KALMAN FILTERING FOR ROBOTIC CALIBRATION

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ABSTRACT

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This thesis is concerned with the use of calibration techniques to increase robot accuracy. It is mainly an overview of some of the problems involved in the identification phase of calibration. A robot error model is developed and Kalman filtering algorithm is used in the identification of robot kinematic error parameters. Computer simulations and examples are used to study the behavior of the Kalman filter and its theoretical advantages in robot calibration.

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CHAPTER 1

INTRODUCTION

1.1 Background

A robot manipulator is, in general, a collection of bodies, connected together in a chain arrangement through joints. Typically, the bodies are quite rigid and the joints are either revolute or prismatic. A revolute joint is essentially a hinge between two connected bodies. A prismatic joint, on the other hand, allows only a sliding motion between the bodies (or links) so that their angular velocities are the same.

Robot manipulators are essentially used to move tools and manufacturing materials from one place to another. In doing so, they often are used to perform assembly operation (or value-add operations). In order for a robot controller to be able to position the robot's tool at a specified point in world coordinates, a model relating the tool's location (position and orientation) to the joints angles is needed. Typically, the robot's controller contains a geometric model that evaluates the tool location as a function of the joints angles. The controller accepts commands to position the tool in world coordinates; a cartesian coordinate frame that can be arbitrarily defined (figure 1). Given such a command, the robot controller uses an inverted kinematic robot model to solve for the joint's angles that would correctly position the tool, and the joint's

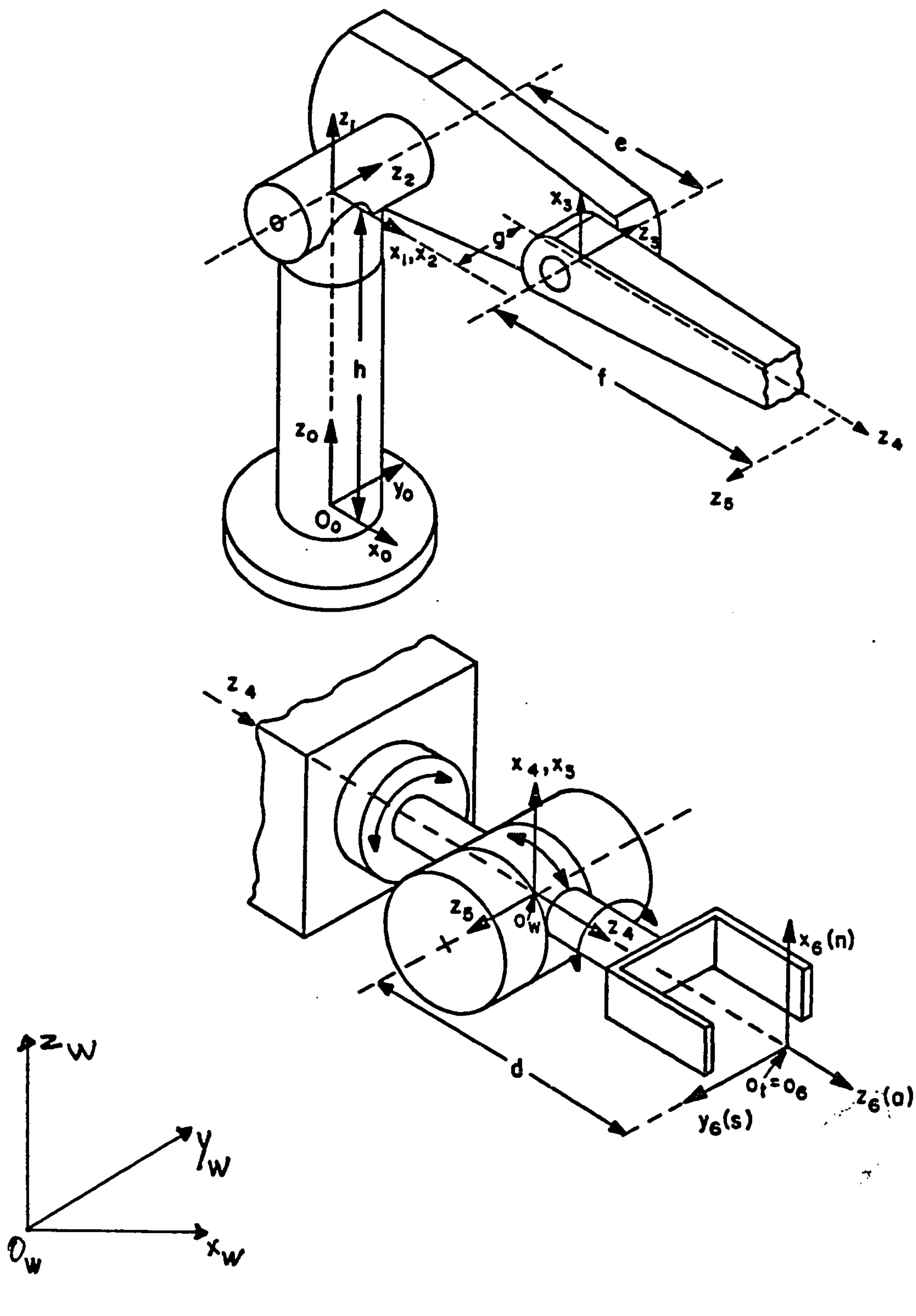


Fig. 1 Robot and World Coordinate Frames Assignments

servos are simultaneously commanded to move to the desired angles. Because each servo loop measures and controls only one joint angle, not the tool location, any errors in the robot model show up as positioning inaccuracies.

Robotic applications usually require a considerable amount of programming effort either in the form of teaching or in a data driven manner. At the present time, the most common means of programming an industrial robot is the "teach by doing" method. The robot is led through the desired task by the operator and critical points of the task are recorded in the memory of the robot controller. Once the task is learned, it can be replayed over and over so that the task is accomplished a number of times. Hence, the ability of the robot to repeatedly perform an application heavily depends on the robot precision, which is the capability of the robot to accurately repeat or execute the sequence of task points.

While this "teach by doing" method has been successful, it has unfortunately significant limitations.

Some typical examples:

1. Accidents during a teaching phase where the robot end-effector may collide with an obstacle (e.g., the work table). This may cause permanent offset in the relative positions of a motor and position sensor mounted on it.
2. Robot replacement (no two robots are exactly identical, due to casting, machining and assembly tolerances).

3. A new task is programmed by stopping the current operation, setting up the new task, and then stepping the robot through the new task points. This can, and does usually, tie up the production line causing expensive delays.

One solution to some of those problems would be to program the new tasks "off line" by either using a similar robot or a computer simulation of the robot. The task would be programmed, debugged, and then down loaded to the robot on the production line with a minimum of interruption to product flow.

Unfortunately, off line programming techniques have all encountered one common problem namely robot accuracy. There are basically two approaches to solving this problem.

The first approach is to decrease the machining and assembly tolerances during the manufacture of the robot itself. This would tend to minimize the variations in geometry from one robot to the next. Of course, this approach would be prohibitively expensive. The second approach is the use of some robot calibration techniques. Calibration techniques can be used to improve robot accuracy through software rather than changing the mechanical structure on machining tolerances of the robot. Calibration can also minimize the risk of having to change the application programs due to slight changes or drifts in the robot kinematic model (such as wear of parts and tightening or replacement of parts effects) of the robotic system.

1.2 Robot Calibrations

Robot calibration is a process by which robot accuracy can be improved by modifying the robot's positioning software rather than by changing or altering the design of the robot or its control system. It involves identifyinga more accurate functional relationship between the point transducer readings and the actual workspace position of the end-effector and using these these identified changes to permanently modify (between two consecutive calibrations) the robot's positioning software. This definition assumes that nominal relationship between the end-effector position and the joint transducer readings is known but that this relationship is not necessarily accurate.

Calibration procedures vary widely in their complexity. For example, some robot calibration procedures only consider the joint transducer information while others may involve changes in the kinematic or dynamic model of the robot. In order to clarify, in a somehow systematic manner, most of the current approaches toward robot calibration, three levels of robot calibration will be defined.

"Level 1" calibration shall be defined as "joint level" calibration. Its goal is to determine the correct relationship between the signal produced by the joint displacement encoder and the actual joint displacement. This usually involves calibration of kinematics of the drive and the sensor mechanisms. "Level 2" calibration is defined as the entire robot kinematic model calibration. At this level, the purpose of the calibration is to determine the kinematic structure of the robot as well as the correct joint angle relationships. "Level 3" calibration is defined as "Non-kinematic" calibration. This level of calibration can only be used in cases where the robot is under dynamic control. It involves determining the dynamic parameters that would indentify the changes in the dynamic model of the robot that would result in improved accuracy.

In general, the calibration process consists of four steps. The first step would be to choose the form of a suitable functional relationship. This step will be referred to as the "modeling" step. The second step would be to collect some data from the actual robot that relates the input of the model to the output. This step will be termed the "measurement" step since it consists of the physical data collection process. The third step would be the mathematical process of using the data collected to identify the coefficients in the model. This step will be called the "identification" step. It is important to note that an essential part of this step is the determination of the expected error in the identified coefficients due to error or noise in the measurement step. The final step would be the implementation of the new model in the position control software of the robot. This will be referred to as the "correction" step [27].

1.3 Scope of Thesis

This thesis will mainly be concerned with the use of calibration techniques to increase the robot accuracy. It is mainly an overview of some the problems involved in the identification phase of calibration and the main concern is to identify some of the pertinent questions that need to be answered.

A robot error model will be developed and a Kalman filtering algorithm will be used in the identification of robot kinematic error parameters.

Limited computer simulations and examples will also be used to give some intuition on the behavior of the Kalman filter and its theoretical advantages in robot calibration.

CHAPTER 2

MATHEMATICAL BACKGROUND

2.1 Homogeneous Transformations:

The study of robot manipulation is concerned with the relationship between objects, and between objects and manipulators. In this section, the necessary representation for the relationships will be developed. For a more detailed discussion the reader is referred to [26].

2.1.1 Orientation Matrix

Let (i-1) and i be any arbitrary coordinate frames, and let $(\underline{x}_{i-1}, \underline{y}_{i-1}, \underline{z}_{i-1})^T$ and $(\underline{x}_i, \underline{y}_i, \underline{z}_i)^T$ be the unit vectors of frames (i-1) and i respectively. And let ${}^{i}\tau^{(i-1)}_{x,y}$ represent the projection of the unity vector of the y-axis in coordinates (i-1) on the x-axis of coordinates i. For simplicity, any obvious or unimportant superscript or subscript will not be written. For instance, ${}^{i}\tau^{(i-1)}_{x,y}$ may be referred to as $\tau_{x,y}$ if the relation is true for any coordinates or the coordinates were mentioned in the text.

The projection of any component of coordinates (i-1) on any axis of coordinates i has three components, and the following nine projection equations can be obtained for the projection of $(\underline{x}_{(i-1)}, \underline{y}_{(i-1)}, \underline{z}_{(i-1)})^T$ on i.

$$\tau_{x,x} = \underline{x}^T_{i-1} \cdot \underline{x}_i \qquad (2\text{-}1)$$

$$\tau_{x,y} = \underline{y}^T_{i-1} \cdot \underline{x}_i \qquad (2\text{-}2)$$

$$\tau_{x,z} = \underline{z}^T_{i-1} \cdot \underline{x}_i \qquad (2\text{-}3)$$

$$\tau_{y,x} = \underline{x}^T_{i-1} \cdot \underline{y}_i \qquad (2\text{-}4)$$

$$\tau_{y,y} = \underline{y}^T_{i-1} \cdot \underline{y}_i \qquad (2\text{-}5)$$

$$\tau_{y,z} = \underline{z}^T_{i-1} \cdot \underline{y}_i \qquad (2\text{-}6)$$

$$\tau_{z,x} = \underline{x}^T_{i-1} \cdot \underline{z}_i \qquad (2\text{-}7)$$

$$\tau_{z,y} = \underline{y}^T_{i-1} \cdot \underline{z}_i \qquad (2\text{-}8)$$

$$\tau_{z,z} = \underline{z}^T_{i-1} \cdot \underline{z}_i \qquad (2\text{-}9)$$

Combining equations 2-1:2-9 yields the following matrix:

$$^{i}\underline{\underline{R}}^{(i-1)} = \begin{bmatrix} \tau_{x,x} & \tau_{x,y} & \tau_{x,z} \\ \tau_{y,x} & \tau_{y,y} & \tau_{y,z} \\ \tau_{z,x} & \tau_{z,y} & \tau_{z,z} \end{bmatrix} = \begin{bmatrix} \underline{x}^T_{i-1} \cdot \underline{x}_i & \underline{y}^T_{i-1} \cdot \underline{x}_i & \underline{z}^T_{i-1} \cdot \underline{x}_i \\ \underline{x}^T_{i-1} \cdot \underline{y}_i & \underline{y}^T_{i-1} \cdot \underline{y}_i & \underline{z}^T_{i-1} \cdot \underline{y}_i \\ \underline{x}^T_{i-1} \cdot \underline{z}_i & \underline{y}^T_{i-1} \cdot \underline{z}_i & \underline{z}^T_{i-1} \cdot \underline{z}_i \end{bmatrix}$$

(2-10)

This ${}^{i}\underline{\underline{R}}^{(i-1)}$ matrix is often called the "Orientation Matrix" and is independent of the displacement between the coordinate frames. ${}^{i}\underline{\underline{R}}^{(i-1)}$ is sometimes referred to as the "matrix of directional cosines").

2.1.2 Displacement Vector:

Let $\underline{P}_{i-1,i}$ represent a vector joining the origin of frame (i-1) to that of frame i. By convention, the first subscript of $\underline{P}$ identifies the frame in which the vector is represented (figure 2).

The displacement components from the origin of frame (i-1) to that of frame i are given by the following equations.

$$ {}_{i-1}\underline{P}_x = (\underline{P}_{i-1,i})_x = \underline{x}^T_{i-1}\cdot\underline{x}_i({}_{i-1}\underline{P}_x) + \underline{y}^T_{i-1}\cdot\underline{x}_i({}_{i-1}\underline{P}_y) + \underline{z}^T_{i-1}\cdot\underline{x}_i({}_{i-1}\underline{P}_z) \quad (2\text{-}11) $$

$$ {}_{i-1}\underline{P}_y = (\underline{P}_{i-1,i})_y = \underline{x}^T_{i-1}\cdot\underline{y}_i({}_{i-1}\underline{P}_x) + \underline{y}^T_{i-1}\cdot\underline{y}_i({}_{i-1}\underline{P}_y) + \underline{z}^T_{i-1}\cdot\underline{y}_i({}_{i-1}\underline{P}_z) \quad (2\text{-}12) $$

$$ {}_{i-1}\underline{P}_z = (\underline{P}_{i-1,i})_z = \underline{x}^T_{i-1}\cdot\underline{z}_i({}_{i-1}\underline{P}_x) + \underline{y}^T_{i-1}\cdot\underline{z}_i({}_{i-1}\underline{P}_y) + \underline{z}^T_{i-1}\cdot\underline{z}_i({}_{i-1}\underline{P}_z) \quad (2\text{-}13) $$

where

$(\underline{P}_{i-1,i}) = {}_{i-1}\underline{P}_x$ = x-component of $\underline{P}$ in coordinate frame (i-1). Similarly for $(\underline{P}_{i-1,i})_y$ and $(\underline{P}_{i-1,i})_z$.

Referring to equation 2-10 and 2-11:2-13, the "displacement vector" can be expressed as

$$ \underline{P}_{i,i-1} = -{}^{i}\underline{\underline{R}}^{i-1}\cdot\underline{P}_{i-1,i} \quad (2\text{-}14) $$

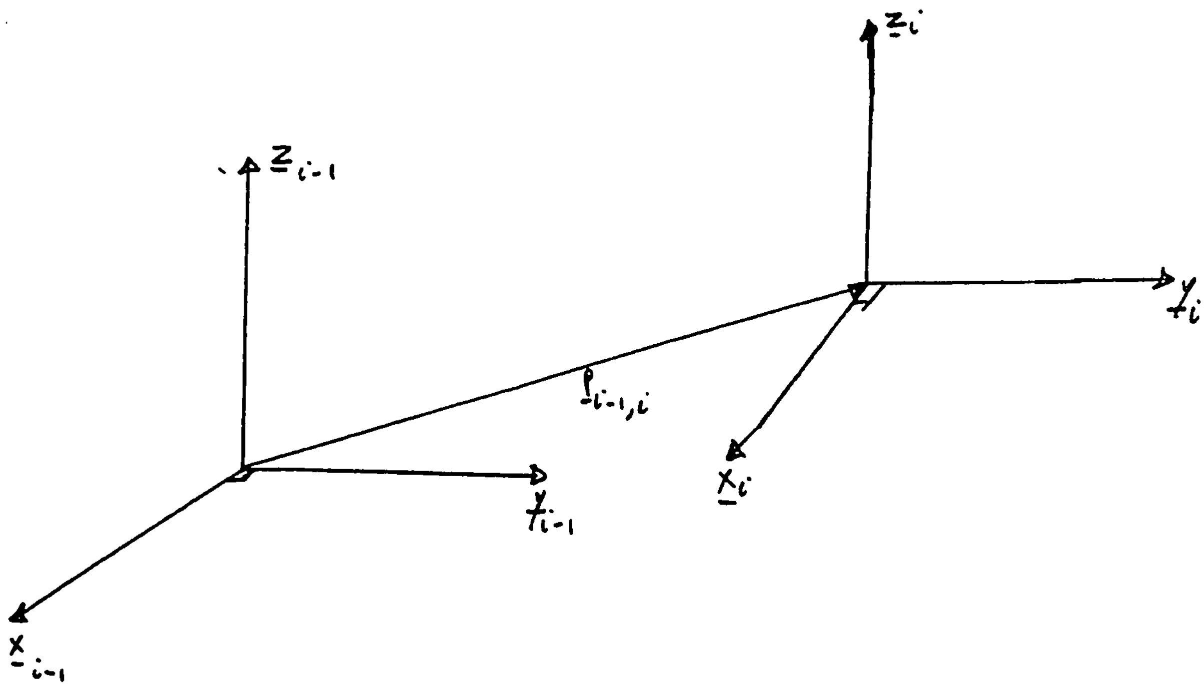


Fig. 2 Displacement Vector

2.1.3 Homogeneous Transformation Matrix:

Using the previous results, a method for transferring the reference from one coordinate to another can now be developed according to "Chalses' Theorem". The most general displacement of a rigid body, and thus a fixed reference frame on it, can be characterized as a screw displacement which is a translation along and a rotation about a screw axis. This screw axis in this context is the vector joining the origin of frame i-1 to that of frame i.

From the above discussion, it can be concluded that a homogeneous transformation between two coordinate frames is given by [26].

