $t \neq s$, then there exists a largest nonnegative integer M such that if $t_i = s_i$ for $-M \leq i \leq M$ and $t_{M+1} \neq s_{M+1}$. This implies that $d({\sigma_n}^{M+1}(s), {\sigma_n}^{M+1}(t)) \geq 1$ which means that σ_n has sensitive dependence on initial conditions with sensitivity constant equal to one.

We have shown that σ_n is chaotic-like, and since Σ_n is compact, we have that σ_n is chaotic. ■

As we will see now, one can build a larger collection of symbolic dynamical systems from Σ_n, but first we need the following definition.

Definition 2.16. *An $n \times n$ matrix with entries of the form $t_{ij} = 0, 1$ is called a* ***transition matrix*** *if there exists a nonzero element in every column j and there exists a nonzero element in every row i.*

Given a transition matrix T we can define the subset of sequences $\Sigma_T \subset \Sigma_n$ to be

$$\Sigma_T := \left\{ a \in \Sigma_n \mid t_{a_k a_{k+1}} = 1 \text{ for } k \in \mathbb{Z} \right\}.$$

Example 2.17. For the space $\Sigma_T \subset \Sigma_3$ defined by the transition matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

we have the possible transitions of $1 \to 1, 1 \to 3, 3 \to 2$ and $2 \to 3$ which has two periodic points $(\ldots 2, 3, 2, 3, \ldots)$ and $(\ldots 3, 2, 3, 2, \ldots)$ and only one fixed point

$(\ldots, 1, 1, 1 \ldots)$.

One can easily see that the map σ_n restricted to Σ_T maps into Σ_T which leads us to the following definition.

Definition 2.18. *The space Σ_T along with the map $\sigma_T := \sigma_n|\Sigma_T$ is a dynamical system which we call the* ***subshift of finite type for the transition matrix*** *T.*

At this point we can ask whether or not the dynamical system (Σ_T, σ_T) is chaotic, but first we need to determine when σ_T is transitive. We are of course in a more restrictive environment where transitions are determined by a transition matrix, and so it is not necessarily possible to concatenate all possible sequences in Σ_T to obtain a point $\mathbf{q}$ that has a dense forward orbit as in Proposition 2.15. We need some additional information from our transition matrix in order to conclude transitivity.

Definition 2.19. *A transition matrix T is called* ***irreducible*** *if for each $1 \leq i, j, \leq n$ there exists a k depending on i, j such that $(T^k)_{ij} > 0$.*

An irreducible transition matrix should be interpreted as saying for any pair i and j, there exists some finite sequence, allowed by the transition matrix, from i to j. To show that a transition matrix is irreducible it is not wise to take the matrix and actually compute powers of the matrix. Instead, construct the graph with the i's as vertices and with an edge from i to j if $(T)_{ij} = 1$. Then the property of being irreducible is equivalent to saying the graph is strongly connected. Finding the connected components of a graph is easily computed in linear time using depth-first search algorithms. Now we come to the following lemma.

Lemma 2.20. *Let T be a transition matrix. The map σ_T is transitive if and only if T is irreducible.*

Proof. First we show that σ_T being transitive implies T is irreducible. Suppose that T is not irreducible so that there exists at least two strongly connected components of the graph of T. Select two components and label them A and B. Without loss of generality we assume that there does not exist a path from A to B. Let $q := (\ldots, q_0, q_1, \ldots)$ be a point in Σ_T with a dense forward orbit. Since A is strongly connected there exists a fixed point or periodic point with sequence values from A. Since the forward orbit of q must accumulate on this point this implies that there must exist a $k > 0$ such that $q_k \in A$. This means that for all values $n > k$ we have $q_n \notin B$. However there also exists either a fixed point or periodic point with sequence elements inside of B on which the forward orbit of q must accumulate. This contradicts that for all $n > k$ we have $q_n \notin B$. Therefore if σ_T is transitive, then T is irreducible.

Now we show the converse. Suppose that T is irreducible. Then for each i and j there exists a finite allowable sequence from i to j which we denote by s_{ij}. As in the proof of Proposition 2.15, we denote all possible finite sequences given by T by $b_{k,l}$, which means the l-th sequence of length k. For $b_{k,l}$ and $b_{k,(l+1)}$ there exists a s_{ij} connecting the last sequence element of $b_{k,l}$ denoted by $e(b_{k,l})$ to the first sequence element of $b_{k,(l+1)}$ denoted by $v(b_{k,(l+1)})$, and we can also find a connecting sequence for $b_{k,l}$ and $b_{(k+1),l}$. Now starting with $q_0 = b_{1,1}$ we can concatenate all possible finite sequences given by T.

$$q := \left(\ldots, b_{1,1}, s_{e(b_{1,1})v(b_{1,2})}, b_{1,2}, \ldots, b_{1,m}, s_{e(b_{1,m})v(b_{2,1})}, b_{2,1}, \ldots\right).$$

For the backward orbit of q we can take any allowable sequence. Thus the forward orbit of q is dense in Σ_T. ■

At this point we have two options to choose from in showing that σ_T is chaotic. We can use similar arguments as in Proposition 2.12 to show that Σ_T is compact and then directly show sensitive dependence on initial conditions, or we could show that the periodic points are dense and that we have more than one periodic point. Let us choose the latter option.

Proposition 2.21. *If T is irreducible, then the periodic points of Σ_T for the map σ_T are dense.*

Proof. Let

$$x = (\ldots, x_{-N}, x_{-N+1}, \ldots, x_0, \ldots, x_{N-1}, x_N, \ldots)$$

be any point in Σ_T, and let N be any positive integer. Since T is irreducible, there exists an allowable sequence from x_N to x_{-N} which we denote by $s_{x_N x_{-N}}$. Let us denote the finite sequence of $x_{-N}, x_{-N+1}, \ldots, x_0, \ldots, x_{N-1}, x_N$ by b_n. Now we can construct the periodic sequence q_N by

$$q_N := \left(\ldots, b_n, s_{x_N x_{-N}}, x_{-N}, x_{-N+1}, \ldots, x_0, \ldots, x_{N-1}, x_N, s_{x_N x_{-N}}, b_n, \ldots\right).$$

The periodic sequence q_N is such that $(q_N)_i = (x)_i$ for $-N \leq i \leq N$. As $N \to \infty$ we have q_N converging to x. Therefore the periodic points are dense. ■

Now one needs to check if there is more than one periodic point. But this is an easy check of the matrix T. If a row or a column has more than one nonzero entry, then this leads to more than one periodic point. Therefore we have the following theorem.

Theorem 2.22. *If T is an irreducible transition matrix with at least one row or at least one column having more than one nonzero entry, then the map σ_T on Σ_T is* *chaotic.*

2.3 Entropy

Given a sequence of iterates of a map or a time series of data, one would like to measure how chaotic or unpredictable the map or data is. One way of doing this is to compute what is called the entropy of the map or time series. Before we give the precise definitions and notations of entropy, we first give a brief description of entropy due to F. Takens [8].

Suppose we are given a finite time series $\{x_n\}_{n=0}^{N}$. We ask whether we can predict x_{N+1} without knowing the map that generates the $\{x_n\}$, hence we need a prediction procedure. Let $\epsilon > 0$ and let k be a positive integer less than N. Now consider all $m < N$ such that $|x_m - x_N| < \epsilon, |x_{m-1} - x_{N-1}| < \epsilon, \ldots |x_{m-k} - x_{N-k}| < \epsilon$. Then for each of these values m we can form an ϵ-neighborhood around x_{m+1} and take the union of these neighborhoods over all of the m. We can then predict that x_{N+1} will be in this union.

This prediction can be bad or labeled as a failure if the region of prediction is too large. This can happen if the region of prediction does not shrink with ϵ. Let us try to quantify some meaning to this statement. Suppose we are given two different points in the time series x_m and $x_{m'}$. We look at the length k-orbit of x_m which is $x_m, x_{m+1}, \ldots, x_{m+k}$ and compare it to the length k-orbit of $x_{m'}$ which is $x_{m'}, x_{m'+1}, \ldots, x_{m'+k}$. We say the two orbits are ϵ-distinguishable if for one of the iterates $0 \leq j \leq k$, we have $|x_{m+j} - x_{m'+j}| > \epsilon$. We say a list of length k orbits is ϵ-distinguishable if every orbit on the list is ϵ-distinguishable with every other orbit on the list. For a given k and a time-series $\{x_n\}$, there can be many different ϵ-distinguishable lists with the additional property that, if given any k-orbit in the time-series then there exists a k-orbit on the list with which it is not ϵ-distinguishable i.e. there exists a k-orbit on the list so that all the iterates are within ϵ distance of this orbit on the list. Given (k, ϵ, N), we define $r(k, \epsilon, N)$ as the cardinality of the smallest list of k-orbits that is ϵ-distinguishable with the additional property just mentioned. Basically, $r(k, \epsilon, N)$ is the smallest number of ϵ-distinguishable k-orbits for the sequence $\{x_n\}_{n=0}^{N}$ with the additional property. Suppose that the time-series is known for all $n \in N$. Since $r(k, \epsilon, N)$ is a non-decreasing function of the variable N, then we can take the limit as N goes to infinity which we denote by $r(k, \epsilon) = \lim_{N \to \infty} r(k, \epsilon, N)$. A straightforward argument will show that this limit is finite if the space containing the time series is bounded and finite-dimensional.

Now we come back to the failure of our prediction procedure. To reiterate, suppose we want to predict x_{N+1} by taking a k-orbit from the past that is not ϵ-distinguishable from the orbit $x_{N-k-1}, x_{N-k}, \ldots, x_{N-1}, x_N$, that is, we have a k-orbit $x_{m-k-1}, x_{m-k}, \ldots, x_{m-1}, x_m$ such that $|x_m - x_N| < \epsilon, |x_{m-1} - x_{N-1}| < \epsilon, \ldots, |x_{m-k} - x_{N-k}| < \epsilon$. We want to predict that x_{N+1} is in an ϵ neighborhood around x_{m+1}. Now we can say that there are roughly $r(k, \epsilon)$ number of ϵ-distinguishable k-orbits and that $x_{m-k-1}, x_{m-k}, \ldots, x_{m-1}, x_m$ is non-distinguishable with one of them. Likewise $x_{m-k-1}, x_{m-k}, \ldots, x_{m-1}, x_m, x_{m+1}$ is a $k+1$ orbit and is non-distinguishable with one of the $r(k+1, \epsilon)$ number of ϵ-distinguishable $k+1$-orbits. If $r(k+1, \epsilon)$ is much larger than $r(k, \epsilon)$, then it means that k-orbits can be extended to $k+1$-orbits in many possible ways. So if we were to take any k-orbit from the past that is non-distinguishable from $x_{N-k-1}, x_{N-k}, \ldots, x_{N-1}, x_N$, take an ϵ-neighborhood around the next iterate, and take the union all of these neighborhoods, then it is possible that this union could be very large; hence we do not have a good prediction. Thus if $r(k, \epsilon)$ grows too fast with respect to k, then this impedes our ability to predict.

We measure the expansive growth rate of $r(k, \epsilon)$ with respect to k by

$$h(\epsilon) = \limsup_{k \to \infty} = \frac{\ln(r(k, \epsilon))}{k}.$$

Now we said that our prediction procedure ultimately fails if our prediction neighborhood does not shrink with ϵ going to zero. We now define the **entropy** of the time series as

$$h = \lim_{\epsilon \to 0} h(\epsilon).$$

It can easily be shown that if we are in a compact space, then the entropy calculation h is finite. A positive entropy $h > 0$ will imply unpredictability and estimate chaotic behavior of the time series.

Now we give the definition of entropy for continuous maps.

Definition 2.23. *Let $f : X \to X$ be a continuous map on a compact metric space X with metric d. For a positive integer q, let*

$$d_{q,f}(x, y) = \sup_{0 \le j < q} d(f^j(x), f^j(y)).$$

Let n be a positive integer and let $\epsilon > 0$. A set $S \subset X$ is said to (n, ϵ)-span X provided for each $x \in X$ there exists a $y \in S$ such that $d_{n,f}(x, y) \le \epsilon$. We denote $r_{span}(n, \epsilon, X, f)$ as being the smallest number of elements in any subset $S \subset X$ which (n, ϵ)-span X. Then we define $h_{span}(\epsilon, X, f)$ as

$$h_{span}(\epsilon, X, f) := \limsup_{n \to \infty} \frac{\log(r_{span}(n, \epsilon, X, f))}{n}.$$

*Now we define the **entropy** of the map f on X which we denote by $h_{span}(X, f)$ as*

$$h_{span}(X, f) := \lim_{\epsilon \to 0^+} h_{span}(\epsilon, X, f).$$

From here we will denote the entropy by $h(f) = h_{span}(X, f)$.

We proceed with some important definitions and theorems involving entropy. For proofs of theorems see Robinson [25].

Definition 2.24. *For a continuous map $f : X \to X$, we call a point p a **nonwandering** provided for every neighborhood U of p there is a positive integer n and a point $q \in U$ such that $f^n(q) \in U$. The set of nonwandering points for f is called the **nonwandering set** and is denoted by $\Omega(f)$.*

Theorem 2.25 ([25])**.** *Let $f : X \to X$ be a continuous map on a compact metric space X. Then, the entropy of f equals the entropy of f restricted to its nonwandering set, $h(f) = h(f|\Omega)$.*

We would like to say that the positive entropy is equivalent to chaos, but there are obvious examples of maps with positive entropy that do not exhibit chaos. The next theorem shows that it does imply chaos if given transitivity.

Theorem 2.26. *If $h(f) > 0$ and there exists a $p \in X$ that has a dense forward orbit then f is chaotic.*

Proof. We need to show f has sensitive dependence on initial conditions. We argue by contradiction. Suppose f does not have sensitive dependence on initial conditions. Then, for any $\delta > 0$ there exists an $x \in X$ and a neighborhood $B_\epsilon(x)$ such that for all $y \in B_\epsilon(x)$ we have that $f^n(y) \in B_\delta(f^n(x))$ for all future time n. Since p has a dense forward orbit, there exists an n such that $f^n(p) \in B_\epsilon(x)$ and that $d(f^n(p), x) < \delta/2$.

Now take any $q \in X$. There exists a time m such that $d(f^m(f^n(p)), q) < \delta/2$. This implies that $d(f^m(x), q) \leq d(f^m(x), f^m(f^n(p))) + d(f^m(f^n(p)), q) < \delta/2 + \delta/2 = \delta$. Therefore all of X is covered by the balls of radius δ around the forward orbit of x. Since X is compact, it is covered by finitely many of these balls. Let N be the highest iterate of f for the finite cover. Then for all $n \geq N$ we have that $r_{span}(n, \delta, X, f) = 1$ which implies that $\log(r_{span}(n, \delta, X, f)) = 0$ so that $h_{span}(\delta, X, f) = 0$. Since this is true for all $\delta > 0$, we have that $h_{span}(X, f) = 0$, which is a contradiction to the assumption that $h(f) > 0$. ∎

Definition 2.27. *Given a continuous map $k : X \to Y$ we say that k is **uniformly finite to one** if for each $y \in Y$ we have that $k^{-1}(y)$ is finite and there exists a C such that $k^{-1}(y) \leq C$ for all $y \in Y$.*

Theorem 2.28 ([25])**.** *Assume $f : X \to X$ and $g : Y \to Y$ are continuous maps on compact metric spaces X and Y. Assume that $k : X \to Y$ is a semi-conjugacy from f to g that is onto and uniformly finite to one. Then $h(f) = h(g)$.*

Theorem 2.29 ([25])**.** *(a) Let $\sigma : \sigma : \Sigma_n \to \Sigma_n$ be the full shift on N symbols. Assume $X \subset \Sigma_n$ is a closed invariant subset, so $\sigma|X$ is a subshift. Let w_n be the number of words of length n in X, i.e.*

$$w_n = \sharp \{(s_0, \ldots, s_{n-1}) : s_j = x_j \text{ for } 0 \leq j < n \text{ for some } x \in X\}.$$

Then,

$$h(\sigma|X) = \limsup_{n\to\infty} \frac{\log(w_n)}{n}.$$

(b) Let T be an irreducible transition matrix on N symbols. Let $\sigma_T : \Sigma_T \to \Sigma_T$ be the associated subshift of finite type. Then, $h(\sigma_T) = \log(\lambda_1)$, where λ_1 is the real eigenvalue of T such that $\lambda_1 \geq |\lambda_j|$ for all other eigenvalues λ_j of T.

The last two theorems are used to establish a lower bound on the entropy of a system. If one can establish a uniformly finite to one semiconjugacy on a subset of the state space from a continuous map f to the shift map, then Theorem 2.28 says the entropy is the same on that subset. Moreover, Theorem 2.29 tells us that we only have to calculate the eigenvalues of the transition matrix associated with the shift map to compute the entropy. This entropy calculation then gives a lower bound on the entropy over the state space of f.

3 Simplicial Homology

In this chapter we present all definitions and theorems for a specific homology theory we use in this thesis. Our presentation is not complete or general. For a more general description of homology see Munkres [23] or Hatcher [15].

3.1 Simplices in the Plane

Definition 3.1. *Let V be a set of points $V = \{v_0, v_1, \ldots, v_n\}$ in $\mathbb{R}^N$. We say that V is* ***geometrically independent*** *if for any real scalars λ_i, we have that*

$$\sum_{i=0}^{n} \lambda_i = 0 \text{ and } \sum_{i=0}^{n} \lambda_i v_i = 0$$

imply that $\lambda_0 = \lambda_1 = \ldots = \lambda_n = 0$. That is a set $\{v_0, \ldots, v_n\}$ is geometrically independent if and only if the set of vectors $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent.

From here on we restrict ourselves to $\mathbb{R}^2$, but much of what is done is dimension independent. The previous definition tells us that a set consisting of one point is always geometrically independent, a set consisting of two points is geometrically independent if and only if the two points are distinct, and a set consisting of three points is geometrically independent if and only if the three points are non-collinear.

Definition 3.2. *Given a set of three points $\{v_0, v_1, v_2\}$ in $\mathbb{R}^2$ that are geometrically independent, we define the* **2*-simplex*** *spanned by $\{v_0, v_1, v_2\}$ to be the set of points $x \in \mathbb{R}^2$ such that $x = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2$ where $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and $\lambda_i \geq 0$. For a set* *of two points $\{v_0, v_1\}$ that are geometrically independent, we define the* ***1-simplex*** *spanned by $\{v_0, v_1\}$ to be the set of points $x \in \mathbb{R}^2$ such that $x = \lambda_0 v_0 + \lambda_1 v_1$ where $\lambda_0 + \lambda_1 = 1$ and $\lambda_i \geq 0$. For a set consisting of just one point $\{v_0\}$, we define the* ***0-simplex*** *spanned by $\{v_0\}$ to just be the point $\{v_0\}$.*

For the 1-simplex and the 2-simplex, the definition of geometric independence implies that for a given x in those simplices, the coefficients λ_i in the sum are unique, and we call the coefficients λ_i the **barycentric coordinates** of x in the simplex. From this definition it is easy to see that 0-simplices are points in the plane, 1-simplices are line segments between two distinct points, and 2-simplices are triangles along with the interior points of the triangle. Therefore by this geometric description we have that simplices as sets in $\mathbb{R}^2$ are compact, convex, and are the convex hulls of the points they span. The n-points that span an n-simplex are called the **vertices** of the simplex. Any simplex that is spanned by a subset of the vertices of a given simplex is called a **face** of that given simplex, and if it is a proper subset of the vertices, it is called a **proper face** of the given simplex. For example the proper faces of a 2-simplex would consist of each of the three edges that make up the triangle along with each of the three vertices of the triangle, and for a 1-simplex the proper faces would consist of the two vertices of the line segment.

