CURVE SHORTENING IN SECOND-ORDER LAGRANGIAN SYSTEMS

by

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ABSTRACT

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A second-order Lagrangian system is a generalization of a classical mechanical system for which the Lagrangian action depends on the second derivative of the state variable. Recent work has shown that the dynamics of such systems can be substantially richer than for classical Lagrangian systems. In particular, topological properties of the planar curves obtained by projection onto the lower-order derivatives play a key role in forcing certain types of dynamics. However, the application of these techniques requires an analytic restriction on the Lagrangian that it satisfy a twist property. In this dissertation we approach this problem from the point of view of curve shortening in an effort to remove the twist condition. In classical curve shortening a family of curves evolves with a velocity which is normal to the curve and proportional to its curvature. The evolution of curves with decreasing action is more general, and in the first part of this dissertation we develop some results for curve shortening flows which shorten lengths with respect to a Finsler metric rather than a Riemannian metric. The second part of this dissertation focuses on analytic methods to accommodate the fact that the Finsler metric for second-order Lagrangian systems has singularities. We prove the existence of simple periodic solutions for a general class of systems without requiring the twist condition. Further, our results provide a framework in which to try to further extend the topological forcing theorems to systems without the twist condition.

DEDICATION

To my family and my friends for their unending support.

CURVE SHORTENING IN SECOND-ORDER LAGRANGIAN SYSTEMS

CHAPTER 1

INTRODUCTION

Second-order Lagrangian systems are used as models in various areas of physics, namely nonlinear elasticity, nonlinear optics, and mechanics. These models arise as fourth-order differential equations obtained variationally as the Euler-Lagrange equations of an action functional which depends on the second derivative of the state variable u'' as well as its lower derivatives. One important class of differential equations is $u'''' - \beta u'' + f(u) = 0$, known as the Swift-Hohenberg equation for $\beta \leq 0$ and the extended Fisher-Kolmogorov (eFK) equation for $\beta > 0$. There has been numerous results concerning these equations, see for instance [12, 16, 19, 20, 28, 29]. A second order Lagrangian system is, under suitable assumptions on the u''-dependence on L, equivalent to a Hamiltonian system on $\mathbb{R}^4$. Trajectories of the Lagrangian system, and thus Hamiltonian system, lie on three dimensional sets $M_E = \{H = E\}$, where H is the Hamiltonian (the conserved quantity). The sets M_E are smooth manifolds for all regular values E of H, i.e. $\nabla H\big|_{M_E} \neq 0$, and are non-compact for all $E \in \mathbb{R}$. A central feature of Hamiltonian systems that come from second-order Lagrangians, one can find a natural two dimensional section $\{u' = 0\} \cap M_E$ which bounded trajectories have to intersect finitely often. This section will be denoted by Σ_E and $\Sigma_E = N_E \times \mathbb{R}$, where N_E is a one dimensional set defined by

$$N_E = \left\{ (u, u'') \mid \frac{\partial L}{\partial u''} u'' - L(u, 0, u'') = E \right\};$$

see Section 3.2 for further details.

The Hamiltonian flow induces a return map to the section Σ_E, and closed trajectories (closed characteristics) correspond to fixed points of iterates of this map. In the twist case mentioned below the return map is an analogue of a monotone area-preserving twist map. The existence of closed characteristics for second-order Lagrangian systems has been studied throughout the last few decades, in [5] the authors show that under the twist assumption the existence of a closed characteristic can be proved by discretizing the variational problem to study closed characteristics using recurrence relations. To accomplish this, an extra assumption on the system must be made which is called the twist condition. This condition requires the existence of a minimizer $u(x, u_1, u_2)$ of the functional J for all u_1, u_2 with $u_1 \neq u_2$ and u is C^1. This can be a difficult assumption to verify. The twist property allows a reduction of the problem to a finite-dimensional recurrence relation constructed from plugging the minimizer into the functional $S_E(u_1, u_2) = J_E\left[z(x; u_1, u_2)\right]$. One then shows the existence of a critical point of S_E to obtain existence of a simple closed characteristic.

In [12] the authors use degree theory and Lagrangian intersection theory to show existence of a simple closed characteristic without the twist condition via continuation to a twist system. In [20] a type of Morse theory is used in relation with braiding and knotting of trajectories to obtain forcing theorems of closed characteristics for second-order Lagrangian systems of twist type; the authors show that the set of closed characteristics can have a rich structure. In [12], it is shown that for regular energy manifolds, the number of closed characteristics can be bounded below by the second Betti number of M_E, which in turn can be computed from the superlevel sets of the potential function $L(u, 0, 0) + E \geq 0$. In the case of a first-order Lagrangian system $L(u) = \frac{1}{2}(u')^2 + F(u)$ where $H = \frac{1}{2}(u')^2 - F(u)$. An energy manifold M is one-dimensional, if M is regular then each compact component of M consists of a single periodic solution. Thus the number of closed characteristics is exactly $\dim H_1(M)$.

For second-order Lagrangians $\dim H_2(M)$ is only a lower bound. There are examples of systems with infinitely many different closed characteristics, see [20].

In the broad scope of studying second order Lagrangians our interest is in the existence of periodic solutions from a curve shortening perspective, potentially allowing us to bypass the restrictions inherent with the "twist condition". The curve shortening problem is to analyze the long-term behavior of smooth curves, immersed in a Riemannian surface, which evolve by their curvature vectors. The flow is called "Curve Shortening" because the flow lines in the space of closed curves are tangent to the gradient for the length functional; the curve is shrinking as fast as it can using only local information. The flow is determined by the heat equation, as long as the heat operator is computed in the metric induced by the immersion.

Curve shortening processes can be used to find embedded geodesics on surfaces, especially spheres. To produce closed simple geodesics, a curve shortening method must (a) keep embedded curves embedded, (b) exist for all time, or at least until the curve shrinks to a point, and (c) shorten non-geodesics fast enough so that only geodesics survive in the limit. In [15], the Grayson shows that Simple closed curves in the plane become convex in finite time and shrink to a point. Here we consider an evolution that is determined by a Finsler metric, therefore the evolution becomes a parabolic equation to which the results of [22, 23, 11] apply. It turns out this evolution represents the gradient flow for the action functional given by the Lagrangian, this allows us to view critical points of the functional as geodesics of a Finsler metric.

CHAPTER 2

FINSLER CURVE SHORTENING

2.1 CURVES EVOLVING WITH PARABOLIC VELOCITY

The space of unparametrized C^2 immersed curves with orientation is defined as

$$\Omega(\mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}_+(S^1),$$

where $\mathrm{Imm}(S^1, \mathbb{R}^2) = \{\gamma \in C^2(S^1, \mathbb{R}^2) : |\gamma'(x)| \neq 0 \; \forall x \in S^1\}$ and $\mathrm{Diff}_+(S^1)$ is the group of all orientation preserving diffeomorphisms of $S^1 = \mathbb{R}/\mathbb{Z}$, two such immersions which only differ by an orientation-preserving reparameterization are considered to be identical. We will abuse notation by writing γ to represent both a parametrization and its corresponding equivalence class in $\Omega(\mathbb{R}^2)$.

The space $\Omega(\mathbb{R}^2)$ is locally homeomorphic to $C^2(S^1)$. To see this let $\gamma \in \Omega(\mathbb{R}^2)$, extend γ to a local C^2 diffeomorphism $\sigma : S^1 \times (-r, r) \to \mathbb{R}^2$ for some sufficiently small $r > 0$. For example one could take $\sigma(x, r) = \exp_{\gamma(x)}(r\mathbf{N(x)})$. Let $z \in C^2(S^1)$ be a C^1 small function and consider the curve $\gamma_z(x) = \sigma(x, z(x)) \in \Omega(\mathbb{R}^2)$. Define the family of sets $U_r = \{z \in C^2(S^1) : |z(x)| < r\}$ then the map $\Phi : U_r \to \Omega(\mathbb{R}^2), z \mapsto \gamma_z$ is a homeomorphism onto $\Phi(U_r)$ which is an open neighborhood of γ. Letting γ vary yields a covering of $\Omega(\mathbb{R}^2)$ which implies that $\Omega(\mathbb{R}^2)$ is a Banach manifold.

Any curve $\alpha \in \Omega(\mathbb{R}^2)$ yields a constant speed parametrization $\alpha : S^1 \to \mathbb{R}^2$. Consider a family of curves of the form $\alpha : [0, T) \times S^1 \to \mathbb{R}^2$, we denote the unit tangent and arclength parametrization by

$$\mathbf{T} = \frac{\alpha_x}{|\alpha_x|}, \quad \frac{\partial}{\partial s} = \frac{1}{|\alpha_x|}\frac{\partial}{\partial x}, \quad ds = |\alpha_x|\, dx.$$

The curvature vector of α is

$$\frac{\partial^2 \alpha}{\partial s^2} = \alpha_{ss} = \kappa \mathbf{N},$$

where κ is the geodesic curvature of the curve $\alpha(t, \cdot)$ in $\mathbb{R}^2$ measured by the Euclidean metric g. We assume that $\{\mathbf{T}, \mathbf{N}\}$ is a positively oriented basis of $T_{\alpha(s)}\mathbb{R}^2$. One can decompose the time derivative as $\alpha_t(t, s) = v^{\parallel}\mathbf{T} + v^{\perp}\mathbf{N}$. The second component $v^{\perp}$ is the *normal velocity* of this family of curves, which is independent of the chosen parametrization of each $\alpha(t, \cdot)$.

Denote the unit tangent bundle of $\mathbb{R}^2$ by $S^1(\mathbb{R}^2) = \{\xi \in T(\mathbb{R}^2) \mid g(\xi, \xi) = 1\}$. $S^1(\mathbb{R}^2)$ is a smooth 3-dimensional submanifold of $T(\mathbb{R}^2)$. Now the tangent bundle to the unit tangent bundle can be split into the Whitney sum of the bundle of horizontal vectors and vertical vectors see Section 2.4. The bundle of horizontal vectors can be identified with the pullback $\tau^* T(\mathbb{R}^2)$ where $\tau : S^1(\mathbb{R}^2) \to \mathbb{R}^2$ is the projection map. The bundle of vertical vectors is naturally isomorphic to the following subbundle of $\tau^* T(\mathbb{R}^2)$

$$\text{Vert} = \{(u, v) \in \tau^* T(\mathbb{R}^2) \mid u \perp v\}.$$

The Whitney sum $TS^1(\mathbb{R}^2) = \tau^*(\mathbb{R}^2) \oplus \text{Vert}$ allows us to decompose the connection ∇ on $S^1(\mathbb{R}^2)$ into two components corresponding to differentiation in the horizontal direction, ∇^h and its vertical direction ∇^v, i.e. $\nabla = \nabla^v \oplus \nabla^h$.

Given a function $V : S^1(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}$ and a curve $\alpha_0 \in \Omega(\mathbb{R}^2)$, we compose the following initial value problem. Find a family of curves $\alpha(t) \in \Omega(\mathbb{R}^2)$ for $0 \leq t < t_{\text{Max}}$ such that for $t > 0$ $\alpha(t)$ has continuous curvature and is parametrized such that $\alpha_s \perp \alpha_t$. We can then formulate the following parabolic PDE where the normal velocity satisfies

$$v^{\perp} = V(\mathbf{T}, \kappa).$$

That is

$$\alpha_t = v^{\perp}\mathbf{N} \tag{2.1}$$

with initial value $\alpha|_{t=0} = \alpha_0$.

Lemma 2.1.1. Let $r = |\alpha_x|$ and $\alpha(t,\cdot)$ be a family of curves evolving normally through Equation (2.1). Then the following hold:

(i) $\frac{\partial r}{\partial t} = -v^{\perp}\kappa r$

(ii) $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \kappa v^{\perp}\frac{\partial}{\partial s}$

Proof.

(i) $$\frac{\partial}{\partial t}|\alpha_x| = \frac{\alpha_{xt}\cdot\alpha_x}{|\alpha_x|} = \frac{\frac{\partial}{\partial x}\left(v^{\perp}\mathbf{N}\right)\cdot\alpha_x}{|\alpha_x|} = \frac{\left(v_x^{\perp}\mathbf{N} + v^{\perp}\mathbf{N}_x\right)\cdot\alpha_x}{|\alpha_x|} = \frac{v^{\perp}\mathbf{N}_x\cdot\alpha_x}{|\alpha_x|} = -v^{\perp}\kappa r.$$

(ii) $$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\alpha = \frac{\partial}{\partial t}\left(\frac{1}{r}\alpha_x\right) = \frac{1}{r}\frac{\partial}{\partial x}\alpha v^{\perp}\kappa + \frac{1}{r}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\alpha = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\alpha + v^{\perp}\kappa\frac{\partial}{\partial s}\alpha.$$

■

Lemma 2.1.2. Let $\alpha(t,\cdot)$ be a solution to Equation (2.1). Then the following hold:

(i) $T_t = v_s^{\perp}\mathbf{N}$

(ii) $N_t = -v_s^{\perp}\mathbf{T}$

(iii) $\kappa_t = v_{ss}^{\perp} + k^2 v^{\perp}$

Proof. For (i)

$$\frac{\partial}{\partial t}\mathbf{T} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\alpha + \kappa v^{\perp}\alpha_s = \frac{\partial}{\partial s}\left(v^{\perp}\mathbf{N}\right) + \kappa v^{\perp}\mathbf{T} = v_s^{\perp}\mathbf{N}.$$

For $(ii), (iii)$

$$
\begin{aligned}
&\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{T} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{T} - \kappa v^{\perp}\mathbf{T}_s \\
&\Rightarrow \frac{\partial}{\partial s}\left(v_s^{\perp}\mathbf{N}\right) = \frac{\partial}{\partial t}\left(\kappa\mathbf{N}\right) - \kappa v^{\perp}\mathbf{T}_s \\
&\Rightarrow v_{ss}^{\perp}\mathbf{N} - v_s^{\perp}\kappa\mathbf{T} = \kappa_t\mathbf{N} + \kappa\mathbf{N}_t - \kappa^2 v^{\perp}\mathbf{N} \\
&\Rightarrow \left(\kappa_t - v_{ss}^{\perp} - \kappa^2 v^{\perp}\right)\mathbf{N} + \left(\kappa v_s^{\perp}\mathbf{T} + \kappa\mathbf{N}_t\right) = 0
\end{aligned}
$$

it follows from Theorem 4.1 in [22] that $\kappa_t = v_{ss}^{\perp} + k^2 v^{\perp}$. Therefore $\mathbf{N}_t = -v_s^{\perp}\mathbf{T}$. ∎

Remark 2.1.3. We impose the following assumptions on the normal velocity $v^{\perp} : S^1(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}$ introduced in ([23], [11]):

(V1) $v^{\perp}(\mathbf{T}, \kappa)$ is $C^{2,1}$

(V2) $\lambda^{-1} \leq \frac{\partial v^{\perp}}{\partial \kappa} \leq \lambda$

(V3) $\left|v^{\perp}(\mathbf{T}, 0)\right| \leq \mu$ for all $\mathbf{T} \in S^1(\mathbb{R}^2)$

(V4) $\left|\nabla^h v^{\perp}\right| + |\kappa|\left|\nabla^v v^{\perp}\right| \leq \nu\left(1 + \kappa^2\right)$

(S) $v^{\perp}(-\mathbf{T}, -\kappa) = -v^{\perp}(\mathbf{T}, \kappa)$,

for positive constants λ, μ, ν and $1 \leq \chi < \infty$.

A remark on notation. By $\nabla v^{\perp}$ we mean the gradient with respect to the first argument $\mathbf{T} \in S^1(\mathbb{R}^2)$. $\nabla^h v^{\perp}$ and $\nabla^v v^{\perp}$ correspond to diferentiation in the horizontal and vertical directions respectively holding the second argument of $v^{\perp}$ fixed, see [22]. The vertical direction can be thought of as "rotation" and the horizontal direction can be thought as "parallel translation" of a vector $\mathbf{T} \in S^1(\mathbb{R}^2)$. The local existence and regularity results for (2.1) can be found in [22, 23]. There Angenent proves that for curves evolving with velocity $v^{\perp}$, the number of self-intersections cannot increase and singularities are due to contracting a loop of at least 180° in which a self-intersection would disappear or forming a kink of at least 360°. In Section 2.3 we show for a normal velocity derived from a Finsler metric F, in the plane singularities only form by contracting a loop thereby losing a self-intersection. In [11], the Oaks proves that simple closed curves either shrink to a point in finite time or exist for infinite time.

2.2 CURVE SHORTENING

In this section we state and prove some observations about curve shortening on the plane where the normal velocity is derived from a Finsler length (2.8), see [25]. The below results can be extended to a Riemannian surface. First recall the definition of a Finsler metric. A Finsler structure on a C^{∞} n-dimensional manifold M is a function

$$F : TM \to [0, \infty)$$

with the following properties:

(i) Regularity: F is C^{∞} on the slit tangent bundle $TM \setminus 0$.

(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

(iii) Strong convexity: The $n \times n$ Hessian matrix

$$(g_{ij}) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of $TM \setminus 0$.

(E) If the Finsler structure F satisfies the criterion $F(x, -y) = F(x, y)$, we have absolute homogeneity: $F(x, \lambda y) = |\lambda| \, F(x, y)$ for all $\lambda \in \mathbb{R}$.

It's worth pointing out that

$$g_{ij}(y) := \left(\frac{1}{2}F^2\right)_{y^i y^j}(y) = \left[FF_{y^i y^j} + F_{y^i}F_{y^j}\right](y). \tag{2.2}$$

Let us recall Euler's Theorem, see [4].

Theorem 2.2.1. Suppose a real-valued function H on $\mathbb{R}^n$ is differentiable away from the origin of $\mathbb{R}^n$. Then the following two statements are equivalent:

- H is positively homogeneous of degree r. That is,

$$H(\lambda y) = \lambda^r H(y) \quad \text{for all } \lambda > 0.$$

- The radial directional derivative of H is r times H. Namely,

$$y^i H_{y^i}(y) = rH(y).$$

Applying Euler's theorem to g yields

$$g_{ij}(y)y^i y^j = F^2(y). \tag{2.3}$$

The fibers of TM are n dimensional vector spaces and thus linearly isomorphic to $\mathbb{R}^n$, hence the strong convexity of F implies that $g_{ij}(y)y^i y^j$ is strictly positive on $\mathbb{R}^n \setminus 0$, since at each point $y \in \mathbb{R}^n$, (g_{ij}) defines an inner product, we have the Cauchy-Schwarz type inequality

$$\left[g_{ij}(y)\xi^i \eta^j\right]^2 \leq \left[g_{ij}(y)\xi^i \xi^j\right]\left[g_{kl}(y)\eta^k \eta^l\right] \quad \forall \xi,\ \eta \in \mathbb{R}^n, \tag{2.4}$$

where equality is obtained if and only if ξ and η are collinear. Set $\eta = y$ and use (2.3) to obtain

$$\left[g_{ij}(y)\xi^i y^j\right]^2 \leq F^2(y)\left[g_{ij}(y)\xi^i \xi^j\right] \quad \forall \eta \in \mathbb{R}^n. \tag{2.5}$$

Using (2.2.1) yields

$$F_{y^i y^j}(y)\xi^i\xi^j = \frac{1}{F^3(y)}\left[F^2(y)g_{ij}(y)\xi^i\xi^j - \left(g_{ij}(y)y^i\xi^j\right)^2\right], \tag{2.6}$$

which combined with (2.5) gives

$$F_{y^i y^j}\xi^i\xi^j \geq 0 \quad \forall\xi \in \mathbb{R}^n, \tag{2.7}$$

where equality holds if and only if ξ and y are collinear. Therefore if F is a strongly convex Finsler metric with ξ and y not collinear, then on a compact set $F_{y^i y^j}\xi^i\xi^j$ is bounded above and below by positive constants.

The Finsler length of any rectifiable curve $\gamma : [a, b] \to M$ is given by the length functional

$$\ell[\gamma] = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt. \tag{2.8}$$

Remark 2.2.2. Note that if M is compact, then $S^1(M)$ is compact, and for a Riemannian metric g we define the function $f : S^1(M) \to \mathbb{R}$, by $f(v) = \frac{\sqrt{g(v,v)}}{F(v)}$. The map f is continuous, and therefore bounded above and below. Hence there is a positive constant C such that for any arc α measured in the Euclidean length $L(\alpha)$ and the Finsler length $\ell(\alpha)$ we have

$$C^{-1}\ell(\alpha) \leq L(\alpha) \leq C\ell(\alpha).$$

First Variation

Computing the first variation of ℓ yields

$$\frac{d}{dt}\ell\left(\alpha\left(t,s\right)\right) = \frac{d}{dt}\int F(\alpha, \alpha_s)\, ds = \int F_{\alpha^i} v^{\perp} N^i + F_{\alpha^i_s}\left(\alpha^i_s\right)_t - v^{\perp}\kappa F\, ds.$$

Here we point out that by (2.1.1ii), $\mathbf{N}_s = -\kappa\mathbf{T}$, also note F being a homogeneous Finsler form of degree one on each tangent space implies by Euler's indentity that

$\alpha_s^i F_{\alpha_s^i} = F$, and integrating by parts we get

$$\begin{aligned}\frac{d}{dt}\ell\left(\alpha\left(t,s\right)\right) &= \int \left\{F_{\alpha^i}N^i - \left(F_{\alpha_s^i}N^i\right)_s - \kappa F\right\} v^\perp \, ds \\ &= \int \left\{F_{\alpha^i}N^i - F_{\alpha_s^i\alpha^j}N^iT^j - F_{\alpha_s^i\alpha_s^j}N^iN^j\kappa - F_{\alpha_s^i}N_s^i - \kappa F\right\} v^\perp \, ds \\ &= \int \left\{F_{\alpha^i}N^i - F_{\alpha_s^i\alpha^j}N^iT^j - F_{\alpha_s^i\alpha_s^j}N^iN^j\kappa\right\} v^\perp \, ds\end{aligned}$$

We obtain a gradient flow for ℓ by setting

$$v^\perp = -F_{\alpha^i}N^i + F_{\alpha_s^i\alpha^j}N^iT^j + F_{\alpha_s^i\alpha_s^j}N^iN^j\kappa. \tag{2.9}$$

We also see that from equation (2.9) that geodesics of the Finsler metric F are smooth curves which satisfy $v^\perp = 0$, i.e.

$$\kappa = \frac{F_{\alpha^i}N^i - F_{\alpha_s^i\alpha^j}N^iT^j}{F_{\alpha_s^i\alpha_s^j}N^iN^j},$$

where $F_{\alpha_s^i\alpha_s^j}N^iN^j$ is bounded above and below by positive constants. Note that if F is induced by a Riemannian metric g

$$F(\mathbf{x};\mathbf{y}) = \sqrt{g\left(\mathbf{y},\mathbf{y}\right)}$$

then the $F_{\alpha^i}, F_{\alpha_s^i\alpha^i}$ vanish and

$$F_{\alpha_s^i\alpha_s^j}N^iN^j = \left[1-(\alpha_s^1)^2\right](\alpha_s^2)^2 + 2\left(\alpha_s^1\alpha_s^2\right)^2 + \left[1-\left(\alpha_s^2\right)^2\right]\left(\alpha_s^1\right)^2 = 1,$$

our normal velocity reduces to that of the standard curve shortening flow. We say that a family of curves $\alpha(t)$ evolve by (Finsler) curve shortening on M (with background metric g) if they satisfy (2.1) with normal velocity given by (2.9).

Proposition 2.2.3. If $\alpha(s,t)$ is a one-parameter family of curves which evolve by curve shortening, then $\frac{d}{dt}\ell(\alpha(s,t)) \leq 0$.

Proposition 2.2.4. The normal velocity $v^\perp : S^1\left(\mathbb{R}^2\right) \times \mathbb{R} \to \mathbb{R}$ which was derived from a Finsler metric (2.9) satisfies the hypotheses in Angenent [22] and Oaks [11], see Remark 2.1.3.

Proof.

Recall that $v^{\perp} = F_{\alpha_s^i \alpha_s^j} N^i N^j \kappa + F_{\alpha_s^i \alpha^j} N^i T^j - F_{\alpha^i} N^i$ and F is C^{∞} on $TM \setminus 0$. It is clear that $v^{\perp}$ satisfies (V_1). Now $\frac{\partial v^{\perp}}{\partial \kappa} = F_{\alpha_s^i \alpha_s^j} N^i N^j$ and due to the strong convexity of F this term was shown to be bounded above and below by positive constants satisfying (V_2). For (V_3) a straightforward calculation yields

$$\left|v^{\perp}(\mathbf{T}, 0)\right| \leq \sum_{i,j=1}^{2} \left|F_{\alpha_s^i \alpha^j}\right| + \sum_{i=1}^{2} \left|F_{\alpha^i}\right|.$$

By our assumption that solutions are contained in a compact set, we see that $F_{\alpha_s^i \alpha^j}$, F_{α^i} are bounded in time. Next

$$\begin{aligned}\left|\nabla^h v^{\perp}\right| + |\kappa| \left|\nabla^v v^{\perp}\right| \leq & \sum_{i,j,l=1}^{2} \kappa^2 \left|F_{\alpha_s^i \alpha_s^j \alpha_s^l}\right| + \kappa \left(\left|F_{\alpha_s^i \alpha^j \alpha_s^l}\right| + \left|F_{\alpha_s^i \alpha_s^j \alpha_s^l}\right|\right) + \left|F_{\alpha_s^i \alpha^j \alpha^l}\right| \\ & + \sum_{i,j=1}^{2} 2\kappa^2 \left|F_{\alpha_s^i \alpha_s^j}\right| + 2\kappa \left|F_{\alpha_s^i \alpha^j}\right| + \left|F_{\alpha^i \alpha^j}\right| + \kappa \left|F_{\alpha^i \alpha_s^j}\right| + \frac{\kappa}{2} \left|F_{\alpha^i}\right|.\end{aligned}$$

We know that the partial derivatives of F are bounded, if $\kappa < 1$ then $\left|\nabla v^{\perp}\right| < \hat{\mu}$ for some constant $\hat{\mu}$. Suppose $\kappa \geq 1$, then the dominating curvature term is κ^2 therefore we obtain the required bound in (V_4). Condition (S) follows from the assumption (E) on F. ■

2.3 THE δ-WHISKER LEMMA

The δ-whisker lemma is a crucial tool ensuring that a curve cannot get to close to itself along subarcs which turn through at least π. Given a positively oriented curve $C(t)$ in the plane we can label the critical points of the height function $y(\cdot)$ of the curve with a "+" if the interior of the curve lies below the critical point (the tangent vector points to the left) and " $-$ " if the interior of the curve is above the critical point (the tangent vector points to the right). We will utilize the following Maximum Principle.