$\underline{\underline{A}}$ = [translation] · [Rotation]

Let ${}^{i}\underline{\underline{A}}^{i-1}_{(i-1)}$ be a homogeneous transformation from coordinate frame (i-1) to frame i written with respect to the (i-1) frame. Using equation 2-10 and 2-14 we can express ${}^{i}\underline{\underline{A}}^{i-1}_{(i-1)}$ as

$$ {}^{i}\underline{\underline{A}}^{i-1}_{(i-1)} = \left[\begin{array}{c|c} {}^{i}\underline{\underline{R}}^{i-1} & {}^{i}\underline{\underline{R}}^{i-1} \cdot \underline{P}_{i-1,i} \\ \hline 0\ 0\ 0 & 1 \end{array} \right]_{i-1} = \left[\begin{array}{c|c} {}^{i}\underline{\underline{R}}^{i-1} & \underline{P}_{i,i-1} \\ \hline 0\ 0\ 0 & 1 \end{array} \right]_{i-1} \qquad (2\text{-}15) $$

2.1.4 Properties of the Orientation Matrix:

In order to reduce the complexity of the notation, define

$$ {}^{i}\underline{\underline{R}}^{i-1} = \begin{bmatrix} \tau_{x,x} & \tau_{x,y} & \tau_{x,z} \\ \tau_{y,x} & \tau_{y,y} & \tau_{y,z} \\ \tau_{z,x} & \tau_{z,y} & \tau_{z,z} \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \qquad (2\text{-}16) $$

Since $({}^{i}\underline{\underline{R}}^{i-1})^{-1} \cdot {}^{i}\underline{\underline{R}}^{i-1} = {}^{i-1}\underline{\underline{R}}^{i} \cdot {}^{i}\underline{\underline{R}}^{i-1} = \underline{\underline{I}}$ substitute equations 2-14 and 2-16 into 2-1:2-9 and obtain

$$ a_x \cdot a_x + a_y \cdot a_y + a_z \cdot a_z = \underline{a}\cdot\underline{a} = 1 \qquad (2\text{-}17) $$

$$ a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z = \underline{a}\cdot\underline{b} = 0 \qquad (2\text{-}18) $$

$$ a_x \cdot c_x + a_y \cdot c_y + a_z \cdot c_z = \underline{a}\cdot\underline{c} = 0 \qquad (2\text{-}19) $$

$$ b_x \cdot a_x + b_y \cdot a_y + b_z \cdot a_z = \underline{b}\cdot\underline{a} = 0 \qquad (2\text{-}20) $$

$$ b_x \cdot b_x + b_y \cdot b_y + b_z \cdot b_z = \underline{b}\cdot\underline{b} = 1 \qquad (2\text{-}21) $$

$$ b_x \cdot c_x + b_y \cdot c_y + b_z \cdot c_z = \underline{b}\cdot\underline{c} = 0 \qquad (2\text{-}22) $$

$$ c_x \cdot a_x + c_y \cdot a_y + c_z \cdot a_z = \underline{c}\cdot\underline{a} = 0 \qquad (2\text{-}23) $$

$$ c_x \cdot b_x + c_y \cdot b_y + c_z \cdot b_z = \underline{c}\cdot\underline{b} = 0 \qquad (2\text{-}24) $$

$$ c_x \cdot c_x + c_y \cdot c_y + c_z \cdot c_z = \underline{c}\cdot\underline{c} = 1 \qquad (2\text{-}25) $$

These nine equations indicate that the three axes of the coordinate frame are mutually orthogonal. The cartesian coordinate system fulfills the requirements of equations 2-17:2-25 and is suitable for modelling the Kinematics of robot manipulators. Now, the computation of the orientation matrix in the cartesian coordinate system can be performed.

Suppose the coordinate i-1 rotates an angle θ in the direction of the x-axis and becomes the coordinate i (figure 3). Then, by definition

$a_x = 1$

$b_x = 0$

$c_x = 0$

for the unity vector $\underline{x}$ on the x-axis; and

$a_y = 0$

$b_y = \cos\theta$

$c_y = \sin\theta$

for the unity vector $\underline{y}$ on the y-axis; and

$a_z = 0$

$b_z = \sin\theta$

$c_z = \cos\theta$

For the unity vector $\underline{z}$ on the z-axis. Hence

$$\text{Rot}\ (x,\theta) = {}^{i}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}^{i-1} \qquad (2\text{-}26)$$

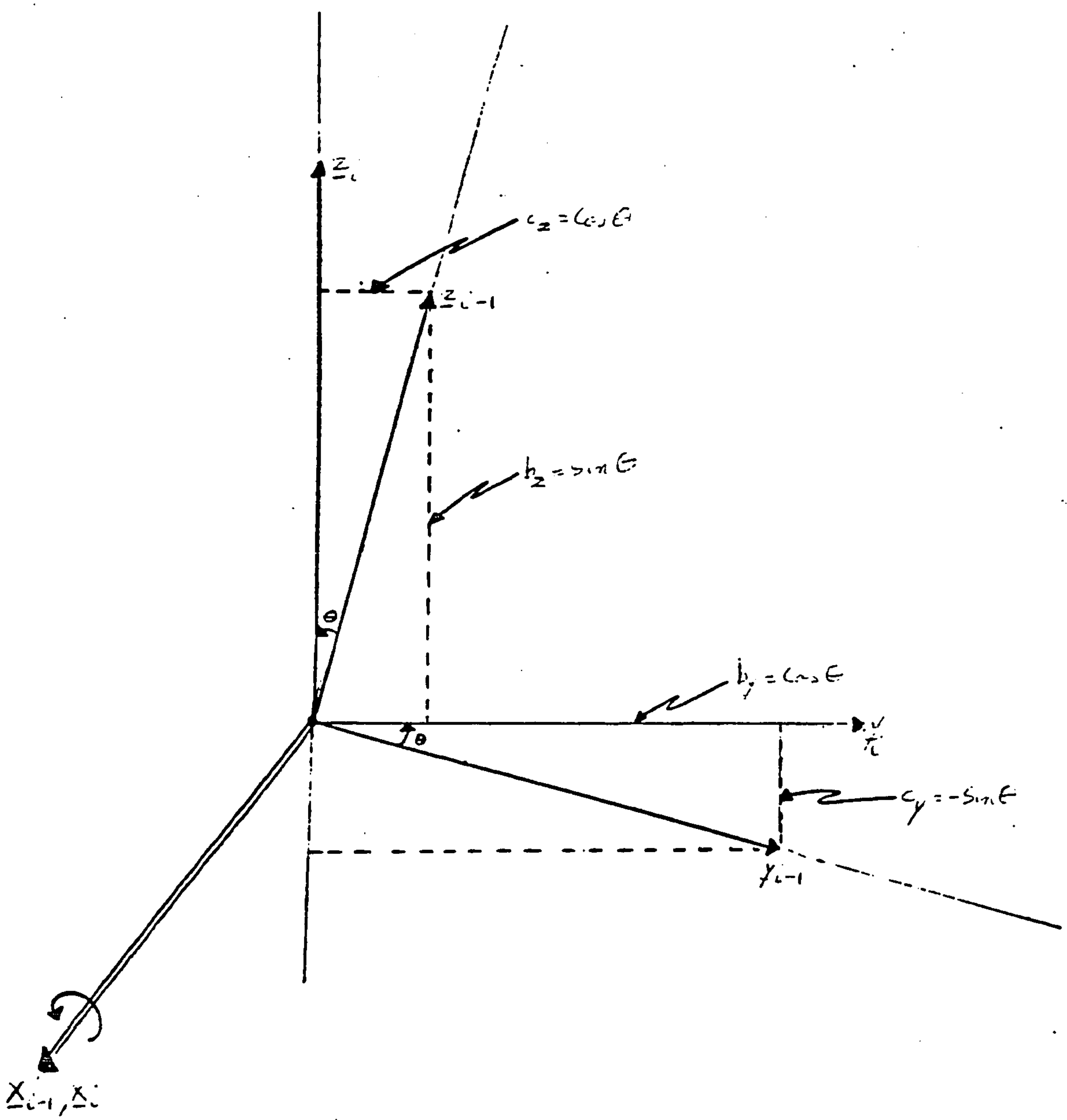


Fig. 3 Rotation by an Angle θ Around the x-axis

similarly

$$^{i}\text{Rot}(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}^{i-1} \qquad (2\text{-}27)$$

and

$$^{i}\text{Rot}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{i-1} \qquad (2\text{-}28)$$

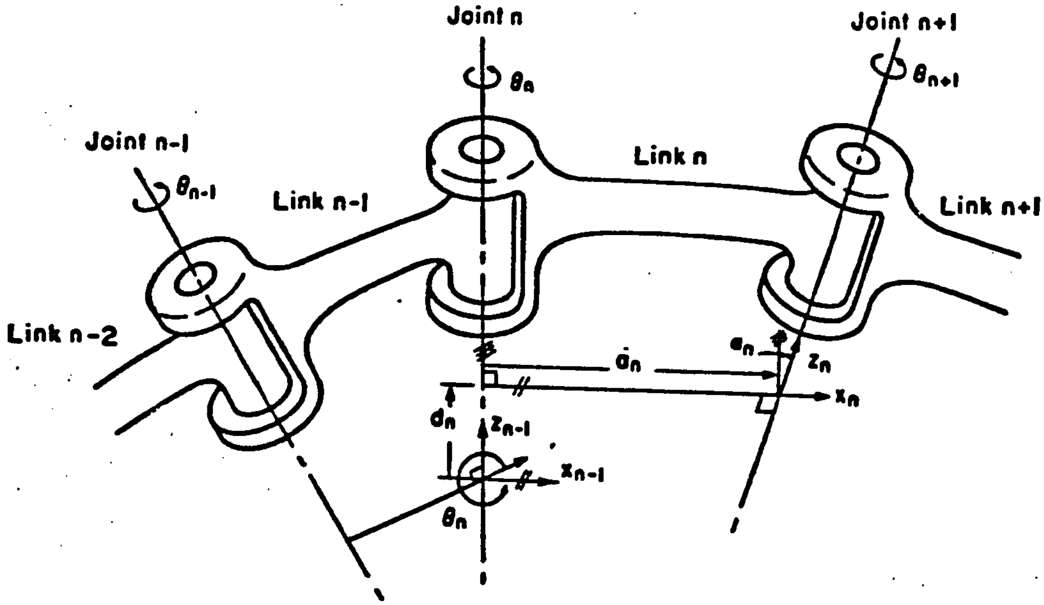
Note that rotation operations are not communative.

2.1.5 Denavit-Hartenberg Matrix Specifications:

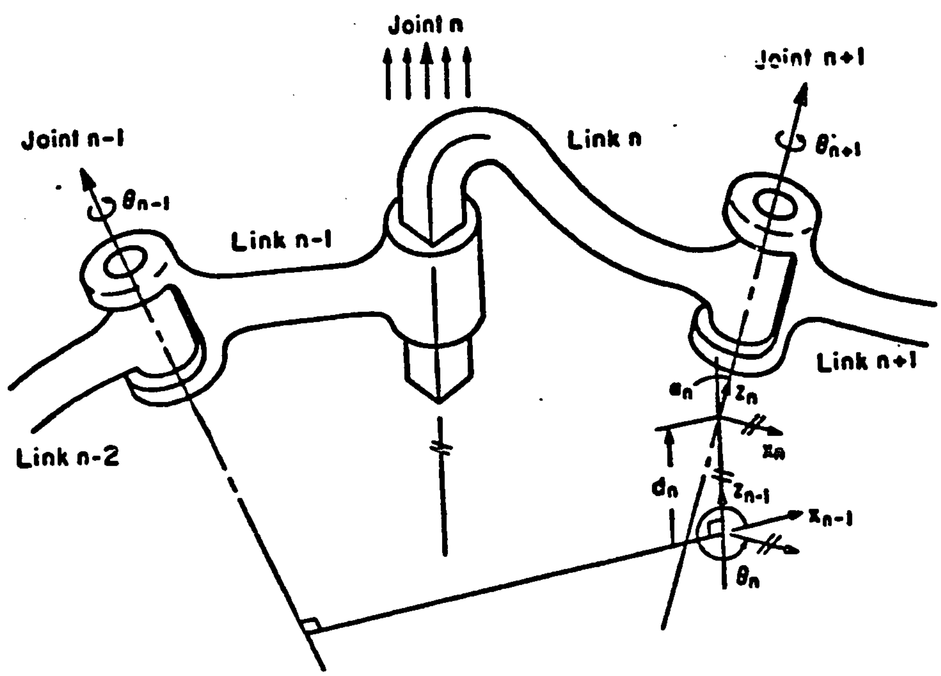
Historically, the homogeneous transformation describing the relation between one link and the next has been called the "Denavit-Hartenberg" matrix [26].

Any two neighboring links can be characterized each by two dimensions: the common normal distance, a_i, and the angle, α_i, between the axis in a plane perpendicular to a_i. It is customary to call a_i the length and α_i the twist of the link. Generally, two links are connected at each joint axis. (figure 4).

The joint axis will have two normals to it, one for each neighboring link. The relative position of two such consecutive common normals is given by d_i, the distance between the normals along the joint i axis, and θ_i, the angle between the normals, measured in a plane normal to the axis. d_i and θ_i are called the distance and the angle between the links, respectively.



a) Revolute Joint



b) Prismatic Joint

Fig. 4 Denavit-Hartenberg Link Parameters

In order to describe the relationship between links, it will be necessary to assign coordinate frames to each link:

1) The axes of motion should be found and numbered starting from the base towards the end-effector. ($i=1,\ldots,N$)

2) All the common normals a_i's and the distance d_i's should be located. In case of parallel axes, the common normal a_i is placed in such a way that $d_{i+1} = 0$.

3) The base coordinate frame is chosen. It is preferable that the z_0 axis coincides with the first axis of motion.

4) A coordinate frame for each link is assigned. It is done as follows:

- The origin of frame $F_i = [\underline{x}_i, \underline{y}_i, \underline{z}_i]$ is placed at the intersection of a_i and the i+1 axis of motion.

- The z_i-axis is aligned with the i+1 axis of motion.

- The x_i-axis is aligned with a_i pointing from joint i to joint i+1. If $a_i = 0$, i.e., common or intersecting axis of motion, $\underline{x}_i$ is chosen to be parallel to $\underline{z}_{i-1} \times \underline{z}_i$ (The cross product of $\underline{z}_{i-1}$ and $\underline{z}_i$).

- Finally, $\underline{y}_i$ is chosen such that $\underline{y}_i = \underline{z}_i \times \underline{x}_i$ (right hand system).

It is worth noting that the direction of $\underline{z}_i$ is arbitrary, but once the positive direction of z_i is defined then the positive sense of the joint variable is implied. Also, if $\underline{z}_{i-1}$ and $\underline{z}_i$ are parallel (i.e., $\underline{z}_{i-1} \times \underline{z}_i = 0$), x_i can be chosen arbitrarily.

5) The rotation angle, θ_i is determined. θ_i is the rotation of $\underline{x}_{i-1}$ about $\underline{z}_{i-1}$ to align its direction with $\underline{x}_i$.

6) The angle of twist, α_i, is determined. After aligning the direction of $\underline{x}_{i-1}$ and $\underline{x}_i$, the origin of F_{i-1} is moved to that of F_i. Therefore, α_i is the rotation of $\underline{z}_{i-1}$ about $\underline{x}_{i-1} = \underline{x}_i$ to align it with $\underline{z}_i$.

Having assigned coordinate frames to all links according to the preceding scheme, the relationship between two successive frames, (i-1) and i, can be established by the following rotations and translations:

1) Rotate about $\underline{z}_{i-1}$, an angle θ_i

2) Translate along $\underline{z}_{i-1}$, a distance d_i

3) Translate along rotated $\underline{x}_{i-1} = \underline{x}_i$, a length a_i

4) Rotate about $\underline{x}_i$, a twist angle α_i

This may be expressed as the product of four homogeneous transformations relating coordinate frame of link i to that of link i-1.

$$ {}^{i}\underline{\underline{A}}^{i-1} = \text{Rot}\,(z,\theta).\ \text{Trans}\,(0,0,d).\ \text{Trans}\,(a,0,0).\ \text{Rot}\,(x,\alpha) \qquad (2\text{-}29)$$

$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}30)$$

The description of the end of the manipulator, (link coordinate frame N) with respect to link coordinate frame i-1 is given by ${}^{i-1}\underline{\underline{T}}^{N}$ where

$$ {}^{i}\underline{\underline{T}}^{N} = {}^{i}\underline{\underline{A}}^{i+1} \cdots\cdots {}^{N-1}\underline{\underline{A}}^{N} \tag{2-31} $$

2.1.6 Inverse Homogeneous Transformation Matrix:

The homogeneous transformation transfers the reference back to the original coordinate. In general, given a transformation $\underline{\underline{T}}$ with elements

$$ \underline{\underline{T}} = \begin{bmatrix} a_x & b_x & c_x & p_x \\ a_y & b_y & c_y & p_y \\ a_z & b_z & c_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-32} $$

then the inverse is

$$ \underline{\underline{T}}^{-1} = \begin{bmatrix} a_x & a_y & a_z & -\underline{a}\cdot\underline{p} \\ b_x & b_y & b_z & -\underline{b}\cdot\underline{p} \\ c_x & c_y & c_z & -\underline{c}\cdot\underline{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-33} $$

where $\underline{a}\cdot\underline{p} = a_x \cdot p_x + a_y \cdot p_y + a_z \cdot p_z$ is the scalar product of the vectors $\underline{a}$ and $\underline{p}$. Similarly for $\underline{b}\cdot\underline{p}$ and $\underline{c}\cdot\underline{p}$.

This result is easily verified by post-multiplying equation 2-32 by equation 2-33.

In the case of the Denavit-Hartenberg matrix equation 2-30, it can be easily shown that

$$({}^{i}\underline{\underline{A}}^{i-1})^{-1} = {}^{i-1}\underline{\underline{A}}^{i} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\sin\theta_i\cos\alpha_i & \cos\theta_i\cos\alpha_i & \sin\alpha_i & -d_i\cos\alpha_i \\ \sin\theta_i\sin\alpha_i & -\cos\theta_i\sin d_i & \cos\alpha_i & -d_i\sin\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2\text{-}34)$$

2.2 Differential Homogeneous Transformations:

Using the result of [26], the rotation part of the homogeneous transformation, equation 2-15 and 2-34 can be assumed to be the result of the rotation of frame i about a unit vector, $\underline{k} = (k_x, k_y, k_z)^T$ by an angle θ. Hence,

$${}^{i}\underline{\underline{R}}^{i-1} = \mathrm{Rot}(\underline{k},\theta) =$$

$$\begin{bmatrix} k_x^2\ \mathrm{vers}\theta+\cos\theta & k_yk_x\mathrm{vers}\theta-k_z\sin\theta & k_zk_x\mathrm{vers}\theta+k_y\sin\theta \\ k_xk_y\mathrm{vers}\theta+k_z\sin\theta & k_y^2\mathrm{vers}\theta+\cos\theta & k_zk_y\mathrm{vers}\theta-k_y\sin\theta \\ k_xk_y\mathrm{vers}\theta-k_y\sin\theta & k_yk_z\mathrm{vers}\theta+k_x\sin\theta & k_z^2\ \mathrm{vers}\theta+\cos\theta \end{bmatrix} \quad (2\text{-}35)$$

where vers θ is defined or versθ = 1-cosθ.

For a differential change dθ, the limits of the corresponding trigonometric functions become

A.1: $\lim_{\theta\to 0} \sin\theta = d\theta$

A.2: $\lim_{\theta\to 0} \cos\theta = 1$

A.3: $\lim_{\theta\to 0} \mathrm{vers}\theta = 0$

and the differential rotation matrix becomes

$$\mathrm{Rot}(\underline{k},d\theta) = \begin{bmatrix} 1 & k_z d\theta & k_y d\theta \\ k_z d\theta & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{bmatrix} \quad (2\text{-}36)$$

Using the above result and assuming differential translation of the reference, $\underline{d}$, the translational part of the homogeneous transformation, i.e., ${}^{i}\underline{\underline{R}}^{i-1} \cdot \underline{d}_{i-1,i}$ is given by

$${}^{i}\underline{\underline{R}}^{i-1} \cdot \underline{d}_{(i-1,i)} = \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta \\ k_z dr & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{bmatrix} \cdot \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$= \begin{bmatrix} d_x - k_z \cdot d_y \cdot d\theta + k_y \cdot d_z \cdot d\theta \\ k_z \cdot d_x \cdot d\theta + d_y - k_x \cdot d_z \cdot d\theta \\ k_y \cdot d_x \cdot d\theta + k_x \cdot d_y d\theta + dz \end{bmatrix} \quad (2\text{-}37)$$

Neglecting the second order terms, it can be easily seen that

$${}^{i}\underline{\underline{R}}^{i-1} \cdot \underline{d}_{i-1,i} = \underline{d}_{i-1,i} \quad (2\text{-}38)$$

In order to reduce the complexity of the notation, the following quantities will be defined.