Definition 3.3. *A* ***simplicial complex*** *K in $\mathbb{R}^2$ is a collection of simplices such that*

(1) *Every face of a simplex of K is in K,and*

(2) *The intersection of any two simplices in K is a face of each of them.*

Definition 3.4. *Given a simplicial complex K in $\mathbb{R}^2$, the union of all simplices of K is called a* ***polytope*** *of K and is denoted by $|K|$. A subset $P \subset \mathbb{R}^2$ is called a*

polyhedron *if P is the polytope of some simplicial complex K. In this case K is called a* ***triangulation*** *of P.*

We now put an algebraic structure on our simplicial complexes by defining a vector space. To reiterate, the following definitions are not the general definitions but only for the specific homology used here.

Definition 3.5. *Let K be a simplicial complex. A p-**chain** on K is a function c from the set of p-simplices of K to $\mathbb{Z}_2$ such that $c(\sigma) = 0$ for all but finitely many p-simplices σ. If σ is a simplex, the* ***elementary chain*** *c corresponding to σ is the function defined by having $c(\sigma) = 1$ and $c(\tau) = 0$ for all other simplices τ.*

For notational purposes we identify the symbol for a p-simplex σ with its elementary chain. The p-chains form a vector space by addition of functions with coefficients in $\mathbb{Z}_2$ which we call the **vector space of p-chains** of K which we denote by $C_p(K)$. Note that each p-chain of K can be written as a unique finite linear combination of elementary chains with coefficients in $\mathbb{Z}_2$. Therefore the elementary chains are a basis for $C_p(K)$. For $p < 0$, we define $C_p(K)$ to be the trivial vector space.

Definition 3.6. *We define a linear map $\partial_p : C_p(K) \to C_{p-1}(K)$ called the* ***boundary operator*** *as follows:*

(1) *If $p = 0$, then $\partial_0(v) = 0$ where v is a 0-simplex.*

(2) *If $p = 1$, then $\partial_1(e) = v_1 + v_2$ where e is a 1-simplex, and v_1 and v_2 are its vertices.*

(3) *If $p = 2$, then $\partial_2(s) = e_1 + e_2 + e_3$ where s is a 2-simplex, and e_1, e_2,and e_3 are its edges.*

Now extend ∂_p linearly to get a linear map on $C_p(K)$.

For example, if e_1 is a 1-simplex with vertices v_0 and v_1, and if e_2 is a 1-simplex with vertices v_1 and v_2, then $\partial_1(e_1+e_2) = \partial_1(e_1)+\partial_1(e_2) = v_0+v_1+v_1+v_2 = v_0+v_2$ since our coefficients are in $\mathbb{Z}_2$.

Lemma 3.7. $\partial_{p-1} \circ \partial_p = 0$

Proof. First we show it is true for simplices, and then we will show it is true for chains. For 0-simplices it is trivially true since ∂_0 maps everything to 0, and ∂_{-1} is defined to be the 0 map. For a 1-simplex e, we have $\partial_0(\partial_1(e)) = \partial_0(v_1+v_2) = \partial_0(v_1)+\partial_0(v_2) = 0+0 = 0$. For a 2-simplex s, we have $\partial_1(\partial_2(s)) = \partial_1(e_1+e_2+e_3) = v_1+v_2+v_2+v_3+v_3+v_1 = 0$. Now let $c = c_1+c_2+\ldots+c_n$ be any chain. Then $\partial_{p-1}(\partial_p(c)) = \partial_{p-1}(\partial_p(c_1)+\partial_p(c_2)+\ldots+\partial_p(c_n)) = \partial_{p-1}\partial_p(c_1)+\partial_{p-1}\partial_p(c_2)+\ldots+\partial_{p-1}\partial_p(c_n) = 0$. ∎

Definition 3.8. *The kernel of the boundary map $\partial_p : C_p(K) \to C_{p-1}(K)$ which we denote by $Z_p(K)$ is called the p-**cycles**. The image of the boundary map $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ which we denote by $B_p(K)$ is called the p-**boundaries**. From the previous lemma we have that $B_p(K) \subset Z_p(K)$, and we define the p-th **homology vector space** which we denote by $H_p(K)$ to be*

$$H_p(K) = Z_p(K)/B_p(K).$$

For the zero-homology vector space we have the following theorem that facilitates computability.

Theorem 3.9 ([23])**.** *Let K be a simplicial complex. The vector space $H_0(K)$ over $\mathbb{Z}_2$ has a basis consisting of the homology classes of one vertex from each connected component of $|K|$.*

So the dimension for $H_0(K)$ equals the number of connected components of $|K|$.

Given a subcomplex K_0 of a simplicial complex K, we can define homology vector spaces of K modulo the subcomplex K_0.

Definition 3.10. *If K_0 is a subcomplex of K, then the vector space $C_p(K, K_0) := C_p(K)/C_p(K_0)$ is called the space of* ***relative*** *p-**chains of** K* ***modulo*** *K_0. The vector space $C_p(K, K_0)$ has basis $\{\sigma_i + C_p(K_0)\}$ where σ_i ranges over all p-simplices of K which do not intersect K_0. The boundary operator ∂ on C_p, induces a linear map on $C_p(K, K_0)$ to $C_{p-1}(K, K_0)$, denoted also by ∂, by $\partial(\sigma_i + C_p(K_0)) = \partial(\sigma_i) + C_{p-1}(K_0)$. From this we define*

$$Z_p(K, K_0) = \ker \partial : C_p(K, K_0) \to C_{p-1}(K, K_0),$$

$$B_p(K, K_0) = \text{im } \partial : C_{p+1}(K, K_0) \to C_p(K, K_0),$$

$$H_p(K, K_0) = Z_p(K, K_0)/B_p(K, K_0),$$

which we call respectively, the vector space of ***relative*** *p-**cycles**, the vector space of* ***relative*** *p-**boundaries**, and the* ***relative homology vector space*** *in dimension p of K modulo K_0.*

Definition 3.11. *Let $\epsilon : C_0(K) \to \mathbb{Z}_2$ be the surjective linear map defined by setting $\epsilon(v) = 1$ for each vertex v of K. If c is a 0-chain, then $\epsilon(c)$ is the sum of the coefficients of c mod 2 and $\epsilon(\partial d) = 0$ if d is a 1-chain. We can now define a different homology vector space for dimension 0 denoted by $\tilde{H}_0(K)$ by $\tilde{H}_0(K) = \ker \epsilon / \operatorname{im} \partial_1$. We call this the* ***reduced homology vector space*** *of dimension 0 and we define the* ***reduced homology vector space*** *of dimension p by $\tilde{H}_p(K) := H_p(K)$ when $p \neq 0$.*

We treat ϵ as a linear map $\epsilon : C_0(K) \to C_{-1}(K)$ where we identify $C_{-1}(K)$ as the group $\mathbb{Z}_2$.

Theorem 3.12. *The vector space $\tilde{H}_0(K)$ satisfies $\tilde{H}_0(K) \oplus \mathbb{Z}_2 \cong H_0(K)$.*

This theorem tells us that $\tilde{H}_0(K)$ is the zero vector space if K is connected and if K is not connected then $v_\alpha - v_{\alpha_0}$ form a basis for $\tilde{H}_0(K)$ where $\{v_\alpha\}$ consist of one vertex from each connected component of K and α_0 is a fixed index.

Definition 3.13. *We say that a simplex K is* ***acyclic*** *if $\tilde{H}_p(K) = 0$ for all p.*

We come to the question of whether we can compute the homology vector spaces H_p. If we have a finite simplicial complex K, the answer to this is affirmative as we now show.

Theorem 3.14 ([23, 17])**.** *Let G and G' be finite dimensional vector spaces with coefficients in Z_2. Let $f : G \to G'$ be a linear map. Then there exist bases for G and G' such that relative to these bases the matrix f has the form*

$$\begin{bmatrix} b_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & b_k & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

where $b_i = 1$. The matrix is called the ***normal form*** *of f and we call k the* ***normal index*** *for the map f.*

Taking a matrix to its normal form consists of a series of elementary row operations in an algorithm called the reduction algorithm. If we let $G = C_p$ and $G' = C_{p-1}$ then we can write the boundary homomorphism ∂_p in normal form with respect to some base $e_1, e_2, \dots, e_k, e_{k+1}, \dots, e_n$ for C_p where e_j is a p-chain. If k is the normal index for the matrix, then we get that $e_{k+1}, \dots, e_n$ is a basis for Z_p. If we repeat this for the boundary map ∂_{p+1} with normal index m, then with $g_1, g_2, \dots, g_m, g_{m+1}, \dots, g_j$ being the basis for C_p corresponding to the matrix ∂_{p+1} in normal form we get that $g_1, \dots, g_m$ is a basis for B_p. We can then compute $H_p = Z_p/B_{p+1}$ by checking if each of the elements in Z_p are homologous to a linear combination of the $g_1, \dots, g_m$. The reason we chose to do homology with coefficients in $\mathbb{Z}_2$ was because the operations in the reduction algorithm and the last step of checking whether elements are homologous, were easy to code for computations. So far the homology presented is standard classical simplicial homology theory, and we do not deviate from the classical theory until the next section.

3.2 Homology of Maps

In this section we discuss maps between homology spaces which are induced by continuous maps. Homology can be used to study the action of continuous maps. Suppose you are given a simplicial complex K and a simplicial complex L, with a continuous map $f : |K| \to |L|$. We want to define a homomorphism between $H(K)$ and $H(L)$ which we denote by $f_* : H(K) \to H(L)$. In the previous section we used standard simplicial homology theory, but in this section we deviate from the standard theory when defining maps on homology for computational reasons.

Definition 3.15. *Let $C = \{C_p, \partial_p\}$ and $C' = \left\{C'_p, \partial_p\right\}$ be chain complexes. A* ***chain map*** *is a family of homomorphisms $\phi_p : C_p \to C'_p$ such that $\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p$.*

From here on we will always define $\phi_{-1} : C_{-1} \to C'_{-1}$ to be the identity so that we have $\epsilon' \circ \phi_0 = \phi_{-1} \circ \epsilon$. It is easy to see that a chain map maps cycles to cycles and boundaries to boundaries, which means that ϕ induces a linear map between the vector spaces, which we denote by ϕ_*. So if we want to construct homomorphisms between homology groups then we often first construct chain maps.

Definition 3.16. *If $\phi, \varphi : C_p(K) \to C_p(L)$ are chain maps such that for each p there exists a homomorphism $D_p : C_p(K) \to C_{p+1}(L)$ such that*

$$\partial_{p+1} D_p + D_{p-1}\partial_p = (\varphi)_p - (\phi)_p,$$

*then we say that ϕ and φ are **chain homotopic** and that D is a **chain homotopy** between ϕ and φ.*

Theorem 3.17. *If there is a chain homotopy between ϕ and φ then ϕ_* and φ_* are equal for the reduced and ordinary homology.*

Proof. Let z be a p-cycle of K. Then

$$\varphi(z) - \phi(z) = \partial D z + D\partial z = \partial D z + 0$$

which implies that $\varphi(z)$ and $\phi(z)$ are in the same homology class so that $\varphi_*(\{z\}) = \phi_*(\{z\})$. ∎

The previous theorem gives us a condition for determining when two maps on homology are the same. We just need to verify the existence of a chain homotopy. Before we introduce how to induce maps on homology through continuous maps, we need the following definition and theorem.

Definition 3.18. *If K and L are two simplicial complexes, and Φ is a map between K and L, then we say Φ is an **acyclic carrier** between K and L if Φ maps each simplex σ of K to a subcomplex $\Phi(\sigma)$ of L such that the following two conditions hold:*

(1) $\Phi(\sigma)$ *is nonempty and acyclic, and*

(2) *if* τ *is a face of* σ*, then* $\Phi(\tau) \subset \Phi(\sigma)$*.*

Furthermore, if ϕ *is a chain map* $\phi : C_p(K) \to C_p(L)$*, then we say that* ϕ *is* ***carried*** *by* Φ *if for each* p*-simplex* σ *of* K*, the chain* $\phi(\sigma)$ *is contained in the subcomplex* $\Phi(\sigma)$*.*

Theorem 3.19 (Acyclic Carrier Theorem [23])**.** *Suppose* Φ *is an acyclic carrier from* K *to* L*. Then*

(1) *If* ϕ *and* φ *are chain maps from* $C(K)$ *to* $C(L)$ *that are carried by* Φ*, then there exists a chain homotopy* D *of* ϕ *to* φ *that is also carried by* Φ*.*

(2) *There exists a chain map from* $C(K)$ *to* $C(L)$ *that is carried by* Φ*.*

The triangulations that we consider are triangulations made up of 2-simplices and their faces, so there will be no 0-simplices and 1-simplices in the complex that are not a face of a 2-simplex. Therefore we only consider acyclic carriers that map to complexes consisting of 2-simplices and their faces. We call these acyclic carriers **simplicial acyclic carriers**. With the use of the acyclic carrier theorem we are now ready to define the induced homology maps from continuous maps. First, given a continuous map $f : |K| \to |L|$, we say that a simplicial acyclic carrier Φ **carries** f if the following properties hold:

(1) if v is a vertex, then $\Phi(v)$ consists of all 2-simplices that intersect $f(v)$, and

(2) for any simplex σ, we have $f(|\sigma|) \subset |\Phi(\sigma)|$.

We say that an acyclic carrier Φ **carries** another acyclic carrier Γ if for each σ in the triangulation we have that $|\Gamma(\sigma)| \subset |\Phi(\sigma)|$. We say that f has a **minimal simplicial acyclic carrier** denoted by Φ_f, if Φ_f carries f, and if Φ is any simplicial acyclic carrier that carries f, then Φ carries Φ_f. It is not necessarily true that a map has a minimal simplicial acyclic carrier on a fixed pair of complexes K and L.

Definition 3.20. *Let $f : |K| \to |L|$ be a continuous map, and suppose that f has a minimal acyclic carrier Φ_f. Let Φ be an acyclic carrier that carries f. Then we define the* ***map of homology induced by the continuous map*** *f, denoted by $f_* : H(K) \to H(L)$, to be the map of homology induced by any chain map carried by Φ.*

The acyclic carrier theorem tells us there exists a chain map that is carried by Φ, and this chain map is chain homotopic to any other chain map carried by Φ, which means the induced homology is the same for all chains carried by Φ. If Φ_1 and Φ_2 are acyclic carriers that carry f, then $|\Phi_f(\sigma)| \subset |\Phi_1(\sigma)|$ and $|\Phi_f(\sigma)| \subset |\Phi_2(\sigma)|$, and there exists a chain map ϕ that is carried by Φ_f as well as Φ_1 and Φ_2. This implies that chain maps that are carried by Φ_1 are chain homotopic to chain maps carried by Φ_2. This proves that our map of homology is well-defined.

Suppose that f has a minimal acyclic carrier Φ_f. The acyclic carrier theorem implies that any chain map carried by Φ_f induces the same homology map. We now construct a chain map $f_\sharp$ carried by Φ_f. First, for a 0-simplex v of K we define $f_\sharp(v)$ to be any 0-simplex contained in $\Phi_f(v)$. Next, for a 1-simplex e with vertices v_1 and v_2, we know that the subcomplex $\Phi_f(e)$ is acyclic and that $f_\sharp(v_1)$ and $f_\sharp(v_2)$ are vertices in $\Phi_f(e)$. This means that $f_\sharp(v_1)$ and $f_\sharp(v_2)$ are homologous and so there exists a 1-chain c in $\Phi_f(e)$ such that $f_\sharp(v_1) = f_\sharp(v_2) + \partial_1 c$. We define $f_\sharp(e) = c$. Finally, for a 2-simplex s with edges e_1, e_2, and e_3, we know that $\Phi_f(s)$ is acyclic and that the 1-chain $f_\sharp(e_1) + f_\sharp(e_2) + f_\sharp(e_3)$ is a cycle in $\Phi_f(s)$. This means that the 1-chain $f_\sharp(e_1) + f_\sharp(e_2) + f_\sharp(e_3)$ is either equal to 0 in which case we define $f_\sharp(s) = 0$, or it is homologous to a 2-chain d in $\Phi_f(s)$ in which case we define $f_\sharp(s) = d$. By construction we have that $f_\sharp$ is carried by Φ_f and that $f_\sharp$ was not unique. This is still well-defined since any chain map carried by Φ_f induces the same homology map.

The above outline also holds for relative homology except for some small differences. Suppose K_0 is a subcomplex of K and L_0 is a subcomplex of L, we define a **relative simplicial acyclic carrier** to be a simplicial acyclic carrier $\Phi : K \to L$ with the additional property that if σ is a simplex of K_0, then $\Phi(\sigma) \subset L_0$. Suppose we have a continuous map $f : |K| \to |L|$ such that $f(|K_0|) \subset |L_0|$. We say that a relative simplicial acyclic carrier Φ **carries** f if for each simplex $\sigma \in K$ we have $f(|\sigma|) \subset |\Phi(\sigma)|$ and if v is a vertex such that $f(v) \subset |L_0|$, then $\Phi(v)$ will consist only of the 2-simplices in L_0 that intersect $f(v)$. We also say that a relative simplicial acyclic carrier Φ **carries** another relative simplicial acyclic carrier Γ if for each simplex $\sigma \in K$ we have $|\Gamma(\sigma) \mod L_0| \subset |\Phi(\sigma) \mod L_0|$. Finally, we define the **minimal relative simplicial acyclic carrier** for f denoted by Φ_f to be a relative simplicial acyclic carrier that carries f such that if Φ is any other relative simplicial acyclic carrier that carries f, then Φ carries Φ_f.

Definition 3.21. *Let $f : |K| \to |L|$ be continuous such that for some subcomplex $K_0 \subset K$ and subcomplex $L_0 \subset L$ we have that $f(|K_0|) \subset |L_0|$. Suppose that f has a relative acyclic carrier Φ_f. If Φ is any relative simplicial acyclic carrier that carries f, then we define the* ***map of relative homology induced by the continuous map*** *f, denoted by $f_* : H(K, K_0) \to H(L, L_0)$, to be the map of homology induced by any chain map carried by $\bar{\Phi}$.*

With these definitions of maps of homology we can prove that we get the Lefschetz Fixed Point Theorem. In order to prove it we need to use barycentric subdivisions.