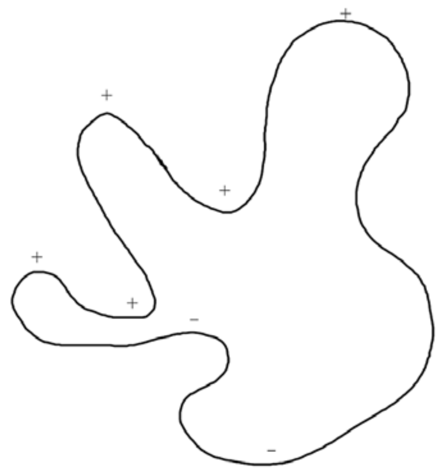

Figure 2.1: $C(t)$ *with critical points labeled.*

We recall the notion of a parabolic operator. The operator

$$D \equiv \sum_{i,j=1}^{n} a_{ij}(\mathbf{x},t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i(\mathbf{x},t)\frac{\partial}{\partial x_i} - \frac{\partial}{\partial t}$$

is said to be uniformly parabolic in a domain E in $(\mathbf{x},t)$ space if

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},t)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2$$

holds with the same number $\mu > 0$ for all $(\mathbf{x},t)$ in E and all $\xi \in \mathbb{R}^n$. Consider the curves in cartesian cordinates, and require that points move to fix their $x-$coordinates, then we get a different flow with the same point-sets as solutions, but different time-derivatives for curvature. We use $'$ to denote differentiation with respect to x.

Lemma 2.3.1. [9] Suppose that $L(x,t) : [0,\epsilon] \times [t_0, t_0+\epsilon] \to \mathbb{R}$ and that

(i) The time evolution of L is given by a uniformly parabolic differential equation.

(ii) $L(x,t_0) \geq 0$, but not $= 0$.

(iii) $L(0,t) \geq 0$, $L(\epsilon,t) \geq 0$.

Then $L(x,t) > 0$ for $x \in (0,\epsilon)$ and $t \in (t_0, t_0+\epsilon]$.

Lemma 2.3.2. Choose Cartesian coordinates so that $\alpha(t_0)$ is locally a graph in $\mathbb{R}^2$. Then the evolution of y fixing x is given by

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{y''}{(1+(y')^2)^2}\left[(y')^2 F_{\alpha_s^1\alpha_s^1} - 2y' F_{\alpha_s^1\alpha_s^2} + F_{\alpha_s^2\alpha_s^2}\right] \\ &+ \frac{1}{\sqrt{1+(y')^2}}\left[y'\left(F_{\alpha_s^1\alpha^1} - F_{\alpha_s^2\alpha^2}\right) + (y')^2 F_{\alpha_s^1\alpha^2} - F_{\alpha_s^2\alpha^1}\right] + \frac{F_{\alpha^2} - y' F_{\alpha^1}}{\sqrt{1+(y')^2}}.\end{aligned}$$

Proof. Let $\theta = \tan^{-1}(y'(x,t))$. The speed of the curve in the vertical direction is

$$\frac{\partial y}{\partial t} = \left(\kappa F_{\alpha_s^i\alpha_s^j} N^i N^j + F_{\alpha_s^i\alpha^j} N^i T^j - F_{\alpha^i} N^i\right)\sec(\theta),$$

where

$$\kappa = \frac{y''}{\left[1+(y')^2\right]^{3/2}} \quad \text{and} \quad \sec(\theta) = \left(1+(y')^2\right)^{1/2}.$$

From this the formula for the evolution of y follows. ∎

The following lemma implies that a pair of tangent arcs which do not cross must separate instantly.

Lemma 2.3.3. Suppose the graphs of $y_1(x,t)$, $y_2(x,t)$ evolve according to (2.1) for $x \in [0,\epsilon]$, $t \in [t_1, t_2]$. Assume the following:

(i) $|y'|$, $|y''|$ are bounded for $i = 1,2$.

(ii) $y_2(x,t_1) \geq y_1(x,t_1)$, but not equal.

(iii) $y_2(0,t) \geq y_1(0,t)$, $y_2(\epsilon,t) \geq y_1(\epsilon,t)$.

Then $y_2(x,t_2) > y_1(x,t_2)$ for $x \in (0,\epsilon)$.

Proof. We define the following operator

$$D(f) = 2\frac{\partial f}{\partial t} - \left[\frac{F_{\alpha_s^i\alpha_s^j} N^i N^j|_{y_1'}}{1+(y_1')^2} + \frac{F_{\alpha_s^i\alpha_s^j} N^i N^j|_{y_2'}}{1+(y_2')^2}\right] f'' - \frac{(y_1''+y_2'')(y_1'+y_2')\left(F_{\alpha_s^1\alpha_s^1}|_{y_1'} + F_{\alpha_s^1\alpha_s^1}|_{y_2'}\right)}{\left[(1+y_1'^2)(1+y_2'^2)\right]^2} f' + g(x,t).$$

Where

$$\begin{aligned} g(x,t) &= \\ &- 2\left[\sqrt{1+(y_2')^2}\left(F_{\alpha_s^i\alpha^j} N^i T^j - F_i N^i\right)|_{y_2'} + \sqrt{1+(y_1')^2}\left(F_i N^i - F_{\alpha_s^i\alpha^j} N^i T^j\right)|_{y_1'}\right] \\ &+ \left[\frac{y_2'^2 F_{\alpha_s^1\alpha_s^1}|_{y_1'} - y_1'^2 F_{\alpha_s^1\alpha_s^1}|_{y_2'}}{\left(1+y_1'^2\right)^2\left(1+y_2'^2\right)^2} + \frac{F_{\alpha_s^2\alpha_s^2}|_{y_2'} - 2y_2' F_{\alpha_s^1\alpha_s^2}|_{y_2'}}{\left(1+y_2'^2\right)^2} - \frac{F_{\alpha_s^2\alpha_s^2}|_{y_1'} - 2y_1' F_{\alpha_s^1\alpha_s^2}|_{y_1'}}{\left(1+y_1'^2\right)^2}\right] \\ &\times \left(y_1''+y_2''\right). \end{aligned}$$

Using Lemma 2.3.2 we observe that $D(y_2 - y_1) = 0$. The function $g(x,t)$ can be viewed as a function of four variables $G(x_1,x_2,x_3,x_4)$ evaluated at (y_1,y_2,y_1',y_2'); computing the Taylor approximation of G and noting that $G(y_1,y_1,y_1',y_1')=0$ gives

$$\begin{aligned} G(y_1,y_2,y_1',y_2') &= \frac{\partial G(y_1,y_1,y',y')}{\partial x_2}(y_2-y_1) + \frac{\partial G(y_1,y_1,y_1',y_1')}{\partial x_4}(y_2'-y_1') \\ &+ h(y_1,y_2,y_1',y_2')(y_2-y_1)(y_2'-y_1'), \end{aligned}$$

this leads to

$$\begin{aligned} D(f) &= 2\frac{\partial f}{\partial t} - \left[\frac{F_{\alpha_s^i\alpha_s^j} N^i N^j|_{y_1'}}{1+(y_1')^2} + \frac{F_{\alpha_s^i\alpha_s^j} N^i N^j|_{y_2'}}{1+(y_2')^2}\right] f'' \\ &- \left[\frac{(y_1''+y_2'')(y_1'+y_2')\left(F_{\alpha_s^1\alpha_s^1}|_{y_1'} + F_{\alpha_s^1\alpha_s^1}|_{y_2'}\right)}{\left[(1+y_1'^2)(1+y_2'^2)\right]^2} + \frac{\partial G(y_1,y_1,y_1',y_1')}{\partial x_4}\right] f' \\ &+ h(y_1,y_2,y_1',y_2')(y_2-y_1) f' + \frac{\partial G(y_1,y_1,y',y')}{\partial x_2} f \end{aligned}$$

Since F is strictly convex we see that D is strictly parabolic. The Maximum Principle implies the result. ∎

Definition 2.3.4. We consider a *subarc* of $C(t)$ as a family of arcs $\alpha(t) \subset C(t)$, with endpoints varying continuously. We say that $\alpha(t_0)$ is *nice* with respect to a vector $\mathbf{v}$ if the inward pointing tangent vectors at the endpoints of $\alpha(t_0)$ point in the same direction as $\mathbf{v}$. This means that if $\mathbf{v}$ is horizontal the endpoints are critical points with opposite signs.

We follow $\alpha(t_0)$ back in time to obtain subarcs $\alpha(t)$ for $t < t_0$. Define $\beta(t) = C(t) \setminus \alpha(t)$, translate these two arcs away from each other horizontally and let $d(t)$ be the maximal distance, possibly infinite, which they can be translated before intersection.

Lemma 2.3.5. *$d(t)$ is a nondecreasing function.*

Proof. Suppose $d(t)$ is realized at an interior point of both $\alpha(t)$ and $\beta(t)$. Since F is spatially dependent, translation and evolution commute up to a factor which is a function of translation. Smoothness of F forces for a small translation of an arc, the commutation of evolution and translation to be small. With out loss we can assume that $\alpha(t)$ and $\beta(t)$ are sufficiently close. By Lemma 2.3.3 d is increasing. Consider the case where $d(t)$ is realized at an endpoint of either $\alpha(t)$ or $\beta(t)$. Since the endpoints are not inflection points for $t < t_0$, in some small neighborhood of the point of tangency to the endpoints $C(t)$ does not cross the tangents. Hence if you translate $\alpha(t)$ by $d(t)$ and the translate $\bar{\alpha(t)}$ is tangent to an endpoint of $\beta(t)$, then the two arcs $C(t)$ and $\bar{\alpha(t)}$ are tangent and do not cross. Therefore the separation lemma applies and $d(t)$ is increasing. Similarly if an endpoint of $\bar{\alpha(t)}$ is tangent to an endpoint of $\beta(t)$. ∎

Lemma 2.3.6. (δ-Whisker Lemma) For $C(0)$ smooth, there is a $\delta > 0$ such that if

(*i*) $C(t)$ exists for all $t < T$,

(*ii*) $\alpha(t_0)$ is a nice subarc of $C(t_0)$ for some $t_0 < t$,

(iii) L is a line segment of length δ based at a point p on $\alpha(t_0)$ pointing in the same direction as the inward tangent vectors to the endpoints of $\alpha(t_0)$,

then L is disjoint from $\beta(t_0) = C(t_0) \setminus \alpha(t_0)$.

Proof. For two disjoint subarcs $\alpha_1(t_0)$, $\alpha_2(t_0)$ of $C(t_0)$ which are nice with respect to vectors v_1, v_2 respectively, we can follow the endpoints of the subarcs backwards in time to obtain a family of subarcs $\alpha_i(t)$. From [23] self intersections are nonincreasing, therefore subarcs do not intersect, remaining disjoint. Define $d_{12}(t)$ to be the maximum distance the two curves can be translated (in the direction of the inward tangents to their endpoints) before intersecting the curve $C(t) \setminus \{\alpha_1(t) \cup \alpha_2(t)\}$, or a translation of the other curve by at most $d(t)$. $d_{12}(t)$ is nondecreasing for the same reasons $d(t)$ is, the first points of contact are at points of tangency where the curves do not cross. Therefore they are separating, and at future times can be slightly translated some more. Set δ to be the minimum of $d_{12}(0)$ over all possible disjoint nice $\alpha_1(0)$ and $\alpha_2(0)$.

Theorem 2.3.7. Let $C(\cdot, t) : S^1 \times [0, T) \to \mathbb{R}^2$ be a smooth solution of (2.1). If $\bar{\omega} > \pi$, the curve shrinks to a point.

First we need a definition.

Definition 2.3.8. Let $\bar{\omega}$ be the supremum of all angles ω such that for any $\epsilon > 0$, there is a $t_0 < T$ such that some subarc $\beta(t_0)$ of $C(t_0)$ has the following properties:

(i) $\int_{\beta(t_0)} \kappa(s) ds = \omega$,

(ii) the diameter of $\beta(t_0)$ is less than ϵ,

(iii) $\left|\kappa_{\beta(t_0)}\right| < K$, where K is the maximum of κ restricted to $\beta(t_0)$.

When the curvature blows up along arcs with total curvature greater than π, a situation that one might suspect to occur is that of a spiral with many turns collapsing in on the inflection point trapped in the middle. The δ-whisker lemma allows us to rule this case out.

Proof. Given $\epsilon > 0$, there is an arc $\beta(t_0)$ with total curvature $\omega \geq \frac{\bar{\omega}+\pi}{2}$ and diameter less than ϵ. If we consider the outward pointing tangents at the endpoints to $\beta(t_0)$, we notice that they either cross themselves or $\beta(t_0)$. In particular this occurs in a ball B_r of radius r about the endpoints, where $r = \max\{\epsilon, \epsilon/(\bar{\omega} - \pi)\}$. The curve $C(t_0)$ is simple, therefore it must avoid any self-intersections, see [22]. By hypothesis the curve does not shrink to a point forcing there to exist an arc $\alpha(t_0)$ intersecting B_r on at least one side of the endpoints with curvature of large magnitude and sign opposite to that of the prevailing curvature of $\beta(t_0)$. We extend $\beta(t_0)$ through $\alpha(t_0)$ up to a point of inflection and choose ϵ small enough, and by [23] the total curvature of $\alpha(t_0)$ will exceed π. So $\alpha(t_0)$ must contain a nice subarc adjacent to $\beta(t_0)$ and hence there is a δ-whisker tangent to the inflection point between $\beta(t_0)$ and $\alpha(t_0)$ pointing away from $\beta(t_0)$. Since ϵ can be chosen smaller than δ, the curve $C(t_0)$ must curve away at the other end of $\beta(t_0)$, similarly we can conclude that there are whiskers tangent to both endpoints pointing away from $\beta(t_0)$. This conflicts with the initial suggestion that the tangent to the endpoints of $\beta(t_0)$ either cross the curve $C(t_0)$ or each other. By Lemma 2.3.6 this cannot occur. ∎

In [23] Theorem 7.1, Angenent proves that the only way a classical solution of (2.1) can become singular is by forming a kink of at least 180° and losing a self-intersection or forming a kink of at least 360°. The latter case is taken care of by the above theorem.

2.4 VERTICAL AND HORIZONTAL SUBSPACES OF $TS^1(M)$

We recall a few basic facts from [7]. Let $\pi : S^1(M) \to M$ be the projection map, the pushforward $d\pi : TS^1(M) \to TM$ is a smooth map. For $(p,u) \in S^1(M)$ we denote the kernel of $d\pi$ by

$$\mathcal{V}_{(p,u)} = \ker \left(d\pi \left|_{(p,u)} \right. \right)$$

and call it the vertical subspace of $T_{(p,u)}S^1(M)$ at (p,u). A curve $X : I \to S^1(M)$ is called vertical if its tangent satisfies $X'(t) \in \mathcal{V}_{X(t)}$ for all $t \in I$. Note that for the inclusion map $i : S^1_p(M) \to S^1(M)$ where $i(u) = (p,u)$, we have $di\left(T_u S^1_p(M)\right) = \mathcal{V}_{(p,u)}$.

Definition 2.4.1. Let V be an open neighborhood of p in M such that the exponential map $\exp_p : S^1_p(M) \to M$ maps a neighborhood V' of 0 in $S^1_p(M)$ diffeomorphically onto V. Let $\tau : \pi^{-1}(V) \to S^1_p(M)$ be the parallel translation map which sends $Y \in \pi^{-1}(V)$ in a parallel manner from the point $q = \pi(Y)$ to p along the unique geodesic in V connecting p and q. Futhermore let $R_u : S^1_p(M) \to S^1_p(M)$ be the translation map defined by $R_u(X) = X - u$. Then the connection map

$$K_{(p,u)} : T_{(p,u)}S^1M \to T_pM$$

of the Levi-Cevita connection ∇ on M is defined to be

$$K(Z) = d\left(\exp_p \circ R_u \circ \tau\right)(Z)$$

for $Z \in T_{(p,u)}S^1(M)$.

If we think of a vector field $Z \in C^\infty S^1(M)$ as a map $Z : M \to S^1(M)$ and let $X_p \in S^1(M)$, then $K\left(dZ_p(X_p)\right) = \left(\nabla_X Z\right)_p$.

Definition 2.4.2. The horizontal subspace $\mathcal{H}_{(p,u)}$ of the tangent space $T_{(p,u)}S^1(M)$ of the unit tangent bundle $S^1(M)$ at (p,u) is defined to be

$$\mathcal{H}_{(p,u)} = \ker\left(K_{(p,u)}\right).$$

A curve $X : I \to S^1(M)$ is said to be horizontal if its tangent X' satisfies $X' \in \mathcal{H}_{X(t)}$ for all $t \in I$. Notice that if we have a curve $\gamma : I \to M$ with $\gamma(0) = p, \gamma'(0) = u$ and take $X : I \to S^1M$ to be a vector field along γ i.e. $X(t) = (\gamma(t), U(t))$ for $U(t) \in S^1_{\gamma(t)}(M)$, then K maps $X' \mapsto (\nabla_{\gamma'}U(t))(0)$. This gives us a correspondence between horizontal curves in $S^1(M)$ and parallel vector fields on M.

Remark 2.4.3.

$$T_{(p,u)}S^1(M) = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)}.$$

We recall the parrallel translation map $\tau : \pi^{-1}(V) \to S^1_p(M)$, for any vertical tangent vector A, $d(i \circ \tau)A = A$. Also notice that $d(\exp \circ R_u)$ is an isomorphism of $T_uS^1_p(M)$, and since the horizontal space is the kernel of K a vertical tangent vector cannot be in the kernel of K. Therefore $\mathcal{H}_{(p,u)} \cap \mathcal{V}_{(p,u)} = \{0\}$. Next we define the horizontal and vertical lifts and it will be clear that the dimensions agree for the direct sum.

Definition 2.4.4. Let $X \in S^1_p(M)$, the horizontal lift of X to $(p,u) \in S^1(M)$ is the unique vector $X^h \in \mathcal{H}_{(p,u)}$ such that $d\pi(X^h) = X$. The vertical lift of X to (p,u) is the unique vector $X^v \in \mathcal{V}_{(p,u)}$ such that $X^v(df) = X(f)$ for all functions f on M, where $df(p,u) = u(f)$.

Definition 2.4.5. The horizontal lift of a vector field $X \in C^\infty(S^1(M))$ is the vector field $X^h \in C^\infty(TS^1(M))$ which value at a point (p,u) is the horizontal lift of $X(p)$ to (p,u). The vertical lift is defined similarly.

For a vector field X on M the horizontal and vertical lifts X^h, X^v are the uniquely determined vector fields on TM satisfying

$$d\pi(X^h)_Z = X_{\pi(Z)}, \quad KX^h_Z = 0_{\pi(Z)}$$

and

$$d\pi(X^v)_Z = 0_{\pi(Z)}, \quad KX^v_Z = X_{\pi(Z)}$$

for all $Z \in S^1(M)$. The maps $X \to X^h$, and $X \to X^v$ are vector space isomorphisms between $S^1_p(M)$ and the subspaces $\mathcal{H}_{(p,u)}$ and $\mathcal{V}_{(p,u)}$ respectfully. If $\hat{Z} \in T_{(p,u)}S^1(M)$, then $\hat{Z}$ can be expressed as

$$\hat{Z} = X^h + Y^v$$

where $X, Y \in S^1_p(M)$ are uniquely determined by $X = d\pi(\hat{Z})$ and $Y = K(\hat{Z})$. For a given smooth function $f : M \to \mathbb{R}$, $X^h(f \circ \pi) = X(f) \circ \pi$ and $X^v(f \circ \pi) = 0$, for all vector fields $X \in C^\infty(M)$. In local coordinates the horizontal and vertical lifts take the following form

$$(X^v)_Z = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial v_{m+i}}$$

and

$$\left(X^h\right)_Z = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial v_i} - \left(\sum_{i,j,k=1}^{m} \xi_j \eta_k \Gamma^i_{jk} \right) \frac{\partial}{\partial v_{m+i}}$$

where $X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}$, $Z = \sum_{i=1}^m \eta_i \frac{\partial}{\partial x_i}$ and Γ^i_{jk} are the Christoffel symbols of the Levi-Cevita connection ∇ on (M, g).

2.5 LONG TERM BEHAVIOR

Suppose γ is a family evolving according to (2.1), we assume throughout that γ remains in a compact subset of $\mathbb{R}^2$. In this Section we prove that if the length of a simple closed curve does not converge to zero, then its normal velocity must converge to zero. See Grayson's papers [15, 14] for the analogous results on standard curve shortening. The following results carry over to a riemannian surface M, in particular for $\chi(M) \leq 0$ we assume the universal cover $\hat{M} = \mathbb{R}^2$, in the case $\chi(M) = 0$ for $M = \mathbb{T}$ or the poicare disk when $\chi(M) < 0$. We treat γ as a lift into $\mathbb{R}^2$ see [25].

Lemma 2.5.1. Let $\gamma_0 : S^1 \to \mathbb{R}^2$ be a simple C^2 curve, then the solution $\gamma(x,t)$ to (2.1) such that $\gamma(x,0) = \gamma_0$ either shrinks to a point in finite time or its (Finsler) length approaches some positive limit as $t \to \infty$.

Proof. From Oaks [11], we see that γ shrinks to a point in finite time or exists for infinite time. Since the length of $\gamma(s,t)$ is decreasing, that is $\frac{d}{dt} l\left(\gamma(s,t)\right) = -\int \left(v^{\perp}\right)^2 ds$, and hence it has a limit. Therefore for infinite time existence $l\left(\gamma(s,t)\right) \xrightarrow{t} l_\infty \geq 0$. To see that $l_\infty > 0$, we establish the following

Claim 2.5.2. If the area enclosed by $\gamma(0)$ is sufficiently small, then the area enclosed by the curve $\gamma(t)$ decreases at a rate bounded away from zero.

Proof. Let $D(t)$ be the region enclosed by $\gamma(\cdot,t) = (x(\cdot,t), y(\cdot,t))$ with area form dS. We integrate by parts and use 2.1.2, applying Green's theorem where we use the outward normal velocity $-\mathbf{N}$ the area $A(t)$ satisfies

$$\begin{aligned}
\frac{dA(t)}{dt} &= \frac{d}{dt}\iint_D dS = \frac{1}{2}\frac{d}{dt}\int_{\partial D} xy_s - yx_s ds = -\frac{1}{2}\frac{d}{dt}\int_{\partial D} \gamma \cdot \mathbf{N} ds \\
&= -\frac{1}{2}\int_{\partial D} \gamma_t \cdot \mathbf{N} + \gamma \cdot \mathbf{N}_t - \gamma \cdot \mathbf{N} v^{\perp}\kappa ds = -\frac{1}{2}\int_{\partial D} v^{\perp} - v^{\perp}_s \gamma \cdot \mathbf{T} - \gamma \cdot \mathbf{N} v^{\perp}\kappa ds \\
&= -\frac{1}{2}\int_{\partial D} v^{\perp} + v^{\perp}\left(\gamma_s \cdot \mathbf{T} + \kappa\gamma \cdot \mathbf{N}\right) - \gamma \cdot \mathbf{N} v^{\perp}\kappa ds = -\int_{\partial D} v^{\perp} ds.
\end{aligned}$$

Since $v^{\perp}$ was choosen so that we have a gradient flow (2.9), we can write

$$\frac{dA(t)}{dt} = \int_{\partial D} -F_{\alpha_s^i \alpha_s^j} N^i N^j \kappa + F_{\alpha^i} N^i - F_{\alpha_s^i \alpha^j} N^i T^j ds.$$

F is a strictly convex Finsler metric, therefore $F_{\alpha_s^i \alpha_s^j} N^i N^j$ is bounded above and below by positive constants. We can employ the Gauss-Bonnet theorem to obtain an upper bound on the rate of change of $A(t)$.

$$\begin{aligned}\frac{dA(t)}{dt} &< \int_{\partial D} -C\kappa + F_{\alpha^i} N^i - F_{\alpha_s^i \alpha^j} N^i T^j ds = \int_{\partial D} -C\kappa ds \\ &+ \int_{\partial D} \underbrace{\left\langle -F_{\alpha^1} + F_{\alpha_s^1 \alpha^1} T^1 + F_{\alpha_s^1 \alpha^2} T^2, -F_{\alpha^2} + F_{\alpha_s^2 \alpha^1} T^1 + F_{\alpha_s^2 \alpha^2} T^2 \right\rangle}_{\mathbf{G}} \cdot (-\mathbf{N}) ds \\ &= -\int_{\partial D} C\kappa ds + \iint_D (\nabla \cdot \mathbf{G})\, dS < -2\pi C + \sup_D |\nabla \cdot \mathbf{G}|\, A(t).\end{aligned}$$

Choose

$$\epsilon = \frac{C\pi}{\sup_D |\nabla \cdot \mathbf{G}|}.$$

Therefore when γ encloses area less than ϵ, $\frac{dA(t)}{dt} < -C\pi$. ∎

(continuing with the proof of Lemma 4.1) We see that for infinite time existence the curve cannot converge to a point.

∎

Lemma 2.5.3. If the curve exists for infinite time, the L^2 norm of $v^{\perp}$ converges to zero as $t \to \infty$.

