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Approximate Isometries as an Eigenvalue Problem and Angular Momentum

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In relativistic physics, a precise definition of a black hole's angular momentum is possible only when its horizon possesses an axial symmetry. Unfortunately most black hole horizons have no such symmetry. However, it is possible to pose an eigenvalue problem that has solutions corresponding to any manifold's "approximate Killing fields." This allows one to generalize formulae requiring symmetry to cases where no symmetry is present and thus define, for example, the spin of an arbitrary black hole. This talk will discuss work using perturbation theory of a horizon to quantify the stability of quantities generalized in this way. We will present precise conditions for the stability of solutions to the eigenvalue problem, and discuss potential applications to numerical relativity.

Black Hole Spin and Approximate Symmetry Stability

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Abstract In relativistic physics, a precise definition of a black hole's angular momentum is possible only when its horizon possesses an axial symmetry. Unfortunately most black hole horizons have no such symmetry. However, it is possible to pose an eigenvalue problem that has solutions corresponding to any manifold's "approximate symmetries". This allows one to generalize formulae that require symmetry to situations for which no symmetry is present; i.e. the spin of a black hole. This work, using perturbation theory, quantifies how reliable an equation generalized in this way is. Applications to numerical relativity are mentioned.

Introduction

A Killing vector field (KVF), ξ^a, solves

$$\mathcal{L}_\xi g^{ab} = -2\nabla^{(a}\xi^{b)} \equiv 0\ ,$$

and generates a continuous symmetry of a Riemannian manifold (M, g_{ab}). See Fig. 1.

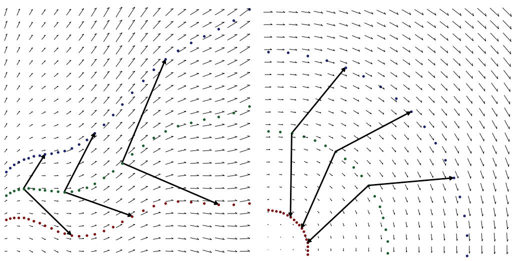
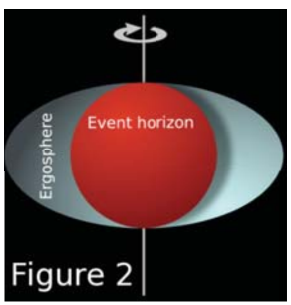


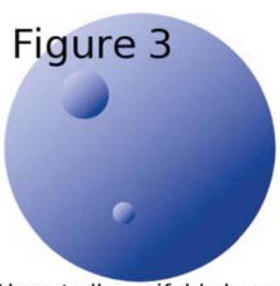
Figure 1 - (Left) General vector flow; (Right) Killing Vector (KV) flow

Only symmetric geometries (Fig. 2) permit a well defined temperature, mass and spin.

Very few manifolds admit symmetries (Fig. 3)!



A black hole with symmetry



Almost all manifolds have no continuous symmetries

How should KVFs be approximated when they are not allowed by a spacetime?

Approximate Killing Vector Fields (AKVFs)

The lowest eigenmode of the equation

$$\Delta_K \xi^a := \nabla_b \mathcal{L}_\xi g^{ab} = \kappa \xi^a$$

minimizes the deviation from Killing's equation over an entire Riemannian manifold. Vector field solutions of this "Killing Laplacian" are AKVFs.

The solution ξ^a is a genuine KVF iff κ = 0. The eigenvalue quantifies the asymmetry of a manifold.

Key question: Do solutions of the above eigenproblem behave predictably under perturbation?

Bounding perturbed AKVFs

The Killing Laplacian, under a one-parameter family of perturbations is expressible as

$$\dot\Delta_K \phi^a = Z^a{}_B(\nabla\nabla\phi)^B + Y^a{}_M(\nabla\phi)^M + X^a{}_b\phi^b\ .$$

Coefficients of this perturbed Killing Laplacian combine metric perturbations and their derivatives.

The norm of the perturbed AKVF is

$$||\dot\xi|| = ||[\Delta_K - \kappa]^{-1}_{\xi^\perp}[\dot\Delta_K - \dot\kappa]\xi||\ .$$

Therefore the perturbed AKVF will be bounded if

$$||\dot\Delta_K\phi|| \le C||\Delta_K\phi|| + ||l.o.t.||$$

for some constant C and arbitrary vector field ϕ^a.

The triangle inequality can be used to write

$$||\dot\Delta_K\phi|| \le ||Z(\nabla\nabla\phi)|| + ||Y(\nabla\phi)|| + ||X\phi||\ .$$

The coefficients act as bounded operators.

Results

The 2nd order Sobolev norm bounds the norm of the perturbed Killing Laplacian.

$$||\dot\Delta_K\phi||^2 \le C(\partial\partial\dot g, \partial\dot g, \dot g)||\phi||_{W_2^2}$$

The 1st order norm of a vector field is bounded by the Killing Laplacian.

$$||\nabla\phi||^2 \le \langle\phi|\Delta_K|\phi\rangle$$

The 2nd order norm of a vector is also bounded by the Killing Laplacian.

$$||\nabla\nabla\phi||^2 \le ||\Delta_K\phi|| + C(\partial\partial g, \partial g, g)||\phi||_{W_1^2}$$

The eigenfields of the Killing Laplacian change in a predictable and continuous manner.

$$||\dot\Delta_K\xi|| \le \kappa C(\partial\partial\dot g, \partial\partial g, \partial\dot g, \partial g, \dot g, g)$$

Applications

1. Calculating observables of gravitational fields.
   a. The angular momentum of a black hole.
   b. Calculating energy deposited via tidal heating.
2. Cosmology - Identifying approximately homogenous and isotropic spacetimes
3. Utilization on Riemannian manifolds; statistical and information geometries.

References

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