Definition 3.22. *Let K be a finite simplicial complex. A complex K' is a* ***subdivision*** *of K if*

(1) *each simplex of K' is contained in a simplex of K, and*

(2) *each simplex of K equals the union of finitely many simplices of K'.*

Note that the second condition implies that K' is also finite.

Definition 3.23. *Given a p-simplex $\sigma = [v_0, \ldots, v_p]$, we define the* ***barycenter*** *of σ denoted by $\widehat{\sigma}$ to be the point*

$$\widehat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i,$$

that is the point whose barycentric coordinates are equal.

Definition 3.24. *The* ***barycentric subdivision*** *of a 0-simplex v is itself v. The* ***barycentric subdivision*** *of a 1-simplex $e = [v_0, v_1]$ is the set of 1-simplices $[v_0, \widehat{e}]$ and $[\widehat{e}, v_1]$. The* ***barycentric subdivision*** *of a 2-simplex $s = [v_0, v_1, v_2]$ is the set of simplices $[\widehat{s}, v_0, \widehat{e_0}], [\widehat{s}, v_1, \widehat{e_0}], [\widehat{s}, v_1, \widehat{e_1}], [\widehat{s}, v_2, \widehat{e_1}], [\widehat{s}, v_2, \widehat{e_2}], [\widehat{s}, v_0, \widehat{e_2}]$ where $\widehat{e_0}$ is the barycenter for edge $e_0 = [v_0, v_1]$, $\widehat{e_1}$ is the barycenter for edge $e_1 = [v_1, v_2]$, and $\widehat{e_2}$ is the barycenter for edge $e_2 = [v_2, v_0]$. The* ***barycentric subdivision of a complex*** *K which we denote by $\mathrm{bsd}(K)$ is the complex derived from K by first taking all barycentric subdivisions of all simplices of K.*

Given the a complex K, we can take the barycentric subdivision of $\mathrm{bsd}(K)$, which we denote by $\mathrm{bsd}^2(K)$, and in general we can take the n-th barycentric subdivision of K denoted by $\mathrm{bsd}^n(K)$. The following lemma is straight forward.

Lemma 3.25. *Given a finite simplicial complex K, let $\epsilon > 0$, then there is an N such that each simplex of $\mathrm{bsd}^N(K)$ has diameter less than ϵ.*

Let K' be a subdivision of K. If σ is a simplex of K then we denote by $K(\sigma)$ the subcomplex of σ and all its faces. We denote by $K'(\sigma)$ the subcomplex of K' whose underlying space is σ.

Theorem 3.26. *Let $K' = \mathrm{bsd}^n(K)$ and let $i : |K'| \to |K|$ be the identity map. Then $i_* : H_p(K') \to H_p(K)$ is an isomorphism.*

Proof. We first prove the theorem for $K' = \mathrm{bsd}(K)$. Let $\Lambda : K \to K'$ and $\Theta : K' \to K$ be the minimal simplicial acyclic carriers induced by the identity map i. Note that it is easy to see that they exist. By the acyclic carrier theorem there exists a chain map λ carried by Λ and a chain map θ carried by Θ. Now let $\Psi : K \to K$ be the minimal simplicial acyclic carrier induced by the identity map. It is easy to see that Ψ carries the identity chain map $i : C_p(K) \to C_p(K)$, and it also carries the chain map $\theta \circ \lambda$ so that $\theta \circ \lambda$ is chain homotopic to the identity chain map.

We define the carrier $\Phi : K' \to K'$ as follows, if τ is a simplex in K', then we let σ_τ be the subcomplex of K consisting of all 2-simplices that intersect τ and their faces. Then we set $\Phi(\tau) = K'(\sigma_\tau)$. It is easy to see that Φ is a simplicial acyclic carrier and that it carries the identity chain map along with the chain map $\lambda \circ \theta$ which implies that $\lambda \circ \theta$ is chain homotopic to the identity chain map. We now have shown that θ and λ are chain homotopic inverses of each other which means that θ_* and λ_* are isomorphisms. However θ_* is by definition the identity map on homology, therefore $i_* : H_p(K') \to H_p(K)$ is an isomorphism.

The arguments also work to show that $i_* : H_p(K') \to H_p(K)$ is an isomorphism where $K' = bsd^{n+1}(K)$ and $K = bsd^n(K)$. Therefore by induction we have the proof. ∎

3.3 Lefschetz Fixed Point Theorem

Let G be a vector space with coefficients in $\mathbb{Z}_2$. Let $f : G \to G$ be a linear map and let A be the matrix representation of f for some particular basis. We define

$$\operatorname{tr} f = trA.$$

Recall that the trace of f is independent of any choice of basis, since if $B : G \to G$ is a change of basis from the basis for A, then the matrix of f in the new basis would be $B^{-1}AB$, and we have

$$\operatorname{tr} B^{-1}AB = \operatorname{tr} B^{-1}(AB) = \operatorname{tr}(AB)B^{-1} = \operatorname{tr} A.$$

In our setting, since we can find a basis for C_p, then for a chain map $\phi_p : C_p \to C_p$ we can represent ϕ_p as a matrix, and we denote the trace by $\operatorname{tr}(\phi, C_p(K))$. Furthermore this is also true for the induced homomorphism $(\phi_*)_p$. The following theorem is a specific case of the Hopf trace theorem for our setting of vector spaces with coefficients in $\mathbb{Z}_2$. The proof is included for completness.

Theorem 3.27 (Hopf Trace Theorem [23])**.** *Let K be a finite simplicial complex and $\phi : C_p \to C_p$ a chain map. Then*

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p(K)) = \sum_p (-1)^p \operatorname{tr}(\phi_*, H_p(K)).$$

Proof. Let G be a vector space with coefficients in $\mathbb{Z}_2$. Let H be a subspace of G and ϕ a homomorphism $\phi : G \to G$ that carries H to itself. We let ϕ' be the homomorphism $\phi' : G/H \to G/H$ induced by ϕ that is, if $[x]$ is an equivalence class in G/H then we take an element $y \in [x]$ and we define $\phi'([x]) := [\phi(y)]$. By taking an element y_i out of each nonzero equivalence class we form a basis $\{\alpha_1 + H, \ldots, \alpha_n + H\}$ for G/H. Furthermore, we let ϕ'' be the restriction of ϕ to H, so that ϕ'' is a homomorphism from H to itself. Let $\{\beta_1, \ldots, \beta_p\}$ be a basis of H. We show that

$$\operatorname{tr}(\phi, G) = \operatorname{tr}(\phi', G/H) + \operatorname{tr}(\phi'', H).$$

Let A and B be the matrices of ϕ' and ϕ'' with respect to their bases, then we have

$$\phi'(\alpha_j + H) = \sum_i a_{ij}(\alpha_i + H);$$

$$\phi''(\beta_j) = \sum_i b_{ij}\beta_i.$$

Standard arguments show that $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p\}$ is a basis for G and by definition G/H we have by the above equation that $\phi'(\alpha_j + H) = [\sum_i a_{ij}\alpha_i]$ which implies that

$$\phi(\alpha_j) = \sum_i a_{ij}\alpha_i + (\text{some element in } H);$$

$$\phi(\beta_j) = \sum_i b_{ij}\beta_i.$$

From this we get the matrix for ϕ is of the form

$$C = \begin{bmatrix} A & 0 \\ * & B \end{bmatrix},$$

which gives $\operatorname{tr} C = \operatorname{tr} A + \operatorname{tr} B$ as we wanted to show.

Applying this to the chain group $C_p(K)$, using the fact that $B_p \subset Z_p \subset C_p$, and that the chain map maps cycles to cycles and boundaries to boundaries, we have

$$\operatorname{tr}(\phi, C_p) = \operatorname{tr}(\phi, C_p/Z_p) + \operatorname{tr}(\phi, Z_p/B_p) + \operatorname{tr}(\phi, B_p).$$

If we restrict the range of ∂_p to get a new homomorphism $\partial'_p : C_p \to B_{p-1}$ then Z_p is the kernel, and ∂' induces an isomorphism of C_p/Z_p with B_{p-1}. The map ϕ commutes with this isomorphism, since it commutes with the original homomorphism

∂. Therefore we have

$$\operatorname{tr}(\phi, C_p/Z_p) = \operatorname{tr}(\phi, B_{p-1}).$$

From this we have

$$\operatorname{tr}(\phi, C_p) = \operatorname{tr}(\phi, B_{p-1}) + \operatorname{tr}(\phi_*, H_p) + \operatorname{tr}(\phi, B_p).$$

Multiplying both sides by $(-1)^p$ and summing over all p we get cancelation over the first and last terms and obtain the resulting formula. ∎

Definition 3.28. *Let K be a finite complex and let $h : |K| \to |K|$ be a continuous map. We define the* ***Lefschetz number of*** *h to be*

$$\Lambda(h) = \sum (-1)^p \operatorname{tr}(h_*, H_p(K)).$$

Lemma 3.29. *Let K be a finite simplicial complex and with $K' = \operatorname{bsd}^N(K)$ for sone $N > 0$. Let $h_1 : |K| \to |K|$ be a continuous map and define $h_2 : |K'| \to |K'|$ where $h_2(x) = h_1(x)$. If Φ_{h_1} and Φ_{h_2} are acyclic, then $\Lambda(h_1) = \Lambda(h_2)$.*

Proof. Let $i : |K'| \to |K|$ be the identity map. We have that $i_* : H_p(K') \to H_p(K)$ is an isomorphism and that $(h_2)_* = {i_*}^{-1} \circ (h_1)_* \circ i_*$. Therefore $(h_2)_*$ and $(h_1)_*$ will have the same trace for each summand p. ∎

The following theorem, the Lefschetz Fixed-Point Theorem, is classical, but we need prove it for the specific homology that we have defined.

Theorem 3.30 (Lefschetz Fixed-Point Theorem)**.** *Let K be a finite complex and $h : |K| \to |K|$ be a continuous map. Suppose that Φ_h exists. If $\Lambda(h) \neq 0$, then h has a fixed point.*

Proof. Let $||x|| = \sup |x_i|$. Suppose that h has no fixed points. Let $\epsilon = \min ||x - h(x)|| > 0$. By uniform continuity there exists a δ such that if $|x - y| < \delta$ then $|h(x) - h(y)| < \epsilon/2$. Choose N so that for all simplices σ of $K' = \text{bsd}^N(K)$, the diameter of σ is less than $\epsilon/2$ and δ. Let h' be the map $h' : |K'| \to |K'|$ with $h'(x) = h(x)$ for $x \in |K'|$. We first show that $\Phi_{h'}$ exists. We construct a simplicial acyclic carrier Φ that carries h'. For a vertex v, we define $\Phi(v)$ to be all 2-simplices in K' that intersect $h(v)$. It is easy to see that the subcomplex $\Phi(v)$ is acyclic. For an edge e, we need to construct a subcomplex that is acyclic and contains the image of e. First we take all the 2-simplices that intersect $h(e)$. If the collection of those simplices are acyclic, then we define $\Phi(e)$ to be that collection For the case where the collection is not acyclic, the image is a connected set in $|K'|$ so the non-acyclicity will not come from the 0-homology but rather the 1-homology, which means that the collection contains some holes. If the holes cannot be filled in by 2-simplices, then this would contradict the assumption that Φ_h exists. Therefore the holes are made up of 2-simplices in K'. We add them to the collection so that we have acyclicity, and we define $\Phi(e)$ to be this collection. For a 2-simplex s, we take the collection of all 2-simplices that intersect $h(|s|)$ along with $\Phi(e)$ for edges $e \in s$, add all 2-simplices that are required to fill in any 1-holes. We define $\Phi(s)$ to be this collection. The resulting map Φ is a simplicial acyclic carrier that carries h', and it is easy to see that $\Phi = \Phi_{h'}$.

Given a chain map ϕ carried by $\Phi_{h'}$ we show the diagonal entries of the matrices representing ϕ are all zero. Consider $\phi_0 : C_0(K') \to C_0(K')$. Let v be a vertex that is a basis element for C_0. Since $|h(v) - v| > \epsilon$, and since the diameter of simplices in K' is less than $\epsilon/2$, we have that $\Phi(v)$ cannot contain any 2-simplices in which v is contained, which implies that $\phi(v) \neq v$. Therefore the diagonal for the matrix of ϕ_0

has all zero entries.

Now we consider the case for $\phi_1 : C_1(K') \to C_1(K')$. Let e be an edge that is a basis element for C_1. Let m be the midpoint of the edge e. By the triangle inequality, the image $h(e)$ has to lie outside of the circle with center m and radius $\epsilon/2$. The circle with center m and radius $\epsilon/2$ will contain all 2-simplices that can possibly intersect the edge e. We now show that $\Phi(e)$ cannot contain any 2-simplex that intersects e. Suppose not. Then the 2-simplices inside the circle were filled in to eliminate non-acyclicity which means that the subcomplex of 2-simplices that $h(e)$ intersects wraps around the exterior of the circle and has a 1-dimensional hole. Let a be a point in $h(e)$. The line through m perpendicular to the line segment between m and a divides the plane into two half-planes. The half-plane containing a must contain all of $h(e)$, because the distance between a and m is larger than $\epsilon/2$ and the distance from a to any other point $b \in h(e)$ is less than $\epsilon/2$. This is a contradiction because the 2-simplices intersecting $h(e)$ must wrap around the circle of radius $\epsilon/2$ and the diameter of each simplex is less than $\epsilon/2$. Therefore $\Phi(e)$ cannot contain any 2-simplex that intersects e which also implies that $e \notin \phi_1(e)$ so the diagonal of ϕ_1 is made up of zeros. For $\phi_2 : C_2(K') \to C_2(K')$ similiar arguments to the case for edges also show that the diagonal of ϕ_2 is made up of zeros. Therefore by the Hopf trace theorem we get that $\Lambda(h') = 0$ and by the previous lemma we have $\Lambda(h) = 0$. ■

Suppose that we have a complex K consisting of 2-simplices and their faces with a subcomplex K_0 also composed of 2-simplices and their faces. Let us consider a continuous map $h : |K| \to |K|$ where for $x \in |K| \setminus |K_0|$ we have $h(x) \in |K|$ and for $x \in |K_0|$ we have $h(x)$ lies either in $|K_0|$ or outside $|K|$. We denote such a map by $h : (|K|, |K_0|) \to (|K|, |K_0|)$. What we would like to know is whether the map h has a fixed point on the domain $|K| \setminus |K_0|$. If we consider relative homology on K mod

K_0, and if we are able to compute $h_* : H(K, K_0) \to H(K, K_0)$ then the following definition and theorem show we have a Lefschetz number and Lefschetz Fixed Point Theorem that extends to maps of relative homology. This then provides such a fixed point theorem for h.

Definition 3.31. *For a continuous map $h : (|K|, |K_0|) \to (|K|, |K_0|)$. We define the **Lefschetz number of** h to be*

$$\Lambda(h) = \sum (-1)^p \operatorname{tr}(h_*, H_p(K, K_0)).$$

Theorem 3.32. *Let K be a finite complex and let $h : (|K|, |K_0|) \to (|K|, |K_0|)$ be a continuous map and suppose that Φ_h is acyclic-valued. If $\Lambda(h) \neq 0$, then h has a fixed point.*

Proof. The proof is essentially the same as the regular Lefschetz Fixed Point Theorem. Subdivide K by barycentric subdivisions so that the matrix representing the chain map $\phi : C_p(K', K_0') \to C_p(K', K_0')$ has zeros along the diagonal and it follows from Hopf Trace Theorem. ∎

This concludes the discussion of the simplicial homology we use for our computations. In the next chapter we explain how Theorem 3.32 is applied in order to show the existence of periodic orbits.

4 Summary of Computational Methods

In this chapter we give a summary of the use of homology and the Lefschetz Fixed Point Theorem used in our computations. Suppose we are given a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and some triangulation of a compact polyhedron in $\mathbb{R}^2$, which gives a finite simplicial complex Δ made up of 2-simplices and their faces. We use the notation $|\Delta|$ to mean the underlying space of the simplicial complex Δ.

First we create a piecewise linear map $\widehat{f}$ that approximates f. Given a 2-simplex S_1 with vertices v_1, v_2, v_3, we map the vertices forward by f to get $w_1 = f(v_1), w_2 = f(v_2)$, and $w_3 = f(v_3)$. We let $\widehat{f}(v_1) = w_1, \widehat{f}(v_2) = w_2, \widehat{f}(v_3) = w_3$, and then we extend linearly for all points in S_1. See Figure 4.2.

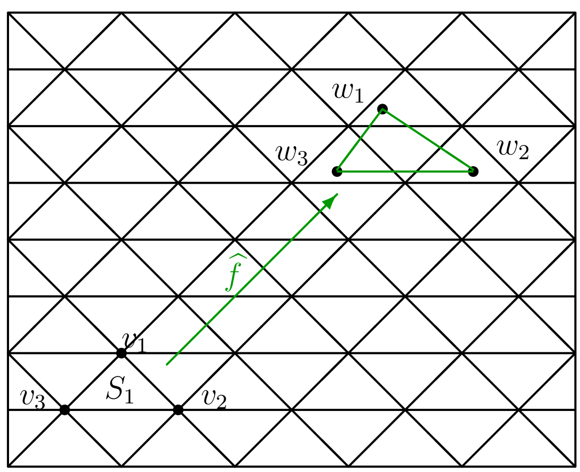


Figure 4.2: The piecewise linear map $\widehat{f}$ on S_1.

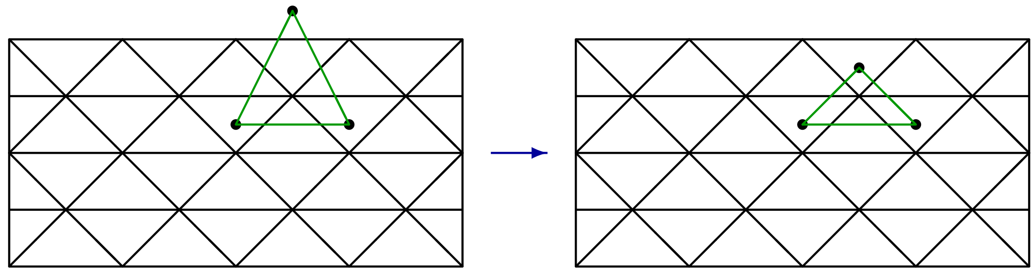


Figure 4.3: The image of the vertex is moved to the nearest barycenter.