Proof. The time derivative of l is $\frac{d}{dt} l(\gamma(x,t)) = -\int \left(v^{\perp}\right)^2 ds$, from Lemma 2.5.1 we know that if $\lim_{t\to\infty} \frac{d}{dt} l(\gamma)$ exists then the limit must be zero. It's enough to prove that we can control the time derivative to ensure it converges to zero. Using Lemma

2.1.1 and Lemma 2.1.2 we compute.

$$\begin{aligned}\frac{d}{dt}\int\left(v^{\perp}\right)^{2}\,ds &= \int 2v^{\perp}\frac{\partial}{\partial t}v^{\perp}-\left(v^{\perp}\right)^{3}\kappa\,ds\\&= \int 2v^{\perp}\left[\nabla^{h}v^{\perp}\cdot\gamma_{t}+\nabla^{v}v^{\perp}\cdot\gamma_{st}+v^{\perp}_{\kappa}\kappa_{t}\right]-\left(v^{\perp}\right)^{3}\kappa\,ds\\&= \int 2v^{\perp}\left[\nabla^{h}v^{\perp}\cdot\gamma_{t}+\nabla^{v}v^{\perp}\cdot\mathbf{T}_{t}+v^{\perp}_{\kappa}\kappa_{t}\right]-\left(v^{\perp}\right)^{3}\kappa\,ds\\&= \int 2v^{\perp}\left[\nabla^{h}v^{\perp}\cdot\gamma_{t}+\nabla^{v}v^{\perp}\cdot v^{\perp}_{s}\mathbf{N}\right]ds\\&+\int 2v^{\perp}_{\kappa}\left(v^{\perp}v^{\perp}_{ss}+\kappa^{2}\left(v^{\perp}\right)^{2}\right)-\left(v^{\perp}\right)^{3}\kappa\,ds.\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\int\left(v^{\perp}\right)^{2}\,ds &\leq \int 2\left(v^{\perp}\right)^{2}\left|\nabla^{h}v^{\perp}\right|+2v^{\perp}v^{\perp}_{s}\left|\nabla^{v}v^{\perp}\right|ds+2\int v^{\perp}_{\kappa}v^{\perp}v^{\perp}_{ss}ds\\&+2\lambda\int\left(v^{\perp}\right)^{2}\kappa^{2}\,ds-\int\left(v^{\perp}\right)^{3}\kappa ds\\&\leq \int 2\nu\left(v^{\perp}\right)^{2}\left(1+\kappa^{2}\right)+2\nu v^{\perp}v^{\perp}_{s}\sup\left|\nabla^{v}v^{\perp}\right|\,ds-\frac{2}{\lambda}\int\left(v^{\perp}_{s}\right)^{2}ds\\&-2\int v^{\perp}v^{\perp}_{s}\left(v^{\perp}_{\kappa}\right)_{s}ds+2\lambda\int\left(v^{\perp}\right)^{2}\kappa^{2}\,ds-\int\left(v^{\perp}\right)^{3}\kappa\,ds.\end{aligned}$$

We assume without loss of generality that

$$\int\left(v^{\perp}_{s}\right)^{2}ds>1,$$

since otherwise the estimate follows seamlessly. Note that from $(V_2),(V_3)$ we have

$$v^{\perp}\geq\lambda+v^{\perp}(\mathbf{T},0).$$

Then for any $\epsilon>0$ using the Peter-Paul inequality

$$v^{\perp}\kappa\geq\lambda\kappa^{2}-\frac{1}{4\epsilon}\left(v^{\perp}(\mathbf{T},0)\right)^{2}-\epsilon\kappa^{2}.$$

Setting $\epsilon=\lambda$ yields

$$v^{\perp}\kappa\geq-\frac{1}{4\lambda}\left(v^{\perp}(\mathbf{T},0)\right)^{2}\geq-\frac{\mu^{2}}{4\lambda}.$$

$$\frac{d}{dt}\int \left(v^{\perp}\right)^2 \ ds \le 2\nu \sup(1+\kappa^2)\int \left(v^{\perp}\right)^2 ds$$
$$+ 2\nu \sup\left|\nabla^v v^{\perp}\right| \left(\int \left|v^{\perp}\right|^2 ds\right)^{\frac{1}{2}} \left(\int \left|v_s^{\perp}\right|^2 \ ds\right)^{\frac{1}{2}} - \frac{2}{\lambda}\int \left(v_s^{\perp}\right)^2 ds$$
$$- 2\int v^{\perp} v_s^{\perp} \left(v_{\kappa}^{\perp}\right)_s ds + 2\lambda \sup(\kappa^2)\int \left(v^{\perp}\right)^2 ds + \frac{\mu^2}{4\lambda}\int \left(v^{\perp}\right)^2 ds$$
$$\le 2\nu \sup(1+\kappa^2)\int \left(v^{\perp}\right)^2 ds - \left(\int \left|v_s^{\perp}\right|^2 ds\right)\left\{\frac{2}{\lambda} - 2\nu \sup\left|\nabla^v v^{\perp}\right| \left(\int \left|v^{\perp}\right|^2 ds\right)^{\frac{1}{2}}\right\}$$
$$- 2\int v^{\perp} v_s^{\perp} \left(v_{\kappa}^{\perp}\right)_s ds + 2\lambda \sup(\kappa^2)\int \left(v^{\perp}\right)^2 ds + \frac{\mu^2}{4\lambda}\int \left(v^{\perp}\right)^2 ds$$

For our particular choice of normal velocity 2.9, we have

$$(v_{\kappa}^{\perp})_s = (F_{\alpha_s^i\alpha_s^j}N^iN^j)_s = F_{\alpha_s^i\alpha_s^j\alpha^l}N^iN^jT^l + \kappa F_{\alpha_s^i\alpha_s^j\alpha_s^l}N^iN^jN^l + F_{\alpha_s^i\alpha_s^j}(N^iN_s^j + N_s^iN^j)$$
$$= F_{\alpha_s^i\alpha_s^j\alpha^l}N^iN^jT^l + \kappa F_{\alpha_s^i\alpha_s^j\alpha_s^l}N^iN^jN^l + 2\kappa F_{\alpha_s^i\alpha_s^j}N^iT^j.$$

Under assumptions (1), (2) we see that

$$\left|(v_{\kappa}^{\perp})_s\right| \le \left|F_{\alpha_s^i\alpha_s^j\alpha^l}\right| + \kappa\left|F_{\alpha_s^i\alpha_s^j\alpha_s^l}\right| + 2\kappa\lambda$$

is bounded in time. We can estimate

$$\left|\int v^{\perp}v_s^{\perp}(v_{\kappa}^{\perp})_s ds\right| \le \sup\left|(v_{\kappa}^{\perp})_s\right| \left(\int \left|v^{\perp}\right|^2 ds\right)^{\frac{1}{2}} \int \left|v_s^{\perp}\right|^2 ds.$$

$$\frac{d}{dt}\int \left(v^{\perp}\right)^2 ds \le$$
$$- \left(\int \left|v_s^{\perp}\right|^2 ds\right)\left\{\frac{2}{\lambda} - \left(2\nu\sup\left|\nabla^v v^{\perp}\right| + \sup\left|\left(v_{\kappa}^{\perp}\right)_s\right|\right)\left(\int \left|v^{\perp}\right|^2 ds\right)^{\frac{1}{2}}\right\}$$
$$+ \left\{2\lambda\sup(\kappa^2) + \frac{\mu^2}{4\lambda} + 2\nu\sup(1+\kappa^2)\right\}\int \left(v^{\perp}\right)^2 ds.$$

Now

$$\sup\left(v^{\perp}\right)^2 \le \left(\inf\left|v^{\perp}\right| + \int \left|v_s^{\perp}\right| ds\right)^2 \le \frac{2}{l_{\infty}}\int \left(v^{\perp}\right)^2 ds + 2l_0\int \left(v_s^{\perp}\right)^2 ds.$$

That is

$$-\int \left(v_s^{\perp}\right)^2 ds \leq \frac{1}{l_0 l_\infty}\int \left(v^{\perp}\right)^2 ds - \frac{\sup\left(v^{\perp}\right)^2}{2l_0}.$$

Therefore

$$\frac{d}{dt}\int \left(v^{\perp}\right)^2 ds \leq$$
$$\left(\frac{1}{l_0 l_\infty}\int \left(v^{\perp}\right)^2 ds - \frac{\sup\left(v^{\perp}\right)^2}{2l_0}\right)\left\{\frac{2}{\lambda} - \left(2\nu \sup\left|\nabla^v v^{\perp}\right| + \sup\left|\left(v_\kappa^{\perp}\right)_s\right|\right)\left(\int \left|v^{\perp}\right|^2 ds\right)^{\frac{1}{2}}\right\}$$
$$+\left\{2\lambda \sup(\kappa^2) + \frac{\mu^2}{4\lambda} + 2\nu\sup(1+\kappa^2)\right\}\int \left(v^{\perp}\right)^2 ds.$$

This bounds $\left|\frac{d^2\ell}{dt^2}\right|$ to exponential when $\int \left(v^{\perp}\right)^2\ ds$ is sufficiently small. Now if $\gamma(\cdot,t_n)$ is a subsequence of $\gamma(\cdot,t)$ such that $\lim_{n\to\infty}|\ell(\gamma(\cdot,t_n))| \geq \beta > 0$ for $\epsilon > 0$,

$$\ell(t_n) - \ell(t_n+\epsilon) = \int_{t_n}^{t_n+\epsilon}\frac{d\ell}{dt}dt \geq \epsilon\mu\left(\left\{t\in[t_n,t_n+\epsilon]\,|\,\left|\frac{d\ell}{dt}(t)\right| \geq \epsilon\right\}\right)$$

therefore

$$\mu\left(\left\{t\in[t_n,t_n+\epsilon]\,|\,\left|\frac{d\ell}{dt}(t)\right| \geq \epsilon\right\}\right) \to 0.$$

Set $\epsilon = \frac{1}{k}$ and choose N large enough so that $\mu\left(\left\{t\in[t_N,t_N+\epsilon]\,|\,\left|\frac{d\ell}{dt}(t)\right| \geq \frac{1}{k}\right\}\right) < \frac{1}{k}$, hence we can find a sequence $s_k \in [t_{N_k}, t_{N_k}+\epsilon]$ such that $\left|\frac{d\ell}{dt}(s_k)\right| < \frac{1}{k}$. Therefore $|t_{N_k} - s_k| < \frac{1}{k}$ and

$$\left|\frac{d}{dt}\ell(t_{N_k})\right| \leq \left|\frac{d}{dt}\ell(s_k)\right| e^{C(s_k - t_{N_k})} < \frac{1}{k}e^{\frac{C}{k}}.$$

■

Corollary 2.5.4. If the curve exists for infinite time, then $\int \left|v^{\perp}\right| ds$ converges to zero as $t\to\infty$.

Proof. This follows from 2.5.3 and Holder's inequality. ∎

Next we estimate the L^2 norm of $v_s^\perp$.

Lemma 2.5.5.

$$\lim_{t\to\infty}\int\left(v_s^\perp\right)^2 ds = 0.$$

Proof. Suppose $\lim_{t\to\infty}\int\left(v_s^\perp\right)^2 ds \neq 0$, when $\int\left(v_s^\perp\right)^2 ds$ is sufficiently larger than $\int\left(v^\perp\right)^2 ds$ we compute the time derivative of $\int v_s^\perp ds$.

$$\begin{aligned}
\frac{d}{dt}\int\left(v_s^\perp\right)^2 ds &= \int 2v_s^\perp v_{st}^\perp - \left(v_s^\perp\right)^2 v^\perp \kappa ds \\
&= \int 2v_s^\perp \left\{\frac{\partial}{\partial s}\left(\nabla^h v^\perp \cdot v^\perp \mathbf{N} + \nabla^v v^\perp \cdot v_s^\perp \mathbf{N} + v_k^\perp k_t\right) + v^\perp \kappa v_s^\perp\right\} - \left(v_s^\perp\right)^2 v^\perp \kappa ds \\
&= \int -2v_{ss}^\perp \left\{\nabla^h v^\perp \cdot v^\perp \mathbf{N} + \nabla^v v^\perp \cdot v_s^\perp \mathbf{N} + v_k^\perp\left(v_{ss}^\perp + \kappa^2 v^\perp\right)\right\} + \left(v_s^\perp\right)^2 v^\perp \kappa ds.
\end{aligned}$$

Without lose of generality we can assume $\int\left(v_s^\perp\right)^2 ds > C\int\left(v^\perp\right)^2 ds$, with this in mind we notice

$$\int\left(v_s^\perp\right)^2 ds = \int -v^\perp v_{ss}^\perp ds \leq \left[\int\left(v^\perp\right)^2 ds \cdot \int\left(v_{ss}^\perp\right)^2 ds\right]^{\frac{1}{2}},$$

thus

$$\int\left(v_s^\perp\right)^2 < C^{-1}\int\left(v_{ss}^\perp\right)^2 ds. \tag{2.10}$$

For ϵ small we can make $\int\left(v^\perp\right)^2 ds < \epsilon$, this implies

$$\begin{aligned}
\int v^\perp \kappa \left(v_s^\perp\right)^2 ds &\leq \int v^\perp \kappa \sup\left(v_s^\perp\right)^2 ds \leq \sup(\kappa)\int v^\perp \sup\left(v_s^\perp\right)^2 ds \\
&\leq \epsilon \sup(\kappa)\left(\int v_{ss}^\perp ds\right)^2 \leq \epsilon l_0 \sup(\kappa)\int\left(v_{ss}^\perp\right)^2 ds.
\end{aligned}$$

For the penultimate inequality we note

$$\sup\left(v_s^\perp\right)^2 \leq \left(\inf\left|v_s^\perp\right| + \int\left|v_{ss}^\perp\right| ds\right)^2 \leq l_0\int\left(v_{ss}^\perp\right)^2 ds.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\int\left(v_s^\perp\right)^2 ds &= -2\int v_{ss}^\perp v^\perp\left[\left(\nabla^h v^\perp\cdot\mathbf{N}\right)+\kappa^2 v_\kappa^\perp\right]ds - 2\int v_{ss}^\perp v_s^\perp\left(\nabla^v v^\perp\cdot\mathbf{N}\right)ds\\ &\quad -2\int v_\kappa^\perp\left(v_{ss}^\perp\right)^2 ds + \int\left(v_s^\perp\right)^2 v^\perp\kappa ds\\ &\le 2\epsilon^{1/2}\left(\sup\left|\nabla^h v^\perp\right| + \lambda\sup(\kappa^2)\right)\left(\int\left|v_{ss}^\perp\right|^2 ds\right)^{\frac{1}{2}}\\ &\quad + 2\sup\left|\nabla^v v^\perp\right|\left(\int\left|v_s^\perp\right|^2 ds\right)^{\frac{1}{2}}\left(\int\left|v_{ss}^\perp\right|^2 ds\right)^{\frac{1}{2}}\\ &\quad + \left[-\frac{2}{\lambda}+\epsilon l_0\sup(\kappa)\right]\int\left(v_{ss}^\perp\right)^2 ds.\end{aligned}$$

By 2.10 we have

$$\begin{aligned}\frac{d}{dt}\int\left(v_s^\perp\right)^2 ds &\le 2\epsilon^{1/2}\left(\sup\left|\nabla^h v^\perp\right| + \lambda\sup(\kappa^2)\right)\left(\int\left|v_{ss}^\perp\right|^2 ds\right)^{\frac{1}{2}}\\ &\quad + 2C^{-1}\sup\left|\nabla^v v^\perp\right|\int\left|v_{ss}^\perp\right|^2 ds + \left[-\frac{2}{\lambda}+\epsilon l_0\sup(\kappa)\right]\int\left|v_{ss}^\perp\right|^2 ds\\ &= 2\epsilon^{1/2}\left(\sup\left|\nabla^h v^\perp\right| + \lambda\sup(\kappa^2)\right)\left(\int\left|v_{ss}^\perp\right|^2 ds\right)^{\frac{1}{2}}\\ &\quad + \left[-\frac{2}{\lambda}+\epsilon l_0\sup(\kappa) + 2C^{-1}\sup\left|\nabla^v v^\perp\right|\right]\int\left|v_{ss}^\perp\right|^2 ds.\end{aligned}$$

Since

$$-\int\left(v_{ss}^\perp\right)^2 ds < -C\int\left(v_s^\perp\right)^2 ds$$

we see that choosing C sufficiently large implies $\int v_s^\perp ds$ either decays exponentially or is proportional to $\int\left|v^\perp\right|^2 ds$ therefore decreasing to zero. ∎

Theorem 2.5.6. Let $\gamma_0 : S^1 \to \mathbb{R}^2$ be a simple C^2 curve, such that $\gamma(x,t)$ is the solution to (2.1) where $\gamma(x,0)=\gamma_0$. If the solution exists for infinite time, then

$$\lim_{t\to\infty}\sup_{\gamma(t)}\left|v^\perp\right| = 0.$$

Proof. Since we have estimates for the L^2 norm of $v^\perp$ and $v_s^\perp$ we see from the Sobolev inequality that $\sup\left|v^\perp\right|$ decreases to zero. ∎

CHAPTER 3

SECOND-ORDER LAGRANGIANS

3.1 BACKGROUND

Second-order Lagrangian systems are used as models in various areas of physics, namely nonlinear elasticity, nonlinear optics, and mechanics. These models arise as fourth-order differential equations obtained variationally as the Euler-Lagrange equations of an action functional which depends on the second derivative of the state variable u'' as well as its lower derivatives. One important class of differential equations is $u'''' - \beta u'' + f(u) = 0$, known as the Swift-Hohenberg equation for $\beta \leq 0$ and the extended Fisher-Kolmogorov (eFK) equation for $\beta > 0$. There have been numerous results concerning these equations, see for instance [12, 16, 19, 20, 28, 29].

A second-order Lagrangian system is defined variationally by extremizing an action functional of the form

$$J[u] = \int L(u, u', u'')\, dx. \tag{3.1}$$

Computing the Euler-Lagrange equation yields

$$\frac{d^2}{dx^2}\frac{\partial L}{\partial u''} - \frac{d}{dx}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0. \tag{3.2}$$

The Lagrangian action J is invariant under the $\mathbb{R}$ action $x \mapsto x+c$, which by Noether's Theorem yields the conservation law

$$\left(\frac{\partial L}{\partial u'} - \frac{d}{dx}\frac{\partial L}{\partial u''}\right) u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'') = E. \tag{3.3}$$

Under the natural hypothesis that L is convex in u'', that is $\partial_w^2 L(u,v,w) > 0$ for all (u,v,w), the Lagrangian system (L, dx) is equivalent to a Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic coordinates $x = (u, v, p_u, p_v)$ endowed with the symplectic form ω given by $\omega = du \wedge dp_u + dv \wedge dp_v$. The Hamiltonian is

$$H = p_u v + w L_w - L, \tag{3.4}$$

where w and p_v are related by $p_v = L_w(u,v,w)$ and $p_u = L_v - p_v'$. Stationary functions of J satisfy equation (3.3), which is equivalent to $H(x) = E$. For the associated Hamiltonian system (H, ω) this means that the flow lines lie on the 3-dimensional energy manifold $M_E = \{x \in \mathbb{R}^4 \,|H(x) = E\}$. If $\nabla H \neq 0$ on M_E, then E is called a regular value and M_E is a smooth non-compact manifold without boundary. Hence for a regular value E, the vector field $X_H = \mathcal{J}\nabla H$, where $\mathcal{J}$ is defined by

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

is non singular when restricted to M_E. The singular points of X_H are points for which $X_H(x^*) = 0$, which are exactly the critical points of the Hamiltonian and therefore occur at singular energy levels. Singular points are of the form $x^* = (u, 0, p_u, p_v)$ where $p_u = L_v(u,0,0)$, $L_u(u,0,0) = 0$, and $p_v = L_w(u,0,0)$. Equivalently, for a Lagrangian system an energy level is said to be regular if and only if $L_w(u,0,0) \neq 0$ for all $u \in \mathbb{R}$ satisfying $-L(u,0,0) = E$. These energy surfaces are invariant under the flow ϕ of (3.3), and the dynamical behavior of the system can be studied on an individual energy surface, c.f. [12]. A bounded characteristic of a Lagrangian system (L, dt) is a function $u \in C_b^2(\mathbb{R},\mathbb{R})$ for which $\delta \int_I L(u,u',u'') = 0$ with respect to variations $\delta u \in C_c^2(I,\mathbb{R})$ for any compact interval $I \subset \mathbb{R}$. Since the Lagrangian is a C^2-function of (u,v,w), it follows from the Euler-Lagrange equations that $u \in C_b^3(\mathbb{R},\mathbb{R})$, $L_w(\cdot) \in C_b^2(\mathbb{R},\mathbb{R})$, and $(\frac{d}{dt}\frac{\partial L}{\partial w} - \frac{\partial L}{\partial v})(\cdot) \in C_b^2(\mathbb{R})$. The main concern of this dissertation is to explore the existence and structure of bounded and closed characteristics on M_E. We conclude this preliminary section with some relationships between closed characteristics and the geometry of M_E.

We have a projection $\pi : M_E \to \mathbb{R}^2$ given by $\pi(u, v, w, p_u) = (u, v)$ and a differential 1-form

$$\alpha = wdu - vdv \tag{3.5}$$

on M_E. Every function $u : \mathbb{R} \to \mathbb{R}$ defines a unique lifted curve in M, that is

$$\gamma_u(x) = (u(x), u'(x), u''(x), p_u(x)) \tag{3.6}$$

with

$$p_u = \frac{L(u, v, w) - wL_w(u, v, w)}{v}. \tag{3.7}$$

This introduces a singularity when $v = 0$. We avoid this by assuming that at critical points of u

$$L - wL_w = 0. \tag{3.8}$$

Thus γ is an integral curve for α, i.e. $\gamma^*(\alpha) = 0$. Conversely, if γ is an integral curve of the one-form α with $\gamma'(s) \notin \ker d\pi$ for all s, then one can reparametrize γ by introducing

$$x = \int \frac{du}{v} = \int \frac{dv}{w}. \tag{3.9}$$

The condition $\gamma' \notin \ker d\pi$ implies that u' and v' cannot vanish simultaneously. If $v \neq 0$, then $wdu - vdv = 0$ implies that $u' \neq 0$, so that $\frac{du}{v} \neq 0$ there. If $v = 0$ then $H = E$ and (3.4) imply that $wL_w - L = E$. We assume henceforth that $L(u, 0, 0) + E \neq 0$ for all $u \in \mathbb{R}$, i.e. E is a regular value of H. Then it is impossible for w to vanish when $v = 0$, and thus $u' = 0$. Then $v' \neq 0$, and we have $\frac{dv}{w} \neq 0$, so we can define x by integrating dv/w. Reparametrizing γ this way, we retrieve the function $u(x)$.

We make the following simplifying assumptions on the Lagrangian:

$$(H1)\ L(u,v,w) = \tfrac{1}{2}w^2 + K(u,v) \tag{3.10}$$

$$(H2)\ K(u,v) \geq -C(|u|) - C(|u|)\,|v|^{\gamma} \text{ for some } \gamma < 4,$$

where $C(|u|)$ is locally bounded. Note that $(H2)$ is a lower bound on K; an upper bound is not necessary, and the hypothesis on K can be weakened, see section 4 in [12]. To establish the existence of closed characteristics on energy manifolds of second-order Lagrangian systems we use their variational structure.

Given an arbitrary $(2n-1)$-dimensional manifold M embedded in $(\mathbb{R}^{2n}, \omega)$, with ω being the standard symplectic form, one can construct a Hamiltonian H for which $M = H^{-1}(0)$. The choice of H is not instrinsic to finding periodic orbits, it turns out that the geometry of M is enough to describe them. The geometry of M and the symplectic 2-form ω define a characteristic line bundle,

$$\mathcal{E}_M = \{(x,\xi) \in T_xM \setminus \{0\} \mid w_x(\xi,\eta) = 0\ \forall \eta \in T_xM\} \subset TM$$

Given a vector field V of H on M, we see that V is a section of $\mathcal{E}_M$. The trajectory of a periodic orbit can be viewed as a closed characteristic of the line bundle, i.e. an embedding $\gamma : S^1 \to M$ of the circle into M for which

$$T\gamma = \mathcal{E}_M\,|_{\gamma}\ .$$

This gives us a relationship between existence of periodic solutions to (3.2) and topological and geometric properties of its energy surfaces. These closed characteristics can be found as critical points of the action functional. We are motivaed by a novel result by Rabinowitz [18], Weinstein [30] conjectured in the 1970's, that any compact hypersurface $M \in (\mathbb{R}^{2n}, \omega)$, with the additional assumption that

$$\alpha(\xi) \neq 0, \quad 0 \neq \xi \in \mathcal{E}_M,$$

for some 1-form α with $d\alpha = \omega$, i.e. M is of *contact type* relative to ω, has at least one closed characteristic. This was proved later by Viterbo [27]. However this theory cannot be applied to energy manifolds determined by second-order Lagrangian systems, because these manifolds are always non-compact and they are not necessarily of contact type in $(\mathbb{R}^4, \omega)$, as was proved in [21].

3.2 CLOSED CHARACTERISTICS

The existence of closed characteristics for second-order Lagrangian systems has been studied throughout the last few decades; in [5] the authors show that under the twist condition the existence of a closed characteristic can be proved by discretizing the variational problem to study closed characteristics using recurrence relations. To accomplish this, an extra condition on the system must be made which is called the twist condition. This condition requires the existence of a minimizer $u(x, u_1, u_2)$ of the functional J for all u_1, u_2 with $u_1 \neq u_2$ and that u is C^1. This can be a difficult assumption to verify. The twist property allows a reduction of the problem to a finite-dimensional recurrence relation constructed from plugging the minimizer into the functional $S_E(u_1, u_2) = J_E\left[z(x; u_1, u_2)\right]$. One then shows the existence of a critical point of S_E to obtain existence of a simple closed characteristic.

In [12] the authors use degree theory and Lagrangian intersection theory to show existence of a simple closed characteristic without the twist condition via continuation to a twist system. In [20] a type of Morse theory is used on the space of braids to obtain forcing theorems of closed characteristics for second-order Lagrangian systems of twist type, the authors show that the set of closed characteristics can have a rich structure. In [12], it is shown that for regular energy manifolds, the number of closed characteristics can be bounded below by the second Betti number of M_E, which in turn can be computed from the superlevel sets of the potential function $L(u,0,0)+E \geq 0$.

Definition 3.2.1. For $u_1 < u_2$, an *increasing lap* u_+ from u_1 to u_2 is a solution to the Euler-Lagrange equation

$$u'''' - \partial^2_{u'}K(u,u')u'' - \partial^2_{uu'}K(u,u')u' + \partial_u K(u,u') = 0 \tag{3.11}$$

satisfying the boundary conditions $u(0) = u_1$, $u(T) = u_2$, $u'(0) = u'(T) = 0$ and $u'(x) > 0$ for $0 < x < T$ with a similar definition of a *decreasing lap* u_-. A *simple closed characteristic of type* (u_1, u_2) is a periodic solution to (3.11) for which each period is composed of a single increasing lap from u_1 to u_2 and a single decreasing lap from u_2 to u_1.

We consider functions which have a simple profile consisting of two monotone laps, u_+ which increases from some minimal value u_1 to a maximum value u_2 and u_- which decreases from u_2 back to u_1 with $v = u' = 0$ at u_1 and u_2. If u_+ and u_- are solutions to (3.2), then their concatenation $u_+ \# u_-$ is called a "broken geodesic", and the extrema u_1 and u_2 are called concatenation points. A broken geodesic is not necessarily a solution to (3.2) at the concatenation points, since the third derivatives need not agree there, see [5]. As was mentioned above, from the Hamiltonian (3.3) for $v = 0$ solutions satisfy $wL_w - L = E$, and let N denote this level set in the (u,w)-plane. Then every simple closed characteristic intersects N exactly twice. Moreover N is a section of M given by $M \cap \{v = 0\}$, and due to the convexity of L in the w variable, N consists of two graphs in the (u,w)-plane. That is, the projection π of N onto the u-axis can be described by $\pi N = \{u : L(u,0,0) + E \geq 0\}$ and the sets $N \cap \{(u,w)\,|u \geq 0\}$ and $N \cap \{(u,w)\,|u \leq 0\}$ are graphs over πN. A particular connected component of πN will be denoted by I, and referred to as an *interval component*.