1) A cross operator Δ such that

$$\begin{bmatrix} 0 & -\eta_z & \eta_y \\ \eta_z & 0 & -\eta_x \\ \eta_y & \eta_x & 0 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix} \cdot \Delta \qquad (2\text{-}39)$$

2) A 3 x 1 vector of rotational errors, $\underline{\delta}$, such that

$$\underline{\delta} = \begin{bmatrix} k_x d\theta \\ k_y d\theta \\ k_z d\theta \end{bmatrix} = \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} \qquad (2\text{-}40)$$

3) A 3 x 1 vector of translational errors, $\underline{d}_i$, such that

$$\underline{d}_i = (d_x, d_y, d_z)_i^T \qquad (2\text{-}41)$$

Combining equations 2-15 and 2-35:2-41, a differential homogeneous transformation, ${}^{iA}\underline{\underline{dA}}{}^{iN}$, such that

$$ {}^{iA}\underline{\underline{dA}}{}^{iN} = \left[\begin{array}{c|c} I - \underline{\delta}_i \cdot \Delta & -\underline{d}_i \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] \qquad (2\text{-}42)$$

${}^{iA}\underline{\underline{dA}}{}^{iN}$ represents a differential translation and rotation from coordinate frame iN to coordinate frame iA. In the following chapters, the above differential transformation will be used to model the geometrical errors of each link of the manipulator.

2.3 ELEMENTS OF LINEAR FILTERING THEORY

The problem of estimating the state of a stochastic dynamical system from many observations taken on the state is of central importance in engineering. An outline of the major mathematical methods in estimation will be given in this section.

First, however, a definition of the estimation problem in the context of the mathematical model used in this thesis need to be given.

Consider the discrete stochastic dynamical system described by the stochastic vector difference equation

$$\underline{x}(k+1) = \underline{\underline{\phi}}(k) \cdot \underline{x}(k) + \underline{\underline{\Gamma}}(k) \cdot \underline{w}(k) \quad k=0,1,\ldots \qquad (2\text{-}43)$$

where the state at time k is $\underline{x}(k)$, an N-vector, $\underline{\underline{\phi}}$ is an N-vector function, $\underline{\underline{\Gamma}}$ is an N x P matrix and $\{\underline{w}(k), k=0,1,\ldots\}$ is a P-vector, white Gaussian sequence with zero mean and covariance $\underline{\underline{Q}}(k)$. The distribution of the initial condition $\underline{x}(0)$ is assumed given, and $\underline{x}(0)$ is independent of $\{\underline{w}(k)\}$. Let a discrete time, noisy, M-vector observations (measurements) $\underline{z}(k)$ be given by

$$\underline{z}(k) = \underline{\underline{H}}(k) \cdot \underline{x}(k) + \underline{\underline{L}}(k) \cdot \underline{v}(k) \quad k=0,1,\ldots, \qquad (2\text{-}44)$$

where the M x N matrix $\underline{\underline{H}}$ is assumed to be a known function, as well as, the M x S matrix $\underline{\underline{L}}$ and $\{\underline{v}(k),\ k=0,1,\ldots\}$ is an S-vector white Gaussian sequence with zero mean and positive definite covariance matrix $\underline{\underline{R}}(k)$. For simplicity, $\{\underline{w}(k)\}$ and $\{\underline{v}(k)\}$ are assumed independent.

Now let $\underline{Z}$ be the sequence of observations

$$\underline{Z}(j) = \{\underline{z}(1),\ldots,\underline{z}(j)\} \tag{2-45}$$

Given a realization of the sequence of observations $\{\underline{z}(1),\ldots,\underline{z}(j)\}$, that is, given $\underline{Z}(j)$, the "discrete estimation" problem consists of computing an estimate of $\underline{x}(k)$ based on $\underline{Z}(j)$. If $k < j$, the problem is called "discrete smoothing" problem, if $k > j$, the problem is called "discrete prediction" and if $k=j$, the problem is called the "discrete filtering" problem [7, 31].

2.3.1. Introduction to Filtering Theory

Obviously, it not possible to determine $\underline{x}$ uniquely from equation 2-43 due to the presence of the errors vector $\underline{v}$. However, since there are more measurements than unknowns, we can choose an estimator of $\underline{x}$ that minimizes, in some chosen sense, the effect of the errors.

One such estimation is the least square estimation, where the estimator is chosen to minimize the sum of the squares of the errors. More precisely, $\underline{x}_{LS}$ is defined as the least squares estimator of $\underline{x}$ given the data $\underline{Z}$ it minimizes J_{LS}

$$J_{LS} = (\underline{z} - \underline{\underline{H}}\cdot\underline{x})^T \cdot\underline{\underline{W}}\cdot(\underline{Z} - \underline{\underline{H}}\cdot\underline{x}) \qquad (2\text{-}46)$$

where $\underline{\underline{W}}$ is a symmetric, positive-definite weighting matrix whose elements are chosen to emphasize (or de-emphasize) the influence of specific measurements upon the estimate $\underline{x}$.

Equation 2-45 yields a unique least square estimate $\underline{x}_{LS}$ give by

$$\hat{\underline{x}}_{LS} = (\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{Z} \qquad (2\text{-}47)$$

Note that $\underline{x}_{LS}$ is a linear function of the data $\underline{Z}$ and has some properties worth mentioning.

i) The error in the estimator $\underline{x}_{LS}$ is a linear function of the measurements errors $\underline{v}$. This is so, since

$$\begin{aligned}\tilde{\underline{x}}_{LS} \triangleq \underline{x} - \hat{\underline{x}}_{LS} &= \underline{x} - (\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot(\underline{\underline{H}}\cdot\underline{x} + \underline{\underline{L}}\cdot\underline{v})\\ &= \underline{x} - (\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})\cdot\underline{x} - (\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}\cdot\underline{v}\\ &= [-(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}]\cdot\underline{v} \qquad (2\text{-}48)\end{aligned}$$

ii) Using equation 2-48 it follows immediately that the residual $\underline{r}$ can be written as

$$\begin{aligned}\underline{r} \triangleq \underline{z} - \underline{\underline{H}}\cdot\hat{\underline{x}}_{LS} = \underline{\underline{H}}\cdot\tilde{\underline{x}}_{LS} + \underline{\underline{L}}\cdot\underline{v} &= -\underline{\underline{H}}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}\cdot\underline{v} + \underline{\underline{L}}\cdot\underline{v}\\ &= [\underline{\underline{I}} - \underline{\underline{H}}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}]\cdot\underline{\underline{L}}\cdot\underline{v} \qquad (2\text{-}49)\\ &= \underline{\underline{A}}\cdot\underline{\underline{L}}\cdot\underline{v}\end{aligned}$$

Since $\underline{\underline{A}} = [\underline{\underline{I}}-\underline{\underline{H}}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}]$ is symmetric and idempotent (i.e., $\underline{\underline{A}}^2 = \underline{\underline{A}}$) is an orthogonal projection matrix and thus, the residual is orthogonal to the columns of the observation matrix.

Up till now, probabilistic considerations have not yet entered the discussion. However, since the error $\underline{v}$ is assumed unknown, it is natural to regard it as random vector as defined in equation 2-43. With these assumptions, several other properties can be exhibited for $\underline{x}_{LS}$.

iii) The mean value of the error in the estimate is zero as is seen from using equation 2-50 to evaluate

$$\begin{aligned} E[\tilde{\underline{x}}_{LS}] &= -E[(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}\cdot\underline{v}] \\ &= -(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}\cdot E[\underline{v}] = \underline{0} \end{aligned} \quad (2\text{-}50)$$

In general, estimations which satisfy equation 2-49 are said to be "unbiased".

iv) The covariance of the error in the estimation is given by

$$E[\tilde{\underline{x}}_{LS}\cdot \tilde{\underline{x}}_{LS}^T] = \underline{\underline{P}} = (\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T\cdot\underline{\underline{W}}^T\cdot\underline{\underline{H}}\cdot[(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}]^T \quad (2\text{-}51)$$

It should be noted that the error covariance matrix $\underline{\underline{P}}$ doesn't depend on the measurement $\underline{Z}$.

v) It follows directly from equation 2-51 that the residual has zero mean and covariance

$$E[(\underline{z} - \underline{\underline{H}}\cdot\hat{\underline{x}}_{LS})\ (\underline{z} - \underline{\underline{H}}\cdot\hat{\underline{x}}_{LS})^T] =$$

$$[\underline{\underline{I}} - \underline{\underline{H}}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}]\cdot\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T\cdot\underline{\underline{H}}\cdot[\underline{\underline{I}} - \underline{\underline{H}}\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{W}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{\underline{W}}]^T \tag{2-51}$$

The error covariance matrix provides a measure of the behavior of the estimator. It is natural to attempt to determine the estimator that will minimize the error variances. An important performance index, which can be regarded as the probabilistic version of least squares, equation 2-46, is provided by the mean-square error

$$J_{MS} = E[(\underline{x} - \hat{\underline{x}})^T\cdot(\underline{x} - \hat{\underline{x}})] \tag{2-52}$$

To connect the use of this performance index with the least-square index it is convenient to consider an arbitrary linear, unbiased estimation $\hat{\underline{x}}$ of $\underline{x}$. That is, let $\hat{\underline{x}}$ be

$$\underline{x} = \underline{\underline{K}}\cdot\underline{z} = \underline{\underline{K}}\cdot\underline{\underline{H}}\cdot\underline{x} + \underline{\underline{K}}\cdot\underline{\underline{L}}\cdot\underline{v} \tag{2-53}$$

The requirement that $\hat{\underline{x}}$ is unbiased implies that $\underline{\underline{K}}$ must be chosen so that

$$\underline{\underline{K}}\cdot\underline{\underline{H}} = \underline{\underline{I}} \tag{2-54}$$

This implies that

$$\underline{x} - \underline{\hat{x}} = \underline{x} - \underline{\underline{K}} \cdot \underline{z} \qquad (2\text{-}55)$$

so it follows from equation 2-53 that

$$J_{MS} = E[\underline{v}^T \cdot \underline{\underline{L}}^T \cdot \underline{\underline{K}}^T \cdot \underline{\underline{K}} \cdot \underline{\underline{L}} \cdot \underline{v}] = \text{trace } (E[\underline{\underline{K}} \cdot \underline{\underline{L}} \cdot \underline{v} \cdot \underline{v} \cdot \underline{\underline{L}}^T \cdot \underline{\underline{K}}^T])^T$$

$$= \text{trace } (\underline{\underline{K}} \cdot \underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T \cdot \underline{\underline{K}}^T) \qquad (2\text{-}56)$$

The determination of the linear, unbiased estimator that minimizes the mean-square error is equivalent to determining the gain $\underline{\underline{K}}$ that minimize equation 2-57. It can be shown [7] that the solution is

$$\underline{\underline{K}} = [\underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{\underline{H}}]^{-1} \cdot \underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \qquad (2\text{-}57)$$

Then equations 2-54 becomes

$$\underline{\hat{x}}_{LS} = [\underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{\underline{H}}]^{-1} \cdot \underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{z} \qquad (2\text{-}58)$$

Obviously, equation 2-55 is satisfied and the error covariance matrix associated with this estimate is

$$\underline{\underline{P}} = [\underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{\underline{H}}]^{-1} \qquad (2\text{-}59)$$

Comparing equations 2-58 and 2-60 with equation 2-47 and 2-51 it is apparent that the generalized least squares and linear minimum mean square estimators are identical when the weighting matrix $\underline{\underline{W}}$ is chosen as a function of the inverse of the measurement noise covariance matrix, i.e., $\underline{\underline{W}} = (\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}$

Generalized least-squares is regarded, hereafter, as using the natural weighting matrix $(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}$.

Alternatively, one may use the maximum likelihood philosophy, which takes as the state estimate, i.e., $\underline{x}_{ML}$, that value which maximizes the probability of the measurements $\underline{Z}$ that actually occurred, taking into account known statistical properties of $\underline{v}$.

Furthermore, it can be proven that for Gaussian random variables any of the above mentioned methods provide identical results for the linear estimator problem as long as the assumptions are the same in each case [30].

2.3.2 The Kalman Filter

The optimal estimator for the linear dynamical system given in equations 2-43 and 2-44 is given by the following set of equations:

-Between observations,

State estimate extrapolation

$$\hat{\underline{x}}(k+1) = \underline{\underline{\phi}}(k)\cdot\hat{\underline{x}}(k) \tag{2-60}$$

Error variance extrapolation:

$$\underline{\underline{P}}'(k+1) = \underline{\underline{\phi}}(k)\cdot\underline{\underline{P}}(k)\cdot\underline{\underline{\phi}}^T(k) + \underline{\underline{\Gamma}}(k)\cdot\underline{\underline{Q}}(k)\cdot\underline{\underline{\Gamma}}^T(k) \qquad (2\text{-}61)$$

-At observations,

State estimate update:

$$\hat{\underline{x}}(k+1) = \hat{\underline{x}}'(k+1) + \underline{\underline{K}}(k+1)[\underline{z}(k+1) - \underline{\underline{H}}(k+1)\cdot\hat{\underline{x}}'(k+1)] \qquad (2\text{-}62)$$

Kalman Filter gain update:

$$\underline{\underline{K}}(k+1) = \underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1)\cdot[\underline{\underline{H}}(k+1)\cdot\underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1) + \underline{\underline{L}}(k+1)\cdot\underline{\underline{R}}(k+1)\cdot\underline{\underline{L}}^T(k+1)]^{-1} \qquad (2\text{-}63)$$

$$= \underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1)\cdot[\underline{\underline{L}}(k+1)\cdot\underline{\underline{R}}(k+1)\cdot\underline{\underline{L}}^T(k+1)]^{-1} \qquad (2\text{-}64)$$

Error Covariance Update:

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} - \underline{\underline{K}}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\underline{\underline{P}}(k+1)$$

$$= \underline{\underline{P}}(k+1) - \underline{\underline{P}}(k+1)\cdot\underline{\underline{H}}^T(k+1)\cdot[\underline{\underline{H}}(k+1)\cdot\underline{\underline{P}}(k+1) \cdot\underline{\underline{H}}^T(k+1) + \underline{\underline{L}}(k+1)\cdot\underline{\underline{R}}(k+1)\cdot\underline{\underline{L}}^T(k+1)]^{-1} \cdot\underline{\underline{H}}(k+1)\cdot\underline{\underline{P}}(k+1) \qquad (2\text{-}64)$$

This optimal estimator was first described by Kalman in 1960, and the algorithm for recursive filtering which is given by equations 2-61: 2-65 is called the "Kalman Filter" [17].

2.3.3 Filter Divergence and Error Compensation Techniques[16]

The Kalman filter utilizes all available data, including all observations and prior data. It is clearly optimal to base an estimate on all available information. This is predicted, however, on the knowledge of the dynamics, measurement function, and the statistical parameters associated with the dynamical system. If, for example, the dynamics are imprecisely known, then, the filter might learn the state "too well". Because of imprecise dynamics used in the filter, the "good estimate" may be wrong.

This problem is referred to as the problem of "filter divergence". It is particularly acute when the noise inputs to the system are small and when the measurement noise is small. Eventually, the error covariance matrix, $\underline{\underline{P}}(k)$(equation 2-64) becomes very small, the filter gain, $\underline{\underline{K}}(k)$, (equation 2-63) is therefore small, and subsequent observations have little effect on the estimate. But the dynamical system model in the filter is different from the actual system model so that difference between the estimate and the state can diverge. In actual applications, the onset of divergence manifests itself by the inconsistency of the residuals, $\underline{r}(k)$ (equation 2-49), with their predicted statistics. Residuals become biased and larger in magnitude than their RMS values, as predicted by equation 2-52.

Faced with model errors, techniques for synthesizing a filter that works are required. That is, a filter that not only produces bounded error, but also one that gives satisfactory performance.

The basic idea, then, in order to prevent divergence, is to increase the covariance matrix, $\underline{\underline{P}}(k)$. This is justified, since such an increase in the covariance matrix "compensates" for model errors. Model errors in some sense add uncertainty to the system which should reflect itself in some degradations in certainty (increase in $\underline{\underline{P}}(k)$).

Two somewhat different points of view can be taken with respect to model error compensation. One is to provide a (fictitious) noise input to the system or to attribute the errors to inaccuracies in some of the parameters of the system, thus effecting an increase in the covariance matrix directly [5,34].

The other point of view is to overweigh the most recent data in some way. This causes the filter to forget old information, and thus indirectly increases the covariance matrix. The physical justification for forgetting old observations is that old observations, when predicted over long time arcs through an erroneous system, can be valueless.

In addition, divergence problems may be generated by numerical instability encountered during the implementation of the filter, due to the inherently finite nature of the digital hardware. The reader is referred to [7, Chap.8].

2.4 Elements of Linear System Theory

2.4.1 Introduction

In this section, two fundamental concepts of linear system theory which are intimately related to the basic ideas of estimation and control will be considered. These notions are called "observability" and "controllability" [1,3,20].

The formulation and study of the concepts of observability and controllability are based, respectively, on the following two questions which are motivated by obvious physical considerations:

1. Under what conditions is it possible to establish, in a finite interval of time, the time history of the state $\underline{x}$ of a dynamic system given the time history of the measurement vector $\underline{Z}$ over the same time interval?
2. Under what conditions is it possible to transfer the state of a dynamic system from a given initial state to a desired terminal state in a finite amount of time using a control vector $\underline{u}$?

Before proceeding, an illustration of these two notions for a general system with the aid of some simple diagrams might be helpful. Consider a dynamic system with state vector $\underline{x}$, control vector $\underline{u}$ and measurement vector $\underline{Z}$. Furthermore, assume that there are no disturbances or measurement errors.