We define the domain of $\widehat{f}$ by $|D|$ where

$$D := \bigcup_{\sigma \in \Delta} \{\sigma : \widehat{f}(\sigma) \cap |\Delta| \neq \emptyset\},$$

that is, the union of all simplices such that their image under $\widehat{f}$ intersects the given triangulation. If a 2-simplex in the domain has a vertex whose image under $\widehat{f}$ lies outside of $|\Delta|$, then we redefine the image of that vertex to be the nearest barycenter of a 2-simplex in Δ, and then we recompute $\widehat{f}$ on that 2-simplex. If the size of the 2-simplex is small, then the change in image is expected to be a small perturbation of the original image. This step increases the chances of obtaining a simplicial acyclic carrier $\widehat{f}$. See Figure 4.3.

In the end we have constructed a piecewise linear map $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ whose domain is $|D|$. From this map $\widehat{f}$, we construct a combinatorial multivalued map $\mathcal{F} : D \rightrightarrows \Delta$ from the 2-simplices of D to the 2-simplices of Δ. For each 2-simplex $S \in D$ we define $\mathcal{F}$ by

$$\mathcal{F}(S) := \{\sigma \in \Delta : |\widehat{f}(S)| \cap \sigma \neq \emptyset\}.$$

We say that $\mathcal{F}$ is a **combinatorial enclosure** for $\widehat{f}$ if for each $S \in D$ we have

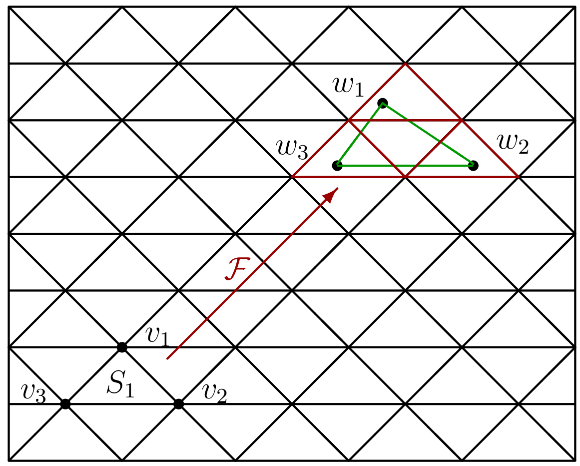


Figure 4.4: The red triangles are the image of $\mathcal{F}$ on the simplex S_1.

the following property

$$\widehat{f}(S) \subset \mathrm{int}(|\mathcal{F}(S)|).$$

If we have a combinatorial enclosure of $\widehat{f}$, then we can focus on the dynamics of the combinatorial map $\mathcal{F}$, from which we can infer some dynamics for the map $\widehat{f}$. Here we give some of the definitions and theorems for dynamics on $\mathcal{F}$.

Given a 2-simplex $\sigma \in D$, we define a **full orbit** with initial simplex σ under $\mathcal{F}$ to be a sequence of 2-simplices $Q_n \in D$ with $n \in \mathbb{Z}$ such that

(1) $Q_0 = \sigma$;

(2) $Q_{n+1} \in \mathcal{F}(Q_n)$.

For example if $S_1 \in D$ with $\mathcal{F}(S_1) = \{S_4, S_5, S_6, S_7\}$ and $S_4 \in D$ with $\mathcal{F}(S_4) = \{S_1, S_2, S_3\}$, then the sequence $\{\ldots, S_4, S_1, S_4, S_1, \ldots\}$ with $Q_0 = S_1$ would be a full orbit with initial simplex S_1, and furthermore we would say this is a period-2 orbit. See Figure 4.5. We use the notation $\gamma_n(S_1)$ to denote the sequence of a full orbit with initial simplex S_1 so that $\gamma_0(S_1) = S_1$.

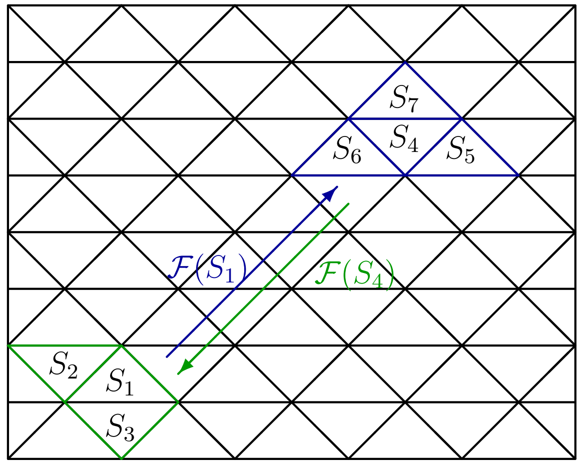


Figure 4.5: The forward orbit of S_1 is $\{S_4, S_5, S_6, S_7\}$ while the forward orbit of S_4 is $\{S_1, S_2, S_3\}$. This implies the existence of the orbit $\{\dots, S_4, S_1, S_4, S_1, \dots\}$ for $\mathcal{F}$.

If we take a finite subset $\mathcal{N}$ of D, then we can define the **combinatorial maximal invariant set** in $\mathcal{N}$ under $\mathcal{F}$ by

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) := \{\sigma \in \mathcal{N} \mid \exists \gamma_n(\sigma) \subset \mathcal{N}\},$$

that is, all $\sigma \in \mathcal{N}$ such that there exists a full orbit with initial simplex σ that is contained in $\mathcal{N}$. One can easily find the combinatorial maximal invariant set by considering a graph where the 2-simplices of $\mathcal{N}$ are the vertices with directed edges correspond to the forward image of $\mathcal{F}$ restricted to $\mathcal{N}$. The strongly connected components of the graph together with the connecting paths between these components correspond to the combinatorial maximal invariant of $\mathcal{N}$. These can easily be computed in linear time by using standard depth-first search algorithms to locate the strongly connected components of the graph [9]. For a given finite subset $\mathcal{N}$ of D we say that $\mathcal{N}$ is a **combinatorial isolating neighborhood** if

$$\{\sigma \in D \mid \sigma \cap |\,\mathrm{Inv}(\mathcal{N}, \mathcal{F})| \neq \emptyset\} \subset \mathcal{N}.$$

Our goal is to take 2-simplices in D that exhibit periodic behavior and try to isolate them. As we show later, an isolated combinatorial invariant set for $\mathcal{F}$ corresponds to an isolating neighborhood for an invariant set of $\widehat{f}$. The following algorithm from Day [10] is used to obtain isolation with input S being a subset of D that consists of periodic simplices under $\mathcal{F}$.

Algorithm-Isolation

1\. $S :=$ invariant set of D.
2\. $S' :=$ empty set.
3\. **while** $(S' \subset S)$
4\. $S' := S$
5\. $\mathcal{N} :=$ all 2-simplices in Δ that intersect $|S'|$.
6\.   **if** $(\mathcal{N} \not\subset D)$
7\.   **return** “Did not isolate”
8\.   **else**
9\. $S := \mathrm{Inv}(\mathcal{N}, \mathcal{F})$
10\. **endwhile**
11\. **return** S and $\mathcal{N}$

For example suppose we have a 2-simplex S_1 that is 1-periodic for $\mathcal{F}$ as in Figure 4.6. In the first step of the algorithm we would take all of the simplices in D that intersect S_1 which would be all of the blue triangles. We then find the maximal invariant set inside the blue triangles, and suppose the maximal invariant set is $S_1 \cup S_2$. This set is not isolated, since there are simplices that intersect S_2 that are not in the blue triangles. We add the simplices that intersect S_2 to the set of blue triangles and then recompute the maximal invariant set of the blue triangles. If the maximal

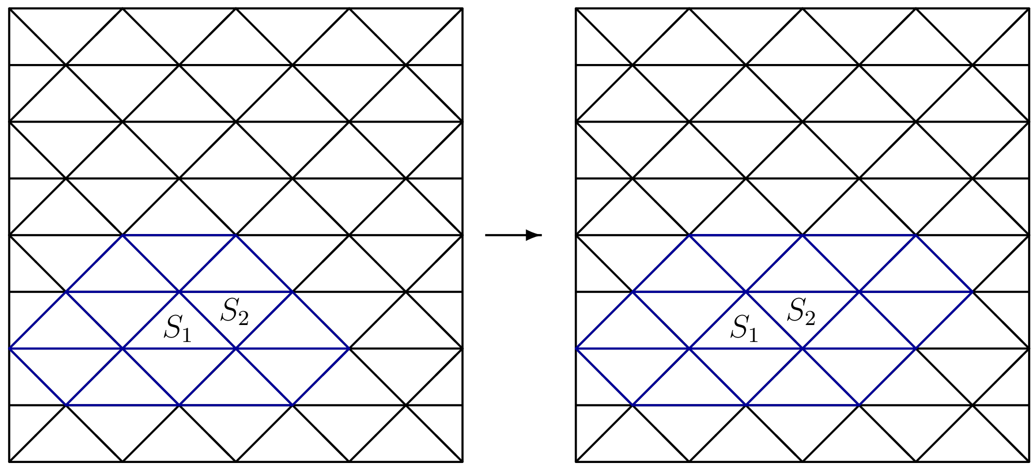


Figure 4.6: Left: S_1 and S_2 are the maximal invariant set but they are not isolated. Right: S_1 and S_2 are now isolated

invariant set is $S_1 \cup S_2$, then we are done, and the set of blue triangles is an isolating neighborhood with maximal invariant set $S_1 \cup S_2$. This is true of course only if the blue triangles are part of D. If at any time one of the blue triangles is not in D, then the algorithm returns "no isolation".

At this point we try to find and isolate many periodic orbits for $\mathcal{F}$, and then find connecting orbits between those periodic orbits with the condition that the connecting orbits can also be isolated. For example suppose that S_1 is a fixed point for $\mathcal{F}$, that is $S_1 \subset \mathcal{F}(S_1)$, and that S_2 and S_3 are a period-2 orbit for $\mathcal{F}$. Furthermore suppose there is an S_4 such that $S_4 \subset \mathcal{F}(S_3)$ and $S_1 \subset \mathcal{F}(S_4)$ and that there exists an S_5 such that $S_5 \subset \mathcal{F}(S_1)$ and $S_2 \subset \mathcal{F}(S_5)$. See Figure 4.7. We then let $S = \{S_1, S_2, S_3, S_4, S_5\}$ and call the isolation algorithm on S. If we can isolate S, then we say we have isolated recurrent dynamics for $\mathcal{F}$. The reason for hooking together periodic orbits with low periods with connecting orbits is so that now we have infinitely many periodic orbits with arbitrarily large periods for $\mathcal{F}$.

Before we discuss the next step, we need to talk about the notion of an index pair. To motivate, consider the following example where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a map given by

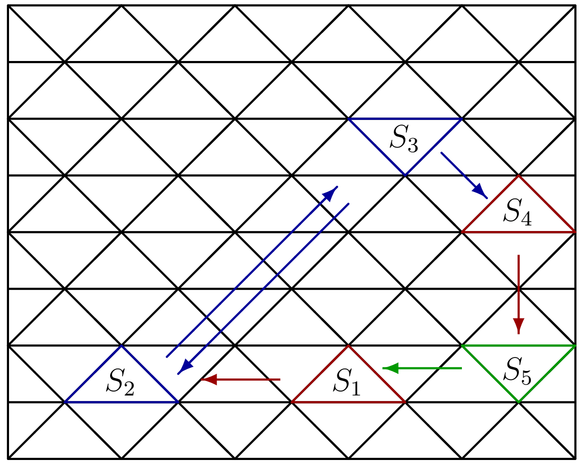


Figure 4.7: We have the period two orbit between S_2 and S_3 with a connecting orbit to S_1 which maps to itself.

the matrix

$$g := \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Define P_1 to be the set $P_1 := [-1,1] \times [-1,1]$ and define $P_0 := (\{-1\} \times [-1,1]) \cup (\{1\} \times [-1,1])$. Under the map g the image of P_1 gets contracted in the y direction by a factor of $1/2$ and expanded in the x direction by 2. The maximal invariant set in P_1 is the fixed point $q := (0,0)$. From Figure 4.8 it is easily seen that the compact sets P_1 and P_0 satisfy the following three properties:

(1) $\mathrm{Inv}(\mathrm{cl}(P_1, P_0), g) \subset \mathrm{int}(\mathrm{cl}(P_1 \setminus P_0))$;

(2) $g(P_0) \cap P_1 \subset P_0$;

(3) $\mathrm{cl}(g(P_1) \setminus P_1) \cap P_1 \subset P_0$.

A pair (P_1, P_0) consisting of two compact sets P_1 and P_0 where $P_0 \subset P_1$ and satisfy the above properties for a given map g is called an **index pair** for g. The set P_0 can be thought of as an "exit set" for P_1. Later we will be computing relative homology groups using index pairs (P_1, P_0) where P_1 is the underlying space of some finite simplicial complex $\mathcal{P}_1$ and P_0 the underlying space of $\mathcal{P}_\prime$, a subcomplex of $\mathcal{P}_1$.

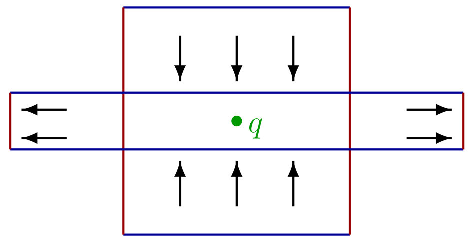


Figure 4.8: P_1 is the unit square with P_0 being the red sides. Expansion in the x-direction and contraction in y-direction with fixed point q.

Now we return to the discussion of the dynamics of the combinatorial map. After isolating recurrent dynamics, we find **index pairs** for the map $\widehat{f}$ using the isolating neighborhoods we computed from $\mathcal{F}$. This will be the first point where the combinatorial dynamics will imply something about the dynamics of $\widehat{f}$. We use the following algorithm from [10] to compute index pairs using a finite set $\mathcal{N} \subset D$ that is an isolating neighborhood for $\mathcal{F}$ with maximal invariant set S.

Algorithm-Combinatorial Index Pair

1. $S := \mathrm{Inv}(\mathcal{N}, \mathcal{F})$;
2. Let the domain of $\mathcal{F}$ be restricted to $\mathcal{N}$;
3. $C := \mathcal{N} \setminus S$;
4. $\mathcal{P}_0 := \mathcal{F}(S) \cap C$;
5. **repeat**
6. $\quad \mathcal{P}_0' := \mathcal{P}_0$;
7. $\quad \mathcal{P}_0 := \mathcal{F}(\mathcal{P}_0) \cap C$;
8. $\quad \mathcal{P}_0 := \mathcal{P}_0 \cup \mathcal{P}_0'$;
9. **until**$(\mathcal{P}_0 = \mathcal{P}_0')$;
10. $\mathcal{P}_1 := S \cup \mathcal{P}_0$;
11. **return**$(\mathcal{P}_1, \mathcal{P}_0,)$;

Theorem 4.1 ([17]). *Given a combinatorial enclosure $\mathcal{F}$ of a continuous map $\widehat{f}$:*

$\mathbb{R}^2 \to \mathbb{R}^2$ *with a finite subset* $\mathcal{N} \subset D$ *and* $S = \mathrm{Inv}(\mathcal{F}, \mathcal{N})$ *given by the Isolation algorithm, then the Combinatorial Index Pair algorithm returns finite subsets of* Δ, $\mathcal{P}_1, \mathcal{P}_0$, *with which we define* $P_1 := |\mathcal{P}_1|, P_0 := |\mathcal{P}_0|$, *that have the following properties:*

(1) $|S|$ *is an isolating neighborhood for* g;

(2) (P_1, P_0) *is an index pair for* g *which isolates* $\mathrm{Inv}(|S|, \widehat{f})$;

Proof. First, note that in each pass of the repeat loop we have that $\mathcal{P}_0 \subset \mathcal{N}$, and since $\mathcal{N}$ is a finite subset of D, eventually the repeat loop must stop.

We have $\mathcal{P}_1 := S \cup \mathcal{P}_0 \subset \mathcal{N}$. We take the directed graph with vertices corresponding to 2-simplices in $\mathcal{N}$ and directed edges between them corresponding to the images of map $\mathcal{F}$. We form the subgraph with vertices corresponding to $\mathcal{P}_1$. From here on we do not distinguish between the vertices of the graph and the 2-simplices in $\mathcal{N}$, as it should be clear from the context. The vertices of the set S form the strongly connected components and the connections between the strongly connected components of the directed graph on $\mathcal{N}$ and of the directed subgraph restricted to $\mathcal{P}_1$. The construction of the set $\mathcal{P}_0$ in the algorithm implies that for each vertex of $p \in \mathcal{P}_0$ there exists a vertex $s \in S$ such that there is a directed path from s to p. On the other hand, for each vertex in $p \in \mathcal{P}_0$ there can not exist a directed path from p to a simplex $s \in S$. Otherwise p would be in a strongly connected component or a connection between stongly connected components so that $p \in S$, which is a contradiction since by construction $\mathcal{P}_0 \cap S = \emptyset$.

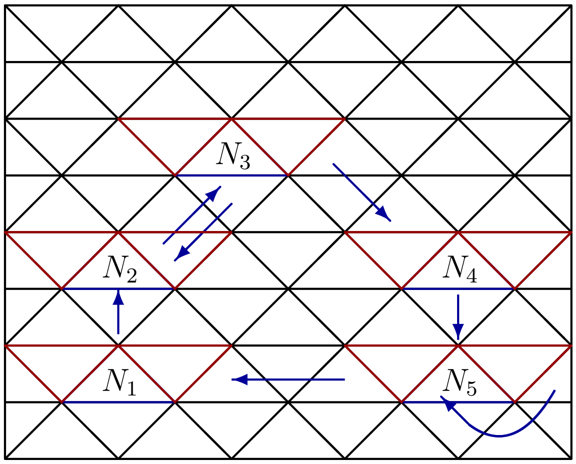
To prove (1), suppose there exists an $x \in \mathrm{Inv}(|S|, \widehat{f})$ such that x is not in the interior of $|S|$ which means x is on the boundary of $|S|$ denoted by $\mathrm{bd}|S|$. This means that x lies on an outer edge of a 2-simplex in S so that x lies in a simplex $\sigma \subset \mathcal{N}$ but $\sigma \notin S$. Now since $x \in \mathrm{Inv}(|S|, \widehat{f})$, there exists an $x_1 \in |S|$ such that $f(x_1) = x$ and $x_2 \in |S|$ such that $\widehat{f}(x) = x_2$. Let $S_1 \in S$ such that $x \in S_1$ and let $S_2 \in S$

such that $x_2 \in S_2$. Since $\mathcal{F}$ is a combinatorial enclosure of $\widehat{f}$, we have $\sigma \in \mathcal{F}(S_1)$ and $S_2 \in \mathcal{F}(\sigma)$. This implies that there exists a directed path in the graph from S through σ and back to S so that σ is in the strongly connected component of the graph, which is a contradiction. Therefore x must be in the interior of $|S|$ so that $|S|$ is an isolating neighborhood for $\widehat{f}$.