Consider a interval component I and define $B = \{(u_1, u_2) \in I^2 \,|u_1 < u_2\}$, our goal is to find points $(u_1, u_2) \in B$ for which there exists a simple closed characteristic which is the concatenation of an increasing and a decreasing lap. For an increasing lap u_+ from u_1 to u_2 we let $p^+_{u_1}$ and $p^+_{u_2}$ be the p_u-values at the concatenation points, and also for a decreasing lap u_- we let $p^-_{u_1}$ and $p^-_{u_2}$ be the corresponding p_u values. If u is the concatenation $u = u_+\#u_-$, then necessary conditions for u to be a solution of (3.11) are $p^+_{u_1} = p^-_{u_1}$ and $p^+_{u_2} = p^-_{u_2}$ see [5]. Since u_+ and u_- are solutions to (3.11), their intersection with N determines the values of $u'' = p_{u'}$ uniquely from u_1 and u_2, and we denote these values by $p_{u'}(u_1)$ and $p_{u'}(u_2)$. Thus the necessary compatibility conditions on the p_u-values are also sufficient.

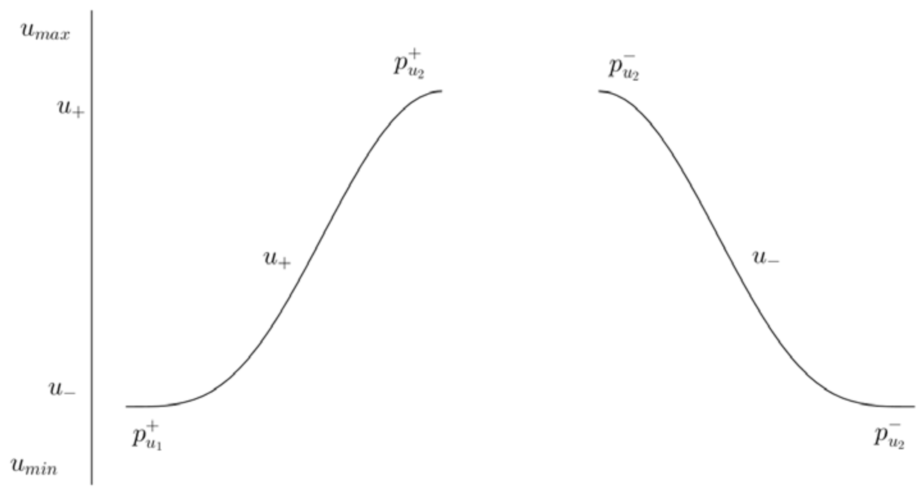

Figure 3.1: *The $p_{u'}$-values at the endpoints of each lap are determined by the minimum and maximum values u_1 and u_2, but the p_u values are not, which gives a necessary and sufficient condition for the concatenation $u_-\#u_+$ to be a simple closed characteristic, $p^+_{u_2} = p^-_{u_2}$ and $p^+_{u_1} = p^-_{u_2}$.*

3.3 FINSLER CURVE SHORTENING FOR SECOND-ORDER LAGRANGIANS

In the broad scope of studying second order Lagrangians our interest is in the existence of periodic solutions from a curve shortening perspective (see [24]), potentially allowing us to bypass the restrictions inherent with the "twist condition", c.f. [5]. To this end the idea of applying curve-shortening techniques was proposed in [20]. Parametrizing with respect to arclength in the (u, v)-plane allows us to study simple closed characteristics as embedded curves γ in $\Omega(\mathbb{R}^2)$ which are unions of graphs in the upper/lower half planes of $\mathbb{R}^2$, and up to a reparameterization $\gamma(0) = \gamma(\pi) = 0$. We denote this subspace by $\widehat{\Omega}$. In terms of arclength, the Lagrangian has the form

$$L(u, v, w) = L\left(u, v, v_s v/u_s\right) = \frac{1}{2}\left(\frac{v_s v}{u_s}\right)^2 + K(u, v). \tag{3.12}$$

Therefore the action integral can be expressed in terms of arclength

$$\int_0^T L(u, v, w)\, dx = \int_\alpha L\left(u, v, \frac{v}{u_s} v_s\right) \frac{u_s}{v} ds = \int_\alpha \frac{1}{2}\frac{v v_s^2}{u_s} + \frac{K(u, v)}{v} u_s\, ds. \tag{3.13}$$

Given a curve $\alpha(t, s) = (u(t, s), v(t, s)) \in \widehat{\Omega}$, for $v \neq 0$ we can view the latter integral as an arclength functional

$$\ell\left(\alpha(t, s)\right) = \int_\alpha F\left(\alpha, \alpha_s\right)\, ds \tag{3.14}$$

where

$$F\left(u, v; u_s, v_s\right) = L\left(u, v, \frac{v}{u_s} v_s\right) \frac{u_s}{v} = \frac{1}{2}\frac{v v_s^2}{u_s} + \frac{K(u, v)}{v} u_s. \tag{3.15}$$

Lemma 3.3.1. F satisfies property (ii) and (iii) of a Finsler metric.

Proof. The homogeneity of F is immediate,

$$F(u, v; \lambda u_s, \lambda v_s) = \frac{1}{2}\frac{v\lambda^2 v_s^2}{\lambda u_s} + \frac{K(u, v)\lambda u_s}{v} = \lambda F(u, v; u_s, v_s).$$

For (iii) computing the Hessian yields

$$\left(\left[\frac{1}{2}F^2\right]_{y^iy^j}\right) = \begin{bmatrix} \frac{3}{4}\frac{v^2v_s^4}{u_s^4} + \frac{K^2}{v^2} & -\frac{v^2v_s^3}{u_s^3} \\ -\frac{v^2v_s^3}{u_s^3} & \frac{3}{2}\frac{v^2v_s^2}{u_s^2} + K \end{bmatrix}.$$

From Sylvester's criterion we see that the strong convexity condition holds. ∎

We can therefore treat F as a singular Finsler metric on $\Omega(\mathbb{R}^2)$, and hence critical points of the Lagrangian action coincide with geodesics of a singular Finsler metric. In fact F is indeed a Finsler metric on the upper/lower half planes $\mathbb{R}\times\mathbb{R}_+$, $\mathbb{R}\times\mathbb{R}_-$ and the results from Section (2.5) apply. This that the singularity causes a geodesic to behave like a saddle instead of an attractor. We compute the gradient flow of ℓ using (2.9):

$$\mathbf{T} = \langle u_s, v_s\rangle\,,\ \mathbf{N} = \langle v_s, -u_s\rangle$$

$$\begin{aligned} F_{\alpha^i}N^i &= \frac{K_u}{v}u_sv_s - \frac{v_s^2}{2} - \left(\frac{vK_v - K}{v^2}\right)u_s^2 \\ F_{\alpha_s^i\alpha^j}N^iT^j &= \frac{K_u}{v}u_sv_s - \frac{v_s^4}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right)v_s^2 - v_s^2 \\ F_{\alpha_s^i\alpha_s^j}N^iN^j &= \frac{vv_s^4}{u_s^3}\kappa + 2\frac{vv_s^2}{u_s}\kappa + u_sv\kappa. \end{aligned}$$

Therefore

$$\begin{aligned}
v^{\perp} &= -\frac{K_u}{v}u_s v_s + \frac{v_s^2}{2} + \left(\frac{vK_v - K}{v^2}\right)u_s^2 + \frac{K_u}{v}u_s v_s - \frac{v_s^4}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right)v_s^2 \\
&- v_s^2 + \frac{vv_s^4}{u_s^3}\kappa + 2\frac{vv_s^2}{u_s}\kappa + u_s v\kappa \\
&= \frac{-v_s^2}{2} + \left(\frac{vK_v - K}{v^2}\right) - \frac{v_s^4}{2u_s^2} + \frac{vv_s^4}{u_s^3}\kappa + 2\frac{vv_s^2}{u_s}\kappa + u_s v\kappa \\
&= \frac{-v_s^2}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right) + \frac{vv_s^2}{u_s^3}\kappa + \frac{vv_s^2}{u_s}\kappa + u_s v\kappa \\
&= \frac{-v_s^2}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right) + \frac{vv_s^2}{u_s^3}\kappa + \frac{v}{u_s}\kappa \\
&= \frac{v}{u_s^3}\kappa - \frac{v_s^2}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right). \qquad (3.16)
\end{aligned}$$

Consequently

$$\begin{aligned}
&\frac{d}{dt}\ell\left(\alpha\left(t,s\right)\right) = \\
&-\int_\alpha \left\{\frac{vv_s^4}{u_s^3}\kappa + 2\frac{vv_s^2}{u_s}\kappa + vu_s\kappa + \frac{v_s^2}{2} + \frac{v_s^4}{2u_s^2} - \left(\frac{vK_v - K}{v^2}\right)v_s^2 - \left(\frac{vK_v - K}{v^2}\right)u_s^2\right\}^2 ds \\
&= -\int_\alpha \left\{\frac{v}{u_s^3}\kappa - \frac{v_s^2}{2u_s^2} + \left(\frac{vK_v - K}{v^2}\right)\right\}^2 ds \leq 0.
\end{aligned}$$

If we consider the Hamiltonian (3.3) for (3.10) we obtain the relation

$$-u'u''' + \frac{1}{2}(u'')^2 + K_{u'}u' - K(u,v) = 0.$$

Without loss of generality we assume E is a regular value of H. Locally each curve can be expressed as a function $v(u)$ in the (u,v)-plane, for $v \neq 0$ we have the following:

$$w = v_u v \Rightarrow w' = v^2 v_{uu} + v_u^2 v$$

and

$$v_s = u_s v_u \Rightarrow v_{ss} = u_{ss}v_u + u_s^2 v_{uu} \Rightarrow v_u u = \frac{u_s v_{ss} - u_{ss}v_s}{u_s^3} = \frac{-\kappa}{u_s^3}.$$

This yields

$$
\begin{aligned}
&-v\left(v^2\frac{-\kappa}{u_s^3}+\frac{vv_s^2}{u_s^2}\right)+\frac{1}{2}(v\frac{v_s}{u_s})^2+K_v v-K(u,v)\\
&=\frac{v^3\kappa}{u_s^3}-\frac{v^2v_s^2}{2u_s^2}+vK_v-K=0.
\end{aligned}
$$

The geodesics of the Finsler form (3.15), that is curves for which $v^{\perp}=0$, satisfy

$$
\kappa=\frac{u_s v_s^2}{2v}-(vK_v-K)\frac{u_s^3}{v^3}.
$$

are solutions to the Euler-Lagrange equation for the Lagrangian system (L,dx). If we scale (3.16) by $\frac{u_s^3}{v}$ we obtain the following normal velocity

$$
v^{\perp}=\kappa-\frac{v_s^2u_s}{2v}+(vK_v-K)\left(\frac{u_s}{v}\right)^3,
$$

which allows one to study the corresponding evolution

$$
\begin{aligned}
\alpha_t&=v^{\perp}(\kappa,\mathbf{T})\mathbf{N} \qquad (3.17)\\
\alpha(0)&=\alpha_0.
\end{aligned}
$$

Equation (3.17) is clearly parabolic for $v\neq 0$. For $v(s_0)=0$ we consider the degenerate velocity obtained as follows

$$
\begin{aligned}
\lim_{s\to s_0}\left(\kappa-\frac{u_s v_s^2}{2v}+(vK_v-K)\frac{u_s^3}{v^3}\right)&=\kappa(\alpha(t,s_0))-\frac{\kappa(\alpha(t,s_0))}{2}-\kappa^3(\alpha(t,s_0))K\\
&=\frac{\kappa(\alpha(t,s_0))}{2}-\kappa^3(\alpha(t,s_0))K.
\end{aligned}
$$

This gives us a candidate for $v^{\perp}$ at $v=0$.

$$
v^{\perp}=\begin{cases}\kappa-\frac{u_s v_s^2}{2v}+(vK_v-K)\frac{u_s^3}{v^3} & v\neq 0\\ \frac{\kappa}{2}-\kappa^3K & v=0\end{cases}.
$$

Unfortunately when $v=0$ the parabolicity of (3.17) is lost, since for $v=0$

$$
\frac{\partial}{\partial\kappa}v^{\perp}=\frac{1}{2}-3\kappa^2K.
$$

Therefore the Finsler setting for curve-shortening in the plane developed in Section (2.2) will not work.

In the next chapter we return to simple closed curves and view them as graphs in the upper/lower half planes $\{v > 0\}$, $\{v < 0\}$. We can formulate an evolution of these curves by a gradient flow for J which provides an evolution of the graphs in each half plane and also an evolution of the endpoints where $v = 0$.

CHAPTER 4

GRADIENT FLOW OF THE ACTION FUNCTIONAL

We mentioned how under the twist condition the functional J can be minimized to obtain existence of closed characteristics via the method of broken geodesics. Throughout the rest of the dissertation we do not assume the twist condition and show existence of closed characteristics through a gradient flow. Under an infinitesimal change of the (u, v) curve we get

$$\begin{aligned}
\delta J &= \delta \int \left\{ \frac{1}{2} v^2 v_u^2 + K(u,v) \right\} \frac{du}{v} \\
&= \delta \int \left\{ \frac{1}{2} v v_u^2 + \frac{K(u,v)}{v} \right\} du \\
&= \int \left\{ v v_u \delta v_u + \frac{v_u^2}{2} \delta v + \frac{v K_v - K}{v^2} \delta v \right\} du \\
&= \int \left\{ -(v v_u)_u + \frac{v_u^2}{2} + \frac{v K_v - K}{v^2} \right\} \delta v du \\
&= \int \left\{ -v v_{uu} - \frac{v_u^2}{2} + \frac{v K_v - K}{v^2} \right\} \delta v du.
\end{aligned}$$

The steepest descent flow is given by

$$\frac{\partial v}{\partial t} = A \cdot \left\{ v v_{uu} + \frac{v_u^2}{2} - \frac{v K_v - K}{v^2} \right\} \tag{4.1}$$

This equation is obviously singular when $v = 0$; we change variables and set $z = v^{3/2}$ and compute the variation of J in terms of z, see below. To study (4.1) as a well posed problem we first fix the domain to avoid a free boundary problem by setting $x = \frac{u - u_1}{u_2 - u_1}$. To simplify notation we will suppress the E when referring to an interval component I of $\{K(u, 0, 0) + E \geq 0\}$ and simply write $K(u, 0, 0)$ instead of

$K(u,0,0)+E$. The evolution of u_1 and u_2 is given by computing the total variation of the action functional $J : H_0^1(\Omega)\times I^2 \to \mathbb{R}$ as a function of three variables z, u_1, u_2

$$J[z,u_1,u_2] = \int_\Omega \frac{2}{9(u_2-u_1)} z_x^2 + \frac{(u_2-u_1)K((u_2-u_1)x+u_1, z^{2/3})}{z^{2/3}} dx, \qquad (4.2)$$

where $\Omega=(0,1)$, I is an interval component, and K is a smooth function such that K is strictly positive on the interior of $[u_-,u_+]$ and zero on the boundary. Computing the total variation yields

$$\begin{aligned}\delta_z J[z,u_1,u_2] &= \int_\Omega \frac{4}{9(u_2-u_1)} z_x\delta z_x + (u_2-u_1)\frac{z^{2/3}\partial_2 K\frac{2}{3}z^{-1/3}\delta z - \frac{2}{3}Kz^{-1/3}\delta z}{z^{4/3}} dx\\ &= \int_\Omega \frac{-4}{9(u_2-u_1)} z_{xx}\delta z + \frac{2}{3}(u_2-u_1)\frac{z^{2/3}\partial_2 K - K}{z^{5/3}}\delta z dx,\end{aligned}$$

$$\delta_{u_1} J[z,u_1,u_2] = \int_\Omega \frac{2}{9(u_2-u_1)^2} z_x^2 + \frac{(u_2-u_1)(1-x)\partial_1 K - K}{z^{2/3}} dx\cdot\delta u_1,$$

$$\delta_{u_2} J[z,u_1,u_2] = \int_\Omega \frac{-2}{9(u_2-u_1)^2} z_x^2 + \frac{(u_2-u_1)x\partial_1 K + K}{z^{2/3}} dx\cdot\delta u_2.$$

Rescaling by positive constants yields

$$\begin{aligned}(u_2-u_1)\delta_z J[z,u_1,u_2] &= \int_\Omega \frac{-4}{9} z_{xx} + \frac{2}{3}(u_2-u_1)^2\frac{z^{2/3}\partial_2 K - K}{z^{5/3}} dx\cdot\delta z\\ (u_2-u_1)^2\delta_{u_1} J[z,u_1,u_2] &= \int_\Omega \frac{2}{9} z_x^2 + (u_2-u_1)^2\frac{(u_2-u_1)(1-x)\partial_1 K - K}{z^{2/3}} dx\cdot\delta u_1,\\ (u_2-u_1)^2\delta_{u_2} J[z,u_1,u_2] &= \int_\Omega \frac{-2}{9} z_x^2 + (u_2-u_1)^2\frac{(u_2-u_1)x\partial_1 K + K}{z^{2/3}} dx\cdot\delta u_2.\end{aligned}$$

The rescaled steepest descent flow is

$$\begin{cases} z_t = \dfrac{4}{9} z_{xx} - \dfrac{2}{3} (u_2(t) - u_1(t))^2 \dfrac{z^{2/3} \partial_2 K - K}{z^{5/3}}, \\ \\ (u_1)_t = - \displaystyle\int_\Omega \dfrac{2(z_x)^2}{9} dx - (u_2(t) - u_1(t))^2 \displaystyle\int_\Omega \dfrac{(u_2 - u_1)(1-x)\partial_1 K - K}{z^{2/3}} dx, \\ \\ (u_2)_t = \displaystyle\int_\Omega \dfrac{2(z_x)^2}{9} dx - (u_2(t) - u_1(t))^2 \displaystyle\int_\Omega \dfrac{(u_2 - u_1) x \partial_1 K + K}{z^{2/3}} dx. \end{cases}$$

Remark 4.0.2. From the functional we can obtain a bound on $(u_1)_t$ as follows,

$$\begin{aligned} |(u_1)_t| &\leq \int_\Omega \frac{2(z_x)^2}{9} + \frac{(u_2(t) - u_1(t))^2 K}{z^{2/3}} dx + (u_2(t) - u_1(t))^3 \int_\Omega \frac{(1-x)\,|\partial_1 K|}{z^{2/3}} dx \\ &\leq (u_2(t) - u_1(t)) J\left[z_0, u_1(0), u_2(0)\right] + (u_2(t) - u_1(t))^3 \sup |\partial_1 K| \int_\Omega \frac{(1-x)}{z^{2/3}} dx. \end{aligned}$$

In the next section we will see that the $\int_\Omega \frac{1}{z^{2/3}} dx$ term is bounded which allows us to conclude that $(u_1)_t$ is bounded. Similarly we can say the same for $(u_2)_t$.

4.1 EXISTENCE OF SOLUTIONS

Here we show well posedness of the IBVP. Let us consider the case where

$$K((u_2 - u_1)x + u_1, z^{2/3}) = \frac{\beta}{2} z^{4/3} + F((u_2 - u_1)x + u_1).$$

We have a coupled system consisting of a singular semilinear parabolic equation and two ordinary differential equations. For notational convenience we write $l(t) = u_1(t), r(t) = u_2(t)$ and $\Sigma_T = (0, T) \times \partial\Omega$ where $T > 0$.

$$(S_t) \begin{cases} z_t = \dfrac{4}{9} z_{xx} + (r-l)^2 \left(\dfrac{2}{3} \dfrac{F((r-l)x + l)}{z^{5/3}} - \dfrac{\beta}{3z^{1/3}} \right) \quad \text{in } Q_T, \\ z = 0 \ \text{ on } \Sigma_T, \ \ z > 0 \text{ in } Q_T, \ \ z(0,x) = z_0(x) \text{ in } \Omega, \\ \\ l_t = -\displaystyle\int_\Omega \frac{2z_x^2}{9} dx - (r-l)^2 \int_\Omega \frac{(r-l)(1-x)F' - F}{z^{2/3}} - \frac{\beta}{2} z^{2/3} dx, \\ \\ r_t = \displaystyle\int_\Omega \frac{2z_x^2}{9} dx - (r-l)^2 \int_\Omega \frac{(r-l)xF' + F}{z^{2/3}} + \frac{\beta}{2} z^{2/3} dx, \\ l(0) = l_0, \ r(0) = r_0. \end{cases}$$

To study (S_t) we use a semi-discretization in time with implicit Euler method. The key points in the proof are to show that z belongs to a cone $\mathcal{C}$ of functions with suitable profile as $z \to 0$ near $x = 0$ and to show by the weak comparison principle that $\frac{1}{z^{5/3}}, \frac{1}{z^{1/3}} \in L^\infty(0, T; H^{-1}(\Omega))$ and $\frac{1}{z^{2/3}} \in L^\infty(0, T; L^1(\Omega))$, so the singularity is controlled on $\mathcal{C}$.

In the proof of Theorem 4.1.6 we give a necessary bound on β to guarantee monotonicity of the operator

$$\frac{4}{9} z_{xx} + (r-l)^2 \left(\frac{2}{3} \frac{F((r-l)x + l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}} \right). \tag{4.3}$$

To handle the singular term in the parabolic equation one considers solutions in a conical shell $\mathcal{C}$

$$\mathcal{C} = \left\{\chi \in L^\infty(\Omega) \,|\quad C_1 d(x)^{3/4} \le \chi(x) \le C_2(d(x)^{3/4} + d(x))\right\} \quad \forall x \in \Omega,$$

for some C_1, C_2 with $0 < C_1 < C_2$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$. We say $z \in \mathcal{C}$ uniformly if $z(t) \in \mathcal{C}$ for all time t. We follow methods used in [1], though here the singular term is more general and we have a coupled system. For $\epsilon > 0$ sufficiently small we assume the functions $l(t)$ and $r(t)$ take values in I^ϵ where $I^\epsilon = [u_- + \epsilon, u_+ - \epsilon]$. We consider weak solutions in the space $\mathbf{V}(Q_T)$.

Definition 4.1.1.

$\mathbf{V}(Q_T) = \left\{z : z \in L^\infty(Q_T), z_t \in L^2(Q_T), z \in L^\infty(0,T; H_0^1(\Omega))\right\}$ where $Q_T = (0,T) \times \Omega$.

Definition 4.1.2. A *weak solution* to (P_t) is a function $z \in \mathbf{V}(Q_t)$, satisfying

(i) $\mathrm{ess\,inf}_K z > 0$ for every compact $K \subset Q_T$.

(ii) for every test function $\phi \in \mathbf{V}(Q_T)$

$$\int_{Q_T} \left(\phi\frac{\partial z}{\partial t} + \frac{4}{9} z_x \phi_x - \frac{(r-l)^2}{3}\left(\frac{2F((r-l)x+l)\phi}{z^{5/3}} - \frac{\beta\phi}{z^{1/3}}\right)\right) dxdt = 0,$$

(iii) $z(0,x) = z_0(x)$ a.e. in Ω.

Remark 4.1.3. If $\frac{1}{z^{5/3}} \in L^\infty(0,T; H^{-1}(\Omega))$ then (ii) is well-defined.

For a fixed l, r it will be evident in the proof that the problem (P_t) has a solution for all $T > 0$. The two ODE's introduce a constraint on the choice of T. Following Remark 4.0.2 we can a posteriori determine T. We first consider the PDE with l and r fixed.

$$(P_t) \begin{cases} z_t = \dfrac{4}{9} z_{xx} + (r-l)^2 \left(\dfrac{2}{3}\dfrac{F((r-l)x+l)}{z^{5/3}} - \dfrac{\beta}{3z^{1/3}}\right) & \text{in } Q_T, \\ z = 0 \ \text{ on } \Sigma_T, \ \ z > 0 \text{ in } Q_T, \ \ z(0,x) = z_0(x) \text{ in } \Omega, \end{cases}$$

Theorem 4.1.4. Let $z_0 \in H_0^1(\Omega) \cap \mathcal{C}$ and l, $r \in I^\epsilon$ fixed. Then for any $T > 0$, there exists a unique weak solution z to (P_t) such that $z(t) \in \mathcal{C}$ uniformly for $t \in [0, T]$ and $z \in C([0,T], H_0^1(\Omega))$.

We follow the methods used in [1], though here the PDE has an extra lower order singularity, the $\frac{1}{z^{5/3}}$ term has a function in the numerator, and we couple the PDE with two ODE's. We also provide a stabilization result.

Theorem 4.1.5. Let $z_0 \in \mathcal{C} \cap H_0^1(\Omega)$, there exists a unique $z_\infty \in H_0^1(\Omega) \cap \mathcal{C} \cap C_0(\overline{\Omega})$ satisfying

$$(P) \begin{cases} -\dfrac{4}{9} z_{xx} = (r-l)^2 \left(\dfrac{2}{3} \dfrac{F((r-l)x+l)}{z^{5/3}} - \dfrac{\beta}{3z^{1/3}} \right) & \text{in } \Omega, \\ z_\infty = 0 \ \text{ on } \partial\Omega. \end{cases}$$

The solution $z(t)$ to (P_t) defined on $(0,\infty) \times \Omega$ satisfies $z(t) \to z_\infty$ as $t \to +\infty$ in $L^\infty(\Omega) \cap H_0^1(\Omega)$. In particular $z(t)$ in bounded in $H^{9/8}(\Omega)$ for $t > 0$.