First, suppose that the system block diagram can be put in the form shown in figure 5 where $\underline{y}$ is a vector whose components are some or all of the elements of $\underline{x}$. Because of the system's structure there

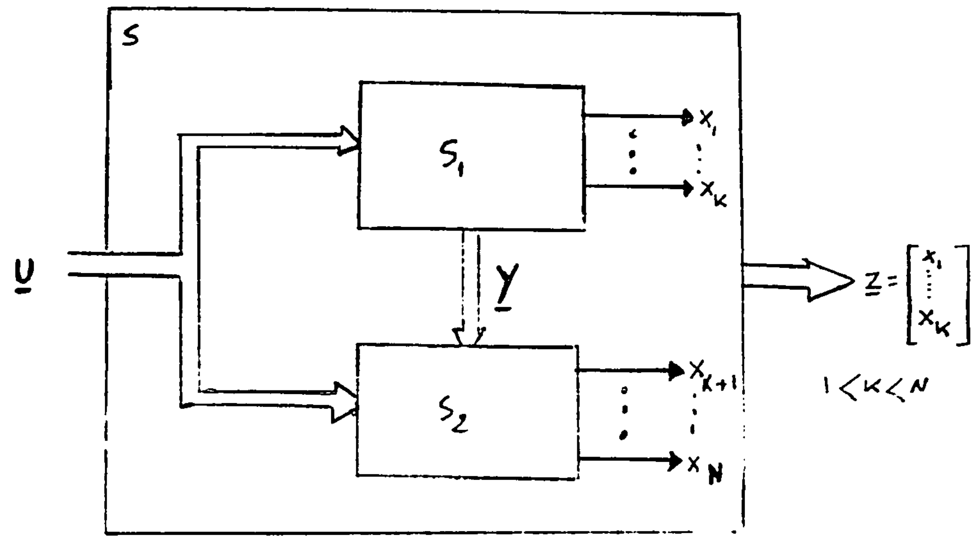


Fig. 5 An Unobservable System

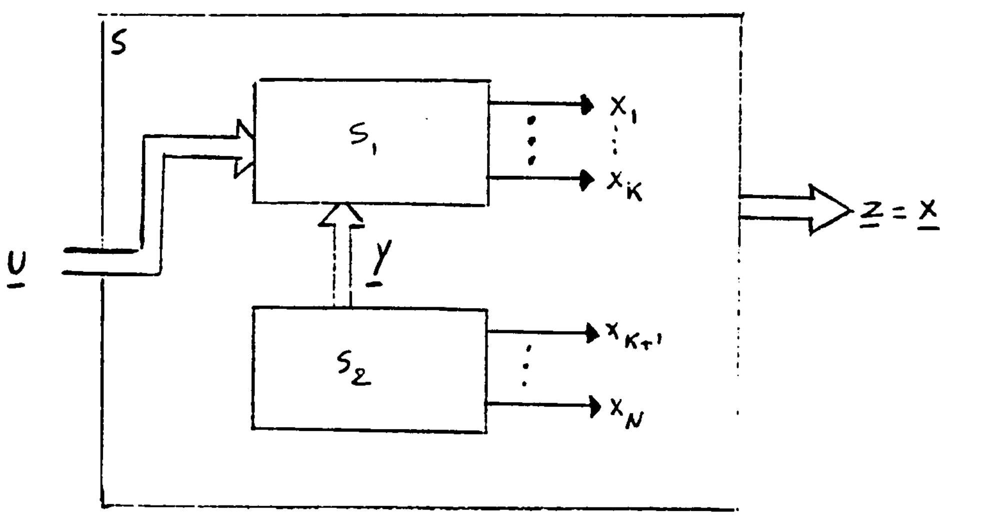


Fig. 6 An Uncontrollable System

is no way that the values $x_{k+1},\ldots,x_n$ can be determined from an examination of $\underline{z}$ since those values do not affect $x_1,\ldots,x_k$, nor do they appear in $\underline{z}$. Hence, the system is unobservable. On the other hand, if $\underline{u}$ affects all the elements of $\underline{x}$, the system is controlable.

In a similar way, the system in figure 6 is observable, but it is uncontrollable since $\underline{u}$ affects only the variable $x_1,\ldots,x_k$.

All theorems in this section will be stated without proof. For further details, the reader is referred to [3,20].

2.4.2 Observability of Discrete Linear Systems:

Suppose the system under consideration is described by the following two equations

$$\underline{x}(k+1) = \underline{\underline{\phi}}(k)\cdot\underline{x}(k) + \underline{\underline{\Gamma}}(k)\cdot\underline{u}(k) \qquad (2\text{-}67)$$

and

$$\underline{z}(k+1) = \underline{\underline{H}}(k+1)\cdot\underline{x}(k+1) \qquad (2\text{-}68)$$

where the initial state, $\underline{x}(0)$, is unknown, and the inputs, $\underline{u}(k)$'s are assumed given for any $k\geq 0$.

The discrete linear system of equations 2-67:2-68 is said to be observable if $\underline{x}(0)$ can be determined from the set of measurements $\{\underline{z}(1),\ldots,\underline{z}(N)\}$ for some finite time N. If this is true for any initial time (k=0 corresponds to t_o), the system is said to be "completely observable".

Theorem I: The discrete linear system of Equations 2-67:2-68 is completely observable if and only if the symmetric n x n matrix

$$\underline{\underline{M}}_d(o,N) = \sum_{k=1}^{N} \underline{\underline{\phi}}^T(k,0)\cdot\underline{\underline{H}}^T(k)\cdot[\underline{\underline{L}}(k)\cdot\underline{\underline{R}}(k)\cdot\underline{\underline{L}}^T(k)]^{-1}\cdot\underline{\underline{H}}(k)\cdot\underline{\underline{\phi}}(k,0) \qquad (2\text{-}69)$$

is positive definite for some N>0, where

$$\underline{\underline{\phi}}(k,0) = \underline{\underline{\phi}}(k,k-1)\ldots\underline{\underline{\phi}}(0,1), \text{ for } k = 1,\ldots,N.$$

Corollary I: The constant coefficient discrete linear system

$$\underline{x}(k+1) = \underline{x}(k) \qquad (2\text{-}70)$$

$$\underline{z}(k+1) = \underline{\underline{H}}\cdot\underline{x}(k+1) \qquad (2\text{-}71)$$

where $k \geq 0$, is completely observable if and only if the n x mn matrix $\underline{\underline{\Omega}}$

$$\underline{\underline{\Omega}} = [\underline{\underline{H}}^T, \underline{\underline{\phi}}^T\cdot\underline{\underline{H}}^T,\ldots, (\underline{\underline{\phi}}^T)^{n-1}\cdot\underline{\underline{H}}^T]^T \qquad (2\text{-}72)$$

rank n.

Corollary II: The discrete linear system of equation 2-67:2-68 is uniformly completely observable if there exist a positive integer N and positive constants α and β such that

$$\alpha\underline{\underline{I}} \leq \sum_{i=k-N}^{k} \underline{\underline{\phi}}^T(i,k)\cdot\underline{\underline{H}}^T(i)\cdot[\underline{\underline{L}}^T(i)\cdot\underline{\underline{R}}(i)\cdot\underline{\underline{L}}^T(i)]^{-1}\cdot\underline{\underline{H}}(i)\cdot\underline{\underline{\phi}}(i,k) \leq \beta\underline{\underline{I}} \quad (2\text{-}73)$$

for all $k \geq N$ and were $\underline{\underline{\phi}}(i,k)$ was defined previously.

2.4.3 Controllability of discrete linear system:

Once again consider the discrete linear system given by

$$\underline{x}(k+1) = \underline{\underline{\phi}}(k)\cdot\underline{x}(k) + \underline{\underline{\Gamma}}(k)\cdot\underline{u}(k) \qquad (2\text{-}74)$$

where $k \geq 0$ and $\underline{x}(0)$ is assumed to be known.

The discrete linear system of equation 2-74 is said to be controllable at time k=0 (corresponding to an initial time t_o) if there exists a control sequence $\{\underline{u}(0),\underline{u}(1),\ldots,\underline{u}(N-1)\}$ depending on $\underline{x}(0)$ and the initial time for which $\underline{x}(N) = 0$ where N is finite. If this is true for all $\underline{x}(0)$ and initial times, the system is completely controllable.

Theorem II: The discrete linear system of equation 2-74 is completely controllable if and only if the symmetric n x n matrix

$$\underline{\underline{W}}_d(0,N) = \sum_{k=1}^{N} \underline{\underline{\phi}}(0,k)\cdot\underline{\underline{\Gamma}}(k,k-1)\cdot\underline{\underline{Q}}(k)\cdot\underline{\underline{\Gamma}}^T(k,k-1)\cdot\underline{\underline{\phi}}^T(0,k) \qquad (2\text{-}75)$$

is positive definite for some finite $N > 0$ where

$\underline{\underline{\phi}}(0,k) = \underline{\underline{\phi}}(0,1)\ \ldots\ \underline{\underline{\phi}}(k-1,k)$, for $k = 1,2,\ldots N$.

Corollary III: The constant coefficient discrete linear system

$$\underline{x}(k+1) = \underline{\underline{\phi}} \cdot \underline{x}(k) + \underline{\underline{\Gamma}} \cdot \underline{u}(k) \tag{2-76}$$

where $k \geq 0$, is completely controllable if and only if the n x nr matrix $\underline{\underline{C}}$,

$$\underline{\underline{C}} = [\underline{\underline{\Gamma}},\ \underline{\underline{\phi}} \cdot \underline{\underline{\Gamma}}, \ldots, \underline{\underline{\phi}}^{n-1} \cdot \underline{\underline{\Gamma}}] \tag{2-77}$$

has rank n.

Corollary IV: The discrete linear system of equation 2-74 is uniformly completely controllable if there exists a positive integer N and positive constants α and β such that

$$\alpha \underline{\underline{I}} \leq \sum_{i=k-N}^{k-1} \underline{\underline{\phi}}(k,i+1) \cdot \underline{\underline{\Gamma}}(k,i+1) \cdot \underline{\underline{Q}}(i) \cdot \underline{\underline{\Gamma}}^T(k,i+1) \cdot \underline{\underline{\phi}}^T(k,i+1) \leq \beta \underline{\underline{I}} \tag{2-78}$$

for some value of $N > 0$, and where $\underline{\underline{\phi}}(k,i)$ was defined previously.

2.4.4 Stability of Discrete Filters:

The main objective of this section is to state the conditions for uniform asymptotic stability of the discrete Kalman filter. In this connection, it is interesting to note that optimality of the filter does not imply stability. The stability needs to be proved.

Lemma I: If the dynamical discrete linear system of equations 2-67: 2-68 is uniformly completely observable and uniformly completely controllable and if $P_o \geq 0$, then $\underline{\underline{P}}(k)$ is uniformly bounded from above for all $k \geq N$,

$$\underline{\underline{P}}(k) \leq \underline{\underline{M}}_d^{-1}(k,k-N) + \underline{\underline{W}}_d(k,k-N) \leq \left(\frac{1+\alpha\beta}{\alpha}\right)\cdot\underline{\underline{I}} \qquad (2\text{-}79)$$

Lemma II: If the dynamical discrete system of equations 2-67:2-68 is uniformly completely observable and uniformly completely controllable, and if $P_o > 0$, then $\underline{\underline{P}}(k)$ is uniformly bounded form below for $k \geq N$,

$$\underline{\underline{P}}(k) \geq [\underline{\underline{M}}_d(k,k-N) + \underline{\underline{W}}_d^{-1}(k,k-N)]^{-1} \geq \left(\frac{\alpha}{1+\alpha\beta}\right)\cdot\underline{\underline{I}} \qquad (2\text{-}80)$$

It is interesting to note that if the system is uniformly completely observable and noise free ($\underline{\underline{W}}_d = \underline{\underline{0}}$), then

$$\underline{\underline{P}}(k) \leq \underline{\underline{M}}_d^{-1}(k,0), \text{ for } k \geq N \qquad (2\text{-}81)$$

Theorem III: If the dynamical discrete linear system of equation 2-67:2-68 is uniformly completely observable and uniformly completely controllable, and if $\underline{\underline{P}}_o \geq 0$, then the discrete linear filter of section 2.3.2. is uniformly asymptotically stable.

CHAPTER 3

ERROR MODEL

3.1 Introduction

There are a number of different approaches to developing the kinematic model for a robot manipulator. The most popular method has been the procedure established by Denavit and Hartenberg [6] that is based on homogeneous transformation matrices. This procedure consists of establishing coordinate systems on each joint motion axis. Each coordinate system is then related to the next through a specific set of coefficients in the homogeneous transformation matrices. This technique is used by Wu [40] to examine robots with small variations in their kinematic arrangement. Unfortunately, a problem arises when two revolute joint axes are parallel. One of the parameters in the Denavit-Hartenberg model is the length of the common normal between two consecutive axes. If two axes are parallel, there are an infinite number of common normals all of the same length. The problem arises when one of the axes becomes slightly misaligned. In this case the length of the common normal can assume virtually any positive value. This variation may not be continuous as the axes become misaligned. The result is that the coefficients in the kinematic model do not vary proportionally with the degree of misalignment. There can be discontinuities or very large variations in the coefficients for small degrees of misalignment which create severe numerical difficulties in the identification process. For this reason, a number of investigators have chosen to use a modified form of the Denavit-Hartenberg formulation for all axes except those which are parallel or nearly parallel [11,12]. Parallel axes are treated as special cases with a different formulation that allows the parameters to vary in proportion to the degree of misalignment. Similar approaches have been used by Hsu and Everett [14] and Whitney, Lozinski, and Rourke [39] who modified the basic procedure described by Denavit and Hartenberg. Ibarra and Perreira [15] utilize the Denavit-Hartenberg procedure and then modify each transformation matrix with a differential screw matrix for small misalignments. Mooring and Tang [22] used the general spatial rigid body displacement equation to develop a model to handle manipulators with variations in the kinematic model.

The main issues in modelling seem to be the adequacy and the numerical stability of the representation. To implement a Kalman filter, an accurate state space model of the robotic system to be estimated must be formulated. This is the most challenging part of the estimation process and requires good engineering judgement. The system state vector is the set of variables that completely describes the system. If more variables are specified, they are redundant. If too few are specified, then some aspects of the system are ignored and this may affect both control and estimation tasks adversely. Therefore, an adequate representation should be chosen that has the ability to characterize changes in the kinematics of the robot in terms of a finite set of parameters. The other criterion the model has to satisfy is numerical stability. A stable representation implies that small changes in the actual kinematics of the robot will result in similarly small changes in the kinematic model of the robot. The goal of this chapter is to provide such a representation in a form compatible with Kalman filters.

3.1 Error Mapping

In the following sections, it will be necessary to map each link error sources to a base coordinate frame. Those error mapping equations will be developed in this section.

Let ${}^{iN}\underline{\underline{A}}^{(i-1)A}$ denote a transformation from the i-1 coordinate frame to the i coordinate frame. The A superscript (for Actual) signifies that the position of the coordinate frame i-1 is known and the N superscript (for Nominal) specifies that the exact position of frame i is unknown due to joints errors.

Also define ${}^{iA}\underline{\underline{dA}}_{i}{}^{iN}$, equation 2-42, as an error transformation such that when premultiplied by ${}^{iN}\underline{\underline{A}}^{(i-1)A}$ gives the exact (and actual) position of the coordinate frame i (figure 7) with respect to the actual frame (i-1), and also define ${}^{iA}\underline{\underline{dA}}_{(i-1)}{}^{iN}$ as the error transformation such that postmultiplied by ${}^{iN}\underline{\underline{A}}^{(i-1)A}$ gives the exact position of frame i with respect to the actual frame i-1.

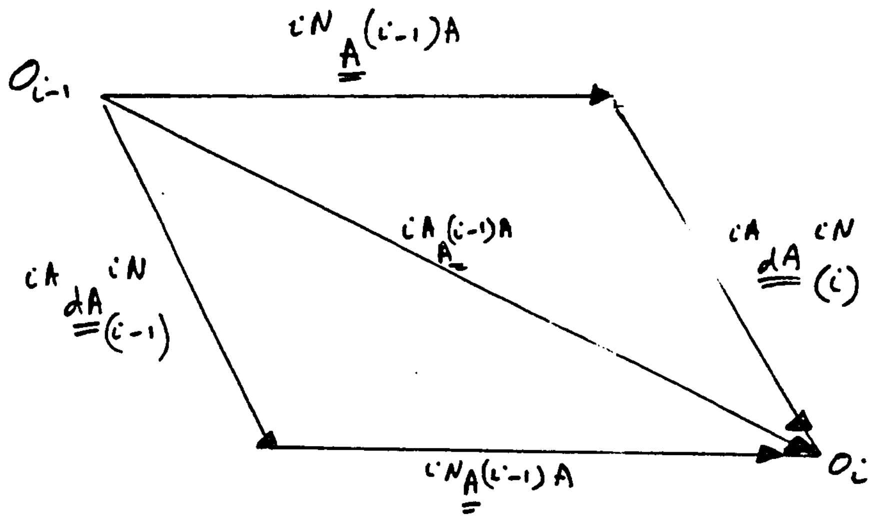


Fig. 7 Differential Error Transformation Matrix Definition

Then

$$ {}^{iA}\underline{\underline{A}}^{(i-1)A} = {}^{iA}\underline{\underline{dA}}^{iN}_{(i-1)} \cdot {}^{iN}\underline{\underline{A}}^{(i-1)A} $$

$$ = \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_i \Delta & -\underline{d}_i \\ \hline 0\ 0\ 0 & 1 \end{array}\right]_{(i-1)} \cdot {}^{iN}\underline{\underline{A}}^{(i-1)A} \qquad (3\text{-}1) $$

where $\underline{\delta}_i$ and $\underline{d}_i$ are small rotational and translational error vectors represented in frame iN.

One is concerned, in the error mapping, with the representation of an error vector in a frame preceding the one this error vector is represented in. The motivation behind this mapping will become clear, as all the error vectors will have to be referred to a unified and well defined base frame.

Referring to figure 7 for a second time, if the equivalent error transformation in frame (i-1)A is denoted by ${}^{(i-1)A}\underline{\underline{dA}}^{iN}_{i-1}$, the relationship between ${}^{(i-1)A}\underline{\underline{dA}}^{iN}_{i-1}$ and ${}^{iA}\underline{\underline{dA}}^{iN}_{i-1}$ is sought. Equation 3-1 can be rewritten as

$$ {}^{iA}\underline{\underline{A}}^{(i-1)A} = {}^{iN}\underline{\underline{A}}^{(i-1)A} \cdot {}^{(i-1)A}\underline{\underline{dA}}^{iN}_{i-1} \qquad (3\text{-}2) $$

in other words

$$ {}^{(i-1)A}\underline{\underline{dA}}^{iN}_{i-1} = \left({}^{iN}\underline{\underline{A}}^{(i-1)A} \right)^{-1} \cdot {}^{iA}\underline{\underline{dA}}^{iN}_{i} \cdot {}^{iN}\underline{\underline{A}}^{(i-1)A} $$

$$ = {}^{(i-1)A}\underline{\underline{A}}^{iN} \cdot {}^{iA}\underline{\underline{dA}}^{iN}_{i} \cdot {}^{iN}\underline{\underline{A}}^{(i-1)A} \qquad (3\text{-}3) $$

If the error vectors $\underline{\delta}$ and $\underline{d}$ are represented by a composite 6 x 1 vector

$$\underline{z} = [d^T, \delta^T]^T \tag{3-4}$$

it can then be shown that $(\underline{z}_i)_{i-1}$ and $(\underline{z}_i)_i$ are related by (see Appendix A).

$$(\underline{z}_i)_{i-1} = \left[\begin{array}{c|c} {}^{(i-1)}\underline{\underline{A}}_R{}^{iN} & (\underline{p}_{(i-1,i)} \cdot \Delta)\,{}^{(i-1)}\underline{\underline{A}}_R{}^{iN} \\ \hline 0 \quad 0 \quad 0 & {}^{(i-1)}\underline{\underline{A}}_R{}^{iN} \end{array} \right] \cdot (z_i)_i$$

$$= \left[\begin{array}{c|c} {}^{(i-1)}\underline{\underline{A}}_R{}^{iN} & {}^{(i-1)}\underline{\underline{A}}_R{}^{iN}\,(\underline{p}_{(i,i-1)} \cdot \Delta) \\ \hline 0 \quad 0 \quad 0 & {}^{(i-1)}\underline{\underline{A}}_R{}^{iN} \end{array} \right] \cdot (z_i)_i \tag{3-5}$$

If one denotes this Jacobian matrix by ${}^{i-1}\underline{\underline{J}}^i$, then the preceding equation can be rewritten as

$$(\underline{z}_i)_{i-1} = {}^{i-1}\underline{\underline{J}}^i \cdot (\underline{z}_i)_i \tag{3-6}$$

where

$$
{}^{i-1}\underline{\underline{J}}^{i} = \begin{bmatrix}
c\theta_i & s\theta_i & 0 & d_i s\theta_i c\alpha_i + a_i s^2\theta_i s\alpha_i & -d_i c\theta_i c\alpha_i - a_i s\theta_i c\theta_i s\alpha_i & -d_i s\alpha_i + a_i s\theta_i c\alpha_i \\
-s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & d_i c\theta_i - a_i s\theta_i c\theta_i s\alpha_i & d_i s\theta_i + a_i c^2\theta_i s\alpha_i & -a_i c\theta_i c\alpha_i \\
s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -a_i s\theta_i c\theta_i [1 + c\alpha_i] & -a_i [s^2\theta_i - c^2\theta_i \; c\alpha_i] & a_i c\theta_i s\alpha_i \\
0 & 0 & 0 & c\theta_i & s\theta_i & 0 \\
0 & 0 & 0 & -s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i \\
0 & 0 & 0 & s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i
\end{bmatrix} \tag{3-7}
$$

and

$$
\underline{z}_i = \left[d_{x_i},\, dy_i,\, dz_i,\, \delta_{x_i},\, \delta_{y_i},\, \delta_{z_i} \right]^T \tag{3-8}
$$

3.2 LINK ERROR MODELLING

There are a number of different approaches to developing the kinematic model for a robot manipulator. The most widely used is the method developed by Denavit and Hartenberg [4] and it is based on homogeneous transformation matrices as shown in 2.1.5.