To prove (2), we have prove all three properties in the index pair definition. First, it is trivial to see that $\mathrm{cl}(P_1 \setminus P_0) = |S|$ and by (1) we have $\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \widehat{f}) \subset \mathrm{int}(\mathrm{cl}(P_1 \setminus P_0))$. Second we need to prove that $\widehat{f}(P_0) \cap P_1 \subset P_0$. Let $y \in \widehat{f}(P_0) \cap P_1$ and let $x \in P_0$ such that $\widehat{f}(x) = y$ also let $\sigma \in \mathcal{P}_0$ such that $x \in |\sigma|$. If $y \in S_1$ for some $S_1 \in S$ then there would exist a directed edge from σ to S_1 in the subgraph but we showed previously in (1) that this can not happen. So $y \in |\mathcal{P}_1| \setminus |S| \subset |\mathcal{P}_0| \subset P_0$. Finally we need to show that $\mathrm{cl}(\widehat{f}(P_1) \setminus P_1) \cap P_1 \subset P_0$. Let $y \in \mathrm{cl}(\widehat{f}(P_1) \setminus P_1) \cap P_1$ and suppose $y \notin P_0$. This implies that y is contained on the edge of a 2-simplex S_1 for some $S_1 \in S$ and is contained on the edge of a 2-simplex $\sigma \in N \setminus S$ with $\sigma \notin \mathcal{P}_0$ such that there exists a point $z \in \sigma$ and a point $x \in P_1$ with $\widehat{f}(x) = z$. Let $\tau \in \mathcal{P}_1$ such that $x \in \tau$. This means that $\sigma \in \mathcal{F}(\tau)$. If $\tau \in S$ then this would contradict $\sigma \notin \mathcal{P}_0$ and if $\tau \in \mathcal{P}_0$ then this would also contradict $\sigma \notin \mathcal{P}_0$ since the algorithm would have added σ to $\mathcal{P}_0$. Therefore we have that $\mathrm{cl}(\widehat{f}(P_1) \setminus P_1) \cap P_1 \subset P_0$ and we have proven (2). ■

We apply the combinatorial index pair algorithm to get index pairs P_0 and P_1 for the map $\widehat{f}$. We label each of the disjoint sets of $|P_1|$ with the label N_i. We construct a directed graph with the N_i as vertices and a directed edge from a N_i to a N_j exists if there exists a 2-simplex in $N_i \setminus P_0$ that has a forward image under $\mathcal{F}$ into $N_j \setminus P_0$. See Figure 4.9.

What we would like to do now is to show that the map $\widehat{f}$ contains all dynamics



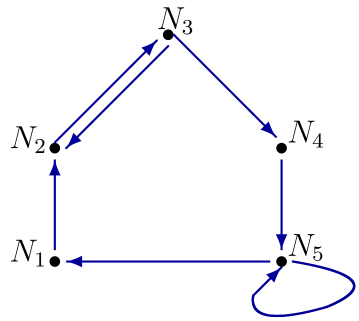


Figure 4.9: Left: The index pair consisting of 5 components. The red triangles indicate the exit set. Right: The transitional graph corresponding to the index pair.

which correspond to the shift dynamics of the transition graph of the N_i. For example, the transition graph above has an orbit $(\ldots, N_3, N_2, N_3, \ldots,)$ that alternates between the vertices N_2 and N_3 and we will want to find a point $s \in N_2$ such that $\widehat{f}(s) \in N_3$ and $\widehat{f}^2(s) \in N_2$ and so on. In fact we establish a stronger result by finding a point $s \in N_2$ such that $\widehat{f}(s) \in N_3$ and $\widehat{f}^2(s) = s$. We first take the image of N_2 under the map $\widehat{f}$. If $\widehat{f}(|N_2|)$ is not contained in N_3, then we check if $\mathcal{F}(N_2)$ is acyclic in the regular homology. If $\mathcal{F}(N_2)$ is acyclic, then there exists a retraction map r_{N_3} that retracts $|\mathcal{F}(N_2)|$ to N_3 by keeping N_3 fixed and retracting the other points to the exit set of N_3. If $\widehat{f}(|N_2|)$ is contained in N_3, then we let r_{N_3} be the identity. We let $\widehat{f}_{N_2 \to N_3} := r_{N_3} \circ \widehat{f}|_{N_2}$. Now we repeat the same arguments to get the continuous map $\widehat{f}_{N_3 \to N_2}$. We define the map $\widehat{f}_{N_2 \to N_2} := \widehat{f}_{N_2 \to N_3} \circ \widehat{f}_{N_2 \to N_3}$ so that $\widehat{f}_{N_2 \to N_2} : (|N_2|, |N_2 \cap P_0|) \to (|N_2|, |N_2 \cap P_0|)$ is a continuous map on index pairs. We check that the map $\widehat{f}_{N_2 \to N_2}$ has a minimal relative acyclic carrier, and then we compute the relative homology map for $\widehat{f}_{N_2 \to N_2}$. From Figure 4.9 we see that dimension $p = 1$ is the only dimension with nontrivial homology and we let $[\widehat{f}_{N_2 \to N_2}]_*$ denote the matrix of the homology map $[\widehat{f}_{N_2 \to N_2}]_* : H_1(N_2, N_2 \cap P_0) \to H_1(N_2, N_2 \cap P_0)$. We apply the

Lefshetz Fixed Point Theorem to the map $\widehat{f}_{N_2 \to N_2}$ so that if the matrix $[\widehat{f}_{N_2 \to N_2}]_*$ has nonzero trace, then there exists an $x \in |N_2| \backslash |P_0|$ such that $\widehat{f}(x) \in |N_3|$ and $\widehat{f}^2(x) = x$. The point x does not have a smaller period since it must travel through the different components. This means that the point x is a periodic point corresponding point to the periodic sequence $(\ldots, N_3, N_2, N_3, \ldots,)$ in the shift space.

We would like to verify that there is a corresponding point in $|P_1| \setminus |P_0|$ for every point in the shift space. There are some difficulties with the verification process. For example, let $(N_2, N_3)^n$ denote the repeating symbol space n-times and take the shift space point $(\ldots, N_1, N_2, N_3, N_2, N_3, (N_2, N_3)^n, N_4, N_5, N_1, N_2, N_3, N_2, N_3, (N_2, N_3)^n, \ldots)$. For the $N_3, N_2, N_3, (N_2, N_3)^n$ part of the sequence we compose the map $\widehat{f}_{N_3 \to N_3}$ n-times and by the functorial properties we have that the matrix $[\widehat{f}_{N_3 \to N_3}]_*^n$ is the matrix of the homology map for $\widehat{f}^n_{N_3 \to N_3}$. To verify a point for the sequence we need to compose homology matrices $[\widehat{f}_{N_5 \to N_1}]_* [\widehat{f}_{N_4 \to N_5}]_* [\widehat{f}_{N_3 \to N_4}]_* [\widehat{f}_{N_3 \to N_3}]_*^n [\widehat{f}_{N_2 \to N_3}]_* [\widehat{f}_{N_1 \to N_2}]_*$ and apply the Lefschetz fixed point theorem. But n is arbitrary, which means there are infinitely many symbols in the shift space that we need to verify, which of course we cannot do, since we can only check finitely many conditions. We are rescued from this difficulty if $[\widehat{f}_{N_3 \to N_3}]_*^k = [\widehat{f}_{N_3 \to N_3}]_*$ for some positive integer k. Now there are only finitely many powers of the matrix which implies that the verification of $(\ldots, N_3, (N_2, N_3)^{j+k}, N_4, N_5, N_1, N_2, N_3, (N_2, N_3)^{j+k}, \ldots)$ will be the same as verifying $(\ldots, N_3, (N_2, N_3)^j, N_4, N_5, N_1, N_2, N_3, (N_2, N_3)^j, \ldots)$. Therefore if the matrix satisfies this property then the verification process is finite. The matrix $[\widehat{f}_{N_3 \to N_3}]_*$ consists of entries from $\mathbb{Z}_2$, which means that either the matrix is nilpotent or it satisfies $[\widehat{f}_{N_3 \to N_3}]_*^k = [\widehat{f}_{N_3 \to N_3}]_*$ for some k. If the matrix is nilpotent, then there exists a k such that $[\widehat{f}_{N_3 \to N_3}]_*^k$ is the zero matrix which means for all $n \geq k$ the composition of the matrices is the zero matrix, which means the trace is zero, and therefore we cannot verify the existence of a periodic point corresponding to that symbol sequence. Suppose d is the number of entries of $[\widehat{f}_{N_3 \to N_3}]_*$. If $[\widehat{f}_{N_3 \to N_3}]_*^{2^d}$ is the zero matrix, then $[\widehat{f}_{N_3 \to N_3}]_*$ is nilpotent, otherwise if it is non-zero then there exists a $k < 2^d$ such that $[\widehat{f}_{N_3 \to N_3}]_*^k = [\widehat{f}_{N_3 \to N_3}]_*$.

The following discussion of showing a semiconjugacy is from Day [11]. Let Σ_T denote the shift space for the transition matrix of the N_i. Define S to be the set $\mathrm{Inv}(|P_1|, \widehat{f})$ which is compact. For $x \in S$ let us denote $x_j = \widehat{f}^j(x)$. We know that $x_j \in N_i$ for some i which we denote by N_{i_j}. We define a map $h : S \to \Sigma_T$ as follows:

$$h(x) := (\ldots, N_{i_{-1}}, N_{i_0}, N_{i_1}, \ldots).$$

It is easy to see that h is continuous. Let $S' := \mathrm{cl}(h^{-1}(\{\text{periodic orbits of } \Sigma_T\})$. The set S' is a compact subset of S. Since h is continuous and S' is compact, we have $h(S')$ is compact. The periodic orbits of Σ_T are contained in $h(S')$, and since the space Σ_T is equal to the closure of the periodic orbits we have $h(S') = \Sigma_T$. Moreover, since $S' \subset S$, we have that h maps S onto Σ_T. For the restriction of $\widehat{f}$ to S, $\widehat{f} : S \to S$, and for the shift map, $\sigma : \Sigma_T \to \Sigma_T$, it easy to see that $\sigma \circ h = h \circ \widehat{f}$. This implies that h is a semiconjugacy from $\widehat{f}$ to σ. Since $\widehat{f}$ is semiconjugate to σ, then the dynamics of σ is a lower bound for the dynamics of $\widehat{f}$. In particular if the shift-map σ is chaotic, then $\widehat{f}$ contains chaotic-behavior, and the entropy of $\widehat{f}$ is greater than or equal to the entropy of σ. In our example the matrix for the shift map σ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix implies that the shift-map is chaotic, and it has a largest positive eigenvalue of 1.4433 with $\log(1.4433) \approx 0.159$. So if we can verify that h is a semiconjugacy, then we can conclude that the map $\widehat{f}$ has chaos and that the entropy is greater than or equal to $\log(1.4433)$.

5 Implementation

Given a time-series of data points d_n that lie in some compact set $M \subset \mathbb{R}^2$, we form a triangulation using a subset of the time-series data points d_n. The triangulation that we use is called the **Delaunay Triangulation**. The Delaunay triangulation has the property that the open circumballs of the triangles contain no other vertices in the triangulation, and it gives the convex hull of the vertices. We chose the Delaunay triangulation for these properties and also because of the fact there are efficient algorithms for inserting and deleting points from the triangulation. For a description of the Delaunay Triangulation and the insertion algorithm see [6]. For the deletion algorithm see [13]. In what follows, we have conditions that we require on our triangulation. We have an algorithm that involves adding and removing appropriate data points to the triangulation until the conditions for the triangulation are met. The final triangulation gives us a finite simplicial complex on which we define a piecewise linear map $\widehat{f}$ and a combinatorial map $\mathcal{F}$ based upon the behavior of the time-series data points. The methods of Chapter 4 are used to make inferences about the dynamics of $\widehat{f}$. If the time-series comes from experimental data, then we conclude that the dynamics present in $\widehat{f}$ approximate the dynamics of the physical system that produced the experimental data. If the time-series comes from an iteration of a continuous map $f : M \to M$ or a projection of an iteration of a continuous map, then we conclude that the dynamics present in $\widehat{f}$ approximate the dynamics of the continuous map f if conditions on embedding hold as in Chapter 2.

We now discuss the implementation of our code. We define a **data point** to be a point from the time-series. If a data point is currently a vertex of the triangulation, then it is refered to as a vertex; otherwise it is called a **sample point**. We start with a rectangle consisting of four points A, B, C, and D, such that the time-series data is contained in the interior of the rectangle. The four points A, B, C, D are called the **exterior points**, and it is important to clarify that the exterior points are not data points. We start with an initial triangulation of the rectangle consisting of the two triangles ABC and CDA. Here we give a list of the steps in the implementation of our code using this initial triangulation.

Implementation Steps

1. Build Interior Method
2. Interchange Method
3. Selection Method
4. Rangemaker Method
5. Index Pair Method
6. Compute Homology Map Method

5.1 Build Interior Method

This method has four arguments, n,m,k, and δ. The argument n denotes how many total data points we have, m denotes how many data points are initially used as vertices in the triangulation, k denotes a transfer rate to be explained later, and δ is a positive real number. We take the n data points and put them on a list called

$$\texttt{temp} := \{z_1, z_2, \ldots, z_n\}$$

For each data point z_i, we add a bounded noise r_i to get $d_i := z_i + r_i$ where r_i is contained in the ball of radius δ centered at 0. We call this action a **blurring** of the `data points`, and we call d_i a **blurred data point**. The reason for the blurring is that the data points often come from a forward orbit on a transitive attractor which typically has fractal dimension that is less than two. A triangulation from these points would not cover the attractor with sufficient area. Therefore we blur randomly to get a triangulation that covers the attractor and approximates an attracting neighborhood. We put the blurred data points on a list called

$$\texttt{allpoints} := \{d_1, \ldots, d_n\}.$$

From here on we refer to the blurred data points as data points. The first m data points from `allpoints` are placed on a list called `trimakers` and the other data points from `allpoints` are placed on a list called `others`. The data points on `trimakers` are now inserted one at a time into the triangulation using a Delaunay Insert algorithm. The data points that are on `trimakers` are now vertices in the triangulation, while the data points on the `others` list are now sample points. We call a simplex in the triangulation an **exterior simplex**, if one of its vertices is an exterior point. A sample point is called **interior** if the vertices of the simplex it is contained in are not exterior points. A vertex is called **interior** if every one of its simplices in its star is not exterior. After the insertion of the data points of `trimakers` into the triangulation, we perform an interior check of each sample point on the `others` list and flag each sample point that is not interior. We do an interior check of each vertex of `trimakers` and flag each vertex that is not interior. The first condition that we want our triangulation to fullfill is to use m data points as vertices for the triangulation such that the remaining sample points are contained in simplices that are not exterior. We achieve this through a series of operations that add non-interior data points to the triangulation and remove interior vertices from the triangulation.

First we remove all interior sample points from the `others` list so that the `others` list consists of non-interior sample points. We then take the first k sample points from the `others` list and add them one at a time to the triangulation. If the `others` list is smaller than k than we add all the points on the list. Next we remove k interior vertices from the triangulation. Removing interior vertices from the triangulation preserves sample points and vertices being interior. If there are less than k interior vertices, then we remove all of them. The interior vertices that were removed now become sample points and are removed from the `trimakers` list. The sample points that were added to the triangulation are now vertices and are removed from the `others` list and put on the `trimakers` list. Now we do an interior check of the vertices and the sample points. The `others` list is updated to contain non-interior sample points. Now we repeat the process of adding and deleting. After each instance of adding and deleting, the `others` list consists of sample points that are non-interior and the algorithm stops when the `others` list is empty. After stopping the algorithm, it could very well be that the number of data points in the triangulation is larger than m. This is due to the fact that at the instance where we remove k interior vertices, there may not be any interior vertices to remove and we just go on to the next step in the algorithm. In the end we construct a triangulation of at least m vertices such that the remaining sample points are interior. The resulting triangulation has the property that all sample points are interior. The algorithm for this step is listed here.

Build Interior Algorithm

1. Make list `temp`$:= \{z_1, \ldots, z_2\}$ consisting of time-series.
2. **for** $i = 1$ to $i = m$

3. $d_i := z_i + r_i$

4. **endfor**

5. Make list `allpoints`:= $\{d_1, \ldots, d_m\}$.

6. Make list `trimakers`:= $\{d_1, \ldots, d_n\}$.

7. Make list `others`:= $\{d_{n+1}, \ldots, d_m\}$.

8. Insert data points of `trimakers` into triangulation.

9. **while** `others` is nonempty

10. Do Internal check on `allpoints`.

11. Update `others` list to contain non-interior sample points.

12. Add k points from others to triangulation.

13. Put the k points on `trimakers` list and remove them from `others`.

14. Do internal check on `allpoints`.

15. Delete k internal vertices from `trimakers`.

16. **endwhile**

5.2 Interchange Method

Our goal is to approximate the dynamics contained in the time-series by using a combinatorial multivalued map $\mathcal{F}$. To get a good approximation, the sizes of triangles in the domain of $\mathcal{F}$ should be small. In this method we refine our triangulation in order to get a better approximation of the dynamics. The refinement is done through a series of adding and deleting points from the triangulation. The idea of adding or deleting points from the triangulation in order to get a better approximation for the linear interpolant was done by Mees [1]. Mees used conditions for adding and deleting based on error estimates of functions. In the following our conditions are based on the geometry of the triangulation.

For each sample point, we take each vertex of the triangle that contains the sample point. If the vertex is interior, then we compute the distance of the sample point to the vertex. We take the minimum of the these distances that were computed and we call it the **min-distance** of the sample point. The the min-distances are computed for all sample points and they are sorted from lowest to highest. The highest min-distance is called the **max-min-distance**. By a series of adding and deleting appropriate points, we make the max-min-distance decrease. The max-min-distance being small implies that any triangle containing a sample point is small. The Interchange method takes two parameters j and k which are positive integers. The Interchange method has a **for** loop and the parameter j gives the number of instances of the loop. For each instance of the loop the following procedure is repeated k-times. We take the sample point with the lowest min-distance and the vertex corresponding to the min-distance is deleted from the triangulation. The sample point corresponding to the max-distance is then added to the triangulation. The min-distances are recomputed for all sample points. If the max-min-distance increased, then we keep adding the sample points with the highest min-distance to the triangulation until the max-min-distance decreases. See Interchange Algorithm.

Interchange Algorithm

1. Compute min-distances for all sample points to interior vertices.
2. Order min-distances from largest to smallest.
3. R :=largest min-distance.
4. r :=smallest min-distance.
5. **for** $i = 1$ to $i = j$
6. $\quad d := R$
7. $\quad$ **for** $h = 1$ to $h = k$

8. Delete closest interior vertex to r.

9. Add the sample point corresponding to R.

10. Recompute min-distances.

11. Order them from largest to smallest.

12. Add the N sample points to triangulation corresponding to the N largest min-distances.

13. Recompute min-distances.

14. **endfor**

15.**endfor**

21. return R.