Proof of 4.1.4. For $N > 1$ define $\Delta t = \frac{T}{N}$ for $0 \le n \le N$ and $t_n = n\Delta t$. As in [2] or following the same procedure as in Theorem 0.5 in [1] with $\lambda = \Delta t, g = z^{n-1} \in L^\infty(\Omega)$, we can define by iteration $z^n \in H_0^1(\Omega) \cap \mathcal{C}$ using the implicit Euler scheme:

$$\begin{cases} \dfrac{z^n - z^{n-1}}{\Delta t} - \dfrac{4}{9} z^n_{xx} - \dfrac{(r-l)^2}{3} \left(\dfrac{2F((r-l)x+l)}{(z^n)^{5/3}} - \dfrac{\beta}{(z^n)^{1/3}} \right) = 0 \quad \text{in } \Omega \\ z^n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

with $z^0 = z_0 \in H_0^1(\Omega) \cap \mathcal{C}$. Define the following functions $z_{\Delta t}, \tilde{z}_{\Delta t}$ for all $n = 1, ..., N$ and $t \in [t_{n-1}, t_n)$

$$z_{\Delta t} := z^n, \tag{4.5}$$

$$\tilde{z}_{\Delta t} := \frac{t - t_{n-1}}{\Delta t}(z^n - z^{n-1}) + z^{n-1}. \tag{4.6}$$

Therefore we have the following relation

$$\frac{\partial \tilde{z}_{\Delta t}}{\partial t} - \frac{4}{9}\frac{\partial^2 z_{\Delta t}}{\partial x^2} - \frac{(r-l)^2}{3}\left(\frac{2F}{z_{\Delta t}^{5/3}} - \frac{\beta}{z_{\Delta t}^{1/3}}\right) = 0. \tag{4.7}$$

We first need to establish some energy estimates for $z_{\Delta t}$ and $\tilde{z}_{\Delta t}$. Multiplying (4.4) by $\Delta t z^n$, summing from $n = 1$ to N, then we get

$$\begin{aligned}&\sum_{n=1}^{N}\int_{\Omega}(z^n - z^{n-1})z^n dx + \Delta t\frac{4}{9}\sum_{n=1}^{N}\|z^n\|^2_{H_0^1(\Omega)}\\ &- \Delta t\frac{(r-l)^2}{3}\sum_{n=1}^{N}\int_{\Omega}\frac{2F}{(z^n)^{2/3}} - \beta(z^n)^{2/3}dx = 0.\end{aligned} \tag{4.8}$$

Also

$$\begin{aligned}\sum_{n=1}^{N}\int_{\Omega}(z^n - z^{n-1})z^n dx &= \frac{1}{2}\sum_{n=1}^{N}\int_{\Omega}\left(\left|z^n - z^{n-1}\right|^2 + |z^n|^2 - \left|z^{n-1}\right|^2\right)dx\\ &= \frac{1}{2}\sum_{n=1}^{N}\int_{\Omega}\left|z^n - z^{n-1}\right|^2 dx + \frac{1}{2}\int_{\Omega}\left|z^N\right|^2 dx - \frac{1}{2}\int_{\Omega}|z_0|^2 dx.\end{aligned} \tag{4.9}$$

Using the weak comparison principle, see Cuesta and Takáč [3], Fleckinger-Pellé and Takáč [17], we can construct lower and upper solutions to the stationary problem for (P_t). That is, $\underline{z}, \bar{z} \in H_0^1(\Omega) \cap \mathcal{C}$ such that $\underline{z} \le z_0 \le \bar{z}$ and

$$-\frac{4}{9}\underline{z}_{xx} - \frac{(r-l)^2}{3}\left(\frac{2F(u)}{\underline{z}^{5/3}} - \frac{\beta}{\underline{z}^{1/3}}\right) \le 0 \text{ in } \Omega, \tag{4.10}$$

$$-\frac{4}{9}\bar{z}_{xx} - \frac{(r-l)^2}{3}\left(\frac{2F(u)}{\bar{z}^{5/3}} - \frac{\beta}{\bar{z}^{1/3}}\right) \ge 0 \text{ in } \Omega. \tag{4.11}$$

Here $u = (r-l)x + l$, let ϕ_1 denote the normalized positive eigenfunction associated with the principal eigenvalue λ_1 of $-\Delta$ with homogeneous Dirichlet boundary conditions. In this case $\phi_1(x) = \sin(\pi x)$ and $\lambda_1 = \pi^2$. Since $\phi_1 \in C^1(\overline{\Omega})$, we have $\phi_1^{3/4} \in \mathcal{C}$.

We define the lower solution

$$\underline{z} = \eta \phi_1^{3/4},$$

for $\eta > 0$ sufficiently small, the choice of η will be made later so that (4.10) is satisfied and $\underline{z} \le z_0$. Next define the upper solution

$$\bar{z} = M \phi_1^{3/4}$$

for $M > 0$ large enough, the choice of M will be made later so that $z_0 \le \bar{z}$ and (4.11) holds.

We determine a necessary estimate in order to construct a supersolution $\bar{z}$, that is

$$-\left(\frac{4}{9}\bar{z}_{xx} - \frac{(r-l)^2\beta}{3\bar{z}^{1/3}} + \frac{2}{3}\frac{(r-l)^2 F((r-l)x+l)}{\bar{z}^{5/3}}\right) \ge 0. \tag{4.12}$$

First

$$\bar{z}_{xx} = \frac{\partial}{\partial x}(M\frac{3}{4}\phi_1^{-1/4}\partial_x\phi_1) = M\frac{3}{4}\phi_1^{-1/4}\partial_{xx}\phi_1 - M\frac{3}{16}\phi_1^{-5/4}(\partial_x\phi_1)^2.$$

Plugging into (4.12) we obtain

$$\frac{M\pi^2\phi_1^{3/4}}{3} + \frac{M}{12}\phi_1^{-5/4}(\partial_x\phi_1)^2 + \frac{(r-l)^2\beta}{3\bar{z}^{1/3}} - \frac{2}{3}\frac{(r-l)^2 F}{\bar{z}^{5/3}} \ge 0$$

Multiplying through by $12M^{5/3}\phi_1^{5/4}$ yields

$$M^{8/3}\left(4\pi^2\phi_1^2 + (\partial_x\phi_1)^2\right) + M^{4/3}4(r-l)^2\beta\phi_1 - 8(r-l)^2 F \ge 0.$$

Treating this as a quadratic in $M^{4/3}$ the two roots are

$$\frac{-2\beta(r-l)^2\phi_1 \pm 2(r-l)\sqrt{(r-l)^2\beta^2\phi_1^2 + 2F\cdot(4\pi^2\phi_1^2 + (\partial_x\phi_1)^2)}}{4\pi^2\phi_1^2 + (\partial_x\phi_1)^2},$$

which by plugging in $\phi_1 = \sin(\pi x)$ can be written as

$$\frac{-2\beta(r-l)^2\sin(\pi x) + 2(r-l)\sqrt{(r-l)^2\beta^2\sin^2(\pi x) + 2F\cdot\left(3\pi^2\sin^2(\pi x)+\pi^2\right)}}{3\pi^2\sin^2(\pi x)+\pi^2}. \tag{4.13}$$

Therefore we take $M = C_2(l,r,\beta)\cdot(r-l)^{3/4}$ where $C_2^{4/3}(l,r,\beta)$ is an upper bound of

$$\frac{-2\beta(r-l)\sin(\pi x) + 2\sqrt{(r-l)^2\beta^2\sin^2(\pi x) + 2F\cdot\left(3\pi^2\sin^2(\pi x)+\pi^2\right)}}{3\pi^2\sin^2(\pi x)+\pi^2}. \tag{4.14}$$

Similarly, $\eta = C_1(l,r,\beta)\cdot(r-l)^{3/4}$ where $C_1^{4/3}(l,r,\beta)$ is a lower bound for (4.14). When $\beta = 0$ this reduces to

$$C_1^{4/3}(l,r) \leq 2\sqrt{\frac{2F}{3\pi^2\sin^2(\pi x)+\pi^2}}$$
$$C_2^{4/3}(l,r) \geq 2\sqrt{\frac{2F}{3\pi^2\sin^2(\pi x)+\pi^2}},$$

In order to apply the comparison principle we need to show that the term $\frac{2}{3}\frac{F((l-r)x+l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}$ is nonincreasing in z. We compute the derivative with respect to z

$$\frac{d}{dz}\left(\frac{2}{3}\frac{F((l-r)x+l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}\right) = \frac{-10}{9}z^{-8/3}F + \frac{\beta}{9}z^{-4/3} \leq 0,$$

which implies

$$\beta \leq \frac{10F}{z^{4/3}} \quad \text{for all } x\in\Omega.$$

To ensure that $\frac{2}{3}\frac{F((l-r)x+l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}$ is nonincreasing in z we take

$$\beta \leq \frac{10F}{\bar{z}^{4/3}} \quad \text{for all } x\in\Omega.$$

Note that the above is immediately satisfied for $\beta \leq 0$. For $\beta > 0$ choosing

$$\beta \leq \frac{10F}{(r-l)C_2^{4/3}(r,l)\sin(\pi x)} \quad \text{for all } x\in\Omega \tag{4.15}$$

guarantees that the operator $\frac{4}{9}z_{xx} + (r-l)^2\left(\frac{2}{3}\frac{F((r-l)x+l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}\right)$ is monotone, and therefore allows us to invoke the weak comparison principle. After iterating the application of the weak comparison principle, we obtain $\underline{z} \leq z^n \leq \bar{z}$ for all $n \in \mathbb{N}$. This leads us to a second cone determined by the upper and lower solutions

$$\mathsf{C}(l,r,\beta) = \left\{z \in H_0^1(\Omega) \mid \underline{z} \leq z \leq \bar{z}\right\}.$$

The above yields

$$\underline{z} \leq z_{\Delta t}, \tilde{z}_{\Delta t} \leq \bar{z}. \tag{4.16}$$

Since $\underline{z} \in \mathcal{C}(l,r,\beta)$

$$\Delta t \sum_{n=1}^{N} \int_\Omega \frac{1}{(z^n)^{2/3}} dx \leq T \int_\Omega \frac{1}{\underline{z}^{2/3}} dx < \infty. \tag{4.17}$$

From (4.8) and (4.9) we observe

$$\begin{aligned}
\frac{4}{9}\|z_{\Delta t}\|^2_{L^2(0,T;H_0^1(\Omega))} &= \frac{4}{9}\Delta t \sum_{n=1}^{N} \|z^n\|^2_{H_0^1(\Omega)} = \frac{(r-l)^2}{3}\Delta t \sum_{n=1}^{N}\int_\Omega \frac{2F}{(z^n)^{2/3}} - \beta (z^n)^{2/3} dx \\
&- \frac{1}{2}\sum_{n=1}^{N}\int_\Omega \left|z^n - z^{n-1}\right|^2 dx - \frac{1}{2}\int_\Omega \left|z^N\right|^2 dx + \frac{1}{2}\int_\Omega |z_0|^2 dx \\
&\leq \frac{(r-l)^2}{3}\Delta t \sum_{n=1}^{N}\int_\Omega \frac{2F}{(z^n)^{2/3}} - \beta (z^n)^{2/3} dx + \frac{1}{2}\int_\Omega |z_0|^2 dx < \infty.
\end{aligned} \tag{4.18}$$

Combining (4.16), (4.17), and (4.18) we get that $z_{\Delta t}, \tilde{z}_{\Delta t} \in \mathcal{C}$ uniformly and are bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$.

There is a second energy estimate that we need. Multiplying (4.4) by $z^n - z^{n-1}$ and summing from $n = 1$ to N

$$\begin{aligned}
\Delta t \sum_{n=1}^{N}\int_\Omega \left(\frac{z^n - z^{n-1}}{\Delta t}\right)^2 dx + \frac{4}{9}\sum_{n=1}^{N}\int_\Omega z_x^n \frac{d}{dx}(z^n - z^{n-1}) dx \\
- \frac{(r-l)^2}{3}\sum_{n=1}^{N}\int_\Omega (z^n - z^{n-1})\left(\frac{2F}{(z^n)^{5/3}} - \frac{\beta}{(z^n)^{1/3}}\right) dx = 0.
\end{aligned} \tag{4.19}$$

Furthermore note that

$$\frac{1}{2}\left(\int_\Omega |z_x^n|^2\,dx - \int_\Omega \left|z_x^{n-1}\right|^2 dx\right) \le \int_\Omega z_x^n \frac{d}{dx}(z^n - z^{n-1})dx. \tag{4.20}$$

Due to the convexity of the $-\frac{1}{1-5/3}\int_\Omega z^{1-5/3}dx$ term

$$\frac{1}{1-5/3}\left[\int_\Omega (z^{n-1})^{1-5/3}dx - \int_\Omega (z^n)^{1-5/3}dx\right] \le -\int_\Omega \frac{z^n - z^{n-1}}{(z^n)^{5/3}}dx. \tag{4.21}$$

combined with (4.19),(4.21) and (4.20) shows

$$\begin{aligned}&\Delta_t \sum_{n=1}^N \int_\Omega \left(\frac{z^n - z^{n-1}}{\Delta t}\right)^2 dx + \frac{2}{9}\int_\Omega \left|z_x^N\right|^2 dx - \frac{2}{9}\int_\Omega |(z_0)_x|^2\,dx \\ &- (r-l)^2\left[\int_\Omega \frac{2F}{z_0^{2/3}} - \frac{2F}{(z^N)^{2/3}}dx\right] \le \frac{(r-l)^2}{3}\sum_{n=1}^N \int_\Omega \left(\frac{-\beta}{(z^n)^{1/3}}\right)(z^n - z^{n-1})dx.\end{aligned} \tag{4.22}$$

Applying Young's inequality to the right side yields

$$\begin{aligned}\frac{1}{3}\sum_{n=1}^N \int_\Omega \left(\frac{-\beta(r-l)^2}{(z^n)^{1/3}}\right)(z^n - z^{n-1})dx &\le \frac{1}{6}\sum_{n=1}^N \int_\Omega \frac{\beta^2(r-l)^4\Delta t}{(z^n)^{2/3}}dx \\ &+ \frac{1}{6}\Delta t\sum_{n=1}^N \int_\Omega \left(\frac{z^n - z^{n-1}}{\Delta t}\right)^2 dx.\end{aligned} \tag{4.23}$$

From the above, (4.17), the boundedness of $(r-l)^4$ on I^ϵ and the fact that $\int_\Omega \frac{1}{(z^n)^{2/3}}dx \le \int_\Omega \frac{1}{\underline{z}^{2/3}}dx$, we get that

$$\frac{\partial \tilde{z}_{\Delta t}}{\partial t} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta t, \tag{4.24}$$

$$z_{\Delta t}, \tilde{z}_{\Delta t} \text{ are bounded in } L^\infty(0,T;H_0^1(\Omega)) \text{ uniformly in } \Delta t. \tag{4.25}$$

Moreover, from this we see that there is a $C > 0$ independent of Δt such that

$$\|z_{\Delta t} - \tilde{z}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \le \max_{n\in\{1,\dots,N\}} \left\|z^n - z^{n-1}\right\|_{L^2(\Omega)} \le C\cdot(\Delta t)^{1/2}. \tag{4.26}$$

Letting $N \to \infty$, (i.e.$\Delta t \to 0^+$) and from (4.24) and (4.25) there exists $\tilde{z}, z \in L^\infty(0,T;H_0^1(\Omega)\cap L^\infty(\Omega))$ such that $\frac{\partial \tilde{z}}{\partial t} \in L^2(Q_t)$, $\tilde{z}, z \in \mathcal{C}$ uniformly and as $\Delta t \to 0^+$,

$$\tilde{z}_{\Delta t} \overset{*}{\rightharpoonup} \tilde{z}, z_{\Delta t} \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,T;H_0^1(\Omega)\cap L^\infty(\Omega)) \tag{4.27}$$

and

$$\frac{\partial \tilde{z}_{\Delta t}}{\partial t} \rightharpoonup \frac{\partial \tilde{z}}{\partial t} \text{ in } L^2(Q_T). \tag{4.28}$$

From (4.26) we see that $\tilde{z} = z$ with $\underline{z} \leq z \leq \bar{z}$, by (4.16) and $z \in \mathbf{V}(Q_T)$.

Since $\frac{\partial \tilde{z}_{\Delta t}}{\partial t}$ is bounded in $L^2(Q_T)$, we get that $\{\tilde{z}_{\Delta t}\}_{\Delta t}$ is equicontinuous in $C(0,T;L^q(\Omega))$, for $1 \leq q \leq 2$. Thus with $\underline{z} \leq z \leq \bar{z}$ and from the interpolation inequality $\|\cdot\|_q \leq \|\cdot\|_\infty^\alpha \|\cdot\|_2^{1-\alpha}$ where $\frac{1}{q} = \frac{\alpha}{\infty} + \frac{1-\alpha}{2}$ we see that $\{\tilde{z}_{\Delta t}\}_{\Delta t}$ is equicontinuous in $C(0,T;L^q(\Omega))$ for $1 < q < \infty$. Furthermore, from the boundedness of $\{\tilde{z}_{\Delta t}\}_{\Delta t}$ in $H_0^1(\Omega)$ which is compactly embedded in $L^q(\Omega)$ for $1 \leq q < \infty$, we get that $\{\tilde{z}_{\Delta t}\}_{\Delta t}$ is bounded in $L^q(\Omega)$. By the Arzela-Ascoli Theorem as $\Delta t \to 0^+$ for $q \geq 2$

$$\tilde{z}_{\Delta t} \to \tilde{z} \text{ in } C(0,T;L^q(\Omega)). \tag{4.29}$$

From (4.26) with the interpolation inequality for $q > 1$

$$z_{\Delta t} \to z \text{ in } L^\infty(0,T;L^q(\Omega)), \tag{4.30}$$

that is $\lim_{\Delta t \to 0} \|z_{\Delta t} - z\|_\infty = 0$.

To show that z is a weak solution of (P_t) we multiply (4.7) by $(z_{\Delta t} - z)$

$$\int_0^T \int_\Omega \frac{\partial \tilde{z}_{\Delta t}}{\partial t}(z_{\Delta t} - z)dxdt - \frac{4}{9}\int_0^T \int_\Omega \frac{\partial^2 z_{\Delta t}}{\partial x^2}(z_{\Delta t} - z)dxdt$$
$$- \frac{2(r-l)^2}{3}\int_0^T \int_\Omega \frac{F}{z_{\Delta t}^{5/3}}(z_{\Delta t} - z)dxdt + \frac{\beta(r-l)^2}{3}\int_0^T \int_\Omega \frac{1}{z_{\Delta t}^{1/3}}(z_{\Delta t} - z)dxdt = 0 \tag{4.31}$$

adding and subtracting $\frac{\partial z}{\partial t}$ using (4.29) and (4.30) and noting that $\frac{\partial z}{\partial t} \in L^2(Q_T)$we get

$$\int_0^T\int_\Omega\left(\frac{\partial\tilde z_{\Delta t}}{\partial t}-\frac{\partial z}{\partial t}\right)(\tilde z_{\Delta t}-z)dxdt-\frac{4}{9}\int_0^T\left\langle\frac{\partial^2 z_{\Delta t}}{\partial x^2},z_{\Delta t}-z\right\rangle dt$$
$$-\frac{2}{3}\int_0^T\int_\Omega\frac{(r-l)^2F}{z_{\Delta t}^{5/3}}(z_{\Delta t}-z)dxdt+\frac{\beta}{3}\int_0^T\int_\Omega\frac{(r-l)^2}{z_{\Delta t}^{1/3}}(z_{\Delta t}-z)dxdt=O(\Delta t).$$

It follows from (4.16) and (4.29) that

$$\frac{2(r-l)^2}{3}\int_0^T\int_\Omega\frac{F}{z_{\Delta t}^{5/3}}(z_{\Delta t}-z)dxdt=O(\Delta t),$$
$$\frac{\beta(r-l)^2}{3}\int_0^T\int_\Omega\frac{1}{z_{\Delta t}^{1/3}}(z_{\Delta t}-z)dxdt=O(\Delta t).$$

Next note from (4.30) we have

$$\int_0^T\int_\Omega\left(\frac{\partial\tilde z_{\Delta t}}{\partial t}-\frac{\partial z}{\partial t}\right)(\tilde z_{\Delta t}-z)dxdt=\frac{1}{2}\int_\Omega\left|\tilde z_{\Delta t}(T)-z(T)\right|^2dx$$

and using the weak star convergence of $z_{\Delta t}$

$$-\int_0^T\left\langle\frac{\partial^2 z_{\Delta t}}{\partial x^2},z_{\Delta t}-z\right\rangle dt=-\int_0^T\left\langle\frac{\partial^2 z_{\Delta t}}{\partial x^2}-\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x^2},z_{\Delta t}-z\right\rangle dt$$
$$=-\int_0^T\left\langle\frac{\partial^2 z_{\Delta t}}{\partial x^2}-\frac{\partial^2 z}{\partial x^2},z_{\Delta t}-z\right\rangle dt-O(\Delta t).$$

Then, from the above and (4.31)

$$\frac{1}{2}\int_\Omega\left|\tilde z_{\Delta t}(T)-z(T)\right|^2dx+\frac{4}{9}\int_0^T\left\langle\frac{\partial}{\partial x}(z_{\Delta_t}-z),\frac{\partial}{\partial x}(z_{\Delta t}-z)\right\rangle dt=O(\Delta t).$$

Thus by (4.30) we get that $z_{\Delta t}\to z$ in $L^2(0,T;H_0^1(\Omega))$ and

$$-\frac{\partial^2 z_{\Delta t}}{\partial x^2}\to-\frac{\partial^2 z}{\partial x^2}\ \text{ in } L^2(0,T;H^{-1}(\Omega)).\tag{4.32}$$

Moreover, for $\phi\in H_0^1(\Omega)$ from (4.16)

$$\left|\frac{2(r-l)^2}{3}\int_\Omega F\frac{\phi}{z_{\Delta t}^{5/3}}\right|\le\frac{2(r-l)^2\max(F)}{3}\int_\Omega\frac{|\phi|}{\underline{z}^{5/3}}$$
$$\le(r-l)^2\frac{2\max(F)}{3}\left(\int_\Omega\left|\frac{d(x)}{\underline{z}^{5/3}}\right|^2dx\right)^{1/2}\cdot\left(\int_\Omega\left|\frac{\phi}{d(x)}\right|^2dx\right)^{1/2}$$

and since $\underline{z} \in \mathcal{C}$

$$\int_\Omega \left| \frac{d(x)}{\underline{z}^{5/3}} \right|^2 dx < \infty.$$

Futhermore, using the Hardy inequality we observe that

$$\left(\int_\Omega \left| \frac{\phi}{d(x)} \right|^2 dx \right)^{1/2} < \infty,$$

this with the Lebesgue dominated convergence theorem yields

$$\frac{(r-l)F((r-l)x+r)}{z_{\Delta t}^{5/3}} \to \frac{(r-l)F((r-l)x+r)}{z^{5/3}} \tag{4.33}$$

in $L^\infty(0,T;H^{-1}(\Omega))$. The convergence of

$$\frac{(r-l)^2}{z_{\Delta t}^{1/3}} \to \frac{(r-l)^2}{z^{1/3}}$$

in $L^\infty(0,T;H^{-1}(\Omega))$ follows from an analogous argument. We can conclude from (4.29), (4.30), (4.32), (4.33) that $z \in \mathbf{V}(Q_T)$ and z satisfies (P_t) for r and l fixed.

For uniqueness suppose that there exist a $\zeta \neq z$ which is a weak solution to the parabolic equation with $\zeta(t) \in \mathcal{C}$ for all $t \in [0,T]$. We have

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial(z-\zeta)}{\partial t}(z-\zeta)dxdt + \frac{4}{9}\int_0^T \int_\Omega (z_x - \zeta_x)(z_x - \zeta_x)dxdt \\
&- \frac{2}{3}\int_0^T \int_\Omega (r-l)^2 F\left(\frac{1}{z^{5/3}} - \frac{1}{\zeta^{5/3}}\right)(z-\zeta)dxdt \\
&+ \frac{1}{3}\int_0^T \int_\Omega (r-l)^2 \beta \left(\frac{1}{z^{1/3}} - \frac{1}{\zeta^{1/3}}\right)(z-\zeta)dxdt = 0.
\end{aligned}$$

Note that from the monotonicity of $\frac{2}{3}\frac{F((l-r)x+l)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}$

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{4}{9}(z_x - \zeta_x)^2 - \frac{2}{3}(r-l)^2 F\left(\frac{1}{z^{5/3}} - \frac{1}{\zeta^{5/3}}\right)(z-\zeta)dxdt \\
&+ \int_0^T \int_\Omega \frac{1}{3}(r-l)^2 \beta \left(\frac{1}{z^{1/3}} - \frac{1}{\zeta^{1/3}}\right)(z-\zeta)dxdt \geq 0
\end{aligned}$$

therefore $z = \zeta$.

We now prove that $z \in C([0,T], H_0^1(\Omega))$ and

$$\|z\|^2_{H_0^1(\Omega)} = \|z(t_0)\|^2_{H_0^1(\Omega)} + \frac{9}{2}(r-l)^2 \int_\Omega \left(\frac{F}{z(t_0)^{2/3}} - \frac{F}{z^{2/3}} \right) dx \\ - \frac{3}{2}\beta(r-l)^2 \int_0^t \int_\Omega \frac{z_t}{z^{1/3}} dxds - \frac{9}{2}\int_0^t \int_\Omega (z_t)^2 dxds. \quad (4.34)$$

We have already established that $z \in C([0,T], L^2(\Omega))$ and $z \in L^\infty(0,T; H_0^1(\Omega))$, therefore $z : [0,T] \to H_0^1(\Omega)$ is weakly continuous and the norm is lower-semicontinuous $\|z(t_0)\|_{H_0^1(\Omega)} \leq \liminf_{t \to t_0} \|z(t)\|_{H_0^1(\Omega)}$ for all $t_0 \in [0,T]$. From (4.22), (4.26) and (4.33) (where we sum from $n = N', .., N$ and let $\Delta t \to 0$) we see that z satisfies for any $t \in [t_0, T]$:

$$\|z\|^2_{H_0^1(\Omega)} \leq \|z(t_0)\|^2_{H_0^1(\Omega)} + \frac{9}{2}(r-l)^2 \int_\Omega \left(\frac{F}{z(t_0)^{2/3}} - \frac{F}{z^{2/3}} \right) dx \\ - \frac{3}{2}\beta(r-l)^2 \int_{t_0}^t \int_\Omega \frac{z_t}{z^{1/3}} dxds - \frac{9}{2}\int_{t_0}^t \int_\Omega (z_t)^2 dxds.$$

It follows from the Lebesgue dominated convergence theorem that

$$\limsup_{t \to t_0^+} \|z(t)\|_{H_0^1(\Omega)} \leq \|z(t_0)\|_{H_0^1(\Omega)}$$

and $z(t) \to z(t_0)$ as $t \to t_0^+$, in $H_0^1(\Omega)$ hence z is right continuous on $[0.T]$. For left-continuity define for $0 < k < t - t_0$, $\tau_k(z)(s) = \frac{z(s+k)-z(s)}{k}$. Multiplying (P_t) by $\tau_k(z)$, integrating over $(t_0, t) \times \Omega$ and using convexity (4.21),(4.22), we get

$$\int_{t_0}^t \int_\Omega z_t \tau_k(z) dxds + \frac{2}{9k} \int_{t_0}^t \int_\Omega z_x^2(s+k) - z_x^2(s) dxds \\ - \frac{2}{k} \int_{t_0}^t \int_\Omega (l-r)^2 F \cdot \left(\frac{1}{z^{2/3}(s+k)} - \frac{1}{z^{2/3}(s)} \right) dxds \\ \geq - \frac{(l-r)^2 \beta}{3} \int_{t_0}^t \int_\Omega \frac{\tau_k(z)}{z^{1/3}(s)} dxds$$

which can be written as

$$\int_{t_0}^{t}\int_\Omega z_t\tau_k(z)-\frac{1}{3}\tau_k^2(z)dxds+\frac{2}{9k}\left[\int_{t+k}^{t}\int_\Omega z_x^2(s+k)dxds-\int_{t_0}^{t_0+k}\int_\Omega z_x^2(s)dxds\right]$$
$$-\frac{2}{k}\left[\int_{t}^{t+k}\int_\Omega \frac{(l-r)^2F}{z^{2/3}(s+k)}dxds-\int_{t_0}^{t_0+k}\int_\Omega \frac{(l-r)^2F}{z^{2/3}(s)}dxds\right]$$
$$\geq -\frac{(l-r)^2\beta}{3}\int_{t_0}^{t}\int_\Omega \frac{\tau_k(z)}{z^{1/3}(s)}dxds. \tag{4.35}$$

By the right-continuity of z in $H_0^1(\Omega)$ and Lebesgue's differentiation theorem letting $k\to 0^+$ yields

$$\frac{1}{k}\int_{t}^{t+k}\int_\Omega |z_x(s)|^2\,dxds\to\int_\Omega |z_x(t)|^2\,dx$$
$$\frac{1}{k}\int_{t_0}^{t_0+k}\int_\Omega |z_x(s)|^2\,dxds\to\int_\Omega |z_x(t_0)|^2\,dx$$
$$\frac{1}{k}\int_{t}^{t+k}\int_\Omega \frac{1}{z^{2/3}(s)}dxds\to\int_\Omega \frac{1}{z^{2/3}(t)}dx$$
$$\frac{1}{k}\int_{t}^{t_0+k}\int_\Omega \frac{1}{z^{2/3}(s)}dxds\to\int_\Omega \frac{1}{z^{2/3}(t_0)}dx.$$

Letting $k\to 0^+$ in (4.35)

$$\|z\|^2_{H_0^1(\Omega)}\geq\|z(t_0)\|^2_{H_0^1(\Omega)}+\frac{9}{2}(r-l)^2\int_\Omega\left(\frac{F}{z(t_0)^{2/3}}-\frac{F}{z^{2/3}}\right)dx$$
$$-\frac{3}{2}\beta(r-l)^2\int_{t_0}^{t}\int_\Omega\frac{z_t}{z^{1/3}}dxds-\frac{9}{2}\int_{t_0}^{t}\int_\Omega (z_t)^2dxds.$$

Consequently we have in fact an equality. This combined with the fact that $t\to\int_\Omega\frac{1}{z^{2/3}(t)}dx$ is continuous implies that $z\in C([0,T],H_0^1(\Omega))$. We obtain (4.34) by setting $t_0=0$.