In the sequel, it will be assumed that small variations in the position and orientation of two consecutive links can be modeled by small variations of the links parameters. As stated earlier,this assumption is violated if the previously mentioned method is used in the link geometry characterization of two consecutive joints that have parallel or near parallel axes. For this reason, a modified form of the Denavit-Hartenberg model will be used for axes that are parallel or near-parallel and the modification procedure is given in Appendix B.

As shown in 2.1.5 the transformation between the Nominal coordinate frame i-1 and the actual coordinate frame i is given by

$$ {}^{iN}\underline{\underline{A}}_i^{(i-1)A} = {}^{(i-1)A}\underline{\underline{A}}_i^{iN} = \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_i \\ -s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & -d_i c\alpha_i \\ s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -d_i s\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3-9} $$

It follows from section 3.1 that the transformation between the actual coordinate frames of i and i-1 are related by

$$ {}^{iA}\underline{\underline{A}}^{(i-1)A}\ (\theta + \delta\theta + \mu_\theta,\ d + \delta d + \mu_d,\ a + \delta a,\ \alpha + \delta\alpha) = $$

$$ {}^{iA}\underline{\underline{dA}}_{(i-1)}^{iN}\ (\delta\theta + \mu_\theta,\ \delta d + \mu_d,\ \delta a,\ \delta\alpha) \cdot {}^{iA}\underline{\underline{A}}^{(i-1)N}(\theta,d,a,\alpha) \tag{3-10} $$

where μ is the encoder resolution (or quantization) errors, $\delta\theta$ and δd are, depending on the type of joints, the measurements biases and δa and $\delta\alpha$ are the link geometric errors.

In order to compute ${}^{iA}\underline{\underline{dA}}^{iN}$, the error mapping techniques of section 3.1 are used. All the error sources of the left side of equation 3-10 are mapped to the (i-1)A coordinate frame.

$$ {}^{iA}\underline{\underline{A}}^{(i-1)A}(\theta + \delta\theta + \mu_\theta,\ d + \delta d + \mu_d,\ a + \delta a,\ \alpha + \delta\alpha) \cdot {}^{(i-1)N}\underline{\underline{A}}^{iA}(\theta,d,a,\alpha) $$

$$ = {}^{iA}\underline{\underline{dA}}^{iN}_{i-1}(\delta\theta+\mu_\theta,\ \delta d + \mu_d,\ \delta a,\ \delta\alpha) \tag{3-11} $$

${}^{iA}dA^{iN}_{(i-1)}$ is a differential matrix and assumptions A.1:A.3 hold. Furthermore, neglecting products of differential angles, the result

$$ {}^{iA}\underline{\underline{dA}}^{iN}_{(i-1)} = \begin{bmatrix} 1 & c\alpha_i\delta\theta'_i & -s\alpha_i\delta\theta'_i & -\delta a_i \\ -c\alpha_i\delta\theta'_i & 1 & \delta\alpha_i & -a_i c\alpha_i\delta\theta'_i - s\alpha_i\delta d_i \\ s\alpha_i\delta\theta'_i & -\delta\alpha_i & 1 & s_i d\alpha_i\delta\theta'_i - c\alpha_i\delta d'_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

or equivalently

$$ \underline{z}_i = [\delta a_i,\ a_i c\alpha_i\delta\theta'_i + s\alpha_i\delta d'_i,\ c\alpha_i\delta d'_i - a_i\delta\alpha_i\delta\theta'_i,\ \delta\alpha_i s\alpha'_i,\ \delta\theta'_i,\ c\alpha'_i\delta\theta'_i] \tag{3-12} $$

where $\delta\theta'_i = \delta\theta_i + \mu_{\theta i}$

and $\delta d'_i = \delta d_i + \mu_{di}$

Equation 3-13 can also be rewritten as

$$\begin{bmatrix} \delta a_i \\ a_i c\alpha_i \delta\theta'_i + s\alpha_i \delta d'_i \\ c\alpha_i \delta d'_i - a_i s\alpha_i \delta\theta'_i \\ \delta\alpha_i \\ \delta\alpha_i \ \delta\theta'_i \\ c\alpha_i \delta\theta'_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a_i c\alpha_i & s\alpha_i & 0 & 0 \\ -a_i s\alpha_i & c\alpha_i & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s\alpha_i & 0 & 0 & 0 \\ c\alpha_i & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta\theta'_i \\ \delta d'_i \\ \delta a_i \\ \delta\alpha_i \end{bmatrix} \qquad (3\text{-}14)$$

such that

$$\underline{z}_i = \underline{\underline{G}}_i \cdot \underline{g}_i \qquad (3\text{-}15)$$

It is interesting to note that $\underline{\underline{G}}_i$ is a constant matrix obtained form the ith link parameters α(the link twist) and a (the link length). It also should be noted that for a revolute joint, the encoder resolution error or bias corresponding to the distance between normals does not exist, i.e., $\delta d'_i = \delta d_i$.

The error model for a prismatic joint can be easily obtained from equation 3-14, by setting the encoder bias corresponding to the angle between normals to zero, i.e., $\delta\theta'_i = \delta\theta_i$ and setting a_i to zero so $\underline{\underline{G}}_i$ matrix becomes

$$\underline{\underline{G}}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & s\alpha_i & 0 \\ 0 & 0 & 1 \\ s\alpha_i & 0 & 0 \\ c\alpha_i & 0 & 0 \end{bmatrix} \qquad (3\text{-}16)$$

and $\underline{g}_i = [\delta\theta, \ \delta d', \ \delta\alpha]$ (3-17)

3.3 OBSERVATION EQUATION

Figure 8 illustrates a serial manipulator having n links. iN and iA, will as before denote the nominal and actual coordinate frames associated with link i. A stationary link 0 has to be introduced to account for the errors associated with the joint 1.

The homogeneous transformation between the frame nA and the base frame B is obtained by successive multiplication of adjacent transformations from right to left. That is

$$ {}^{B}\underline{\underline{T}}{}^{nA} = {}^{B}\underline{\underline{A}}{}^{0N} \cdot {}^{0N}\underline{\underline{A}}{}^{0A} \;\ldots\ldots\; {}^{(n-1)A}\underline{\underline{A}}{}^{nN} \;\; {}^{nN}\underline{\underline{A}}{}^{nA} \qquad (3\text{-}18) $$

But, following the previous assumption that a transformation between a nominal and an actual frame is a differential transformation, then equation 3-18 can be expressed as

$$ {}^{B}\underline{\underline{T}}{}^{nA} = {}^{B}\underline{\underline{A}}{}^{0N} \cdot \left({}^{0A}\underline{\underline{dA}}{}^{0N} \right)^{-1} \ldots\ldots\ldots {}^{(n-1)}\underline{\underline{A}}{}^{nN} \cdot \left({}^{mA}\underline{\underline{dA}}{}^{nN} \right)^{-1} \qquad (3\text{-}19) $$

or, as shown in section 3.1,

$$ {}^{B}\underline{\underline{T}}{}^{nA} = {}^{B}\underline{\underline{A}}{}^{0N} \cdot {}^{0N}\underline{\underline{dA}}{}^{0A} \;\ldots\ldots\ldots\; {}^{(n-1)A}\underline{\underline{A}}{}^{nN} \cdot {}^{nN}\underline{\underline{dA}}{}^{nA} \qquad (3\text{-}20) $$

Using the results of section 3.2, the error transformations may be represented by an equivalent global error transformation in the base coordinate frame B, as

$$ {}^{nA}\underline{\underline{T}}{}^{B} = {}^{nN}\underline{\underline{T}}{}^{B} \cdot \underline{\underline{dA}}_{B} \qquad (3\text{-}21) $$

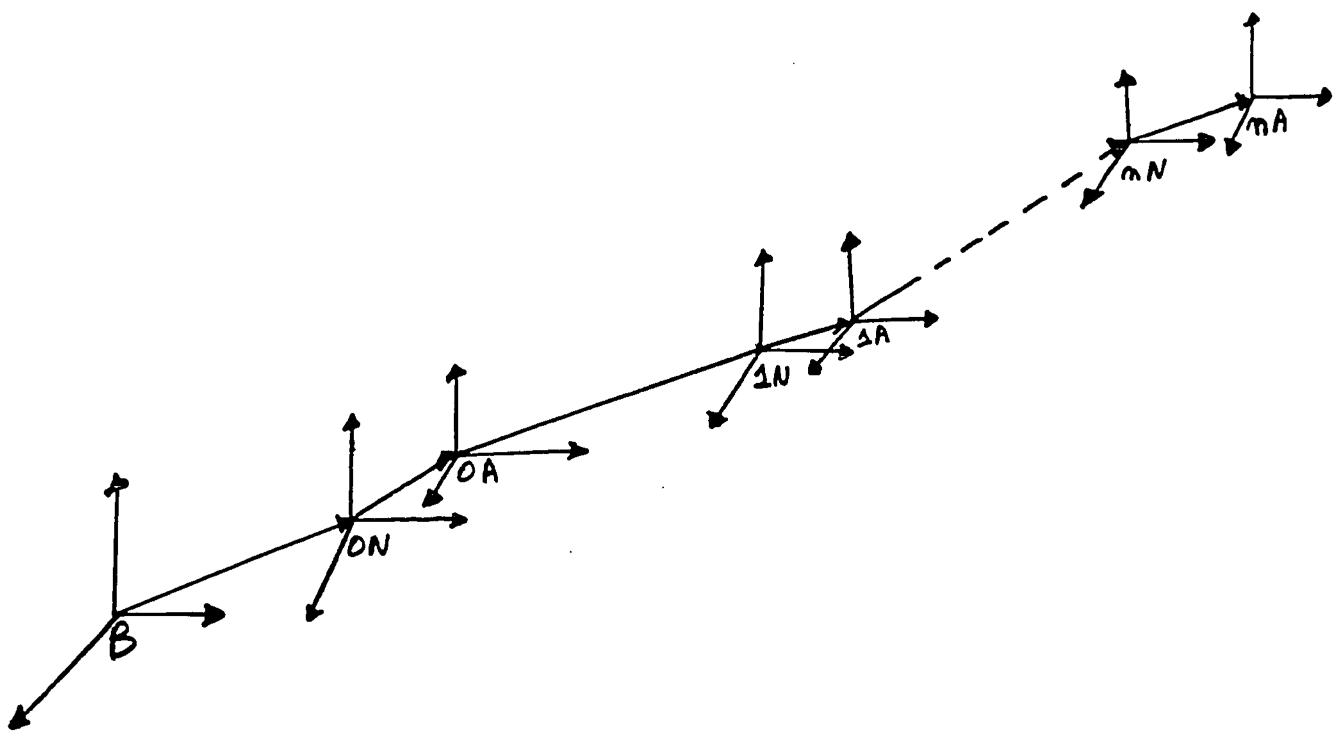


Fig. 8 Actual and Nominal Link Coordinate Frames

The equivalent error vector associated with $d\underline{\underline{A}}_B$, i.e., $\underline{z}_B$, can be computed using equation 3-5 as a function of all the error vector

$$\underline{z}_B = \sum_{i=0}^{n} {}^{B}\underline{\underline{J}}^{i} \cdot \underline{z}_i \qquad (3\text{-}22)$$

where ${}^{B}\underline{\underline{J}}^{i}$ is obtained from (Appendix C)

$${}^{B}\underline{\underline{J}}^{i} = {}^{B}\underline{\underline{J}}^{0} \cdot {}^{0}\underline{\underline{J}}^{1} \cdots {}^{i-1}\underline{\underline{J}}^{i} \qquad (3\text{-}23)$$

In order to estimate the manipulator geometric error, it is necessary to measure the actual position of the last coordinate frame by some means. Here we will not consider any specific measurement method but will assume the same mathematical model described in section 2.3.

If the measurement errors or noises in the Base frame B is represented by an error transformation $d\underline{\underline{A}}_V$, then

$${}^{nA}\underline{\underline{T}}^{B} = {}^{n}\underline{\underline{T}}_m^{B} \cdot d\underline{\underline{A}}_V \qquad (3\text{-}24)$$

where ${}^{n}\underline{\underline{T}}_m^{B}$ is the measured transformation of the final frame relative to the base frame. From equations 3-18 and 3-21 the global error transformation becomes

$$d\underline{\underline{A}}_B = {}^{B}\underline{\underline{T}}^{nN} \; {}^{n}\underline{\underline{T}}_m^{B} \cdot d\underline{\underline{A}}_V \qquad (3\text{-}25)$$

by defining

$$\underline{\underline{T}} = {}^{B}\underline{\underline{T}}^{nN} \cdot {}^{n}\underline{\underline{T}}_m^{B} = \left[\begin{array}{c|c} R & -\rho \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \qquad (3\text{-}26)$$

The error vector $\underline{z}_B = [\underline{d}_B^T, \underline{\delta}_B^T]$ is obtained form equation 3-22 as (see Appendix D)

$$\underline{d}_B = \underline{\underline{R}}\, \underline{d}_V + \underline{\rho} \tag{3-27}$$

$$\underline{\delta}_B \Delta = \underline{\underline{I}} - \underline{\underline{R}}\, (\underline{\underline{I}} - \underline{\delta}_V \Delta) \tag{3-28}$$

Assuming that the measurement transformation ${}^{n}\underline{\underline{T}}_m^B$ is sufficiently accurate so that $\underline{\underline{T}}$ in equation 3-22 has first order off diagonal terms and, in addition, that $\underline{\underline{R}}$ represents a small angle rotation transformation of

$$\underline{\underline{R}} = \begin{bmatrix} 1 & \delta z_m & -\delta y_m \\ -\delta z_m & 1 & \delta x_m \\ \delta y_m & -\delta x_m & 1 \end{bmatrix} = \underline{\delta}_m \Delta \tag{3-29}$$

Then, the error vector $\underline{z}_B$, can be assumed to be the sum of error vector associated with $d\underline{A}_V$ and $\underline{z}_m$ which contains all the measurement information and is given by

$$\underline{z}_m = [\underline{\rho}_m^T, \underline{\delta}_m^T] \tag{3-30}$$

where $\underline{\delta}_m$ is the vector $(\delta_{x_m}, \delta_{y_m}, \delta_{z_m})_m^T$ hence

$$\underline{z}_B = \underline{z}_m + \underline{z}_V \tag{3-31}$$

Now, equation 3-22 can be used to replace $\underline{z}_B$ in equation 3-30 with the joints error vector, $\underline{z}_i$ to obtain

$$\underline{z}_m = \sum_{i=0}^{n} {}^{B}\underline{\underline{J}}^{i} \cdot \underline{z}_i - \underline{z}_v \tag{3-32}$$

The link parameter errors, $\underline{g}_i$'s, are related to $\underline{z}_i$'s by equation 3-15 and thus

$$\underline{z}_m = \sum_{i=0}^{n} {}^{B}\underline{\underline{J}}^{i} \cdot \underline{\underline{G}}_i \, \underline{g}_i - \underline{z}_v \tag{3-33}$$

The vector $\underline{g}_i$, equation 3-15 contains, in addition to the link error parameters, the joint state measurement errors, μ_{θ_i} and μ_{d_i}. Equation 3-33 can be further modified to separate the random errors from the biases as

$$\underline{z}_m = \sum_{i=0}^{n} {}^{B}\underline{\underline{J}}^{i} \cdot \underline{\underline{G}}_i \cdot \underline{x}_i + \sum_{i=0}^{n} {}^{B}\underline{\underline{J}}^{i} \cdot \underline{b}_i \cdot \mu_i - \underline{z}_v \tag{3-34}$$

where, for a revolute joint,

$$\underline{x}_i = (\delta\theta_i,\ \delta d_i,\ \delta a_i,\ \delta\alpha_i)^T \tag{3-35}$$

$$\underline{b}_i = (0,\ \alpha_i,\ c\alpha_i,\ -a_i s\alpha_i,\ 0,\ s\alpha_i,\ c\alpha_i)^T \tag{3-36}$$

and, for a prismatic joint,

$$\underline{x}_i = (\delta'\theta_i,\ \delta d_i,\ \delta\alpha_i)^T \tag{3-37}$$

$$\underline{b}_i = (0,\ s\alpha_i,\ c\alpha_i,\ 0,0,0)^T \tag{3-38}$$

Finally, the observation equation 3-34 can be rewritten as

$$\underline{z}_m = \underline{\underline{H}} \cdot \underline{x} + \underline{\underline{L}} \cdot \underline{v} \tag{3-39}$$

where

$$\underline{x} = (\underline{x}_0^T,\ \underline{x}_1^T, \ldots\ldots ,\underline{x}_n^T) \qquad (3\text{-}40)$$

$$\underline{\underline{H}} = ({}^B\underline{\underline{J}}^{00} \cdot \underline{\underline{G}}_0, \ldots\ldots, {}^B\underline{\underline{J}}^{n} \cdot \underline{\underline{G}}_n) \qquad (3\text{-}41)$$

$$\underline{V} = (\mu_0, \ldots, \mu_{n-1},\ \underline{z}_v^T)^T \qquad (3\text{-}42)$$

$$\underline{\underline{L}} = ({}^B\underline{\underline{J}}^{0} \cdot \underline{b}_0, \ldots\ldots\ldots, {}^B\underline{\underline{J}}^{n-1} \cdot \underline{b}_{n-1}, -\ \underline{\underline{I}}) \qquad (3\text{-}43)$$

CHAPTER 4

IDENTIFICATION OF THE KINEMATIC MODEL PARAMETERS

As outlined in Chapter 2, the recovery of information from measurements corrupted by noise has long been a struggle endured by many investigators. Limited either by the choice of measurements or the nature of the parameters, or both, many techniques have evolved to obtain the best possible estimate of the desired variables. The techniques employed may be merely a simple method of data smoothing or regression techniques, least-squares fit, polynomial approximations, or just plain educated guesswork. The identification of the parameters in a robot kinematic model is a problem that has been addressed by a number of researchers [11, 12, 14, 15,21, 22, 25, 27, 33, 38, 40] using a variety of models and identification algorithms. The aim of this chapter is to cast a variety of identification approaches into a unified mathematical framework while trying to pose and if possible answer some of the yet unresolved theoretical and practical issues.