5.3 Selection Method

In the selection method we now choose the simplices that make up the underlying space on which we define the simplicial multivalued map $\mathcal{F}$. We want the sizes of the simplices in our complex to be relatively small. We want to avoid large simplices, since the multivalued map defined on them would be a bad approximation. We take the triangulation on the rectangle and perform a selection process. After the interchange method is completed, the min-distances are recomputed. The Selection Method has one parameter β which is a positive real number. We define $d := (\beta *$ max-min-distance). The simplices that we choose to keep are put on a list called `Mapmaker`. We have two criterion for selecting a simplex:

(1) if the simplex contains a sample point, then we keep the simplex;

(2) if the sidelengths of the simplex are less than or equal to d, then we keep the simplex.

After the selection process is over, the `Mapmaker` list is final and is not added to or deleted from for the rest of the implementation.

5.4 Rangemaker Method

We now build a simplicial multivalued map $\mathcal{F}$. For each simplex that is on the `Mapmaker` list we construct a forward image of that simplex that consists of simplices from `Mapmaker`. For a given simplex σ in `Mapmaker`, we map each vertex forward by the map f so that if we denote v_1, v_2, and v_3 as the vertices of the simplex we get $w_1 = f(v_1), w_2 = f(v_2)$ and $w_3 = f(v_3)$ where w_1, w_2, and w_3 are points in $\mathbb{R}^2$. With these values we solve the following linear equation

$$w = Av + b$$

for A and b where A is a 2×2 matrix and b is a 1×2 matrix. The matrices A and b are saved for each simplex in `Mapmaker`. After solving for A and b we then define the affine map on the simplex

$$\widehat{f}(x) = Ax + b.$$

We go through all simplices in `Mapmaker` and see which simplices does $\widehat{f}(\sigma)$ intersect. The code we use is based on the algorithms from Moller [22]. All simplices in `Mapmaker` that intersect $\widehat{f}(\sigma)$ are put on a list called the `forward` list of σ. If there is no simplex in `Mapmaker` that intersects $\widehat{f}(\sigma)$, then the `forward` list of σ is left empty.

We want all simplices that have a nonempty `forward` list to satisfy the condition that the image of the vertices under the affine map are contained in simplices in `Mapmaker`. This condition is necessary in order to construct a simplicial acyclic carrier. If a simplex that has a non-empty `forward` list does not satisfy the condition then we put the simplex on a list called `uncovered`. Let σ be a simplex that is on the `uncovered` list. Suppose v is a vertex of σ such that $\widehat{f}(v)$ is not contained in a `Mapmaker` simplex. We compute the barycenters of all the simplices in the `forward` list of σ and from these barycenters let q be the closest barycenter to $\widehat{f}(v)$. The image of v under $\widehat{f}$ is redefined to be q. The matrices A and b along with the `forward` list of σ gets updated. This is done for each simplex on the `uncovered list`. We have now constructed our simplicial multivalued map $\mathcal{F}$. The domain D for the multivalued map $\mathcal{F}$ consists of all mapmaker triangles S that have a nonempty `forward` and such that $\mathcal{F}$ is a combinatorial enclosure for S. To verify that $\mathcal{F}$ is a combinatorial enclosure for S is a simple check using our triangle intersection algorithms. We point out that the check for a combinatorial enclosure is not rigorous due to round-off error.

Now that the simplicial multivalued map $\mathcal{F}$ has been constructed, we compute $\mathcal{F}^{-1}$ for each simplex in `Mapmaker`. For each simplex σ in `Mapmaker` we create a list of simplices called the `backward` list which contains all simplices in `Mapmaker` such that σ is contained on its `forward` list.

For each vertex of a simplex that has a nonempty `forward` list, we take the image of the vertex under the affine map of that simplex. We run through all simplices in `Mapmaker` and check if the image of the vertex intersects each simplex. If a simplex intersects the image of the vertex then we put that simplex on a list called `vertforward`. The list `vertforward` is part of the data structure of the vertex. We repeat the same steps for all edges of simplices with nonempty `forward list`. The simplices that intersect the image of the edge are put on a list called `edgeforward` which is part of the data structure of the edge.

5.5 Index Pair Method

In the Index Pair method we find periodic orbits and connecting orbits under the map $\mathcal{F}$. We then use the Index Pair Algorithm from Section 4. The method has one argument n which is a positive integer. The argument gives the highest period in which we search for a periodic orbit. Starting with period 1 and proceeding to period n, we add periodic orbits to a list called the `worklist` by an algorithm which we now describe.

Periodic Orbit Finder Algorithm

```
k = 0;
for i = 1 to i = n;
    for each simplex S in Mapmaker;
        if S has least period i;
            orbit:= periodic orbit through S of least period i;
            temp:= orbit ∪ worklist;
            call Isolation Algorithm on temp;
            if temp is isolated;
                Iso:= isolating neighborhood for temp;
                 m := number of connected components of Iso.
                 if k > m;
                    Add orbit to worklist;
                    k = m;
                endif;
            endif;
       endif;
     endfor;
```

18. **endfor**;

19. **return** `worklist`;

After finding the periodic orbits and puting them on the `worklist` we now try to find connecting orbits between the periodic orbits. We now describe this process in the following algorithm which takes the `worklist` outputed from the previous algorithm.

Periodic Connector Algorithm

1. $k =$ the number of connected components for isolating `worklist`;
1. **for** $i = 1$ to $i = n$;
2.   `temp1`:= list of periodic orbits of least period i;
3.   **for** each periodic orbit on `temp1`;
4.   `orbit1`:= current periodic orbit from `temp1`;
4.   **for** $j = i + 1$ to $j = n$;
5.   `temp2`:= list of periodic orbits of least period j;
6.   **for** each periodic orbit on `temp2`;
7.   `orbit2`:= current periodic orbit from `temp2`;
7.   `orbit`:= connecting orbit from `orbit1` to `orbit2`;
6.   `temp`:= `orbit` ∪ `worklist`;
7.   call Isolation Algorithm on `temp`;
8.   **if** `temp` is isolated;
9.   `Iso`:= isolating neighborhood for `temp`;
10. $m :=$ number of connected components of `Iso`.
11.   **if** $k > m$;
12.   Add `orbit` to `worklist`;
13. $k = m$;

```
                endif;
            endif;
        endfor;
    endfor;
  endfor;
endfor;
return worklist;
```

After finding and isolating our periodic points and the connections between them we are now ready to construct our index pair P_0 and P_1. We call the Index Pair Algorithm on the `worklist` that was outputed from the previous algorithm. After P_0 and P_1 are found we then give each connected component of P_1 a component number N_i.

5.6 Compute Homology Map Method

We start the last method in our implementation with two checks on homology involving the index pair (P_1, P_0). The first thing we check is to verify that we generate an acyclic simplicial carrier by using the `vertforward`, `edgeforward`, and `forward` lists. By construction, if v is a vertex and e is an edge such that $v \in e$, then the `vertforward` of v is contained in the `edgeforward` of e and also if s is a simplex such that $e \in s$, then `edgeforward` of e is contained in the `forward` of s. For all vertices $v \in P_1 \setminus P_0$, we have by construction that the `vertforward` list for v is acyclic. For all edges $e \in P_1 \setminus P_0$ we do an acyclicity check on the `edgeforward` list of e and for all edges $s \in P_1 \setminus P_0$ we do an acyclicity check on the `forward` list of s. If any of the `edgeforward` or `forward` lists are non-acyclic then the implementation is stopped. If all of the lists are acyclic then we have a minimal acyclic carrier $\Phi_{\widehat{f}}$ where we define $\Phi_{\widehat{f}}(v)$ to be the `vertforward` list of v, $\Phi_{\widehat{f}}(e)$ to be the `edgeforward` list of e, and $\Phi_{\widehat{f}}(s)$ to be the `forward` list of s. Now we move on to the second check. For each connected component N_i of (P_1, P_0), we take the union of all `forward` lists of 2-simplices in N_i and we do an acyclicity check on the union. If it is acyclic, then we know that the retraction maps discussed in Chapter 4 exist.

After the acyclicity check and the verification of $\Phi_{\widehat{f}}$, we now build a chain map carried by $\Phi_{\widehat{f}}$. For vertices, we choose any vertex on the `vertforward` list to be the image under the chain map of a vertex. For an edge e with vertices v_1 and v_2, denote c_1 to be the vertex that is the image of v_1 and c_2 the vertex that is the image of v_2. We construct the matrix for the boundary map $\partial : C_1 \to C_0$ restricted to the image of e under the acyclic carrier and then we do a linear solve for an edge sequence such that the image of the boundary map on the sequence gives $c_1 + c_2$. The edge sequence that is the solution will be the image of e under the chain map for edges. We repeat the same procedure for simplices to get the chain map for simplices. Note that by construction the chain maps are carried by the acyclic carrier.

We compute the generators of the relative homology for the index pair. For a fixed generator, we map the generator forward by using the chain map. It is determined which linear combination of the generators is homologous to the forward map of the generator by a linear solve. We then say the fixed generator maps to this linear combination in the homology map. We do this for all the generators. The matrices of the map of homology are then constructed and outputed along with the directed graph representing the map between the generators and the directed graph representing the map between the components.

6 Convergence Theorems

In this section we state and prove some of the convergence theorems which characterize how well our algorithms approximate the dynamics of a continuous map. If the time-series data is generated from a continuous map f, then we can use the algorithms in Section 4 and Section 5 on the piecewise linear map $\widehat{f}$ to find a lower bound for the dynamics of $\widehat{f}$. We want to infer that this is also a lower bound for the dynamics of f by showing some convergence of $\widehat{f}$ to f. Also we would like to show if f has a transitive attractor, then the region triangulated from the `Mapmaker` list converges to the attractor.

The following proposition is a fundamental proposition in [25], so we exclude the proof.

Proposition 6.1. *A transitive attractor is a compact invariant set.*

We need the following lemma when dealing with continuous maps with transitive attractors.

Lemma 6.2. *Let Λ be a transitive attractor for a continuous map $f : M \to M$ and let $\epsilon > 0$. Then there exists a point $x_0 \in \Lambda$ and a natural number n such that the set of iterates $\{f(x_0), f^1(x_0), \ldots, f^n(x_0)\}$ forms an ϵ-cover of Λ. Futhermore, there exists a natural number k with $k \geq n$ such that for each $j \leq n$ there exists an element $f^i(x_0)$ of $\left\{f(x_0), f^1(x_0), \ldots, f^k(x_0)\right\}$ such that $f^i(x_0) \in B(f^j(x_0), \epsilon)$ and $f^i(x_0) \neq f^j(x_0)$.*

Proof. Since Λ is compact, there exists a finite set of points $\{y_1, \ldots, y_k\}$ contained in Λ that form an $\epsilon/2$-covering of Λ. Since $f|\Lambda$ is transitive, there exists an $x_0 \in \Lambda$

that has a dense forward orbit. Therefore for each y_i there is a natural number n_i such that $d(f^{n_i}(x_0), y_i) < \epsilon/2$. Let n_0 be the maximum over the n_i. By the triangle inequality the set $\{f(x_0), f^1(x_0), \ldots, f^{n_0}(x_0)\}$ form, an ϵ-cover of Λ.

Consider $j \leq n$ and the ball $B(f^j(x_0), \epsilon)$. There exists a natural number $k(j)$ such that $f^{k(j)}(x_0) \in B(f^j(x_0), \epsilon)$ and $f^{k(j)}(x_0) \neq f^j(x_0)$. Let $k = \max_{1 \leq j \leq n}\{k(j)\}$ and the lemma is proved. ■

Corollary 6.3. *If a time-series of data points is generated by an orbit on a transitive attractor Λ with bounded noise, then there is natural number n such that the set $\{z_1, z_2, \ldots, z_n\}$ forms an ϵ-cover of Λ. Futhermore, there exists a natural number k with $k \geq n$ such that for each $j \leq n$ there exists an element z_i of $\{z_1, z_2, \ldots, z_k\}$ such that $z_i \in B(z_j, \epsilon)$ and $z_i \neq z_j$.*

At this point we want to analyze the **Interchange Algorithm** introduced in Chapter 4. Here we introduce the **Optimal Interchange Algorithm** that takes an argument $\gamma > 0$ that stops the **while** loop.

Optimal Interchange Algorithm

1. For each sample point compute min-distance to interior vertices.
2. Order min-distances from largest to smallest.
3. R :=largest min-distance.
4. r :=smallest min-distance.
5. **while** $R > r$ and $R > \gamma$
6. $\quad$ $d := R$
7. $\quad$ Delete closest interior vertex to a sample point with min-distance r.
8. $\quad$ Add a sample point with min-distance R.
9. $\quad$ Recompute min-distances.

Order them from largest to smallest.

d_1 :=largest.

while $d_1 \geq d$

    Add sample point corresponding to d_1.

    Recompute min-distances.

    d_1 :=largest

endwhile

$R = d_1$

r :=smallest.

endwhile

return R.

The difference between the **Interchange** algorithm and the **Optimal Interchange** algorithm is that the **Optimal Interchange** computes the min-distance using all of the vertices in the triangulation and after one instance of adding and deleting, sample points are added until the max-min-distance decreases. The **Optimal Interchange** algorithm adds the least amount of vertices to the triangulation to make the max-min-distance decrease. The problem with using the **Optimal Interchange** algorithm in practice is that it is difficult to design a local efficient check to tell which vertices do not have a change in their min-distances after an addition or deletion, which means that all min-distances must be recomputed after an addition or deletion for all vertices. Thus computations using the **Optimal Interchange** have runtimes that are extremely long. This led us to develop the **Interchange** algorithm which approximates the actions of the **Optimal Interchange** with reasonable runtimes. The problem with **Interchange** is that we cannot prove that it monotonically decreases, which is a necessary assumption for the following theorems. In the computations using **Interchange** we have observed that it decreases monotonically in practice as long as enough new vertices are added. The following theorems are stated using the **Optimal Interchange**.

We now state some theorems on the approximation of the triangulation to the attractor and the approximation of f to $\widehat{f}$.

Lemma 6.4. *For a given $\epsilon > 0$, there exists a $\delta > 0$ such that if the **Optimal Interchange** algorithm returns a **max-min-distance** $< \delta$, then $|\widehat{f}(x) - f(x)| < \epsilon$ for all $x \in D$.*

Proof. Since the domain of f and $\widehat{f}$ is compact, f and $\widehat{f}$ are uniformly continuous so that there exists a $\gamma > 0$ such that if $|x_1 - x_2| < \gamma$, then $|f(x_1) - f(x_2)| < \epsilon$ and $|\widehat{f}(x_1) - \widehat{f}(x_2)| < \epsilon$. Let $\delta < \gamma/2$. By the triangle inequality, the triangles selected to be part of the domain have diameter less than γ. Take any x in the domain $|D|$. Then $x \in S$ for some 2-simplex $S \in D$. Take a vertex x_1 of S. We have that $|x - x_1| < \gamma$ and therefore $|\widehat{f}(x) - \widehat{f}(x_1)| < \epsilon$. Since $\widehat{f}(x_1) = f(x_1)$ we are done. ∎

Theorem 6.5. *Let N be a trapping region and A a transitive attractor with $A \subset N$. Let $\epsilon > 0$ and $x_0 \in N$. There exists k such that for $n > k$ we have that $d(x_n, A) < \epsilon/2$. Furthermore if blurring factor is smaller than $\epsilon/2$, then for $n > k$ the blurred data points satisfy $d(z_n, A) < \epsilon$.*

Proof. Since N is a trapping region for A, $\omega(x_0) \subset A$. Therefore there exists a k such that for $n > k$, $x_n \in B_{\epsilon/2}(A)$ and if the blurring factor is less than $\epsilon/2$ we have that $z_n \in B_\epsilon(A)$. ∎

Theorem 6.6. *Let N be a trapping region with $A = \mathrm{Inv}(N)$ a transitive attractor. Let $\epsilon > 0$ and $x_0 \in N$. Suppose there exists a point $q \in \omega(x_0) \subset A$ such that q has* *a dense forward orbit in A. Let x_k be such that for $n \geq k$ we have $d(x_n, A) < \epsilon/4$. Then there exists j such that for all $r > j$ we have the Hausdorf distance between A and the the set $\{x_k, x_{k+1} \ldots, x_r\}$ is less than $\epsilon/2$. Furthermore, the Hausdorf distance between A and the blurred data is $\{z_k, z_{k+1} \ldots, z_r\}$ is less than ϵ and the triangulation Δ has a Hausdorf distance with A less than 3ϵ.*

Proof. By properties of the ω-limit set, since $\omega(q) = A$ we have that $\omega(x_0) = A$. Since A is compact, there exists a finite subset $\{y_1, \ldots, y_m\} \subset A$ such that A is contained in an $\epsilon/4$ covering of $\{y_1, \ldots, y_m\}$. For each y_i there exists n_i such that $d(x_{n_i}, y_i) < \epsilon/4$. Let $j := \max\{n_i\}$. Therefore the set $\{y_1, \ldots, y_m\}$ is contained in an $\epsilon/4$ covering of $\{x_k, x_{k+1} \ldots, x_r\}$, which implies that A is contained in an $\epsilon/2$ covering of $\{x_k, x_{k+1} \ldots, x_r\}$. Since the blurring factor is less than $\epsilon/2$, we have A is contained in an ϵ covering of $\{z_k, z_{k+1} \ldots, z_r\}$.

Since the $\{y_1, \ldots, y_m\}$ is contained in an $\epsilon/4$ covering of $\{x_{n_1}, \ldots, x_{n_m}\}$, $\{y_1, \ldots, y_m\}$ is covered in a $3\epsilon/4$ covering of $\{z_{n_1}, \ldots, z_{n_m}\}$. Therefore the $\{z_k, z_{k+1} \ldots, z_r\}$ are contained in a 2ϵ covering of $\{z_{n_1}, \ldots, z_{n_m}\}$. This implies that the **max-min-distance** returned by **Optimal Interchange** is less than or equal to 2ϵ. If $x \in |\Delta|$, then x is in some triangle in Δ and x has a distance with a vertex $z(x)$ that is less than 2ϵ. However $d(z(x), A) < \epsilon$ which implies that $d(x, A) < 3\epsilon$. ∎

The previous theorems show that using the **Optimal Interchange** algorithm allows us to reconstruct the phase space by triangulating as close to the attractor as we like. Now we consider whether we can capture the dynamics on the attractor. Suppose that our triangulation covers the attractor of f. We know that if $\mathcal{F}$ is a combinatorial enclosure of f, then a periodic orbit for f induces a periodic orbit for $\mathcal{F}$. The $\mathcal{F}$ that we construct from our algorithms is derived from $\widehat{f}$. If the linear interpolant $\widehat{f}$ approximates f sufficiently, then we expect $\mathcal{F}$ to enclose the dynamics of f. The approximation of f by its linear interpolant $\widehat{f}$ is a well-studied problem. The following theorem from Bramble and Zlamal [7] gives an estimate on the error between f and $\widehat{f}$ on each triangle.