This completes the proof of Theorem 4.1.4. ∎

Theorem 4.1.6. Given $(z_0, l_0, r_0) \in H_0^1(\Omega) \cap \mathcal{C} \times I^\epsilon \times I^\epsilon$ and assuming β is bounded so that (4.3) is monotone, there exists $T(z_0, l_0, r_0) > 0$ such that a unique weak solution (z, l, r) to (S_t) exists and $z(t) \in \mathcal{C}$ uniformly for $t \in [0, T]$. Moreover $(z, l, r) \in C([0, T]\,; H_0^1(\Omega)) \times C([0, T]\,; I^\epsilon) \times C([0, T]\,; I^\epsilon)$.

Proof. For $M > 1$ define $\Delta t = \frac{T}{M}$ for $0 \leq m \leq M$ let $t_m = m\Delta t$, we define by iteration l^m, r^m

$$\begin{cases} \dfrac{l^m - l^{m-1}}{\Delta t} = -\displaystyle\int_\Omega \frac{2(z_x^m)^2}{9} dx - (r^{m-1} - l^{m-1})^2 \int_\Omega \frac{((r^{m-1} - l^{m-1})(1-x)K' - K)}{(z^m)^{2/3}} dx \\ \dfrac{r^m - r^{m-1}}{\Delta t} = \displaystyle\int_\Omega \frac{2(z_x^m)^2}{9} dx - (r^{m-1} - l^{m-1})^2 \int_\Omega \frac{((r^{m-1} - l^{m-1})xK' + K)}{(z^m)^{2/3}} dx \\ l^0 = l_0, r^0 = r_0, \end{cases} \tag{4.36}$$

where z^m is the unique solution of (P_t) with initial conditions l^{m-1}, r^{m-1} on the interval $[t_m, t_{m+1}]$. We introduce the following functions $l_{\Delta t}, \tilde{l}_{\Delta t}, r_{\Delta t}, \tilde{r}_{\Delta t}$ for all $m = 1, ..., M$ and $t \in [t_{m-1}, t_m)$

$$\begin{cases} z_{\Delta t}(t) := z^m(t), \\ l_{\Delta t}(t) := l^{m-1}, \\ \tilde{l}_{\Delta t}(t) := \frac{t - t_{m-1}}{\Delta t}(l^m - l^{m-1}) + l^{m-1}, \\ r_{\Delta t}(t) := r^{m-1}, \\ \tilde{r}_{\Delta t}(t) := \frac{t - t_{m-1}}{\Delta t}(r^m - r^{m-1}) + r^{m-1}. \end{cases} \tag{4.37}$$

From the continuity of z^m, $z_{\Delta t}$ is continuous in t. We have the following relations

$$\frac{d}{dt}\tilde{l}_{\Delta t} = -\frac{2}{9}\int_\Omega (\partial_x z_{\Delta t})^2 dx - (r_{\Delta t} - l_{\Delta t})^2 \int_\Omega \frac{(r_{\Delta t} - l_{\Delta t})(1-x)F' - F}{(z_{\Delta t})^{2/3}} + \frac{\beta}{2}(z_{\Delta t})^{2/3} dx \tag{4.38}$$

$$\frac{d}{dt}\tilde{r}_{\Delta t} = \frac{2}{9}\int_\Omega (\partial_x z_{\Delta t})^2 dx - (r_{\Delta t} - l_{\Delta t})^2 \int_\Omega \frac{(r_{\Delta t} - l_{\Delta t})xF' + F}{(z_{\Delta t})^{2/3}} + \frac{\beta}{2}(z_{\Delta t})^{2/3} dx. \tag{4.39}$$

Under the assumption that $l_0, r_0 \in I^\epsilon$, and from the uniform boundedness of $\|z^m\|_{H_0^1(\Omega)}$ by continuity there is a $T = T(l_0, r_0) > 0$ such that

$$\frac{d}{dt}\tilde{l}_{\Delta t}, \frac{d}{dt}\tilde{r}_{\Delta t} \text{ are bounded in } L^\infty(0,T;\mathbb{R}),$$

in particular T is finite when $\tilde{l}_{\Delta t}(t)$ or $\tilde{r}_{\Delta t}(t)$ leave I^ϵ, thus $\tilde{l}_{\Delta t}(t)$, $\tilde{r}_{\Delta t}(t)$ are bounded for $t \in [0,T)$, this time interval is maximal, see Lemma 4.2.5. This shows that $\left\{\tilde{l}_{\Delta t}\right\}_{\Delta t}$, $\{l_{\Delta t}\}_{\Delta t}$ and $\{\tilde{r}_{\Delta t}\}_{\Delta t}, \{r_{\Delta t}\}_{\Delta t}$ are equicontinuous in $C([0,T];\mathbb{R})$. Therefore by Arzela-Ascoli there exists $l, \tilde{l}, r, \tilde{r} \in C([0,T];\mathbb{R})$ such that up to subsequences as $\Delta t \to 0$

$$\tilde{l}_{\Delta t} \to l, \quad l_{\Delta t} \to \tilde{l} \tag{4.40}$$

$$\tilde{r}_{\Delta t} \to \tilde{r}, \quad r_{\Delta t} \to r. \tag{4.41}$$

Due to the boundedness of $l_{\Delta t}$ and $r_{\Delta t}$, from (4.36) and (4.37) there exists $C' > 0$ satisfying

$$\left|l_{\Delta t} - \tilde{l}_{\Delta t}\right| \leq \max_{m \in \{1,..M\}} \left|l^m - l^{m-1}\right| \leq C'\Delta t, \tag{4.42}$$

thus $l = \tilde{l}$. Similarly we can conclude that $r = \tilde{r}$. We can also deduce that $\frac{d\tilde{l}_{\Delta t}}{dt} \to \frac{dl}{dt}$ and $\frac{d\tilde{r}_{\Delta t}}{dt} \to \frac{dr}{dt}$. Therefore we have convergence in Δt.

Moreover for each Δt, $(z_{\Delta t}, l_{\Delta t}, r_{\Delta t})$ is a solution to (P_t) with $z_{\Delta t} \in C([0,T]; H_0^1(\Omega))$ and $\frac{\partial z_{\Delta t}}{\partial t}$ uniformly bounded in $L^2(Q_T)$, from (4.22) since $|r_{\Delta t} - l_{\Delta t}|$ is uniformly bounded, $z_{\Delta t}$ is uniformly bounded in $H_0^1(\Omega)$ which is compactly embedded in $L^2(\Omega)$. Therefore up to a subsequence $z_{\Delta t} \to z$ in $C([0,T]; L^2(\Omega))$ and $z_{\Delta t} \overset{*}{\rightharpoonup} z$ in $L^\infty(0,T; H_0^1(\Omega) \cap L^\infty(\Omega))$, $\frac{\partial z}{\partial t} \in L^2(Q_T)$. Multiply (P_t) by $z_{\Delta t} - z$ and integrating over Q_T.

$$\int_0^T \int_\Omega \frac{\partial z_{\Delta t}}{\partial t}(z_{\Delta t} - z)dxdt + \int_0^T \int_\Omega \frac{4}{9}\partial_x z_{\Delta t} \partial_x (z_{\Delta t} - z)dxdt$$
$$- \int_0^T \int_\Omega \frac{(r_{\Delta t} - l_{\Delta t})^2}{3} \frac{2F}{z_{\Delta t}^{5/3}}(z_{\Delta t} - z)dxdt + \int_0^T \int_\Omega \frac{\beta}{z_{\Delta t}^{1/3}}(z_{\Delta t} - z)dxdt = 0,$$

$$\int_0^T \int_\Omega (\frac{\partial z_{\Delta t}}{\partial t} - \frac{\partial z}{\partial t})(z_{\Delta t} - z)dxdt + \int_0^T \int_\Omega \frac{4}{9}\partial_x z_{\Delta t}\partial_x (z_{\Delta t} - z)dxdt$$
$$- \int_0^T \int_\Omega \frac{(r_{\Delta t} - l_{\Delta t})^2}{3} \frac{2F}{z_{\Delta t}^{5/3}}(z_{\Delta t} - z)dxdt + \int_0^T \int_\Omega \frac{\beta}{z_{\Delta t}^{1/3}}(z_{\Delta t} - z)dxdt = O(\Delta t),$$

Since $\frac{1}{(z_{\Delta t})^{5/3}} \in H^{-1}(\Omega)$

$$\int_0^T \int_\Omega \frac{(r_{\Delta t} - l_{\Delta t})^2}{3} \frac{2F}{z_{\Delta t}^{5/3}}(z_{\Delta t} - z)dxdt = O(\Delta t),$$
$$\int_0^T \int_\Omega \frac{\beta}{z_{\Delta t}^{1/3}}(z_{\Delta t} - z)dxdt = O(\Delta t),$$

with $z_{\Delta t} \overset{*}{\rightharpoonup} z$ in $L^\infty(0, T; H_0^1(\Omega) \cap L^\infty(\Omega))$ shows that

$$\frac{1}{2}\int_\Omega |z_{\Delta t}(T) - z(T)|^2 \, dx + \int_0^T \int_\Omega \frac{4}{9}\left|\frac{\partial z_{\Delta t}}{\partial x} - \frac{\partial z}{\partial x}\right|^2 dxdt = O(\Delta t),$$

Hence $z_{\Delta t} \to z \in L^2(0, T; H_0^1(\Omega))$. As in the similar part of the proof of Theorem 4.1.4 we get that

$$\frac{(r_{\Delta t} - l_{\Delta t})F((r_{\Delta t} - l_{\Delta t})x + r_{\Delta t})}{z_{\Delta t}^{5/3}} \to \frac{(r - l)F((r - l)x + r)}{z^{5/3}}. \tag{4.43}$$

At this point we know that $z \in C([0, T]; L^2(\Omega))$ and $z \in L^\infty(\Omega)$, we can use equation 4.34

$$\|z\|^2_{H_0^1(\Omega)} = \|z(t_0)\|^2_{H_0^1(\Omega)} + \frac{9}{2}(r_{\Delta t} - l_{\Delta t})^2 \int_\Omega \left(\frac{F}{z(t_0)^{2/3}} - \frac{F}{z^{2/3}}\right) dx$$
$$- \frac{3}{2}\beta(r_{\Delta t} - l_{\Delta t})^2 \int_0^t \int_\Omega \frac{z_t}{z^{1/3}} dxds - \frac{9}{2}\int_0^t \int_\Omega (z_t)^2 dxds.$$

Taking the limit as $\Delta t \to 0$

$$\|z\|^2_{H_0^1(\Omega)} = \|z(t_0)\|^2_{H_0^1(\Omega)} + \frac{9}{2}(r(t) - l(t))^2 \int_\Omega \left(\frac{F}{z(t_0)^{2/3}} - \frac{F}{z^{2/3}}\right) dx$$
$$- \frac{3}{2}\beta(r(t) - l(t))^2 \int_0^t \int_\Omega \frac{z_t}{z^{1/3}} dxds - \frac{9}{2}\int_0^t \int_\Omega (z_t)^2 dxds.$$

Therefore arguing as in the proof of Theorem 4.1.4 we have that $z \in C([0, T]; H_0^1(\Omega))$. At this point we can conclude that (z, l, r) satisfy (S_t).

To obtain uniqueness of (S_t) one appeals to Gronwall's inequality, since we do not have a Lipschitz right hand side of (S_t) the usual techniques involving Gronwall's inequality do not apply, therefore we cannot claim to have a dynamical system. We get around this obstruction by considering a perturbation of (S_t) in ϵ.

Consider the perturbed system

$$(S_{t,\epsilon})\begin{cases} z_t = \frac{4}{9} z_{xx} + (r-l)^2\left(\frac{2}{3}\frac{F((r-l)x+u_1)}{(z+\epsilon)^{5/3}} - \frac{\beta}{3(z+\epsilon)^{1/3}}\right), \\ z = 0 \ \text{ on } \Sigma_T, \quad z > 0 \text{ in } Q_T, \quad z(0,x) = z_0(x) \text{ in } \Omega, \\ \\ l_t = -\displaystyle\int_\Omega \frac{2(z_x)^2}{9} dx - (r-l)^2 \int_\Omega \frac{((r-l)(1-x)K' - K)}{(z+\epsilon)^{2/3}} dx, \\ \\ r_t = \displaystyle\int_\Omega \frac{2(z_x)^2}{9} dx - (r-l))^2 \int_\Omega \frac{((r-l)xK' + K)}{(z+\epsilon)^{2/3}} dx \\ l(0) = l_0, \ r(0) = r_0. \end{cases}$$

If we assume that β satisfies the monotonicity bound we can apply the methods from above. Since $z_0 \in \mathcal{C}$, for any $\epsilon > 0$ there exists $\underline{z}_\epsilon \in C_0(\overline{\Omega}) \cap H_0^1(\Omega)$ in the form $\underline{z}_\epsilon = \eta \sin^{3/4}(\pi x)$ such that $\underline{z}_\epsilon \leq z_0$ and

$$-\frac{4}{9}\partial^2_{xx}\underline{z}_\epsilon - \frac{(r-l)^2}{3}\left(\frac{2F(u)}{\underline{z}_\epsilon^{5/3}} - \frac{\beta}{\underline{z}_\epsilon^{1/3}}\right) \leq 0 \ \text{ in } \Omega.$$

Moreover there exists $\bar{z}$ in the form $\bar{z} = M\sin^{3/4}(\pi x)$ such that $z_0 \leq \bar{z}$ and satisfying

$$-\frac{4}{9}\bar{z}_{xx} - \frac{(r-l)^2}{3}\left(\frac{2F(u)}{\bar{z}^{5/3}} - \frac{\beta}{\bar{z}^{1/3}}\right) \geq 0 \ \text{ in } \Omega.$$

Following the methods and using estimates from the existence proof above, we can prove existence and uniqueness of a positive solution $(z_\epsilon, l_\epsilon, r_\epsilon) \in C([0,T], H_0^1(\Omega)) \times C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$, satisfying $\underline{z}_\epsilon \leq z_\epsilon \leq \bar{z}$ to $(S_{t,\epsilon})$.

4.2 REGULARITY OF SOLUTIONS

We first consider the monotone case.

Theorem 4.2.1. Under the assumptions of (4.1.6), $\forall \eta > 0$ small enough, we have that any solution z to the parabolic equation in (P_t) satisfies $z(t) \in C((0,T]\,; H^{\frac{5}{4}-\eta}(\Omega))$.

In order to obtain this regularity result we use the interpolation theory in Sobolev spaces (cf. Triebel [10]), and the Hardy Inequality. In the following if X, Y are two Banach spaces, $X \subset Y$ denotes the continuous embedding of X into Y. For $\theta \in (0,1)$, $1 \leq q \leq \infty$, let $(X,Y)_{\theta,q}$ denote the interpolation space obtained from X and Y with the real method and for $T \in (0,\infty]$, $q \in [1,\infty]$ define the space

$$W_q(0,T,X,Y) = \{z \in L^q(0,T;X) \mid z_t \in L^q(0,T;Y)\}$$

with norm $\left(\int_0^T \|z\|_X^q + \|z_t\|_Y^q\right)^{1/q}$. Recall that the operator $A = -\frac{4}{9}\frac{d^2}{dx^2}$ has domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, hence for fractional powers of A, A^θ with domain $\mathcal{D}(A^\theta)$ in $L^2(\Omega)$ we have the following fact:

Proposition 4.2.2.

(*i*) $\mathcal{D}(A^\theta) = (\mathcal{D}(-A), L^2(\Omega))_{1-\theta,2}$,

(*ii*) $\mathcal{D}(A^\theta) = H_0^{2\theta}$ if $\frac{1}{4} < \theta < 1$,

(*iii*) A^θ is an isomorphism from $\mathcal{D}(A)$ onto $\mathcal{D}(A^{1-\theta})$ as well as from $L^2(\Omega)$ onto the dual space $(\mathcal{D}(A^\theta))'$.

See Proposition 4.1 in[1], also one can look in [10].

For $1 < q < \infty$, $0 < T \leq \infty$ define the Banach space

$$\mathcal{X}_{q,\theta,T} := W_q(0,T;\mathcal{D}(A^{1-\theta}), (\mathcal{D}(A^\theta))').$$

Lemma 4.2.3. Let $\theta \in [0,1)$ and $q > \frac{2}{1-\theta}$. For $0 < T < \infty$ let L_T be the linear operator defined by $L_T(f) = u$ where u is the solution to

$$\begin{cases} u_t - \frac{4}{9}u_{xx} = f \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Sigma_T, \\ u(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{4.44}$$

Then L_T is a bounded operator from $L^q(0,T;(\mathcal{D}(A^\theta))')$ into $\mathcal{X}_{q,\theta,T}$ as well as from $L^q(0,T;(\mathcal{D}(A^\theta))')$ into $C((0,T],\mathcal{D}(A^{1-\theta-\frac{2}{q}}))$.

In particular we have the following inequality

$$\|L_T f\|_{C((0,T],\mathcal{D}(A^{1-\theta-\frac{2}{q}}))} \leq C \|f\|_{L^q(0,T;L^2(\Omega))}. \tag{4.45}$$

Proof. See Lemma 4.4 in [1].

We make use of the following Hardy type inequality (Lemma 4.5 [1];[10] Par. 3.2.6,Lem.3.2.6.1,p.259).

Lemma 4.2.4. Let $s \in [0,2]$ such that $s \neq \frac{1}{2}$ and $s \neq \frac{3}{2}$. Then the following generalization of Hardy's inequality holds:

$$\left\|d^{-s}g\right\|_{L^2(\Omega)} \leq C \|g\|_{H_0^s(\Omega)} \quad \text{for all } g \in H_0^s(\Omega). \tag{4.46}$$

We now prove Theorem 4.2.1.

Proof. Note that the solution to (S_t) satisfies

$$z(t) = e^{-At}z_0 + L_T\left(\frac{2}{3}\frac{(r(t)-l(t))^2K(x)}{z^{5/3}}\right).$$

We assume that $0 < \eta < \frac{3}{8}$, since $z \in \mathcal{C}$ we know that $\frac{1}{z^{5/3}} = O(\frac{1}{d(x)^{5/4}})$. Therefore by Lemma 4.2.4 $\frac{2}{3}\frac{(r(t)-l(t))^2F}{z^{5/3}} \in C((0,T], (H_0^{\frac{3}{4}+\frac{\eta}{2}}(\Omega))')$, to see this we use the fact that

$z \in \mathcal{C}$

$$\left\|\frac{1}{z^{5/3}}g\right\|^2_{L^2(\Omega)} = \int_\Omega \left|\frac{g}{C_1 d(x)^{5/4}}\right|^2 dx = \frac{1}{C_1^2}\int_\Omega \left|\frac{d(x)^{3/4+\eta/2}}{d(x)^{5/4}}\frac{g}{d(x)^{3/4+\eta/2}}\right|^2 dx$$
$$\leq \frac{1}{C_1^2}\left\|d^{-1/2+\eta/2}\right\|_{L^2(\Omega)}\left\|d(x)^{-(3/4+\eta/2)}g\right\|_{L^2(\Omega)} \leq \frac{1}{C_1^2}\left\|d^{-1/2+\eta/2}\right\|_{L^2(\Omega)}\|g\|_{H^{3/4+\eta/2}}.$$

Furthermore from proposition 4.2.2, $C((0,T], (H_0^{\frac{3}{4}+\frac{\eta}{2}}(\Omega))') = C((0,T], (\mathcal{D}(A)^{\frac{3}{8}+\frac{\eta}{4}})')$. Applying Lemma 4.2.3 with $q = \frac{2}{\eta}$ we see that $L_T\left(\frac{(r(t)-l(t))^2}{3}\left(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\right)\right) \in C((0,T], H^{\frac{5}{4}-\eta}(\Omega))$. Therefore

$$t \to z(t) = e^{-At}z_0 + L_T\left(\frac{(r(t)-l(t))^2}{3}\left(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\right)\right) \in C((0,T], H^{\frac{5}{4}-\eta}(\Omega)).$$

■

Proof of 4.1.5. See Theorem 0.18 in [1].

Next we establish a maximum time interval of existence for problem (S_t).

Lemma 4.2.5. There exists a $T_{\max}$ such that if $T_{\max} < \infty$ then for all $\epsilon > 0$ small, $u_1 \to u_- + \epsilon$ and $u_2 \to u_+ - \epsilon$.

Proof. The time interval of existence relies on the dynamics of $u_1(t)$ and $u_2(t)$. If it is the case that $u_1(t)$, $u_2(t)$ do not converge to $u_- + \epsilon$, $u_+ + \epsilon$ respectively, then $\{(u_1(t), u_2(t), z(t))\}$ forms a bounded set in $I^2 \times H_0^1(\Omega)$. By the compactness result above we can choose a subsequence $\{(u_1(t_n), u_2(t_n), z(t_n))\}_n$ such that $(u_1(t_n), u_2(t_n), z(t_n)) \to (u_1^*, u_2^*, z^*)$. We can then treat (u_1^*, u_2^*, z^*) as initial conditions and continue the flow, this leads to an infinite time of existence contradicting the assumption.

4.3 CHAPTER SUMMARY

For the eFK/SwiftHohenberg system $L(u,v,w) = \frac{1}{2}w^2 + \frac{\beta}{2}v^2 + F(u)$ we proved local existence, uniqueness and regularity of a gradient flow for the action J.

Theorem 4.3.1.

(i) For $\beta \leq \frac{10F}{(r-l)C_2^{4/3}(r,l)}$ the system (S_t) admits a solution, that is for $(z_0, l_0, r_0) \in H_0^1(\Omega) \times I^2$ there exist $T > 0$ and $(z(t), l(t), r(t))$ which solves (S_t). For $\epsilon > 0$ the system $(S_{t,\epsilon})$ is well posed, there exists a unique solution.

(ii) The solution to the parabolic equation stays in the cone $\mathsf{C}(l, r, \beta) = \{z \in H_0^1(\Omega) \mid \underline{z} \leq z \leq \bar{z}\}$. $\bar{z}$, $\underline{z}$ are given by $\bar{z} = C_2(l, r, \beta)(l - r)^{3/4} \sin(\pi x)^{3/4}$ and $\underline{z} = C_1(l, r, \beta)(l - r)^{3/4} \sin(\pi x)^{3/4}$ where $C_2^{4/3}(l, r, \beta)$ is an upper bound and $C_1^{4/3}(l, r, \beta)$ is a lower bound for

$$\frac{-2\beta(r-l)\sin(\pi x) + 2\sqrt{(r-l)^2\beta^2 \sin^2(\pi x) + 2F \cdot \left(3\pi^2 \sin^2(\pi x) + \pi^2\right)}}{3\pi^2 \sin^2(\pi x) + \pi^2} \tag{4.47}$$

over Ω.

(iii) Any solution z to the parabolic equation in (S_t) satisfies $z(t) \in C((0, T]\,; H^{\frac{9}{8}}(\Omega))$. Furthermore for $t > 0$

$$\sup_{[t,\infty]} \|z(t)\|_{H^{9/8}(\Omega)} < \infty.$$

CHAPTER 5

EXISTENCE OF CLOSED CHARACTERISTICS

In this chapter, we use the flow defined at the end of Chapter 4 to establish the existence of simple closed characteristics. Recall that the Lagrangian action is defined by

$$J[z,u_1,u_2] = \int_\Omega \frac{2}{9(u_2-u_1)} z_x^2 + (u_2-u_1)\left[\frac{\beta}{2} z^{2/3} + \frac{F((u_2-u_1)x+u_1)}{z^{2/3}}\right] dx. \quad (5.1)$$

Theorem 4.3.1 implies that the weak solutions of following system generate a dynamical system provided they are unique, and furthermore J is decreasing along orbits of this system. For notational convenience we write $l(t) = u_1(t)$ and $r(t) = u_2(t)$.

$$(S_t)\quad \begin{cases} z_t = \dfrac{4}{9} z_{xx} + (r-l)^2 \left(\dfrac{2}{3}\dfrac{F((r-l)x+l)}{z^{5/3}} - \dfrac{\beta}{3z^{1/3}}\right) \quad \text{in } Q_T, \\ z = 0 \ \text{ on } \Sigma_T, \quad z > 0 \text{ in } Q_T, \quad z(0,x) = z_0(x) \text{ in } \Omega, \\ l_t = -\displaystyle\int_\Omega \frac{2z_x^2}{9} dx - (r-l)^2 \int_\Omega \frac{(r-l)(1-x)F' - F}{z^{2/3}} dx + (r-l)^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx, \\ r_t = \displaystyle\int_\Omega \frac{2z_x^2}{9} dx - (r-l)^2 \int_\Omega \frac{(r-l)xF' + F}{z^{2/3}} dx - (r-l)^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx, \\ l(0) = l_0, \ r(0) = r_0. \end{cases}$$

Recall that in this system l and r take values in a compact interval component I on which $F \geq 0$, and F is strictly positive on the interior of I.