4.1 General Identification Framework

If the vector $\underline{\theta}$ is defined to be the 6-vector that describes the location (position and orientation) of the end-effector in space, then under the assumptions of a "level 2" calibration, $\underline{\theta}$ will be given

$$\underline{\theta} = f(\underline{\mu}, \underline{\gamma}, \underline{a}) \qquad (4\text{-}1)$$

where $\underline{\mu}$ is the vector of joint transducer readings, $\underline{\gamma}$ is the vector of coefficients in the relationships between the transducers and the actual joint displacements, and $\underline{a}$ is the vector of coefficients in the kinematic model that is being used. For a level 2 calibration, therefore, the form of the kinematic model, f(·), must be chosen as seen in chapter 3 and values for the vectors $\underline{\mu}$, $\underline{\gamma}$ and $\underline{a}$ must be determined [27].

Consider two "similar" robots A and B. The identification problem is to estimate the model of robot B given the model of robot A and set of measurements made on robot B. Robot A usually represents the "perfect" or nominal robot (i.e., the model) which essentially provides the functional structure f(·). The problem is then reduced to a parameters identification problem. Specifically, the vectors $\underline{\gamma}_B$ and $\underline{a}_B$ are unknown and must be determined.

Those estimates are to be constructed based on a set of measurement data. The end-effector of robot B is placed at m locations, $[\underline{\theta}_B(1),\ldots,\underline{\theta}_B(N)]$, within the robot workspace. For each of the m positions, the relationship between the workspace positions and the joint displacement transducers will be given by

$$\underline{\theta}_B(k) = f(\underline{\mu}_B(k),\ \underline{\gamma}_B, \underline{a}_B), \qquad k = 1,..,N \tag{4-2}$$

in which end-point sensing is used to determine some of the elements of each vector $\underline{\theta}_B(k)$. Let $\underline{y}_B(k)$ be the subset of $\underline{\theta}_B$ that is measured or determined from the constraint equations [27]. Then

$$\underline{y}_B(k) = f_y(\underline{\mu}_B,\ \underline{\gamma}_B,\ \underline{a}_B), \qquad k = 1,\ldots,N \tag{4-3}$$

where $f_y(\cdot)$ represents the appropriate subset of the kinematic equations.

The measurement equations are modeled as

$$\underline{y}_B(k) = \hat{\underline{y}}_B(k) + \underline{v}_B(k), \qquad K = 1,\ldots,N \qquad (4\text{-}4)$$

where $\underline{y}_B(k)$ are read from the end-point sensors or calculated using the end point sensory data and $\underline{v}_B(k)$ are the measurement error or noise vectors. In addition, at each configuration $\underline{\theta}_B(k)$, the joint displacements $\underline{\mu}_B(k)$ are read through the sensors.

The parameters identification algorithm maps the measurements data and the a priori data (i.e., nominal values of the kinematic parameters and possibly some probabilistic data regarding the measurement noise and the unknown parameters) into a unique estimate of the vectors $\underline{\gamma}_B$ and $\underline{a}_B$. Also, the algorithm may provide for a measure of the estimation error, L, where

$$L = ||\underline{\gamma}_B - \hat{\underline{\gamma}}_B|| + ||\underline{a}_B - \hat{\underline{a}}_B|| \qquad (4\text{-}5)$$

Starting with the ideal situation where the measurement error is zero (i.e., $\underline{v}_B(k) = 0$, $k=1,\ldots,N$), the identification problem reduces to the problem of solving nonlinear algebraic equations of the form

$$\underline{y}_B(k) = \underline{y}(k) = f_y(\underline{\mu}_B(k), \underline{\gamma}_B, \underline{a}_B) \qquad k = 1,\ldots,N \qquad (4\text{-}6)$$

To assure that the number of equations is not smaller than the number of unknowns, the number of measurements m should satisfy the following inequality

$$N.\ \dim(\underline{y}_B) \geq \rho + \zeta \tag{4-7}$$

where ρ is the number of elements in $\underline{\gamma}$ and ζ is the number of elements in $\underline{a}$. The set of equations represented in equation 4-6 is highly nonlinear. If solved numerically, a good initial estimate of the vectors $\underline{\gamma}_B$ and $\underline{a}_B$ is required to assure convergence to the correct solution. Also, more measurements may be needed to resolve possible solution ambiguities.

In the case of the more general solution (i.e., $\underline{v}_B(k) \neq 0$) let the vector $\underline{x}$ be defined as

$$\underline{x} = (\underline{\Delta\gamma}^T, \underline{\Delta a}^T)^T \tag{4-8}$$

since the difference in the two robot models A and B may be expressed as

$$\underline{a}_B = \underline{a}_A + \underline{\Delta a} \tag{4-9}$$

and

$$\underline{\gamma}_B = \underline{\gamma}_A + \underline{\Delta\gamma} \tag{4-10}$$

The relationship between $\underline{x}$ and the measurements $\underline{y}_B(k)$ is of interest. Define a "world coordinate error vector" $\underline{e}(k)$ as follows

$$\underline{e}(k) = f_y(\underline{\mu}_B(k), \underline{\gamma}_B, \underline{a}_B) - f_y(\underline{\mu}_B(k), \underline{\gamma}_A, \underline{a}_A) \tag{4-11}$$

Many researchers have looked at equation 4-11 in one form or another. The common thread in all the works is the assumption that the world coordinate error vector, $\underline{e}(k)$, may be related to the vector of parameter offsets, $\underline{x}$, through a linear transformation. This can be expressed as

$$\underline{e}(k) = \underline{\underline{H}}(k)\cdot\underline{x} \qquad (4\text{-}12)$$

The reader is referred to chapter 3 that deals with the development of the error model. One might then define a "measurement" vector, $\underline{z}(k)$ as follows

$$\underline{z}(k) = \underline{y}_B(k) - f_y(\underline{\mu}_B(k), \underline{\gamma}_A, \underline{a}_A) \qquad (4\text{-}13)$$

where $\underline{z}(k)$ is known since all of the parameters in equation 4-13 are either measured or defined by the nominal model A. From equation 4-4, $\underline{e}(k)$ and $\underline{z}(k)$ can be related by the following expression

$$\underline{z}(k) = \underline{e}(k) - \underline{\underline{L}}(k)\cdot\underline{v}_B(k) = \underline{\underline{H}}(k)\cdot\underline{x} - L(k)\underline{v}_B(k) \qquad k=1.,N \qquad (4\text{-}14)$$

where $\underline{\underline{H}}(k)$ is a matrix that depends on the nominal kinematic parameters and the robot configurations during the robot measurement phase, $\underline{\underline{L}}(k)$ a fixed matrix that depends on the nominal robot model and where $\underline{v}_B(k) = (\underline{\mu}_B{}^T, \underline{v}(k)^T)^T$.

The identification phase now follows a well known path. Let $\underline{Z}$ be the vector of all measurements

$$\underline{Z} = (\underline{z}^T(1), \ldots, \underline{z}^T(N))^T \qquad (4\text{-}15)$$

Equation 4-14 may now be rewritten as

$$\underline{Z} = \underline{\underline{H}} \cdot \underline{x} + \underline{\underline{L}} \cdot \underline{v} \qquad (4\text{-}16)$$

where

$$\underline{\underline{H}} = (\underline{\underline{H}}^T(1),\ldots,\underline{\underline{H}}^T(N))^T \qquad (4\text{-}17)$$

$$\underline{\underline{L}} = \begin{bmatrix} \underline{\underline{L}}(1) & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & \underline{\underline{L}}(N) \end{bmatrix} \qquad (4\text{-}18)$$

and $$\underline{V} = (\underline{v}_B^T(1),\ldots,\underline{v}_B^T(N))^T \qquad (4\text{-}19)$$

$\underline{V}$ and $\underline{x}$ are random vectors and as such have certain probability distribution functions. The modeling of $\underline{V}$ and $\underline{x}$ is a challenging task. Calibration measurement noise, $\underline{V}$, depends on the accuracy and resolution of the end-point sensors, machining tolerances of the calibration fixtures, the method through which the sensor data is processed to provide the values of $\underline{y}_B(k)$, as well as other factors. To properly model $\underline{x}$, one should go through many possible sources that may cause robot B to deviate from robot A. For example, one may wish to consider certain robot links machining tolerances, encoder mounting and quantization noise.

The simplest approach to the identification problem is to ignore the models of both $\underline{x}$ and $\underline{V}$. A unique least squares estimate, $\hat{\underline{x}}$, that minimizes a performance index J, where J is defined as

$$J = (\underline{Z} - \underline{\underline{H}}\cdot\underline{x})^T\cdot(\underline{Z} - \underline{\underline{H}}\cdot\underline{x}) \qquad (4\text{-}20)$$

is given by

$$\hat{\underline{x}} = (\underline{\underline{H}}^T\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{H}}^T\cdot\underline{Z} \qquad (4\text{-}21)$$

under the conditions that $\dim(\underline{z}) \geq \dim(\underline{x})$ and $\underline{\underline{H}}^T \cdot \underline{\underline{H}}$ is nonsingular as shown in section 2.3.1 Such estimator were used by a number of researchers [23,39].

In many practical situations, certain components of $\underline{x}$ may dominate others [40]. For instance, errors in the Denavit-Hartenberg twist and rotation angles may have a larger effect on robot accuracy than error in length parameters. These errors may also become more pronounced for joints closer to the robot base. These points suggest a modification of equation 4-20 by the inclusion of a suitable weighting matrix $\underline{\underline{W}}$ such that

$$J = (\underline{Z} - \underline{\underline{H}} \cdot \underline{x})^T \cdot \underline{\underline{W}} \cdot (\underline{Z} - \underline{\underline{H}} \cdot \underline{x}) \tag{4-22}$$

and the modified estimate becomes

$$\underline{\hat{x}}_{I} = (\underline{\underline{H}}^T \cdot \underline{\underline{W}} \cdot \underline{\underline{H}})^{-1} \cdot \underline{\underline{H}}^T \cdot \underline{\underline{W}} \cdot \underline{Z} \tag{4-23}$$

Least squares estimation may be viewed as a convenient and practical computational technique. It does not, however, offer any framework for investigating other issues. From a practical point of view, the following questions need to be addressed:

1. What is the relationship between the calibration error and robot repeatability?
2. What is the relationship between the calibration error and the accuracy of the calibration sensors?

3. How many measurements are needed to achieve a specified calibration accuracy?
4. Does additional modeling beyond the kinematic modeling (such as the probabilistic characterization of the measurement noise and unknown robot parameters) promise a significant improvement in the calibration accuracy?

4.2 Kalman Filtering Problem Formulation

In an effort to address questions raised in section 4.1, the probabilistic approach to identification will be justified.

A plausible model of the robot calibration is to assume that $\underline{V}$ and $\underline{x}$ are both Gaussian with zero mean and covariance matrices, $\underline{\underline{R}}$ and $\underline{\underline{P}}_o$ respectively. Furthermore, it is assumed that $\underline{V}$ and $\underline{x}$ are statistically independent of each other. It may be shown that a minimum-variance estimate of $\underline{x}$ is given by

$$\hat{\underline{x}} = (\underline{\underline{P}}_o^{-1} + \underline{\underline{H}}^T \cdot [\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T]^{-1} \cdot \underline{\underline{H}})^{-1} \cdot \underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{Z} \qquad (4\text{-}24)$$

and that the error covariance $\underline{\underline{L}}=\underline{\underline{P}}=E[(\underline{x}-\hat{\underline{x}})(\underline{x}-\hat{\underline{x}})^T]$ is given by

$$\underline{\underline{P}} = [\underline{\underline{P}}_o^{-1} + \underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T)^{-1} \cdot \underline{\underline{H}}]^{-1} \qquad (4\text{-}25)$$

as shown in section 2.3.

This important formula relates the calibration error to the measurement noise and the a priori kinematic parameter offsets uncertainty. Note that ignoring the model $\underline{x}$ can be accommodated into equation 4-24 and 4-25 by setting $\underline{\underline{P}}_{o1}$ to zero (see equations 2-58:

2-59). A recursive representation of those two equations is known as the "Kalman Filter". Also observe that $\underline{x}$ in 4-24 is a linear operation on the measurement data. Furthermore, it is proven [19] that for a Gaussian time varying signal, the optimal predictor is a linear predictor. Additionally, as a practical fact, most often all that is known about the characterization of a given random process is its autocorrelation function and hence its expectation and covariance matrix. But there always exists a Gaussian random process possessing the same autocorrelation function. One, therefore, might as well assume that the given random process is itself Gaussian. That is, the two processes are indistinguishable from the standpoint of the amount of knowledge postulated. On the basis of these observations one is led to consider the optimal estimator as a linear operator in most applications and specifically for the purpose of estimating the error in the robot kinematic parameters.

Kalman filtering has several other distinct (but not necessarily exclusive) and practical advantages:

1. Every measurement vector need not be stored. It is operated on once and discarded. Its information is used to update the state estimate vector.
2. Sensors outputs are properly weighted according to their noisiness.
3. The progress toward state estimate convergence to its true value can be determined directly through the error covariance matrix $\underline{\underline{P}}$. Estimation may cease when acceptably small

covariance values are reached. Note that the error covariance equation is "data-independent". The filter convergence may be analyzed a priori.

4. It allows prior knowledge about the system parameters to be included.

At this point, the formulation of the identification as a Kalman filtering problem can be discussed. Consider the linear system described by the following two equations:

$$\underline{x}(k+1) = \underline{x}(k) \qquad (4\text{-}26)$$

$$\underline{z}(k) = \underline{\underline{H}}(k)\cdot\underline{x}(k) + \underline{\underline{L}}(k)\cdot\underline{v}(k) \qquad (4\text{-}27)$$

where $\underline{x}(0) \sim N(0, \underline{\underline{P}}_o)$ and where the noise part of $\underline{v}(k) \sim N(0, \underline{\underline{R}}(k))$ and where N(-,-) denotes a Gaussian distribution with the indicated mean and covariance.

The "process" equation, equation 4-26, is that of a constant random process. The "measurement" equation, equation 4-27 is a rewritten version of equation 4-14 and corresponds to equation 3-33 of the Kinematic model developed in Chapter 3. It is important to note that this system is time varying since $\underline{\underline{H}}(k)$, which represents the robot configurations, and $\underline{\underline{R}}(k)$, which represent the measurement noise, may vary from one measurement to the other. The objective is to find a minimum variance estimate $\underline{x}(k)$ that depends on $\underline{x}(k-1)$ and the new measurement, $\underline{z}(k)$. Under the assumption that $\underline{x}(0)$ and $\underline{v}(k)$ are independent, this problem is equivalent to that of equation 4-16 with the exception of leaving the total number of measurements, N, unspecified. Conjecturing at this point, one might expect that equation 4-26 may be generalized to include robot repeatability effects, as follows:

$$\underline{x}(k+1) = \underline{\underline{\phi}}(k)\cdot\underline{x}(k) + \underline{\underline{\Gamma}}(k)\cdot\underline{\underline{W}}(k) \qquad (4\text{-}28)$$

where $\underline{\underline{W}}(k) \sim N(0,\underline{\underline{Q}}(k))$

Determining suitable "process dynamics", $\underline{\underline{\phi}}(k)$, and "process noise" covariance, $\underline{\underline{Q}}(k)$ is a challenging research task and is beyond the scope of this thesis.

The estimation problem solution (i.e., the Kalman Filter equations) is given in section 2.3.2 and is repeated here for convenience,

$$\hat{\underline{x}}'(k+1) = \underline{\underline{\phi}}(k)\cdot\hat{\underline{x}}(k) \qquad (4\text{-}29)$$

$$\underline{\underline{P}}'(k+1) = \underline{\underline{\phi}}(k)\cdot\underline{\underline{P}}(k)\cdot\underline{\underline{\phi}}^T(k) + \underline{\underline{\Gamma}}(k)\cdot\underline{\underline{Q}}(k)\cdot\underline{\underline{\Gamma}}^T(k) \qquad (4\text{-}30)$$

$$\hat{\underline{x}}(k+1) = \hat{\underline{x}}'(k+1) + \underline{\underline{K}}(k+1)\cdot[\underline{z}(k+1) - \underline{\underline{H}}(k+1)\cdot\hat{\underline{x}}(k+1)] \qquad (4\text{-}31)$$

$$\underline{\underline{K}}(k+1) = \underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1)[\underline{\underline{H}}(k+1)\cdot\underline{\underline{P}}(k+1)\cdot\underline{\underline{H}}^T(k+1) + [\underline{\underline{L}}(k+1)\cdot\underline{\underline{R}}(k+1)\cdot\underline{\underline{L}}^T(k+1)]^{-1} \qquad (4\text{-}32)$$

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} - \underline{\underline{K}}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\underline{\underline{P}}'(k+1) \qquad (4\text{-}33)$$

where the initial estimate is $\underline{x}(0) = \underline{0}$, since it is assumed that $\underline{x}$ is zero-mean and the initial error covariance $\underline{\underline{P}}(0) = \underline{\underline{P}}_o$.

4.3 Calibration Identification Solution and Time Invariant Formulation

To specialize the general solution, equation 4-29 and 4-33, to the filtering problem, equations 4-26 and 4-27, one has to merely substitute the identity matrix for the process dynamics matrix, i.e., $\underline{\underline{\phi}}(k) = \underline{\underline{I}}$ and suppress the process noise covariance matrix, i.e., $\underline{\underline{Q}}(k) = 0$ for all k's.

Now, the measurements, $\underline{z}(k)$, can be ordered in the following fashion with no loss of generality. Let $k_p \geq k_{min}$ be the number of measurement points and $k_r \geq 1$ be the number of repeated measurement at one measurement configuration, then the total number of measurements is given by

$$N = k_p \cdot k_r \tag{4-34}$$

The ordering of the measurements that are fed to the filter is then:

-measurements taken at point p_1:

$\underline{z}(1)$, $\underline{z}(k_p+1), \ldots, \underline{z}((k_r-1)(k_p+1))$

-measurements taken at point p_2:

$\underline{z}(2)$, $\underline{z}(k_p+2), \ldots, \underline{z}((k_r-1)(k_p+2))$

-

-

-

-measurements taken at point P_{k_p}:

$\underline{z}(k_p)$, $\underline{z}(2k_p), \ldots, \underline{z}(k_r k_p) = \underline{z}(N)$

Returning to the indentification problem with no modeling of robot repeatability, i.e., equations 4-26:4-27, a time invariant version of the problem may be formulated by defining

$$\underline{\underline{H}}_{k_p} = (\underline{\underline{H}}^T(1), \ldots ,\underline{\underline{H}}^T(k_p))^T \qquad (4\text{-}36)$$

$$\underline{\underline{R}}_{k_p} = \text{diag } \{\underline{\underline{R}}(k)\} = \begin{bmatrix} \underline{\underline{R}}(1) & & & \underline{\underline{0}} \\ & \underline{\underline{R}}(2) & & \\ & & \ddots & \\ \underline{\underline{0}} & & & \underline{\underline{R}}(k_p) \end{bmatrix} \qquad (4\text{-}37)$$

$$\underline{V}_{k_p} = (\underline{v}^T(1),\ldots,\underline{v}^T(k_p))^T \qquad (4\text{-}38)$$

then where the noise vectors were defined in equations 4-18:4-19

$$\underline{Z}_{k_p}(i) = \underline{\underline{H}}_{k_p} \cdot \underline{x}(i) + \underline{\underline{L}}_{k_p} \cdot \underline{V}_{k_p}(i) \qquad (4\text{-}39)$$

where $i=1,\ldots,k_r$ and $i=1+ \text{int}[k-1/k_p]$ and "int" represents the "largest integer not greater than".