Theorem 6.7 ([7]). *Let $f \in H^2(T)$ where T is a given 2-simplex. Let $\widehat{f}$ be the linear interpolant of f on T. Then we have*

$$||f - \widehat{f}||_{L_2} \leq Ch^2 ||f||_{H^2}$$

where h is the maximum length of the triangle T and C is a constant that does not depend on function f but does depend on the angles of T so that $C \to \infty$ as the minimal angle of T approaches 0. Also we have

$$||f - \widehat{f}||_{C_0} \leq Ch||f||_{H^2}$$

where the constant C depends on f and a lower bound of the angles of T.

For a given triangle T, the image of T under $\widehat{f}$ is a triangle. We can expand the image of T by taking the ball of radius $\delta > 0$ around $\widehat{f}(T)$ denoted by $B_\delta(T)$. We can construct the combinatorial map $\mathcal{F}_{exp}$ using the expanded image of $\widehat{f}$ and then check if $\mathcal{F}_{exp}$ is a combinatorial enclosure of the expanded $\widehat{f}$. If the error between $\widehat{f}(T)$ and $f(T)$ is sufficiently small, then $f(T)$ is contained in the expanded image of $\widehat{f}(T)$ which implies that $\mathcal{F}_{exp}$ is a combinatorial enclosure of f. Therefore periodic orbits of f induce periodic orbits on $\mathcal{F}_{exp}$. This leads to the following theorem.

Theorem 6.8. *Let h be the maximal sidelength of all triangles in* `Mapmaker`*. There exists a $\delta > 0$ such that if we expand the image of $\widehat{f}$ by δ and $\mathcal{F}_{exp}$ is constructed from the expanded image and is a combinatorial enclosure of the expanded image, then $\mathcal{F}_{exp}$ is (within round-off error) a combinatorial enclosure of f.*

Proof. From Theorem 6.7 we set $\delta = C||f||_{H^2}$ and $\delta = \widehat{C}h$. This implies for any triangle T we have that $f(T)$ is contained in the expanded image of $\widehat{f}(T)$, and therefore $\mathcal{F}_{exp}$ is a combinatorial enclosure of f. ∎

Suppose we are able to isolate a periodic orbit for $\mathcal{F}_{exp}$ that corresponds to a periodic orbit of f. Then we want to show that the homology map induced by f is the same as the homology map induced by $\widehat{f}$. Suppose $\widehat{f}$ generates a minimal simplicial acyclic carrier $\Phi_{\widehat{f}}$, and also suppose that f generates a minimal simplicial acyclic carrier Φ_f. We generate a another carrier Φ as follows. For a vertex v we take $\widehat{f}(v)$ and expand it by the $\widehat{C}h$ in Theorem 6.8, and we define $\Phi(v)$ to be all 2-simplices that intersect the expansion of $\widehat{f}(v)$. We repeat the same steps for edges and 2-simplices, and then we check if Φ is acyclic. It is easy to see that Φ carries f and $\widehat{f}$ so that it carries $\Phi_{\widehat{f}}$ and Φ_f. Thus if Φ is acyclic, then the homology map induced by $\widehat{f}$ and f are the same. In practice, we compute an outer approximation $\mathcal{F}$ of $\widehat{f}$ and the minimal carrier $\Phi_{\widehat{f}}$, which we check is acyclic. Algorithmically, we could compute $\mathcal{F}_{exp}$, which is an outer approximation for f, and the carrier Φ, for which acyclicity could also be checked. However, we cannot compute Φ_f, and currently we have no algorithm to check that this minimal acyclic carrier for f exists. If the construction of the triangulation could be performed in a more structured manner, then one might be able to obtain bounds on the difference between $\mathcal{F}$ and $\mathcal{F}_{exp}$ which could establish the existence of an acyclic carrier for f.

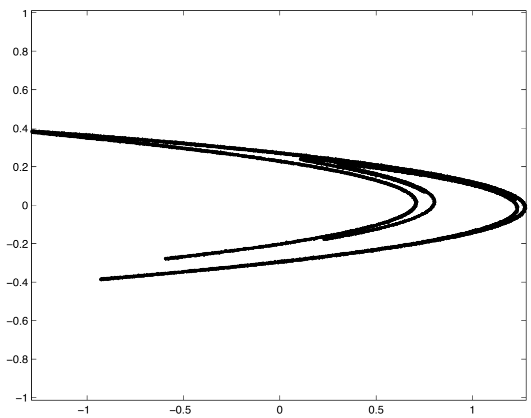
Finally, we do not implement the expansion of $\widehat{f}$ to construct $\mathcal{F}_{exp}$ and the carrier Φ for two reasons. First, the expansion of $\widehat{f}(T)$ could make it difficult to obtain isolation of the periodic orbits. Second, by Theorem 6.7 the L^2 error decreases quadratically in terms of the maximum sidelength, and hence we expect that the expansion of $\widehat{f}$ may not be necessary.

7 Computations

In this chapter we give the results of computations with time-series data coming from an iteration of a map. The first map that we use is the **Henon Map**

$$f(x,y) := (1 - 1.4x^2 + y, 0.3x).$$

The Henon map has a transitive attractor contained in the rectangular region $[-1.6, 1.6] \times [-0.6, 0.6]$ in $\mathbb{R}^2$. The time series is constructed first by taking the forward orbit of the point $(0,0)$, throwing away an arbitrary number of the iterates, and then blurring the remaining points. In this case a time series is constructed by throwing away the first $1,000$ points and keeping the next $100,000$. The interchange method finished with a max-min distance value of 0.00220935, which resulted in a simplicial complex consisting of $95,025$ triangles which is plotted in Figure 7.10. The combinatorial map $\mathcal{F}$ is searched for periodic orbits up to period-6, and one orbit of period-1, one orbit of period-2, one orbit of period-4, and two orbits of period-6 are isolated. Furthermore we are able to isolate some connecting orbits between these periodic orbits. An index pair is computed which contained 64 components. The Figure 7.10 shows the 64 components plotted within the `Mapmaker` triangles and Figure 7.11 shows the index pair plotted without the `Mapmaker` triangles. Each component has an exit set so there are no 0-generators. The index pairs do not have any 2-generators so we only have to compute homology maps for dimension 1. Figure 7.11 shows a close up of some components of the index pair with the red triangles corresponding to the exit set. The graph for the homology map and the transition graph are shown in Figure 7.12.



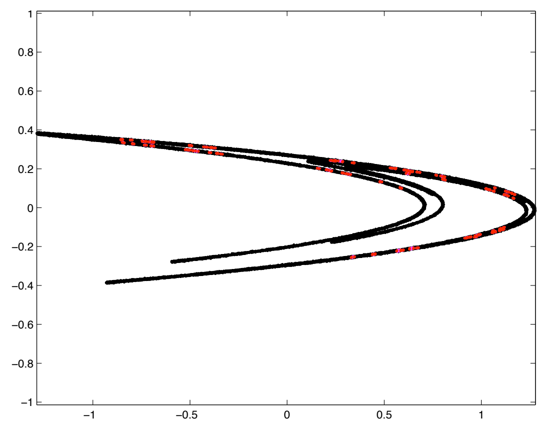


Figure 7.10: Left: The `Mapmaker` triangulation for the Henon computation consisting of $100,000$ data points with $95,025$ trinangles. Right: The components of (P_1, P_0) within the `Mapmaker` triangles.

We now verify the dynamics present in a subgraph of the transition graph. See Figure 7.13. This subgraph has a fixed-point, a period-2 orbit, a period-4 orbit, and a period-6 orbit along with certain connections between them. There are 33 components for this subgraph. We label the vertices of this subgraph as follows. We start with the period-6 orbit, we label N_1 to be the vertex in the period-6 orbit that has the edge coming out of the period-6 orbit. We label the rest of the period-6 orbit N_2, N_3, N_4, N_5, and N_6 by using the directed edges so that N_1 goes to N_2 and N_2 to N_3 and so forth. We label N_7 to be the vertex not in the period-6 orbit such that N_1 has a directed edge to that vertex. There are 6 vertices in the connection with the period-6 orbit and the fixed point. We label the connecting vertices in increasing order by the directed edges so the vertex corresponding to the fixed vertex has label N_{13}. We repeat this for the rest of the graph, increasing the label number according to the direction of the graph.

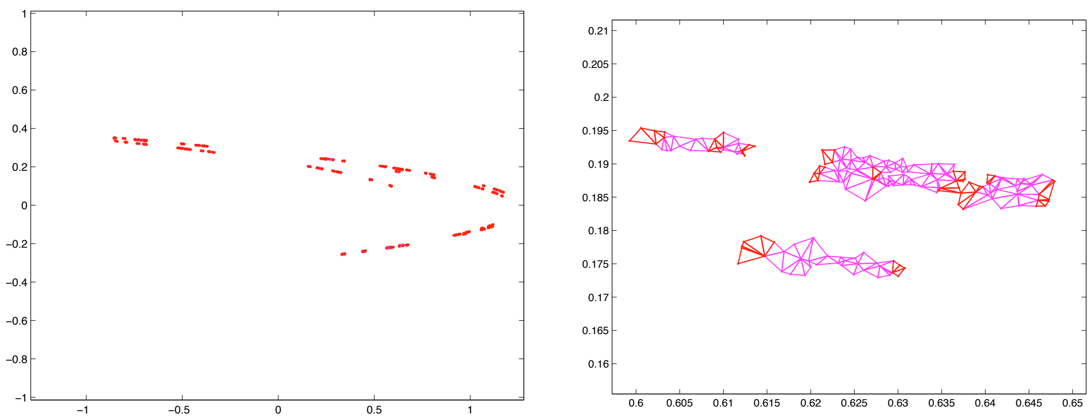
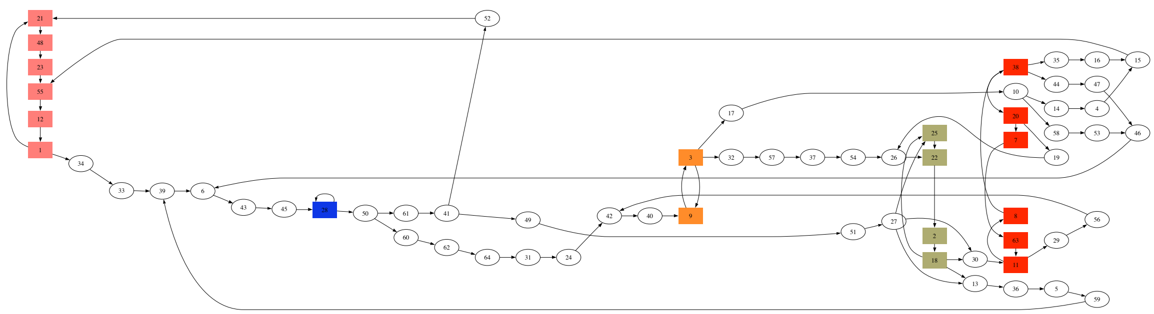


Figure 7.11: Left: The index pair (P_1, P_0) which consists of periodic orbits of length $1, 2, 4$ and 6 and connections between them. Right: A close up of the index pair components red denotes the exit set.

From the discussion at the end of Chapter 4, we need to verify all periodic orbits for the transition graph to get a semiconjugacy. Once we have verified all the periodic orbits, then we calculate the entropy of the transition matrix. In what follows, we use the notation $[N_i \to N_i]$ to mean the H_1 matrix for $[\widehat{f}_{N_i} \to \widehat{f}_{N_i}]_*$. We start with verification of the fixed-point. The homology matrix for $[N_{13} \to N_{13}]$ is

$$[N_{13} \to N_{13}] := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



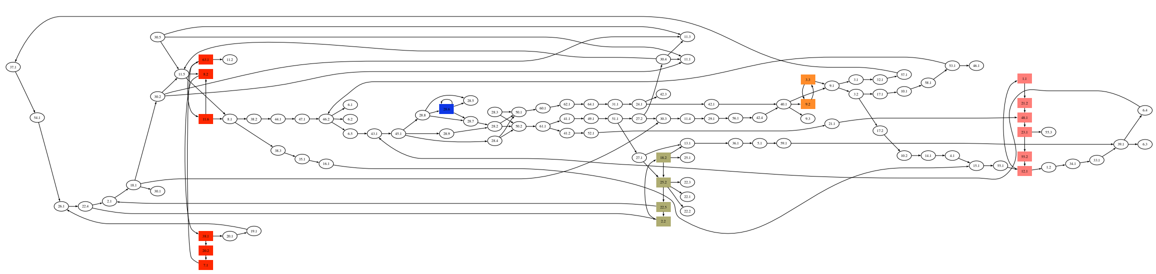


Figure 7.12: Top: The transition graph for the 64 components of the index pair. The period-1 is colored blue, period-2 is colored orange, period-4 is colored gray, one of the period-6 is colored red, and the other period-6 is colored pink. Bottom: the graph representing the map on 1-homology. Each square is labeled first with its component number and generator number. The colors come from the designation in Figure 7.12.

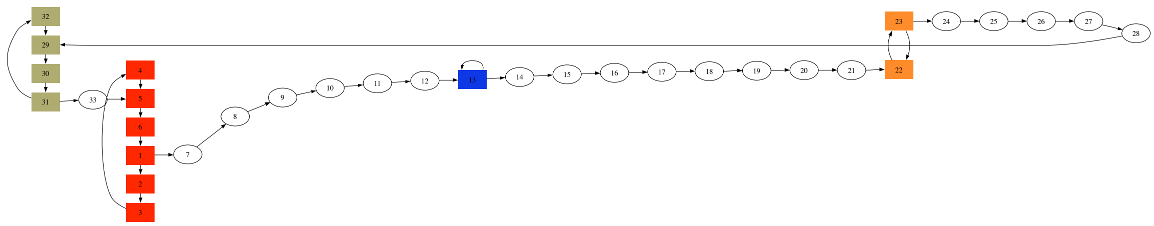


Figure 7.13: The subgraph derived from the graph in 7.12. The colored periodic orbits correspond to the ones in 7.12.

The trace of $[N_{13} \to N_{13}]$ is 1 which implies there exists a fixed-point for $\widehat{f}$. Taking the powers of $[N_{13} \to N_{13}]$ we find that

$$[N_{13} \to N_{13}]^n := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for $n \geq 2$ so that when verifying periodic orbits that travel through N_{13} multiple times n, we use $[N_{13} \to N_{13}]^n$ in the composition of matrices corresponding to the periodic orbit.

For the period-2 orbit, the components are N_{22} and N_{23}, and the homology matrix of $[N_{22} \to N_{22}] := [N_{23} \to N_{22}] * [N_{22} \to N_{23}]$ is

$$[N_{22} \to N_{22}] := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The trace of $[N_{22} \to N_{22}]$ is 1 which implies the existence of a period 2 orbit and we also have

$$[N_{22} \to N_{22}]^n := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

for $n \geq 1$.

The period-4 orbit has components $N_{29} \to N_{30} \to N_{31} \to N_{32} \to N_{29}$ and the homology matrix $[N_{29} \to N_{29}] := [N_{32} \to N_{29}] * [N_{31} \to N_{29}] * [N_{30} \to N_{31}] * [N_{29} \to N_{30}]$ is

$$[N_{29} \to N_{29}] := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The trace of $[N_{29} \to N_{29}]$ is 1 which implies the existence of a period-4 orbit, and also

$$[N_{29} \to N_{29}]^n := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

for all $n \geq 1$. The homology matrix for the period-6 is $[N_5 \to N_5] := [N_4 \to N_5] * [N_3 \to N_4] * [N_2 \to N_3] * [N_1 \to N_2] * [N_6 \to N_1] * [N_6 \to N_5]$ is

$$[N_5 \to N_5] := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The trace of $[N_5 \to N_5]$ is 1 which implies the existence of a period-6 orbit and

$$[N_5 \to N_5]^n := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

for all $n \geq 1$.

Now we verify the rest of the periodic orbits. These periodic orbits travel through all the components. By using the fact that the above matrices have powers that repeat, we show that there 24 matrices to check. We can start with an arbitrary component, so we start with N_1 and list all possible paths from N_1 back to itself. We take the path from N_1 to N_{13} which we denote by $(N_1 \to N_{13})$. From N_{13} to N_{22} there are three possibilities. One can choose $(N_{13} \to N_{22})$, or one can go through the fixed point once and then to N_{22} which we denote by $(N_{13} \to N_{13})(N_{13} \to N_{22})$

or finally, one can go through the fixed point any number of times greater than or equal to 2 and then on to N_{22} denoted by $(N_{13} \to N_{13})^n(N_{13} \to N_{22})$ where $n \geq 2$. The power of the matrix $[N_{13} \to N_{13}]^n$ is the same for $n \geq 2$ and this is why we consider the path $(N_{13} \to N_{13})^n(N_{13} \to N_{22})$ to be the same for all $n \geq 2$. Next we have two choices within the period 2 orbit. We have the path $(N_{22} \to N_{23})$ or we have $(N_{22} \to N_{22})(N_{22} \to N_{23})$ where $(N_{22} \to N_{22}) = (N_{22} \to N_{23})(N_{23} \to N_{22})$ is the path going through the period 2 orbit. Since the powers of $[N_{22} \to N_{22}]^n$ are the same for all n then we only have to consider $(N_{22} \to N_{22})(N_{22} \to N_{23})$. We conclude by writing the rest of the possible paths. We have the path $(N_{23} \to N_{29})$ followed by two choices within the period 4 orbit $(N_{29} \to N_{31})$ and $(N_{29} \to N_{29})(N_{29} \to N_{31})$. Next we have the path $(N_{31} \to N_5)$ followed by two choices within the period 6 orbit, $(N_5 \to N_1)$ and $(N_5 \to N_5)(N_5 \to N_1)$. Counting all of the possible paths, we have 24 paths starting at N_1 and ending at N_1 that travel through all of the periodic orbits. Suppose we choose the path $(N_1 \to N_{13})(N_{13} \to N_{13})^2(N_{13} \to N_{22})(N_{22} \to N_{22})(N_{22} \to N_{23})(N_{23} \to N_{29})(N_{29} \to N_{31})(N_{31} \to N_5)(N_5 \to N_5)(N_5 \to N_1)$, then taking the matrix multiplication $[N_5 \to N_1] \cdot [N_5 \to N_5] \cdot [N_{31} \to N_5] \cdot [N_{29} \to N_{31}] \cdot [N_{23} \to N_{29}] \cdot [N_{22} \to N_{23}] \cdot [N_{22} \to N_{22}] \cdot [N_{13} \to N_{22}] \cdot [N_{13} \to N_{13}] \cdot [N_1 \to N_{13}]$ and we get the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The trace of this matrix is 1, which implies the existence of a periodic point for the chosen sequence path. All 24 possible matrices were computed, and all had their trace equal to 1. This implies that all periodic orbits that are present in the subgraph have been verified. This includes the period-1,the period-2,the period-4, two period-6, a period-33 and all higher periodic orbits in the subgraph. The entropy of the transition graph was computed with a value of 0.156 so that we can conclude that the map $\widehat{f}$ has an approximate lower bound for entropy of 0.156.