For clarity of presentation we will further assume that F is even, and the interval component $I = [u_{\min}, -u_{\min}]$. In this setting we can restrict to the subspace of

$H_0^1(\Omega)\times I^2$ given by $H_0^1(\Omega)\cap C_e(\Omega)\times\triangle$ where $C_e(\Omega)$ is the set of continuous functions on $\Omega=(0,1)$ which are even about $x=1/2$, and $\triangle$ is the anti-diagonal $\{(l,r)\in I^2\mid r=-l\}$. In this setting the system (S_t) reduces to

$$(S_t)\quad\begin{cases} z_t=\frac{4}{9}z_{xx}+4l^2\left(\frac{2}{3}\frac{F(-2lx+l)}{z^{5/3}}-\frac{\beta}{3z^{1/3}}\right) & \text{in } Q_T,\\ z=0\ \text{ on }\Sigma_T,\quad z>0 \text{ in } Q_T,\quad z(0,x)=z_0(x)\text{ in }\Omega, & \\ & \\ l_t=-\int_\Omega\frac{2z_x^2}{9}dx-4l^2\int_\Omega\frac{-2l(1-x)F'-F}{z^{2/3}}dx+(r-l)^2\int_\Omega\frac{\beta}{2}z^{2/3}dx, & \\ l(0)=l_0. & \end{cases}$$

That is, $(z(t),l(t),r(t))$ is a solution with $z(0)\in H_0^1(\Omega)\cap C_e(\Omega)$ and $r(0)=-l(0)$ if and only if $z(t)\in H_0^1(\Omega)\cap C_e(\Omega)$, $r(t)=-l(t)$ for all $t\in[0,T_{\max}]$, and $(z(t),l(t))$ is a solution of the reduced system.

The phase space X of the dynamics of (S_t) is contained in $H_0^1(\Omega)\times[u_{\min},0]$. To define X more precisely, fix $\delta,\sigma>0$ and consider the fiber bundle of cones

$$\mathscr{C}=\bigcup_{l\in[u_{\min}+\delta,-\sigma]}\mathsf{C}(l,\beta)\times\{l\}$$

where

$$\mathsf{C}(l,\beta)=\{z\mid C_1(l,-l,\beta)(-2l)^{3/4}\sin^{3/4}(\pi x)\le z\le C_2(l,-l,\beta)(-2l)^{3/4}\sin^{3/4}(\pi x)\}$$

By Theorem 4.1.6 the cone $\mathscr{C}$ is forward invariant in the z-dynamics; that is, the only way for a solution $(z(t),l(t))$ to leave $\mathscr{C}$ is for $l(t)$ to leave $[u_{\min}+\delta,-\sigma]$. The constants δ,σ will be chosen later.

We will use the notation

$$J^a=\{(z,l)\mid J(z,l)\le a\}\quad\text{and}\quad J_b^a=\{(z,l)\mid b\le J(z,l)\le a\}.$$

Let

$$a > \max_{l \in [u_{\min}, 0]} \max\{J(z,l) \mid \delta_z J(z,l) = 0\}$$

be a regular value larger than all z-critical values of J. Since (S_t) is a scaled gradient flow for J, the set J^a is forward invariant in $\mathscr{C}$, which means that if $(z(0), l(0)) \in \mathscr{C} \cap J^a$ and the solution $(z(t), l(t)) \in \mathscr{C}$ for all $t \in [0, \tau]$, then $(z(t), l(t)) \in \mathscr{C} \cap J^a$ for all $t \in [0, \tau]$.

5.1 A WAZEWSKI SET FOR THE GRADIENT FLOW

For $b < a$ define the set

$$N_b = \mathscr{C} \cap J_b^a.$$

The immediate exit set of N_b is defined by

$$N_b^- = \{(z_0, l_0) \in N_b \mid (z([0,\tau], l([0,\tau])) \not\subset N_b \text{ for all } \tau > 0\},$$

which is the set of points in N_b which immediately leave N_b under the flow. The above observations concerning the flow on $\mathscr{C} \cap J^a$ imply that

$$N_b^- \subset \{(z,l) \in N_b \mid J(z,l) = b\} \cup \{(z,l) \in N_b \mid l = u_{\min} + \delta \text{ or } l = -\sigma\} \subset \partial N_b. \quad (5.2)$$

Theorem 5.1.1. *There exists $\delta, \sigma > 0$ and $b < a$ such that*

$$\{J = b\} \subset N_b^- \subset \{l = u_{\min} + \delta\} \cup \{J = b\}$$

and $\{l = u_{\min} + \delta\} \cap \{J = b\} = \emptyset$.

Proof. We have already shown in equation (5.2) that $N_b^- \subset \{J = b\} \cup \{l = u_{\min} + \delta\} \cup \{l = -\sigma\}$. Moreover, since J is decreasing along solutions, $\{J = b\} \subset N_b^-$. We need to show that for some choice of constants δ, σ, and b, $N_b^- \setminus \{J = b\} \subset \{l = u_{\min} + \delta\}$.

The differential equation for l_t with $l = -\sigma$ can be expressed as

$$l_t = -\frac{2}{9}\|z\|_{H_0^1}^2 + 4\sigma^2 \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx - 8\sigma^3 \int_\Omega \frac{(1-x)F'(2\sigma x - \sigma)}{z^{2/3}}\, dx + 4\sigma^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx.$$

The latter two terms can be estimated for all $z \in \mathsf{C}(-\sigma, \beta)$ as

$$\begin{aligned}\left| -8\sigma^3 \int_\Omega \frac{(1-x)F'(2\sigma x - \sigma)}{z^{2/3}}\, dx + 4\sigma^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx \right| &\le \frac{C\sigma^{5/2}}{C_1^{2/3}(-\sigma,\beta)} \\ &\quad + |\beta| C\sigma^{5/2} C_2^{2/3}(-\sigma,\beta) \\ &\le C\sigma^{5/2}.\end{aligned}$$

Also

$$4\sigma^2 \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx \le \frac{4\sigma^{3/2}}{C_1^{2/3}(-\sigma,\beta)} \int_\Omega \frac{F(2\sigma x - \sigma)}{\sqrt{\sin(\pi x)}}\, dx \le C\sigma^{3/2}$$

for all $z \in \mathsf{C}(l, \beta)$. Therefore we can choose

$$\begin{aligned} b = 2\max&\left\{2\sigma \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx\right\} && (5.3)\\ + \max&\left|4\sigma^2 \int_\Omega \frac{(1-x)F'(2\sigma x - \sigma)}{z^{2/3}}\, dx - 2\sigma \int_\Omega \frac{\beta}{2} z^{2/3} dx\right| && (5.4)\end{aligned}$$

where the maximum is taken over $z \in \mathsf{C}(-\sigma, \beta)$. If $J(z, -\sigma) > b$, then

$$\begin{aligned} J(z,-\sigma) &= \frac{1}{9\sigma}\|z\|_{H_0^1}^2 + 2\sigma \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx \\ &> 2\max\left\{2\sigma \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx\right\} \\ &\quad + \max\left|4\sigma^2 \int_\Omega \frac{(1-x)F'(2\sigma x - \sigma)}{z^{2/3}}\, dx - 2\sigma \int_\Omega \frac{\beta}{2} z^{2/3} dx\right| \end{aligned}$$

so that

$$\begin{aligned} \frac{2}{9}\|z\|_{H_0^1}^2 &> \max\left\{4\sigma^2 \int_\Omega \frac{F(2\sigma x - \sigma)}{z^{2/3}}\, dx\right\} \\ &\quad + \max\left|-8\sigma^3 \int_\Omega \frac{(1-x)F'(2\sigma x - \sigma)}{z^{2/3}}\, dx + 4\sigma^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx\right| \end{aligned}$$

which implies that $l_t < 0$.

Summarizing, we have shown that b can be chosen to satisfy two properties, (1) if $J(z,-\sigma) > b$, then $l_t(z,-\sigma) < 0$, and (2) $b \to 0$ as $\sigma \to 0$. The significance of property (1) is that for $(z_0, l_0) \in N_b \cap \{l = -\sigma\}$ we have $l_t < 0$ so the solution starting at (z_0, l_0) enters N_b if $J(z_0, l_0) \neq b$, and hence $\{l = -\sigma\} \cap N_b^- \subset \{J = b\}$. This proves the first assertion of the theorem that $N_b^- \subset \{l = u_{\min} + \delta\} \cup \{J = b\}$.

We will now use property (2) to show that δ, σ, and b can be chosen so that $\{J = b\} \cap \{l = u_{\min} + \delta\} = \emptyset$. Again estimating for $z \in \mathsf{C}(l, \beta)$ where $l = u_{\min} + \delta$ we have

$$\begin{aligned}
J(l, z) &\geq \frac{1}{9|u_{\min} + \delta|} \|z\|_{H_0^1}^2 \geq \frac{1}{9|u_{\min} + \delta|} \|\underline{z}\|_{L^2}^2 \\
&\geq \frac{1}{9|u_{\min} + \delta|} 2^{3/2} \, |u_{\min} + \delta|^{3/2} \, C_1^2 \int_\Omega \sin^{3/2}(\pi x) \, dx \\
&\geq \frac{2^{3/2}}{9} |u_{\min} + \delta|^{1/2} C_1^2 \int_\Omega \sin^{3/2}(\pi x) \, dx \qquad (5.5)
\end{aligned}$$

where $C_1 = C_1(u_{\min} + \delta, \beta)$ is the constant from the cone condition. Therefore, given $\delta > 0$ we can choose $\sigma > 0$ sufficiently small so that

$$b < \min\{J(z, u_{\min} + \delta) \mid z \in \mathsf{C}(u_{\min} + \delta, \beta)\},$$

which implies that $\{l = u_{\min} + \delta\} \cap \{J = b\} = \emptyset$, which is the second assertion of the theorem. ■

Theorem 5.1.2. *If N_b^- satisfies the conditions of Theorem 5.1.1, and*

$$N_b^- = \{l = u_{\min} + \delta\} \cup \{J = b\}$$

and $\{l = u_{\min} + \delta\} \cap \{J = b\} = \emptyset$, then there exists a stationary point for the flow (S_t) contained in N_b.

Proof. First, note that N_b is a closed and connected subset of $H_0^1(\Omega) \times [u_{\min}, 0]$. The hypotheses imply that the immediate exit set N_b^- is the union of two disjoint, nonempty, closed sets so that N_b^- has at least two connected components. Therefore, N_b^- is not a deformation retract of N_b. Since N_b^- is closed in N_b, it is also closed in the eventual exit set, which implies that N_b is a Wazewski set as in Definition A.1.2. The Wazewski principle stated in Corollary A.1.5 can be applied to conclude that there exists an initial condition $(z_0, l_0) \in N_b$ whose forward orbit $(z(t), l(t))$ stays in N_b for all $t \geq 0$. By the regularity of solutions established in Theorem 4.2.1, we have $\{(z(t), l(t))\}_{t>0}$ is contained in a bounded subset of $H^{9/8}(\Omega) \times [u_{\min}, 0]$ and hence is precompact in $H_0^1(\Omega) \times I^2$. This implies that the omega-limit set $\omega(z_0, l_0)$ is nonempty, and $\omega(z_0, l_0)$ is contained in N_b, since N_b is closed. Since the flow is a scaled gradient flow of J, the ω-limit set of any orbit must contain a stationary point, and hence N_b contains a stationary point. ■

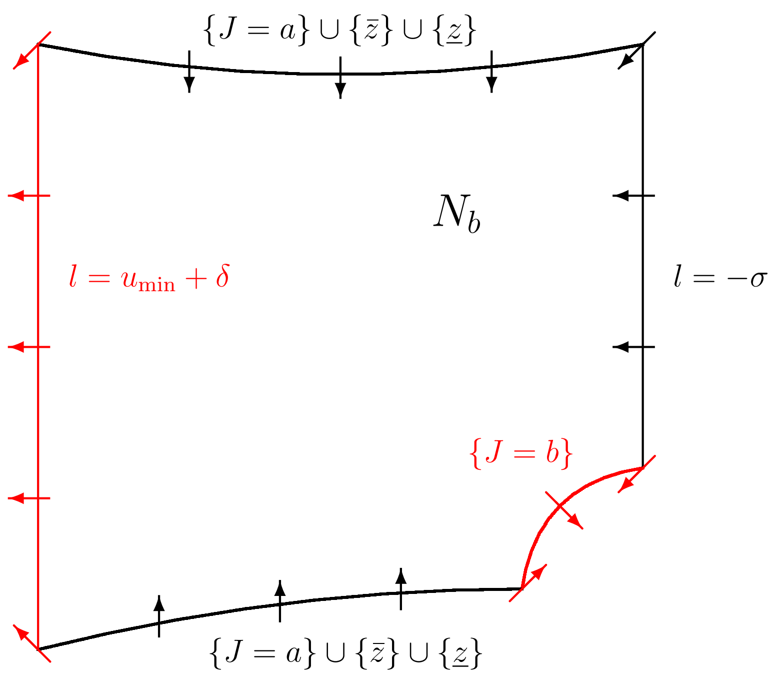

Figure 5.1: Schematic diagram of the Wazewski set N_b.

By Theorem 5.1.2 in order to establish the existence of a critical point of J in N_b we need to show that there exists $\delta > 0$ such that $l_t(z_0, u_{\min} + \delta) < 0$ for all

$z_0 \in \mathsf{C}(u_{\min}+\delta, \beta)$. Recall that the differential equation for l_t with $l = u_{\min}+\delta$ can be expressed as

$$\begin{aligned} l_t = &-\frac{2}{9}\|z\|^2_{H^1_0} + 4(u_{\min}+\delta)^2 \int_\Omega \frac{F(-2(u_{\min}+\delta)x + u_{\min}+\delta)}{z^{2/3}}\,dx \\ &+ 4(u_{\min}+\delta)^2 \int_\Omega \frac{2(u_{\min}+\delta)(1-x)F'(-2(u_{\min}+\delta)x + u_{\min}+\delta)}{z^{2/3}}\,dx \\ &+ 4(u_{\min}+\delta)^2 \int_\Omega \frac{\beta}{2} z^{2/3} dx. \end{aligned}$$

Let

$$g(x) = F(-2(u_{\min}+\delta)x + u_{\min}+\delta) + 2(u_{\min}+\delta)(1-x)F'(-2(u_{\min}+\delta)x + u_{\min}+\delta)$$

and define $g^+ = (g+|g|)/2$ and $g^- = (g-|g|)/2$. Using the inequality (5.5) and cone estimates, where $C_1 = C_1(u_{\min}+\delta,\beta)$ and $C_2 = C_1(u_{\min}+\delta,\beta)$, we obtain

$$\begin{aligned} l_t < &-\frac{\pi^2\sqrt{2}}{8}|u_{\min}+\delta|^{3/2} C_1^2 \int_\Omega \frac{\cos^2(\pi x)}{\sqrt{\sin(\pi x)}}\,dx \\ &+ \frac{4|u_{\min}+\delta|^{3/2}}{C_1^{2/3}\sqrt{2}} \int_\Omega \frac{g^+(x)}{\sqrt{\sin(\pi x)}}\,dx + \frac{4|u_{\min}+\delta|^{3/2}}{C_2^{2/3}\sqrt{2}} \int_\Omega \frac{g^-(x)}{\sqrt{\sin(\pi x)}}\,dx \\ &+ 4|u_{\min}+\delta|^{5/2} \frac{\beta}{\sqrt{2}} C_*^{2/3} \int_\Omega \sqrt{\sin(\pi x)}\,dx \end{aligned}$$

where $C_* = C_1$ when $\beta < 0$ and $C_* = C_2$ when $\beta > 0$, which established the following result.

Proposition 5.1.3. *If*

$$\begin{aligned} &-\frac{\pi^2\sqrt{2}}{8}|u_{\min}+\delta|^{3/2} C_1^2 \int_\Omega \frac{\cos^2(\pi x)}{\sqrt{\sin(\pi x)}}\,dx \\ &+ \frac{4|u_{\min}+\delta|^{3/2}}{C_1^{2/3}\sqrt{2}} \int_\Omega \frac{g^+(x)}{\sqrt{\sin(\pi x)}}\,dx + \frac{4|u_{\min}+\delta|^{3/2}}{C_2^{2/3}\sqrt{2}} \int_\Omega \frac{g^-(x)}{\sqrt{\sin(\pi x)}}\,dx \\ &+ 4|u_{\min}+\delta|^{5/2} \frac{\beta}{\sqrt{2}} C_*^{2/3} \int_\Omega \sqrt{\sin(\pi x)}\,dx < 0, \end{aligned} \tag{5.6}$$

where $C_* = C_1$ *when* $\beta < 0$ *and* $C_* = C_2$ *when* $\beta > 0$*, then* $\{l = u_{\min}+\delta\} \subset N_b^-$.

Finally, we indicated how to utilize Proposition 5.1.3 and Theorem 5.1.2 to establish the existence of a closed characteristic for the second-order Lagrangian system given by J. We exploit another symmetry in the problem, which is $(x, z, l, r) \to (1-x, -z, r, l)$. Indeed

$$J(z, l, r) = \int_0^1 \frac{2}{9} z_x^2(x) + \frac{(r-l)^2 F((r-l)x + l)}{z^{2/3}(x)}\, dx$$

so that

$$\begin{aligned}
J(-z(1-x), r, l) &= \int_0^1 \frac{2}{9} z_x^2(1-x) + \frac{(l-r)^2 F((l-r)x + r)}{z^{2/3}(1-x)}\, dx \\
&= \int_1^0 \frac{2}{9} z_y^2(y) + \frac{(l-r)^2 F((l-r)(1-y) + r)}{z^{2/3}(y)}\, (-dy) \\
&= \int_0^1 \frac{2}{9} z_y^2(y) + \frac{(r-l)^2 F((r-l)y + l)}{z^{2/3}(y)}\, dy \\
&= J(z, l, r).
\end{aligned}$$

This implies that if $\gamma_1(x) = (z(x), l)$ is a critical point of J, i.e. $\delta J = 0$, then $\gamma_2(x) = (-z(1-x), -l)$ is also a critical point. Therefore, if γ_1 is the stationary solution established in Theorem 5.1.2, then the concatenation of γ_1 and γ_2 is a simple closed curve which is critical for J, and hence it is a simple closed characteristic of J.

Since the estimates above are independent of ϵ, for all $\epsilon > 0$ we can apply the above results to obtain a stationary solution $\gamma_\epsilon = (z_\epsilon(x), l_\epsilon)$ of $(S_{t,\epsilon})$. From the weak comparison principle, we get that if $0 \le \tilde{\epsilon} \le \epsilon$ then $z_\epsilon \le z_{\tilde{\epsilon}}$ and $z_{\tilde{\epsilon}} + \tilde{\epsilon} \le z_\epsilon + \epsilon$. To see this note that for $\xi_\epsilon = z_\epsilon + \epsilon$ and $\xi_{\tilde{\epsilon}} = z_{\tilde{\epsilon}} + \tilde{\epsilon}$ we have

$$\begin{cases} \frac{4}{9}\partial_{xx}^2(\xi_{\tilde{\epsilon}} - \xi_\epsilon) + (r-l)^2\frac{2}{3}F((r-l)x+l)\left(\frac{1}{\xi_{\tilde{\epsilon}}^{5/3}} - \frac{1}{\xi_\epsilon^{5/3}}\right) = 0 \\ \qquad -\frac{\beta}{3}\left(\frac{1}{\xi_{\tilde{\epsilon}}^{1/3}} - \frac{1}{\xi_\epsilon^{1/3}}\right) \quad \text{in } \Omega, \\ z_{\tilde{\epsilon}} - z_\epsilon \leq 0 \text{ on } \partial\Omega. \end{cases}$$

From this it follows that $(z_\epsilon)_{\epsilon>0}$ is a cauchy sequence in $L^\infty(\Omega)$ and there exists $z \in L^\infty(\Omega)$ satisfying $\underline{z} = \lim_{\epsilon\to 0^+} \underline{z}_\epsilon \leq z \leq \bar{z}$. From the regularity z_ϵ is bounded in $H^{9/8}(\Omega)$ independent of ϵ, this implies that $z_\epsilon \to z$ in $H_0^1(\Omega)$ as $\epsilon \to 0^+$. Furthermore since l_ϵ is bounded in I we can find a subsequence $l_\epsilon \to l$ in I. Therefore the stationary solutions (z_ϵ, l_ϵ) converge to a stationary solution of (S_t). One then can argue as above to obtain a simple closed characteristic of J.

5.2 EXAMPLE: THE EXTENDED FISHER-KOLMOGOROV / SWIFT-HOHENBERG SYSTEM

As an illustration we compute an isolating neighborhood for the eFK/SwiftHohenberg system where

$(i) K(u,z) = \frac{\beta}{2}z^{4/3} + F(u),$

$(ii) F(u) = \frac{1}{4}\left(u^2-1\right)^2 - \frac{1}{8}.$

We compute an interval component to be $I = \left[-\sqrt{1-\frac{1}{\sqrt{2}}}, \sqrt{1-\frac{1}{\sqrt{2}}}\right]$, where $\sqrt{1-\frac{1}{\sqrt{2}}} \approx 0.5412$. Below we plot the graph of l versus β where the blue curve represents the values of u for which $l_t < 0$ and the magenta curve represents where the monotonicity bound is satisfied. For example taking $u_1 = -.51$ and $\beta < 0.3$ we can conclude that figure 5.1 holds, therefore existence of a simple closed characteristic is guaranteed.

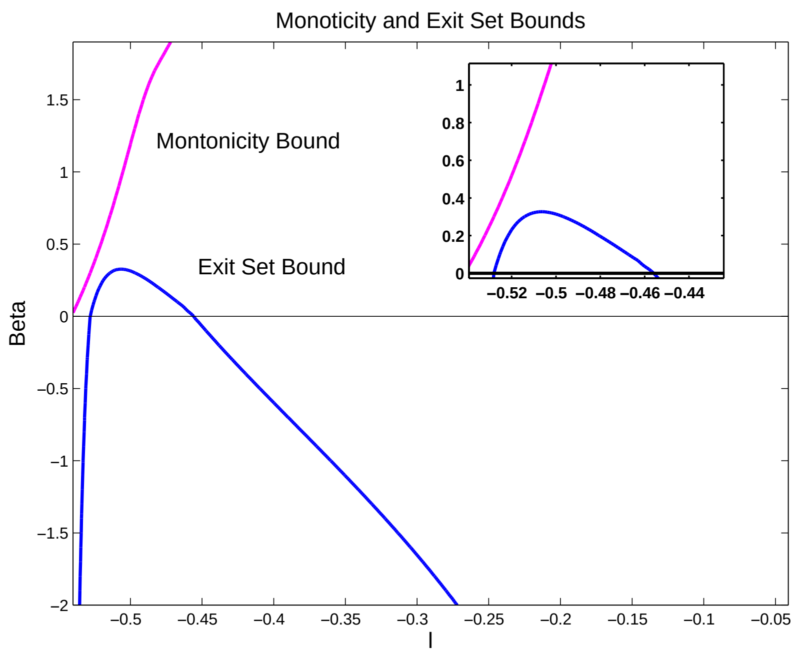

Figure 5.2: Bounds

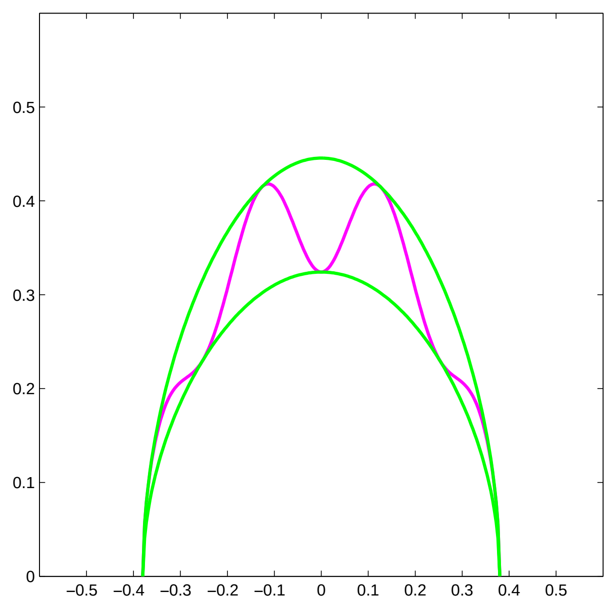

Figure 5.3: *Cone in the* (u, v)*-plane*

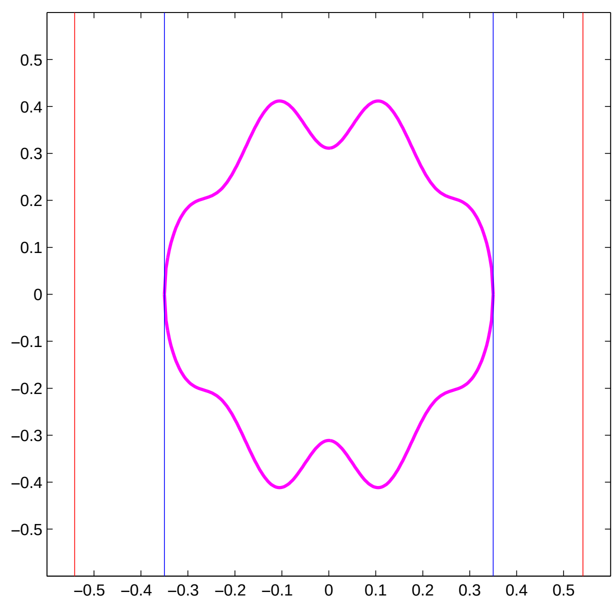

Figure 5.4: *Embedded curve under the flow in the (u, v)-plane*

CHAPTER 6

FUTURE WORK

6.1 EXISTENCE FOR THE NON-MONOTONE CASE

Let us now consider a more general K, to reduce the amount of technical detail we restrict ourselves to the case where K satisfies

$$(H)\ K(x,u_1,u_2,z^{2/3}) \geq -C(x,u_1,u_2) - C(x,u_1,u_2)\,|z|^{\gamma}\,,\quad \gamma < \frac{8}{3},$$

and $C(x,u_1,u_2)$ is locally bounded. In particular we take

$$K(x,u_1,u_2,z^{2/3}) = \frac{\beta}{2}z^{4/3} + F((u_2-u_1)x+u_1), \tag{6.1}$$

where F is a smooth potential function, recall we write $v = u_1$, $w = u_2$. The system then becomes

$$(S_t)\ \begin{cases} z_t = \frac{4}{9}z_{xx} + (w(t)-v(t))^2\left(\frac{2}{3}\frac{F((w-v)x+u_1)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}\right), \\ v_t = -\displaystyle\int_\Omega \frac{2(z_x)^2}{9}dx - (w(t)-v(t))^2\int_\Omega \frac{((w-v)(1-x)K'-K)}{z^{2/3}}dx, \\ w_t = \displaystyle\int_\Omega \frac{2(z_x)^2}{9}dx - (w(t)-v(t))^2\int_\Omega \frac{((w-v)xK'+K)}{z^{2/3}}dx, \end{cases}$$