In order to simplify the notation, the subscript k_p will be dropped in the sequel (e.g., $\underline{\underline{H}}_{k_p} = \underline{\underline{H}}$) when the meaning is clear from the context.

In the case of time invariant system, with identity process dynamics and zero process noise, the Kalman Filter equations, equations 4-29:4-33 are simplified further;

$$\underline{x}(k+1) = \underline{x}(k) + \underline{\underline{K}}(k+1) \cdot [\underline{z}(k+1) - \underline{\underline{H}} \cdot \underline{x}(k)] \qquad (4\text{-}40)$$

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} - \underline{\underline{K}}(k+1) \cdot \underline{\underline{H}}] \cdot \underline{\underline{P}}(k) \qquad (4\text{-}41)$$

$$\underline{\underline{K}}(k+1) = \underline{\underline{P}}(k) \cdot \underline{\underline{H}}^T [\underline{\underline{H}} \cdot \underline{\underline{P}}(k) \cdot \underline{\underline{H}}^T + \underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T]^{-1} \qquad (4\text{-}42)$$

Using equation 4-41:4-42, the error covariance update equation can be rewritten as

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} - \underline{\underline{P}}(k+1) \cdot \underline{\underline{H}}^T \cdot (\underline{\underline{L}} \cdot \underline{\underline{R}} \cdot \underline{\underline{L}}^T) \cdot \underline{\underline{H}}] \underline{\underline{P}}(k) \qquad (4\text{-}43)$$

which yields another expression for $\mathbf{P}(k+1)$ given by

$$\mathbf{P}(k+1)=[\mathbf{I} + \mathbf{H}^T\cdot(\mathbf{L}\cdot\mathbf{R}\cdot\mathbf{L}^T)\cdot\mathbf{H}]^{-1}\cdot\mathbf{P}(k) \qquad (4\text{-}44)$$

This last equation can be solved recursively starting at time k=0,i.e., $\mathbf{P}(0) = \mathbf{P}_o$, to obtain

$$\mathbf{P}(k) = [\mathbf{I} + k(\mathbf{H}^T\cdot(\mathbf{L}\cdot\mathbf{R}\cdot\mathbf{L}^T)^{-1}\cdot\mathbf{H}\cdot\mathbf{P}_o]^{-1}\cdot\mathbf{P}_o \qquad (4\text{-}45)$$

Hence, the state estimate update equation becomes

$$\underline{x}(k+1)=\underline{x}(k)+([\mathbf{P}_o+k(\mathbf{H}^T\cdot(\mathbf{L}\cdot\mathbf{R}\cdot\mathbf{L}^T)^{-1}\cdot\mathbf{H})\cdot\mathbf{P}_o]\cdot\mathbf{H}^T\cdot\mathbf{L}\cdot\mathbf{R}\cdot\mathbf{L}^T[\underline{z}(k+1) -\mathbf{H}\cdot\underline{x}(k)]) \qquad (4\text{-}46)$$

4.4 Number of Measurements

To determine how many measurements are to be taken in order to estimate uniquely the robot kinematic model errors, the error covariance equation (equation 4-45) needs to be studied. the number of measurements, N, as defined in equation 4-33, is that for which the error covariance is sufficiently close to a steady state value, $P(\infty)$, such that

$$||\mathbf{P}(k) - \mathbf{P}(\infty)|| \leq \xi, \text{ for any } k \geq N \qquad (4\text{-}47)$$

where $\xi \geq 0$ and the appropriate norm are determined depending on particular application. Alternatively, the minimum number of measurement needed can be defined with respect to the initial error covariance. That is, a number of measurements for which the error covariance norm reaches a certain acceptable fraction of the initial covariance norm, such that

$$||\mathbf{P}(k)|| \leq \varepsilon \; ||\mathbf{P}_o||, \text{ for every } K \geq N \qquad (4\text{-}48)$$

where $0 \leq \varepsilon \leq 1$ and the norm are determined depending on the application.

It is important to note, though, that the formulation of equation 4-48 is possible only in the context of "level 2" calibration. It is only in the absence of process noise, i.e., $\underline{\underline{Q}} = \underline{\underline{0}}$, that $||\underline{\underline{P}}(k)||$ tend to zero as k tends to infinity as seen from equation 4-45. In the case where the robot repeatability effects are included in the process equation (equation 4-24), the suitable formulation is that of equation 4-47 and $||\underline{\underline{P}}(\infty)||$ should be found first to insure proper convergence. The reader is referred to [2,25,35] for the solution of the Discrete Riccati equation.

Then, at time $k \geq N$, using equation 4-45 and Schwartz norm inequality, the following inequality can be obtained

$$||\underline{\underline{P}}(k)|| < ||\underline{\underline{P}}_o||\cdot||[I + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}|| \leq \varepsilon\ ||\underline{\underline{P}}_o|| \quad (4\text{-}49)$$

Eliminating the norm of $\underline{\underline{P}}_o$ from both sides of equation 4-49 yields

$$||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}|| \leq \varepsilon \quad (4\text{-}50)$$

since

$$||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}[\underline{\underline{I}}+k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]||\geq||\underline{\underline{I}}|| \quad (4\text{-}51)$$

using, Shwartz inequality once more, equation 4-51 becomes

$$||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}||\cdot||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]||\geq||\underline{\underline{I}}|| \quad (4\text{-}52)$$

Combining equation 4-50 and 4-52, one gets

$$\frac{||\underline{\underline{I}}||}{||[\underline{\underline{I}}+k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]||} \leq ||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1} \cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}|| \leq \varepsilon \qquad (4\text{-}53)$$

Using the first and last terms of equation 4-53, leads to

$$||\underline{\underline{I}}|| \leq \varepsilon\ ||[\underline{\underline{I}} + k(\underline{\underline{H}}^T\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1}\cdot\underline{\underline{H}}))\cdot\underline{\underline{P}}_o]|| \qquad (4\text{-}54)$$

Then, defining γ as

$$\gamma = \frac{||\underline{\underline{I}}|| - \varepsilon\ ||\underline{\underline{I}}||}{\varepsilon} \qquad (4\text{-}55)$$

and using equation 4-50, a lower bound on the number of measurements needed can be obtained.

$$\frac{\gamma}{||(\underline{\underline{H}}^T\ \cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^{-1})^T\cdot\ \underline{\underline{H}})\cdot\underline{\underline{P}}_o||} = k_{min} \leq N \qquad (4\text{-}56)$$

Note that if there is no a priori information, the a priori covariance matrix, $\underline{\underline{P}}_o$, is very large and, as can be expected, the minimum number of measurements needed in order to achieve an acceptable accuracy of the robot kinematic model parameters will be larger. The same is true if the magnitude of the statistical parameters of the measurement noise is large, i.e., $(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T)^{-1} \simeq 0$. Those intuitive concepts are illustrated in figures 9:10.

From a practical point of view, the following questions need to be addressed.

1. In an actual measurement process, only one measurement is taken at each configuration, i.e., K_r=1. What is then the minimum number of configurations needed to achieve a predetermined accuracy?

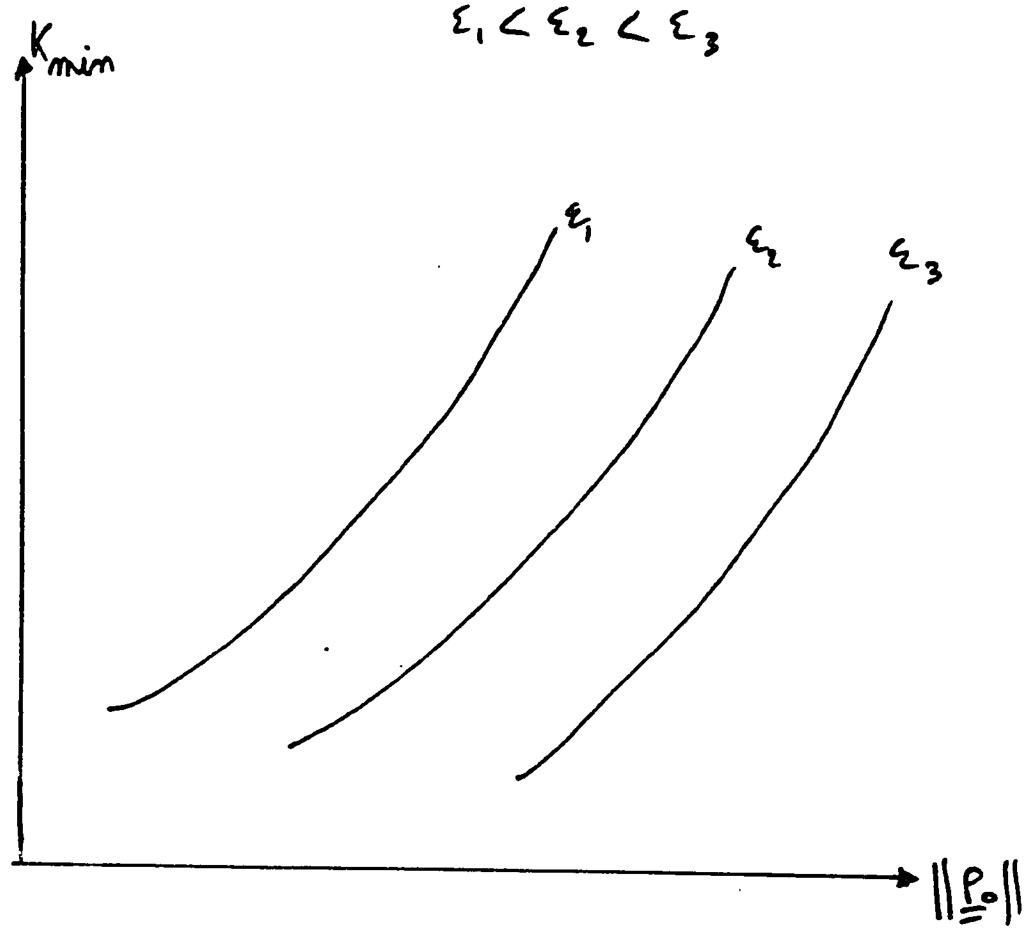


Fig. 9 Effect of the Initial Error Covariance on the Minimum Number of Measurements

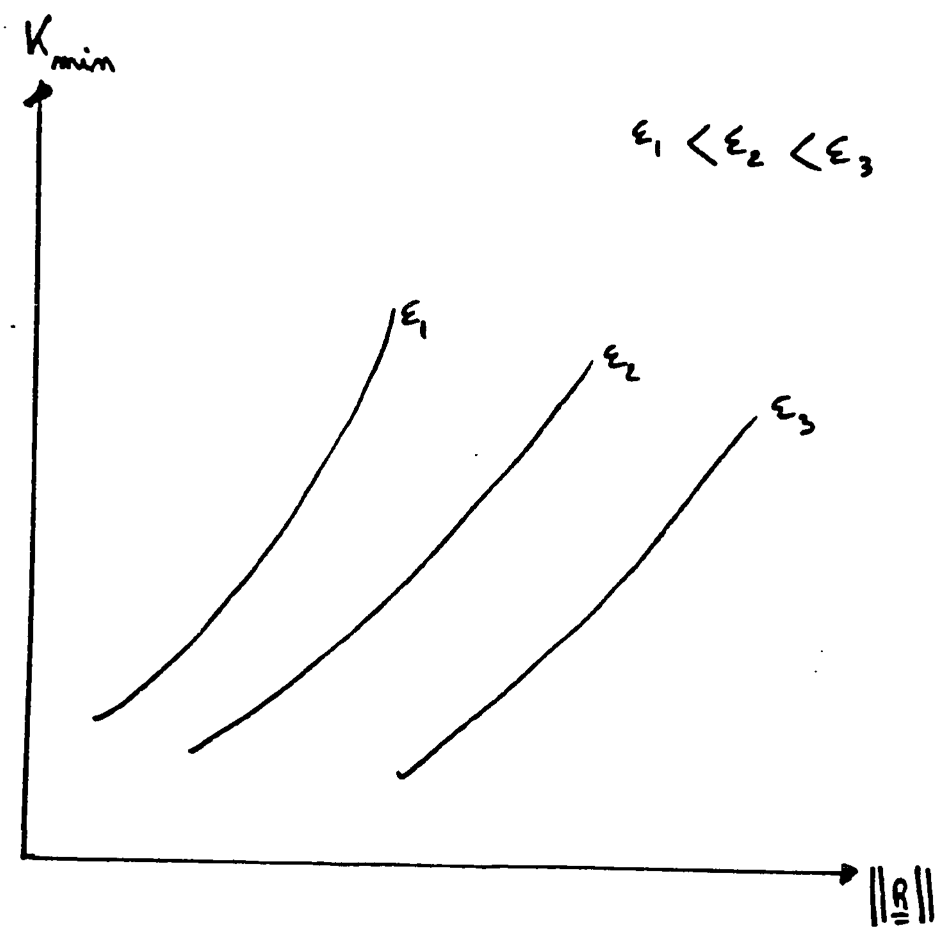


Fig. 10 Effect of Noise Covariance on the Minimum Number of Measurements

2. How tight is the lower bound of equation 4-56? In other words, how close is K_{min}(as determined implicitely by equation 4-50) to the lower bound (equation 4-53).

3. How do different types of matrix norms affect the minimum number of measurements and what is the physical meaning of such choices?

4. How is the minimum number of measurements affected by the uncertainties of the model, i.e., $||\underline{\underline{P}}_o||$, and by the measurement and quanitization noise, i.e., $||\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T||$.

These questions will be addressed in the simulation examples of Chapter Five.

4.5 Kalman Filter Behavior

It is beneficial at this point to look in more detail at the Kalman filter algorithm in order to obtain a deeper understanding of its behavior.

4.5.1 Error Covariance Behavior

The effect of measurement noise on the error covariance of the discrete Kalman Filter is best observed by using the matrix inversion lemma on equation 2-65. This lemma gives an expression for $\underline{\underline{P}}^{-1}(k)$,

$$\underline{\underline{P}}^{-1}(k)= \underline{\underline{P}}'(k)^{-1} +\underline{\underline{H}}^T(k)\cdot[\underline{\underline{L}}(k)\cdot\underline{\underline{R}}(k)\cdot\underline{\underline{L}}^T(k)]^{-1}\cdot\underline{\underline{H}}(k) \qquad (4\text{-}57)$$

which, in the time invariant and identity dynamic process case, becomes

$$\underline{\underline{P}}^{-1}(k+1) = \underline{\underline{P}}^{-1}(k) + \underline{\underline{H}}^{T}\cdot(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^{T})^{-1}\cdot\underline{\underline{H}} \tag{4-58}$$

Large measurement and quantization noise,i.e.,($\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^{T})^{-1}$ is small, provides only a small increase in the inverse of the error covariance when the measurement is used. That is, the error covariance decreases slowly and convergence to a true value of the estimates takes longer and as seen in section 4.4, more measurements are needed. On the other hand, small measurement errors, i.e., large $(\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^{T})^{-1}$, cause the error covariance to decrease considerably whenever a measurement is utilized.

The effects of system disturbances and measurement noises of different magnitudes can be described graphically by considering the standard deviation of the error in the estimate of the a representative state variable. This is presented in figure 11 for a hypothetical system which reaches statistical "steady state".

An important shortcoming, however, exists. Solving recursively equation 4-39, equation 4-40 was obtained. Hence, if $\underline{\underline{H}}_{k_p}=\underline{\underline{H}}$ is nonsingular, as required for observability, and $\underline{\underline{R}}_{k_p}=\underline{\underline{R}}$ is positive definite, as required in the filter formulation, then one might conclude that, as k goes to infinity, the error covariance, $\underline{\underline{P}}(k)$, goes to zero. This, in turn, implies that the calibration can be made infinitely accurate as the number of measurement points is increased. This result is probably a manifestation of the practical inadequacy of the model of equation 4-26:4-27. Practice shows that

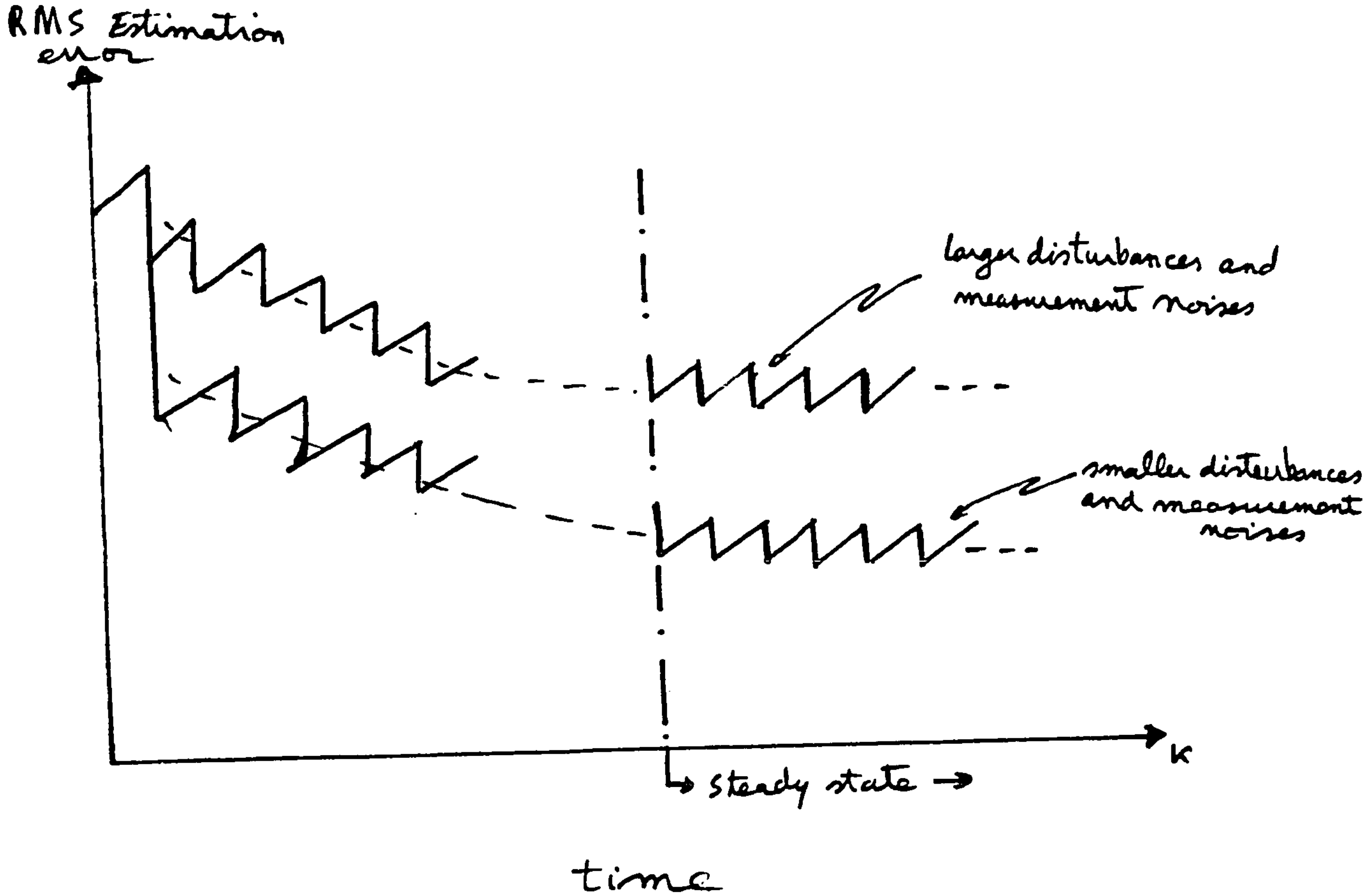


Fig. 11 Behavior of the RMS Error in the Discrete Kalman Filter Estimate of a Particular State Variable

there is a lower bound on the calibration error covariance that is dictated by robot repeatability and calibration measurement error. Again, conjecturing, one might expect that this can be captured by the modified model of equation 4-27:4-28. More research is needed to clarify this aspect of the problem.