Now we give computations for the nonlinear density-dependent **Leslie model** of poulation dynamics,

$$f(x, y) := ((\theta_1 x + \theta_2 y)e^{-0.1(x+y)}, px).$$

This model was studied extensively in [32, 33] and [4]. We investigate this map at two sets of parameter values. In the first set of parameter values we let $\theta_1 = \theta_2 := 27$ and $p := 0.7$. The Leslie map for this set has a transitive attractor consisting of three disconnected components in the region $[0, 100] \times [0, 70]$. The time series is constructed by taking the forward orbit of the point $(0.73, 52.3)$. The first $1,000$ points are discarded and the next $100,000$ are used for the time series. The interchange method finished with a max-min distance value of 0.0349177, which resulted in a simplicial complex consisting of $95,025$ triangles which is plotted in Figure 7.14. The combinatorial map searches for periodic orbits up to period-12, and one orbit of period-3, one orbit of period-6, and one period-12 are isolated along with a connecting orbit from the period-3 to the period-12. An index pair is computed which contains 27 components. Figure 7.14 also shows the 27 components plotted within the `Mapmaker` triangles. The exit set has the same properties as in the Henon computations in that we only need to compute homology for dimension 1. The graph for the homology map and the transition graph are shown in Figure 7.15. There are only three periodic orbits for the transition graph so we do not have chaos from the graph. For the second set of parameter values we let $\theta_1 := 61, \theta_2 := 12.2$ and $p := 0.2$. The Leslie map for

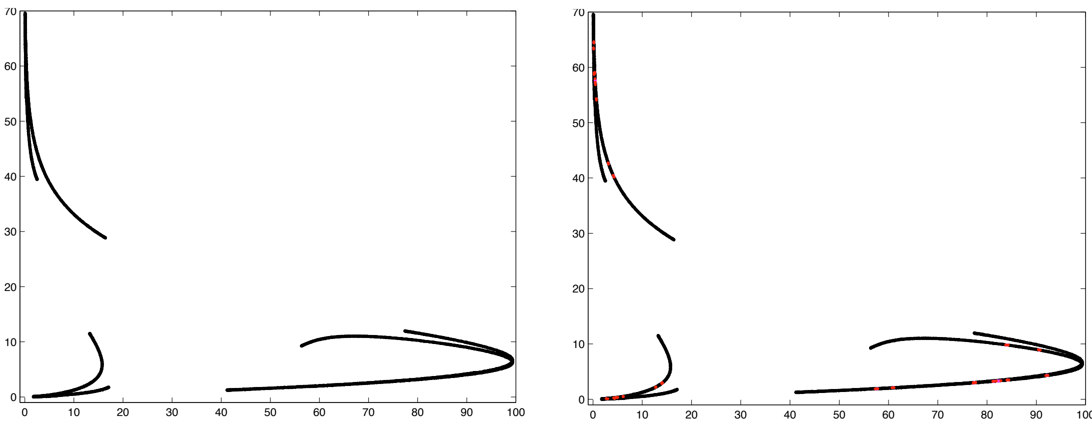


Figure 7.14: Left: The `Mapmaker` triangulation for the first Leslie computation consisting of 100,000 data points with 95,025 triangles. Right: The components of (P_1, P_0) within the `Mapmaker` triangles.

this set has a transitive attractor consisting of one connected component in the region $[0, 100] \times [0, 70]$. We again use the starting point $(0.73, 52.3)$ and also discard the first 1,000 points and use the next 110,000 the time series. The `Mapmaker` triangulation consists of 132,269 triangles and the index pairs consisting of 38 components is shown in Figure 7.16. We isolate a period-1, period-2, and a period-4 orbit along with connecting orbits between the period-1 and the period-2 orbit. The transition graph and the homology map are shown in Figure 7.17. The homology map implies the existence of the period-1 and the period-2 orbit. For the subgraph consisting of the period-1 and the period-2 along with the connections between them, the homology map could not verify all of the periodic orbits in the subgraph and therefore we can not generate a semiconjugacy.

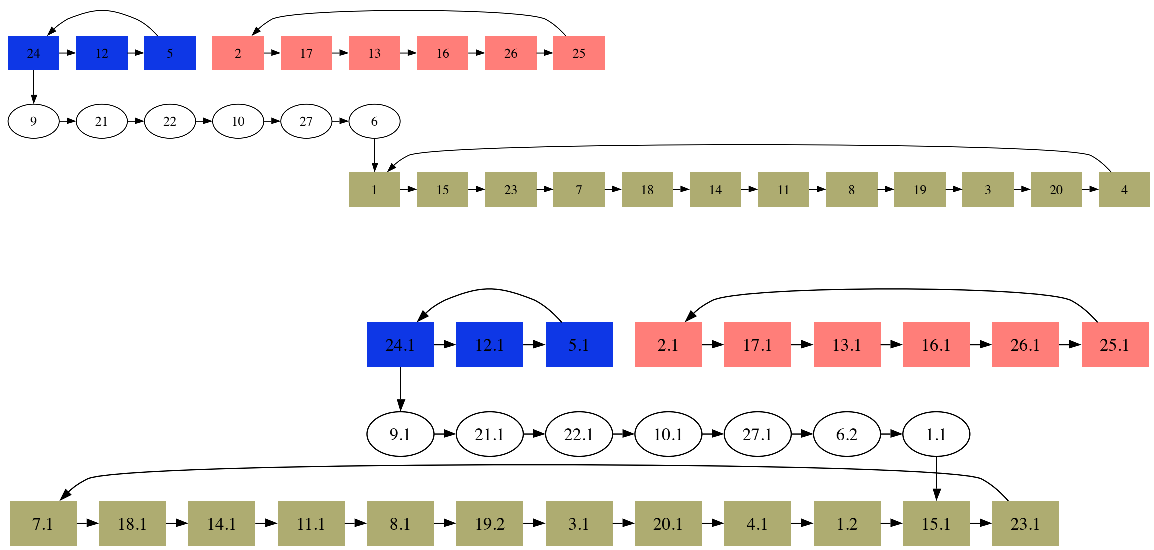


Figure 7.15: Top: The transition graph for the 27 components of the index pair. The period-3 is colored blue, period-6 is colored pink, and the period-12 is colored gray. Bottom: the graph representing the map on 1-homology. Each square is labeled first with its component number and generator number.

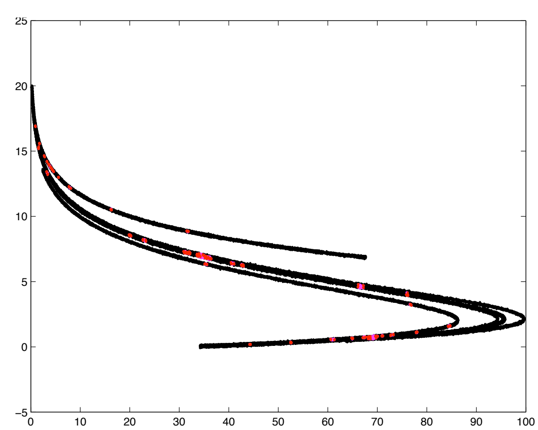


Figure 7.16: The `Mapmaker` triangulation for the second Leslie computation consisting of $110,000$ data points with $132,269$ triangles.

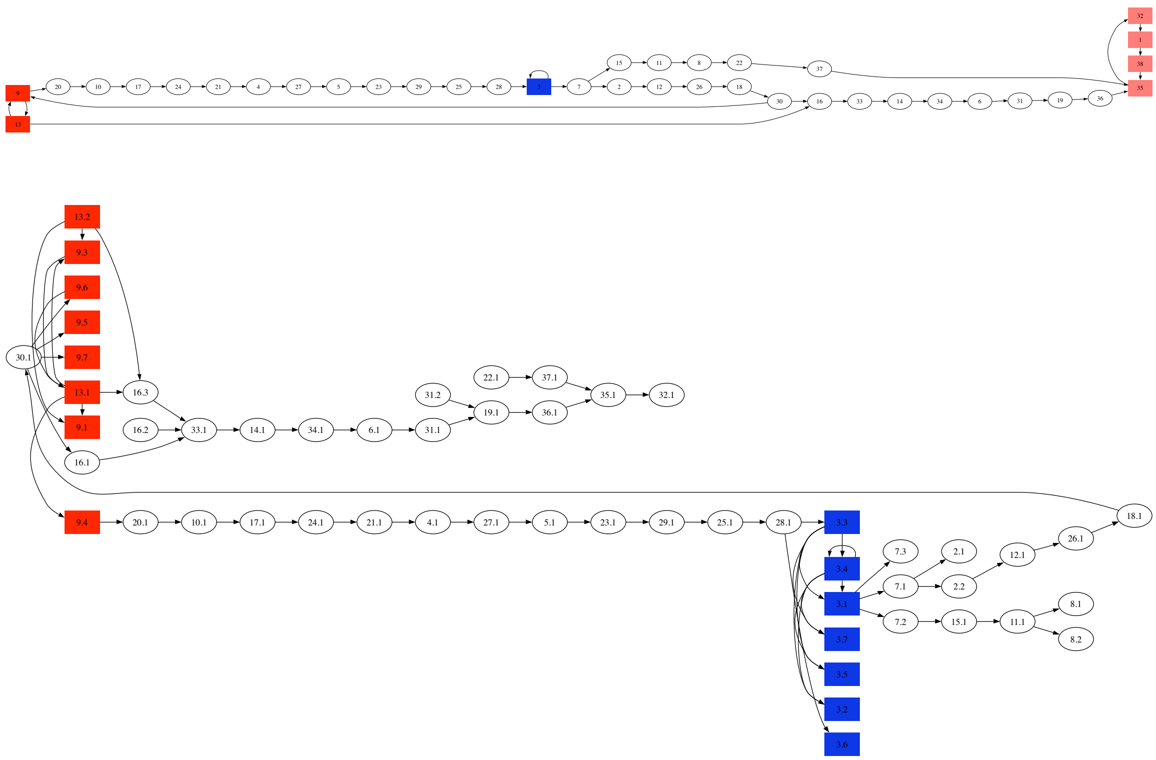


Figure 7.17: Top: The transition graph for the 38 components of the index pair. The period-1 is colored blue, period-2 is red, and period-4 is colored pink. Bottom: the graph representing the map on 1-homology.

8 Conclusions and Future Directions

We have developed algorithms and code to analyze dynamics via phase-space reconstruction and to measure chaos in time series data in 2-dimensions using simplicial homology along with the Lefschetz Fixed Point Theorem. In Chapter 7 we showed computations using the Henon map and the Leslie map. In those computations we are able to locate periodic orbits of low order and high order that were near periodic orbits of the actual Henon map. Also, our computations did not pick up a period-3 orbit or a period-5 orbit which are proven not to exist for the actual Henon map. We were able to compute positive entropies and generate transitional graphs from the semiconjugacies with the shift space. These transitional graphs contained chaos, which allowed us to infer the existence of chaos for $\widehat{f}$ up to round-off error. These results give us confidence that the methods we have used approximate the dynamics very well. We have shown that if a transitive attractor is present, then the Hausdorff distance between the triangulation and the attractor can be made as small as we want by taking sufficiently many time series points. We do not at the moment have a proof that says our triangulation covers the attractor. However we blur the data points randomly in an attempt to cover the attractor in practice. An open question for further study is: does there exist a determined blurring such that the triangulation can be proven to cover the attractor?

Presently, the code can do a time delay reconstruction where the embedding dimension is 2. Suppose we are given a finite sequence of a forward orbit for a dynamical system in $\mathbb{R}^n$ $\{x_1, x_2, \ldots, x_m\}$. If there is a measurement function $\phi : \mathbb{R}^n \to \mathbb{R}$, then we can generate the time-series data $\{\phi(x_1), \phi(x_2), \ldots, \phi(x_m)\}$. This time-series data can be embedded into $\mathbb{R}^2$ by a time-delay reconstruction which gives the data set $\{(\phi(x_1), \phi(x_2)), (\phi(x_2), \phi(x_3)), \ldots, (\phi(x_{m-1}), \phi(x_m))\}$. We treat this data as a forward orbit of some unknown continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$, that is $f((\phi(x_1), \phi(x_2)) = (\phi(x_2), \phi(x_3))$, $f((\phi(x_2), \phi(x_3)) = (\phi(x_3), \phi(x_4))$, and in general $f((\phi(x_{k-1}), \phi(x_k)) = (\phi(x_k), \phi(x_{k+1}))$. We can then implement our code on this data set as was explained in Chapters 4 and 5 and generate a piecewise linear map $\widehat{f}$ in which the dynamics of $\widehat{f}$ approximate a lower bound on the dynamics of f. If the original data points from $\mathbb{R}^n$ come from a chaotic attractor that lies on a 2-dimensional manifold embedded in $\mathbb{R}^n$, then we expect to verify the chaos on the time-delayed data.

The future direction of our work has four stages. First, we want to extend the code and homology theory to dimensions 3 and 4. We believe that this is possible for 3 and 4, but higher dimensions would probably make the triangulation computations as well as simplicial homology too costly. This would require some new algorithms to compute homology in order to make the run times reasonable. Second, we want to extend the code to handle projections of time-series data into 2-dimensions where we can use our techniques. The difficulty lies in determining a systematic way of choosing a projection to use. Third, we want to take measurements on continuous maps and do a delay reconstruction to embed the time-series and then use our techniques to approximate dynamics. We would first embed in 2 dimensions, then in 3 dimensions and so forth. We compare the results between consecutive dimensions and determine if we have picked up more dynamics as the embedding dimension increases. Finally, we want to use our code on time-series generated from experimental data. This requires the first three stages being completed in order for them to be utilized and requires a way to approximate $\widehat{f}$ or f at the vertices of the triangulation. This is our ultimate goal in that we want to approximate dynamics on a physical system using experimental data.

References

[1] S. Allie and A. Mees. *Dynamical systems and tesllations; detecting determinism in data*, International Journal of Bifurcations and Chaos, Vol. No. 4, pp. 771-794, 1991.

[2] S. Allie, A. Mees, K. Judd, and D. Watson. *Reconstructing noisy dynamical systems by triangulations*, Physical Review E, 1997.

[3] K. Alligood, T. Sauer, and J. Yorke. *Chaos; An Introduction to Dynamical Systems.* Springer, New York, 1996.

[4] Z. Arai, W. Kalies, H. Kokubu, K. Miischaikow, P. Pilarczyk, and H. Oka. *Databases for global dynamics of multiparamter systems.* Preprint, 2008.

[5] J. Banks, J. Brooks, G. Cairns, G. Davis, and P.Stacey. *On Devaney's definition of chaos*, Amer. Math. Monthly, 99, pp. 332-334, 1992.

[6] H. Borouchaki, P.L. George, and S.H. Lo. *Optimal Delaunay point insertion*, International Journal for Numerical Methods in Engineering, Vol. 39, pp. 3407-3437, 1996.

[7] J. Bramble, and M. Zlamal. *Triangular elements in the finite element method*, Mathematics of Computation, Vol. 24, No. 112, pp. 809-820, 1970.

[8] H.W. Broer, F. Dumortier, S.J. van Strien, and F. Takens. *Structures in Dynamics.* Elsevier Science Publishing Company Inc, New York, 1991.

[9] T. H. Cormen, C. E. Leiserson, and R. L. Rivest. *Introduction to Algorithms.* The MIT Electrical Engineering and Computer Science Series, MIT Press, Cambridge, MA, 1990.

[10] S. Day. *A Rigorous Numerical Method in Infinite Dimensions.* Ph.D. dissertation, 2003.

[11] S. Day, R. Frongillo, and R. Trevino. *Algorithms for rigorous entropy bounds and symbolic dynamics*, Preprint, 2008.

[12] S. Day, O. Junge, and K. Mischaikow. *A rigorous numerical method for the global analysis of infinite-dimensional discrete dynamical systems*, SIAM J. Appl. Dyn. Syst., 3(2), pp. 117-160, 2004.

[13] O. Devillers. *On deletion in Delaunay triangulations*, International Journal of Computational Geometry and Applications, 12 No.3, pp. 193-205, 2002.

[14] J. Guckenheimer, and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, New York, 1983.

[15] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[16] J.P. Huke. *Embedding Nonlinear Dynamical Systems: A Guide to Takens' Theorem*, MIMS Eprint Manchester, 2006.

[17] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational Homology.* Springer-Verlag, New York, 2004.

[18] H. Kantz and T. Schreiber. *Nonlinear Time Series Analysis, 2nd ed.* Cambridge University Press, Cambridge, 2004.

[19] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, 1995.

[20] K. Mischaikow, M. Mrozek, J. Reiss, and A. Szymczak. *From time series to dynamics: an algebraic topological approach*, Preprint, 1997.

[21] K. Mischaikow, M. Mrozek, J. Reiss, and A. Szymczak. *Constuction of symbolic dynamics from experimental time series*, Physical Review Letters, Vol. 82, No. 6, pp. 1144, Feb. 1999.

[22] T. Moller. *A fast triangle-triangle intersection test.* Journal of Graphics Tools, vol.2, no.2, pp.25-30, 1997.

[23] J. Munkres. *Elements of Algebraic Topology*, Addison-Wesley, 1984

[24] E. Ott, T. Sauer, and J. Yorke. *Coping with Chaos.* Wiley Series in Nonlinear Science, Wiley, New York, 1994.

[25] C. Robinson. *Dynamical Systems.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1999. Stability, symbolic dynamics, and chaos.

[26] C. Robinson *What is a chaotic attractor?*, Preprint, 1997.

[27] T. Sauer, J. Yorke, M. Casdagli. *Embedology*, Journal of Statistical Physics, 65 pp. 579-616, 1991.

[28] A. Szymczak. *A combinatorial procedure for finding isolating neighborhoods and index pairs*, Proc. Roy. Soc. Edinburgh Sect. A, 127(5), pp.1075-1088, 1997.

[29] A. Szymczak. *Index Pairs: from Dynamics to Combinatorics and Back.* PhD. Thesis, Georgia Institute of Technology, 1999.

[30] R. E. Tarjan, *Depth-first search and linear graph algorithms*, SIAM J. Comput. 1, pp. 146-160, 1972.

[31] F. Takens. *Detecting strange attractors in turbulence.* Lecture Notes in Mathematics, Vol.898 Springer-Berlin, pp. 366-381, 1981.

[32] I. Ugarcovici and H. Weiss. *Chaotic dynamics of a nonlinear density dependent population model*, Nonlinearity 17, pp. 1689–1711, 2004.

[33] I. Ugarcovici and H. Weiss. *Chaotic attractors and physical measures for some density dependent Leslie population models*, Nonlinearity 20, pp. 2897–2906, 2007.