By regularizing the pde we can apply semigroup methods [6], the regularized system is

$$
(S_{t,\epsilon})\begin{cases} z_t = \frac{4}{9}z_{xx} + (w(t)-v(t))^2\left(\frac{2}{3}\frac{F((w(t)-v(t))x+u_1)}{(z+\epsilon)^{5/3}} - \frac{\beta}{3(z+\epsilon)^{1/3}}\right), \\ v_t = -\int_\Omega \frac{2(z_x)^2}{9}dx - (w(t)-v(t))^2\int_\Omega \frac{((w(t)-v(t))(1-x)K'-K)}{(z+\epsilon)^{2/3}}dx, \\ w_t = \int_\Omega \frac{2(z_x)^2}{9}dx - (w(t)-v(t))^2\int_\Omega \frac{((w(t)-v(t))xK'+K)}{(z+\epsilon)^{2/3}}dx. \end{cases}
$$

Let e^{-At} denote the semigroup generated by $A = -\frac{4}{9}\frac{d^2}{dx^2}$ in $X = L^2(\Omega)$ with $\mathcal{D}(A) = H_0^1(\Omega)\cap H^2(\Omega)$. Note that a solution $(z_\epsilon(t,x), v_\epsilon(t), w_\epsilon(t))$ to $(S_{t,\epsilon})$ satisfies

$$
(*)\begin{cases} z_\epsilon(t) = e^{-At}z_0 + \int_0^t e^{-A(t-s)}(w(s)-v(s))^2\left(\frac{2}{3}\frac{F((w(s)-v(s))x+v(s))}{(z+\epsilon)^{5/3}} - \frac{\beta}{3(z+\epsilon)^{1/3}}\right)ds \\ v_\epsilon(t) = v_0 - \int_0^t\int_\Omega \frac{2(\partial_x z_\epsilon)^2}{9}dxds \\ \qquad - \int_0^t\int_\Omega (w(s)-v(s))^2\frac{((w(s)-v(s))(1-x)K'-K)}{z_\epsilon^{2/3}}dxds \\ w_\epsilon(t) = u_0 + \int_0^t\int_\Omega \frac{2(\partial_x z_\epsilon)^2}{9}dxds \\ \qquad - \int_0^t\int_\Omega (w(s)-v(s))^2\frac{((w(s)-v(s))xK'+K)}{z_\epsilon^{2/3}}dxds. \end{cases}
$$

Since $\mathcal{D}(A) = X^1 \subset H^2(\Omega)$, for $\alpha > \frac{1}{2}$ there is a continuous inclusion of the fractional space $X^\alpha = \mathcal{D}(A^\alpha) \subset H^1(\Omega)$. The function $f : U \to X$ given by

$$
f(t,z)(x) = (w(t)-v(t))^2\left(\frac{2}{3}\frac{F((w(t)-v(t))x+v(t))}{(z+\epsilon)^{5/3}} - \frac{\beta}{3\,(z+\epsilon)^{1/3}}\right)
$$

is locally Lipschitz in z and locally Holder in t in an open subset $U \subset [0,T]\times X^\alpha$. The lipschitz constant is apparent from the boundedness of the derivative for $\frac{1}{(z+\epsilon)^{5/3}}$ and $\frac{1}{(z+\epsilon)^{1/3}}$, though the bounds blow up in $O(\epsilon)$. Choose $\delta > 0$, $\tau > 0$ and without loss assume

$$
U = \left\{(t,z) | t_0 \le t \le t_0+\tau, \|z - z_0\|_\alpha \le \delta\right\},
$$

and

$$\|f(t,z_1)-f(t,z_2)\| \le L\|z_1-z_2\|_\alpha$$

for (t,z_1), $(t,z_2)\in U$. Set $B=\max_{[0,\tau]}\|f(t,z_0)\|$ where $z_0\,|_{\partial\Omega}=0$ and $z_0\,|_{\Omega}>0$. Moreover take $0<T<\tau$ so that

$$\left\|(e^{-Ah}-\mathrm{I})z_0\right\|_\alpha \le \delta/2 \;\text{ for } 0\le h\le T$$
$$M(B+L\delta)\int_0^T r^{-\alpha}e^{ar}dr \le \delta/2$$

where $\left\|A^\alpha e^{-At}\right\| \le Mt^{-\alpha}e^{at}$ for $t>0$ and $\mathrm{Re}\,\sigma(A)>a$. Let $\mathcal{S}$ denote the set of continuous functions $y:[0,T]\to X^\alpha$ such that $\|y(t)-z_0\|\le\delta$ on $[0,T]$ equipped with the "sup"-norm

$$\|y\|^T=\sup\left\{\|y\|_\alpha : t\in[0,T]\right\},$$

then S is a complete metric space. For $y\in\mathcal{S}$ define $G(y):[0,T]\to X$ by

$$G(y)(t)=e^{-At}z_0+\int_0^t e^{-A(t-s)}(\varphi(s,v_0)-\psi(s,w_0))^2\times \left(\frac{2}{3}\frac{F((\psi(s,w_0)-\varphi(s,v_0))x+\varphi(s,v_0))}{(y+\epsilon)^{5/3}}-\frac{\beta}{3\,(y+\epsilon)^{1/3}}\right)ds.$$

Fix two small neighborhoods $N_1(\rho)$, $N_2(\rho)$ about v_0, $w_0\in[u_-,u_+]$ respectively and let $\mathcal{B}_i=C^{0,1}([0,\tau]\times N_i(\frac{\rho}{2});\overline{N}_i(\rho))$, that is the complete metric space of continuous $\mathbb{R}$ valued functions on $[0,\tau]\times N_i(\frac{\rho}{2})$ with range in $\overline{N}_i(\rho)$ and bounded Lipschitz norm $\|\varphi\|_{\mathcal{B}_i}=\sup\frac{|\varphi(t,x)-\varphi(s,y)|}{|(t,x)-(s,y)|}$. Finally define the operator on $\mathcal{B}_i$

$$\Lambda(\varphi)(t,v)=v-\int_0^t\int_\Omega\frac{2(\partial_x y(s))^2}{9}dxds - \int_0^t\int_\Omega(\psi(s,w)-\varphi(s,v))^2\frac{((w(s)-\varphi(s,v))(1-x)K'-K)}{(y+\epsilon)^{2/3}}dxds.$$

$$\Psi(\psi)(t,w) = w + \int_0^t \int_\Omega \frac{2(\partial_x y(s))^2}{9} dxds$$
$$- \int_0^t \int_\Omega (\psi(s,w) - \varphi(s,v))^2 \frac{((w(s) - \varphi(s,v))xK' + K)}{(y+\epsilon)^{2/3}} dxds.$$

We define $\Phi : \mathcal{S} \times \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{S} \times \mathcal{B}_1 \times \mathcal{B}_2$ by

$$\Phi(y,\varphi,\psi) = (G(y), \Lambda(\varphi), \Psi(\psi)).$$

We need to investigate the above operators further, in particular G maps S to itself and is a strict contraction. First note

$$\|G(y)(t) - z_0\|_\alpha \leq \left\|(e^{-At} - \mathrm{I})z_0\right\|_\alpha + \int_0^t \left\|A^\alpha e^{-A(t-s)}\right\| (B + L\delta) ds$$
$$\leq \delta/2 + M(B + L\delta) \int_0^T (t-s)^{-\alpha} e^{a(t-s)} ds \leq \delta, \text{ for } 0 \leq t \leq T.$$

Since $G(y)$ is continuous from $[0,T]$ to X^α, G maps S into itself. For $y, z \in S$ and $0 < t < T$,

$$\|G(y)(t) - G(z)(t)\|_\alpha \leq \int_0^t \left\|A^\alpha e^{-A(t-s)}\right\| \|f(s,y(s)) - f(s,z(s))\| ds$$
$$\leq ML \|y - z\|^T \int_0^t (t-s)^{-\alpha} e^{a(t-s)} ds.$$

Therefore $\|G(y) - G(z)\|^T \leq \frac{1}{2} \|y - z\|^T$ for all $y, z \in S$. We now show that Λ maps $\mathcal{B}_1$ into itself. By continuity on $[0,\tau] \times N_1$, there is some number $M > 0$ such that

$$\left| \int_\Omega \frac{2(\partial_x y(t))^2}{9} dx + \int_\Omega (\psi(t,w) - \varphi(t,v))^2 \frac{((w(t) - \varphi(t,v))(1-x)K' - K)}{(y+\epsilon)^{2/3}} dx \right| \leq M,$$

for $(t,x) \in [0,\tau] \times N_1(\rho)$. In view of this we have the following inequality

$$|\Lambda(\varphi)(t,v) - v_0| \leq |v - v_0|$$
$$+ \int_0^t \left| \int_\Omega \frac{2(\partial_x y(t))^2}{9} dx + \int_\Omega (\psi(t,w) - \varphi(t,v))^2 \frac{((w(t) - \varphi(t,v))(1-x)K' - K)}{(y+\epsilon)^{2/3}} dx \right| ds$$
$$\leq \frac{\rho}{2} + \tau M < \frac{\rho}{2} + \frac{\rho}{2} = \rho,$$

for τ choosen such that $\tau M < \frac{\rho}{2}$ and $(t, v) \in [0, \tau] \times N_1(\frac{\rho}{2})$. The range of the operator Λ is in $\overline{N}_1(\rho/2)$. Next take φ_1, $\varphi_2 \in \mathcal{B}_1$, then

$$|\Lambda(\varphi_1)(t, v) - \Lambda(\varphi_2)(t, v)| \leq \tau\lambda \left\| \varphi_1 - \varphi_2 \right\|_{\mathcal{B}_1},$$

consequently

$$\left\| \Lambda(\varphi_1) - \Lambda(\varphi_2) \right\|_{\mathcal{B}_1} \leq \tau\lambda \left\| \varphi_1 - \varphi_2 \right\|.$$

Through a similar argument we see that Φ is a strict contraction which maps $\mathcal{B}_2$ into itself. Hence Φ maps $\mathcal{S} \times \mathcal{B}_1 \times \mathcal{B}_2$ to itself, moreover Φ is a strict contraction on the product space which is a complete metric space with norm $\|(y, \varphi, \psi)\|_{\mathcal{S}\times\mathcal{B}_1\times\mathcal{B}_2} = \max\left\{ \|y\|^T, \|\varphi\|_{\mathcal{B}_1}, \|\psi\|_{\mathcal{B}_2} \right\}$. Hence by the contraction mapping theorem Φ has a unique fixed point $(z_\epsilon, \varphi_\epsilon, \psi_\epsilon)$ in $\mathcal{S} \times \mathcal{B}_1 \times \mathcal{B}_2$ which is a continuous solution of $(*)$ and satisfies $(S_{t,\epsilon})$.

If we consider limiting in ϵ need to be careful with the interval of existence. The interval of existence for z_ϵ depends on ϵ, to get a handle on the ϵ dependency we use properties of the functional J^ϵ, see Appendix A.2. Since for each t, $\{z_\epsilon\}_{\epsilon>0}$ is a bounded sequence independent of ϵ and t in $H_0^1(\Omega)$ we get that there exists $z \in H_0^1(\Omega)$ such that $z_\epsilon \rightharpoonup z$ in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, for each $t \in [0, T]$ $z_\epsilon(t, \cdot) \to z(\cdot)$ in $L^2(\Omega)$. Hence we have convergence in $L^\infty(0, T; L^2(\Omega))$. From this we see that both $\{\varphi_\epsilon\}_\epsilon$, $\{\psi_\epsilon\}_\epsilon$ are equicontinuous and therefore converge in ϵ to a solution which solves their respective integral equations in $\mathcal{B}_i$, moreover the solution is differentiable due to the continuity of the right hand side of the v_t and w_t equations.

6.2 OTHER TYPES OF CLOSED CURVES

The next step in proving existence of closed characteristics for the aforementioned second-order Lagrangian systems is to consider simple closed curves which do not have symmetry in the upper and lower half planes, also the symmetry at the endpoints (u-axis) is not imposed. Another step is to consider initial in the space of immersed curves with orientation, that is allowing self-intersections.

APPENDIX A

A.1 WĄZEWSKI PRINCIPLE

Definition A.1.1. Let $W \subset X$. The *immediate exit set* W^- of W is defined bo those points whose forward trajectory immediately leaves W. More precisely,

$$W^- := \{x \in W \mid \text{for all } t > 0, \varphi((0,t),x) \not\subset W\}$$

The *eventual exit set* W^0 of W consists of all points which leave in positive time; that is

$$W^0 := \{x \in W \mid \exists\, t > 0 \text{ such that } \varphi(t,x) \notin W\}.$$

Observe that $W^- \subset W^0 \subset W$.

Definition A.1.2. W is a *Wązewski set* if the following conditions are satisfied.

(i) If $x \in W$ and $\varphi([0,t],x) \subset \mathrm{cl}(W)$, then $\varphi([0,t],x) \subset W$.

(ii) W^- is closed relative to W^0.

Definition A.1.3. Let $A \subset X$. A *deformation retraction* of X onto A is a continuous map $h : X \times [0,1] \to X$ such that

$$\begin{aligned} h(x,0) &= x \quad \text{for all} \quad x \in X, \\ h(x,1) &\in A \quad \text{for all} \quad x \in X, \\ h(a,1) &= a \quad \text{for all} \quad a \in A. \end{aligned}$$

If such an h exists, then A is called a *deformation retract* of X. The map h is called a *strong deformation retraction* and the set A a *strong deformation retract* if the third identity is reinforced as follows:

$$h(a, s) = a \text{ for all } a \in A \text{ and all } s \in [0, 1].$$

Note that the map $r : X \to A$ defined by $r(x) = h(x, 1)$ has the property $r_{|A} = \mathrm{id}_A$. Any continuous map with this property is called a *retraction* and its image A is called a *retract*. Thus a deformation retract is a special case of a retract.

Theorem A.1.4. *If W is a Wazewski set, then W^- is a strong deformation retract of W^0 and W^0 is open relative to W.*

Proof. The first step is to construct the strong deformation retraction

$$r : W^0 \times [0, 1] \to W^0.$$

Define $\tau : W^0 \to \mathbb{R}$ by

$$\tau(x) = \sup \{ t \geq 0 \mid \varphi([0, t], x) \subset W \}.$$

By the definition of W^0, $\tau(x)$ is finite and by the continuity of the flow $\varphi([0, t], x) \subset \mathrm{cl}(W)$. Since W is a Wazewski set, $\varphi(\tau, x) \in W$, and in fact the definition of τ implies that $\varphi(\tau, x) \in W^-$. Observe that $\tau(x) = 0$ if and only if $x \in W^-$.

Assume for the moment that τ is continuous and define r by $r(x, \sigma) = \varphi(\sigma\tau(x), x)$. Now notice that

$$\begin{aligned} r(x, 0) &= \varphi(0, x) = x \\ r(x, 1) &= \varphi(\tau(x), x) \in W^- \end{aligned}$$

and for $y \in W^-$

$$r(y, \sigma) = \varphi(\sigma\tau(x), y) = \varphi(0, y) = y.$$

Therefore r is a strong deformation retraction of W^0 to W^-.

Returning now to the question of continuity we first prove that τ is upper semi-continuous. Let $x \in W^0$ and $\epsilon > 0$, then $\varphi([\tau(x), \tau(x)+\epsilon], x) \not\subset W$. By the first condition there exists $t_0 \in [\tau(x), \tau(x)+\epsilon]$ such that $\varphi(t_0,)x) \notin \text{cl}(W)$. Thus we can choose V be a neighborhood of $x \cdot t_0$ such that $V \cap \text{cl}(W) = \emptyset$. Now let U be a neighborhood of x such that $\varphi(t, U) \subset V$. Then for $y \in U \cap W$, $\varphi(t_0, y) \notin W$. Note that this proves that W^0 is open relative to W and that $\tau(y) < \tau(x) + \epsilon$. Hence, τ is upper semi-continuous.

To prove lower semi-continuity of τ, let $x \in W^0/W^-$ and let $0 < \epsilon < \tau(x)$. Then $\varphi([0, \tau(x)-\epsilon], x) \subset W^0$. Since W^- is closed relative to W^0, $\varphi([0, \tau(x)-\epsilon], x) \cap W^- = \emptyset$ and hence for all $s \in [0, \tau(x)-\epsilon]$ there exists a neighborhood U_s of $\varphi(s, x)$ such that $U_s \cap W^- = \emptyset$. Of course $\{U_s\}$ covers $\varphi([0, \tau(x)-\epsilon], x)$ which is compact and hence a finite number $\{U_{s_i} \mid i = 1, \ldots I\}$ covers $\varphi([0, \tau(x)-\epsilon], x)$. Let $U = \cup_{i=1}^{I} U_{s_i}$, then U is open which implies there exists V a neighborhood of x such that $\varphi([0, \tau(x)-\epsilon], V) \subset U$. Now $U \cap W^- = \emptyset$ implies that for all $y \in V$, $\varphi([0, \tau(x)-\epsilon], y) \cap W^- = \emptyset$. Thus, $\tau(y) \geq \tau(x) - \epsilon$. This implies that τ is lower semicontinuous and hence continuous. ■

The following Corollary is often referred to as the *Wázewski Principle*

Corollary A.1.5. *If W is a Wazewski set and W^- is not a strong deformation retract of W, then $W \setminus W^0 \neq \emptyset$; that is, there exists solutions which stay in W for all positive time. In particular, if W is compact then*

$$\text{Inv}(W, \varphi) \neq \emptyset.$$

A.2 PROPERTIES OF J^{ϵ}.

Here we adapt the results in [12] section 3.1 to the functional

$$J^{\epsilon}[z, u_1, u_2] = \int_{\Omega} \frac{2}{9(u_2 - u_1)} z_x^2 + \frac{(u_2 - u_1)K((u_2 - u_1)x + u_1, z^{2/3})}{(z + \epsilon)^{2/3}} dx. \qquad \text{(A.1)}$$

Lemma A.2.1. Under hypothesis (H), for all $\epsilon > 0$ there exists $C_\delta \geq 0$ such that $K(x, u_1, u_2, z^{2/3}) + \delta^{-1} z^{8/3} \geq -C_\delta z^{2/3}$ for $u_1, u_2 \in [u_-, u_+]$, $x \in \Omega$ and $z \in H_0^1(\Omega)$.

Proof. From (H) we have that $K + E + \delta^{-1} z^{8/3} \geq -C + E - C\,|z|^{\lambda} + \delta^{-1} z^{8/3} \geq -C_\delta^*$. Noting that $K(x, u_1, u_2, 0) + E \geq 0$ and $K_{z^{2/3}}(x, u_1, u_2, 0)$ is bounded, there exist $C_\delta > 0$ such that $K + E + \delta^{-1} z^{8/3} \geq -C_\delta \left|z^{2/3}\right|$.

Lemma A.2.2.

$$\int_0^1 |z'|^2\, dx \geq \int_0^1 |z|^2\, dx.$$

Proof. This is just an application of Poincare's Lemma. For completeness we will sketch the proof. From the absolute continuity of z we have

$$|z(x)| = |z(x) - z(0)| = \left| \int_0^x z' d\mu \right|$$

and then from Holder's

$$\int_0^1 |z(x)|^2\, dx \leq \int_0^1 x \left(\int_0^x |z'|^2\, d\mu \right) dx \leq \int_0^1 x \left(\int_0^1 |z'|^2\, d\mu \right) dx.$$

Applying Fubini's theorem

$$\int_0^1 |z(x)|^2\, dx \leq \left(\int_0^1 x dx \right) \left(\int_0^1 |z'|^2\, d\mu \right).$$

Lemma A.2.3. There exists a constant $C(|u_2 - u_1|) > 0$ such that $J^{\epsilon}[z] \geq -C$ for all $z \in H_0^1(\Omega)$.

Proof.

$$\begin{aligned}
J^\epsilon[z,u_1,u_2] &= \int_\Omega \frac{2}{9(u_2-u_1)} z_x^2 + \frac{(u_2-u_1)K((u_2-u_1)x+u_1, z^{2/3})}{(z+\epsilon)^{2/3}} dx \\
&\geq \int_\Omega \frac{2}{9(u_2-u_1)} z^2 + \frac{(u_2-u_1)K((u_2-u_1)x+u_1, z^{2/3})}{(z+\epsilon)^{2/3}} dx \\
&= \int_\Omega \frac{\frac{2}{9(u_2-u_1)} z^2 (z+\epsilon)^{2/3} + (u_2-u_1)K((u_2-u_1)x+u_1, z^{2/3})}{(z+\epsilon)^{2/3}} \\
&\geq (u_2-u_1) \int_\Omega \frac{\frac{2}{9(u_2-u_1)^2} z^{8/3} + K((u_2-u_1)x+u_1, z^{2/3})}{(z+\epsilon)^{2/3}}.
\end{aligned}$$

By (A.2.1)

$$J^\epsilon[z,u_1,u_2] \geq |u_2-u_1| \int_\Omega \frac{-Cz^{2/3}}{(z+\epsilon)^{2/3}} \geq -C\,|u_2-u_1|\,,$$

therefore J^ϵ is bounded below on $H_0^1(\Omega) \times C^0(\Omega) \times C^0(\Omega)$.

Define the sublevel set

$$J_a^\epsilon = \left\{ z \in H_0^1(\Omega) \times C^0(\Omega) \times C^0(\Omega) : J_a^\epsilon[z,v,w] \leq a \right\}.$$

If $(z_\epsilon, v_\epsilon, w_\epsilon)$ is a solution to $(S_{t,\epsilon})$ then

$$J^\epsilon[z_\epsilon(t), v_\epsilon(t), w_\epsilon(t)] \leq J^\epsilon[z_\epsilon(0), v_\epsilon(0), w_\epsilon(0)] = a,$$

$(z_\epsilon, v_\epsilon, w_\epsilon)$ lies in the sublevel set J_a^ϵ.

Lemma A.2.4. There exists positive constants C_1, C_2 depending on a and $|v-w|$ such that for any $(z,v,w) \in J_a^\epsilon$ we have $\|z_x\|_{L^2} \leq C_1$ and $\|z\|_{L^2} \leq C_2$.

Proof.

$$\begin{aligned} a \geq J^{\epsilon}\left[z, v, w\right] &= \int_{\Omega} \frac{2}{9(w-v)} z_x^2 + \frac{(w-v)K((w-v)x+v, z^{2/3})}{(z+\epsilon)^{2/3}} dx \\ &\geq \int_{\Omega} \frac{2}{9(w-v)} z^2 dx + (w-v) \int_{\Omega} \frac{-C_\delta z^{2/3} - \frac{z^{8/3}}{\delta}}{(z+\epsilon)^{2/3}} dx \\ &\geq \int_{\Omega} \frac{2}{9(w-v)} z^2 dx - \frac{(w-v)}{\delta} \int_{\Omega} z^2 dx - (w-v) \int_{\Omega} \frac{C_\delta z^{2/3}}{(z+\epsilon)^{2/3}} dx, \end{aligned}$$

choosing $\delta = 9(w-v)^2$ yields

$$a \geq \frac{1}{9(w-v)} \int_{\Omega} z^2 dx - C_\delta \left| w - v \right|$$

and

$$\int_{\Omega} z^2 dx \leq 9(w-v)\left(a + C_\delta \left| w-v \right|\right) < 9B(a + C_B B). \tag{A.2}$$

Hence $z \in L^2(\Omega)$ which implies that $z \in H_0^1(\Omega)$. With this in mind we obtain an upper bound on $\int_{\Omega} \frac{1}{z^{2/3}} dx$.

A.3 FREQUENTLY USED RESULTS

A.3.1 Weak comparison principle

Consider the Dirichlet problems

$$-\text{div}(\mathbf{a}(x, \nabla u)) - b(x, u) = f(x) \text{ in } \Omega;\ u = \tilde{f} \text{ on } \partial\Omega, \tag{A.3}$$

$$-\text{div}(\mathbf{a}(x, \nabla v)) - b(x, v) = g(x) \text{ in } \Omega;\ v = \tilde{g} \text{ on } \partial\Omega. \tag{A.4}$$

Theorem A.3.1. (WCP)[3] Assume that $f \leq g$ in $L^{p/(p-1)}(\Omega)$, $\tilde{f} \leq \tilde{g}$ in $W^{1-(1/p),p}(\partial\Omega)$, and u, $v \in W^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems of (A.3) and (A.4) respectively. Then $u \leq v$ hold almost everywhere in Ω provided

(a) $b(x, u)$ is nondecreasing in u for $x \in \Omega \times \mathbb{R}$ and (A.3) has a unique non-negative weak solution $u \in W_0^{1,p}(\Omega)$.

(b) $b(x, \bullet) : \mathbb{R} \to \mathbb{R}$ is nonincreasing for a.e. $x \in \Omega$.

A.3.2 Gronwall's inequality (integral form)

The following version of Gronwall's inequality is taken from Evans [8].

(i) Let $\xi(t)$ be a nonnegative, summable function on $[0, t]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^T \xi(s)ds + C_2 \tag{A.5}$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2(1 + C_1 te^{C_1 t}) \tag{A.6}$$

for a.e. $0 \leq t \leq T$.

(ii) In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0 \text{ a.e.}$$

A.3.3 Arzelá-Ascoli theorem

Theorem A.3.2. [26] Suppose the topological space X is separable and the metric space Y is complete. Let F be a nonvoid equicontinuous subset of $C(X, Y)$ having the property that for each $x \in X$, the closure of the set $\{f(x) \mid f \in F\}$ is a compact subset of Y. Then each sequence $(f_n)_{n=1}^{\infty} \subset F$ has a subsequence that converges pointwise on X to a function $f \in C(X, Y)$. Moreover this convergence is uniform on each compact subset of X.

A.3.4 Lasalle invariance principle

Theorem A.3.3. [13] Let Ω be a bounded closed (compact) set with the property that every solution of $\dot{x} = X(x)$ which begins in Ω remains for all future time in Ω. Suppose there is a scalar function $V(x)$ which has continuous first partials in Ω and is such that $\dot{V} < 0$ in Ω. Let E be the set of all points in Ω where $\dot{V} = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

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