4.5.2. Kalman Gain Matrix

The optimality of the Kalman filter is contained in its structure and in the specification of the gain matrix. There is an intuitive logic behind the equations for the Kalman gain matrix. It can be seen from equation 2-63,

$$\underline{\underline{K}}(k) = \underline{\underline{P}}(k)\cdot\underline{\underline{H}}^T(k)\cdot[\underline{\underline{L}}(k)\cdot\underline{\underline{R}}(k)\cdot\underline{\underline{L}}^T(k)]^{-1} \quad (4\text{-}59)$$

or, in the time invariant case,

$$\underline{\underline{K}}(k) = \underline{\underline{P}}(k)\cdot\underline{\underline{H}}^T\cdot[\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T]^{-1} \quad (4\text{-}60)$$

To better observe the meaning of the expressions, assume that $\underline{\underline{H}}$ is the identity matrix. In this case, both $\underline{\underline{P}}$ and $\underline{\underline{R}}$ are n x n matrices. If $[\underline{\underline{L}}\cdot\underline{\underline{R}}\cdot\underline{\underline{L}}^T]^{-1}$ is a diagonal matrix, no cross-correlation between noise terms or in the case of a time invariant system, measurements are independent, $\underline{\underline{K}}$ results from multiplying each column of the error covariance matrix by the appropriate inverse of the mean square measurement noise. Each element of the filter gain matrix is essentially the ratio between statistical measures of uncertainty in the state estimate and the uncertainty in a measurement.

Thus, the gain matrix is "proportional" to the uncertainty in the estimate, and "inversely proportional" to the measurement noise. If the measurement noise is large and the state estimate errors are small, the measurement residual, equation 2-49

$$\underline{r}(k) = \underline{z}(k) - \underline{\underline{H}}(k)\cdot\underline{\underline{x}}'(k) \tag{4-61}$$

is due chiefly to the noise and only small changes in the state estimates should be made. On the other hand, small measurement noise and large uncertainty in the state estimates suggest that the measurement residual, $\underline{r}(k)$, contains considerable information about errors in the estimates. Therefore, the difference between the actual and the predicted measurement will be used as a basis for strong corrections to the estimates. Hence, the filter gain matrix is specified in a way which agrees with an intuitive approach to improving the estimate.

4.5.3 Controllability

The controllability is important to establishing stability of the filter equations for obtaining a unique steady-state value of the error covariance, $\underline{\underline{P}}(\infty)$. This property can be established as discussed in section 2.4.3. In the case of a system with no process noise, as described by equations 4-26:4-27, this property is of no consequence as the controllability matrix $\underline{\underline{W}}_d = \underline{\underline{0}}$.

4.5.4 Observability

Given a sequence of measurements $\{\underline{z}(0),\ldots,\underline{z}(k)\}$, the observability conditions define the ability to determine the initial system state $\underline{x}(0)$ from those mesurements. The observability conditions are given in section 2.4.2. In the case of the discrete linear system under consideration, i.e., equation 4-26:4-27, the observability matrix given by equation 2-72 reduces to

$$[\underline{\underline{H}}^T,\ \underline{\underline{H}}^T,\ldots,\underline{\underline{H}}^T] \tag{4-62}$$

which is an n x n matrix and should be of rank n, i.e., $\underline{\underline{H}}^T \cdot \underline{\underline{H}}$ is nonsingular matrix.

In the case of a time invariant system as described in section 4.3. The observability matrix is identical equation 4-31,

$$\underline{\underline{M}}_d = \underline{\underline{H}}_{k_p} = (\underline{\underline{H}}^T(1),\ldots,\underline{\underline{H}}^T(k_p))^T \tag{4-63}$$

Then, for the system to be observable,

$$\text{rank }(\underline{\underline{H}}_{k_p}) = \dim(\underline{x}) \tag{4-64}$$

is required, which implies that

$$k_{min} \geq 1 + \text{int}[\frac{\dim(\underline{x})}{\dim(\underline{z})}] \tag{4-65}$$

where "int" represents the "largest integer not greater than".The requirement of equation 4-61 must be simultaneously satisfied along with that of equation 4-52 defining the minimum number of measurements needed to achieve a predefined accruacy in the robot calibration.

From a practical point of view, some of the questions that need to be addressed are:

1. Intuitively, one might expect that at degenerate configuration of a particular robot, the system becomes unobservable. Is that the case?
2. What other, if any, measurement configuration cause the system to become unobservable and, hence, might cause the filter to diverge?
3. If some of the robot's kinematic parameters are not observable, will the Kalman filter still give accurate estimates for the other observable parameters?

4.5.5 Stability of the Kalman Filter

One consideration of both practical and theoretical interest is the stability of the Kalman filter. Stability refers to the behavior of the state extimates when measurements are suppressed in equation 4-27. The "unforced" filter equation takes the form

$$\underline{x}(k+1) = [\underline{\underline{I}} - \underline{\underline{K}}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\hat{\underline{x}}'(k+1) \quad (4\text{-}66)$$
$$= [\underline{\underline{I}} - \underline{\underline{K}}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\underline{\underline{\phi}}(k)\cdot\hat{\underline{x}}(k) \quad (4\text{-}67)$$

It is desirable that the solution of equation 4-62 be asymptotically stable, i.e., loosely speaking, x(k+1) goes to zero as k goes to infinity, for any initial condition x(0). This will insure that any unwanted component of $\underline{x}$ caused by distrubances in equation 4-62, such as computational errors arising from finite word length in a digital computer, are bounded.

As stated before in section 2.4.4, optimality of the Kalman filter does not guarantee its stability. However, one key result exists which assures both stability of the filter and uniquenesss of the behavior of the error covariance matrix, $\underline{\underline{P}}(k)$ for large k, independently of $\underline{\underline{P}}(0)$. This requirement is stated in theorem III. For a system described by equations 4-26:4-27, this requirement reduces to that of equation 2-11. This reliance on observability alone to establish stability of the Kalman filter, makes the answers to questions raised in section 4.5.4 of special importance.

It is well worth noting that, in general, complete observability and controllability requirements are quite restrictive and, in many cases of practical significance, these conditions are not fulfilled; but Kalman filters, designed in a normal way, operate satisfactorily. This is attributed to the fact that hte solution to equation 4-62 frequently tends toward zero over a finite time interval of interest, even though it may not be asymptotically stable in the strict sense of the definition.

CHAPTER 5

EXAMPLES, SIMULATIONS AND RESULTS

5.1 The One Link Manipulator Example:

The aim of this example is to address some of the questions related to the minimum number of measurements needed to achieve a specified accuracy and the dependancy of this lower bound on the type of matrix norm used, the initial conditions and the measurement noise.

5.1.1. Measurement Equation

In this section, the measurement equation for a 1 link manipulator will be developed. To that end, it will be assumed that both the nominal and actual models are given by (see figure 12)

Actual Model

$$P_x = (r + dr) \cdot \cos(\theta + d\theta) \qquad (5\text{-}1)$$

$$P_y = (r + dr) \cdot \sin(\theta + d\theta) \qquad (5\text{-}2)$$

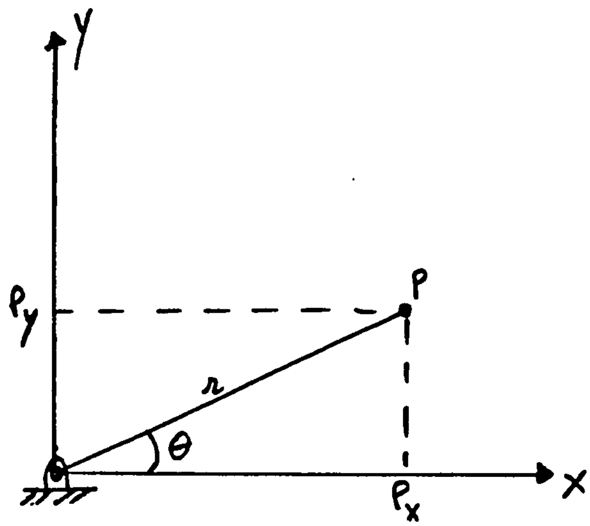
$$P_z = 0 \qquad (5\text{-}3)$$

Nominal Model

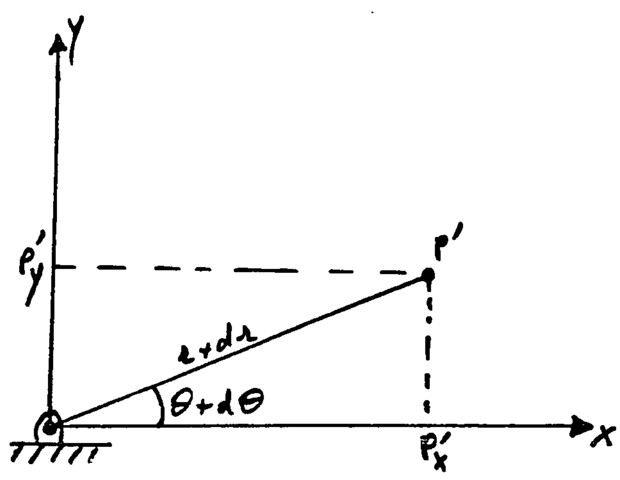
$$P_x = r \cos\theta \qquad (5\text{-}4)$$

$$P_y = r \sin\theta \qquad (5\text{-}5)$$

$$P_z = 0 \qquad (5\text{-}6)$$



a) Nominal Model



b) Actual Model

Fig. 12 One Link Manipulator Models

In addition, the following assumptions are taken

A.1: $\lim_{\theta \to 0} \cos(d\theta) = 1$

A.2: $\lim_{\theta \to 0} \sin(d\theta) = d\theta$

A.3: Double differential error terms are negligable

A.4: A method for measuring the difference between the actual and nominal position of the end point P is available.

Using the equation (5-1:5-6) and assumptions (A.1:A.4) it can be shown that

$$\underline{e}(k) = \begin{bmatrix} e_x \\ e_y \end{bmatrix}_k = \begin{bmatrix} P'_x - P_x \\ P'_y - P_y \end{bmatrix}_k = \begin{bmatrix} -r\,s\theta & c\theta \\ r\,c\theta & s\theta \end{bmatrix}_k \cdot \begin{bmatrix} d\theta \\ dr \end{bmatrix}_k \tag{5-7}$$

Then, the measurements are given by the above difference vector and some additive white Gaussian noise, i.e.

$$\underline{z}(k) = \underline{e}(k) + \underline{v}(k) \qquad k = 1,2,... \tag{5-8}$$

or

$$\underline{z}(k) = \underline{\underline{H}}(k)\cdot\underline{x}(k) + \underline{v}(k) \qquad k = 1,2,... \tag{5-9}$$

where $\underline{v}(k)$ is a white Gaussian sequence with zero mean and positive definite covariance matrix $\underline{\underline{R}}(k)$, i.e., $\underline{v}(k)$ $N(0,\underline{\underline{R}}(k))$, the matrix $\underline{\underline{H}}(k)$ is given by

$$\underline{\underline{H}}(k) = \begin{bmatrix} -r\,s\theta & C\theta \\ r\,c\theta & S\theta \end{bmatrix}_k \qquad k = 1,2,.... \tag{5-10}$$

and where the state vector $\underline{x}(k)$ is

$$\underline{x}(k) = (d\theta,\ dr)^T \qquad (5\text{-}11)$$

5.1.2 Filter equation

The problem is then to estimate the value of the error vector (state vector) $\underline{x}(k) = (d\theta,\ dr)_k^T$, assumed constant (i.e., $\underline{\underline{\phi}}(k) = \underline{\underline{I}}$). with no process noise (i.e., $\underline{\underline{Q}}(k) = 0$), given discrete measurements $\underline{x}$ corrupted by uncorrelated white Gaussain noise $\underline{v}(k)$ as described by equation 5-9. In addition the state vector is assumed initially by Gaussian with zero mean and covariance matrix $\underline{\underline{P}}(0)$ (i.e., $\underline{x}(0) \sim N(0, \underline{\underline{P}}(0)$ and no encoder quantization errors are present (i.e., $\underline{\underline{L}}(k) = \underline{\underline{I}}$).

The Kalman filter equations given in section 2.3.2 (2-60:2-65) reduce to

State estimate extrapolation:

$$\underline{x}(k+1) = \underline{x}(k) \qquad (5\text{-}12)$$

Error covariance extrapolation

$$\underline{\underline{P}}'(k+1) = \underline{\underline{P}}(k) \qquad (5\text{-}13)$$

State estimate update

$$\underline{x}(k+1)=\underline{x}'(k+1) + \underline{\underline{K}}(k+1)[\underline{z}(k+1) - \underline{\underline{H}}(k+1)\cdot\underline{x}(k+1)] \qquad (5\text{-}14)$$

Kalman Filter gain

$$\underline{\underline{K}}(k+1)=\underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1)[\underline{\underline{H}}(k+1)\cdot\underline{\underline{P}}'(k+1)\cdot\underline{\underline{H}}^T(k+1) + \underline{\underline{R}}(k+1)]^{-1} \qquad (5\text{-}15)$$

Error Covariance

Error Covariance

$$\underline{\underline{P}}(k+1)=[\underline{\underline{I}} - \underline{\underline{K}}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\underline{\underline{P}}'(k+1) \quad (5\text{-}16)$$

Using the above equations, the error covariance update equation can be rewritten as

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} - \underline{\underline{P}}(k+1)\cdot\underline{\underline{H}}^T(k+1)\cdot\underline{\underline{R}}^{-1}(k+1)\cdot\underline{\underline{H}}(k+1)]\cdot\underline{\underline{P}}(k) \quad (5\text{-}17)$$

which after a few manipulations yields another expression for $\underline{\underline{P}}(k+1)$

$$\underline{\underline{P}}(k+1) = [\underline{\underline{I}} + \underline{\underline{H}}^T(k+1)\cdot\underline{\underline{R}}^{-1}(k+1)\cdot\underline{\underline{H}}(k+1)]^{-1}\cdot\underline{\underline{P}}(k) \quad (5\text{-}18)$$

This last equation can be solved recursively starting at time to, ie, $\underline{\underline{P}}(k) = \underline{\underline{P}}_o$, to obtain

$$\underline{\underline{P}}(k) = [\underline{\underline{I}} + \sum_{i=1}^{k} \underline{\underline{H}}^T(i)\cdot\underline{\underline{R}}^{-1}(i)\cdot\underline{\underline{H}}(i) \cdot \underline{\underline{P}}_o]^{-1} \quad (5\text{-}19)$$

Hence, the state estimate update equation (5-14) becomes

$$\underline{x}(k+1) = \underline{x}(k) + ([\underline{\underline{P}}_o^{-1} + \sum_{i=1}^{k+1} \underline{\underline{H}}^T(i)\cdot\underline{\underline{R}}^{-1}(i)\cdot\underline{\underline{H}}(i)]^{-1}\cdot\underline{\underline{H}}^T(k+1)$$

$$\cdot\ \underline{\underline{R}}^{-1}(k+1)\cdot[\underline{z}(k+1) - \underline{\underline{H}}(k+1)\cdot\underline{x}(k+1)] \quad (5\text{-}20)$$

Using the result of section 4.3, to reduce the system to a time-invariant one, and assuming that $\underline{\underline{R}}(k+1)=\underline{\underline{R}}(k)=\underline{\underline{R}}$, the above equations 5-19:5-20

$$\underline{\underline{P}}(k) = [\underline{\underline{I}} + k\cdot(\underline{\underline{H}}^T\cdot\underline{\underline{R}}\cdot\underline{\underline{H}})^{-1}\cdot\underline{\underline{P}}_o \quad (5\text{-}21)$$

and

$$\underline{x}(k+1) = \underline{x}(k) + ([\underline{\underline{P}}_o^{-1} + k\ (\underline{\underline{H}}^T\cdot\underline{\underline{R}}^{-1}\cdot\underline{\underline{H}})\cdot\underline{\underline{P}}_o]^{-1}\cdot H^T\cdot R^{-1}$$

$$\cdot\ [\underline{z}(k) - \underline{\underline{H}}\cdot\underline{x}(k)]) \quad (5\text{-}22)$$

Equation 5-21 can be used now to obtain a theoretical lower bound on the number of measurements needed to achieve a predetermined accuracy for the estimate. Hence

$$K_{min} = \frac{\gamma}{||(\underline{\underline{H}}^T \cdot \underline{\underline{R}}^{-1} \cdot \underline{\underline{H}}) \cdot \underline{\underline{P}}_o||} \tag{5-23}$$

where γ is defined in equation 4-51.

5.1.3 Observability

As seen previously, the observability matrix is given by equation 2-71.

$$\underline{\underline{\Omega}} = [\underline{\underline{H}}^T, \underline{\underline{\phi}}^T \cdot \underline{\underline{H}}^T, \ldots, (\underline{\underline{\phi}}^T)^{n-1} \cdot \underline{\underline{H}}^T]^T \tag{5-24}$$

where n is the number of observations. But, the transition matrix $\underline{\underline{\phi}}$ is equal to the identity matrix, therefore the observability matrix of equation 5-24 reduces

$$\underline{\underline{\Omega}} = [\underline{\underline{H}}^T, \ldots, \underline{\underline{H}}^T]^T \tag{5-25}$$

Note that $\underline{\underline{\Omega}}$ is an m x n matrix where n is the dimension of the state vector.

Hence, according to theorems E1 and E2 and corollary I (see Appendix E and Section 2.4.2), the system is observable if the rank of $\underline{\underline{\Omega}}$ is equal to 2 which is the dimension of the state vector $\underline{x} = (d\theta, dr)^T$. For this to be true, at least one of the largest minors of $\underline{\underline{\Omega}}$ or, equivalently, the matrix $\underline{\underline{H}}$ equation 5-10, for one measurement configuration, should be nonsingular. Therefore, the determinant of $\underline{\underline{H}}$ should be nonzero;

$$\det \begin{bmatrix} -rs\theta_i & c\theta_i \\ rc\theta_i & s\theta_i \end{bmatrix} = -r\cdot s^2\theta_i - rc^2\theta_i = -r(s^2\theta_i + c^2\theta_i) = -r \neq 0$$

$$i = 1,\ldots,k \quad (5\text{-}26)$$

The system is therefore always observable for any configuration of the one link manipulator and the measurement point lie a circle of radius r and centered at the origin.

5.1.4 Simulations results

As stated at the beginning of this section, the purpose of this example is to try to answer the following questions:

1. How is the minimum number of measurement affected by the model uncertainty ($||\underline{\underline{P}}_o||$) and by the measurement noise ($||\underline{\underline{R}}||$)?

2. Since in the actual measurement process, only one measurement is taken at each configuraiton, i.e., $K_r = 1$, what is then the mainimum number of configurations needed to achieve a certain accuracy?

3. How do different types of matrix norms affect the minimum number of measurements and what is the physical meaning of each choice?

4. How tight is the lower theoretical bound of equation 5-23?

5. How does the rate of convergence of the filter depend on the model uncertainty and the measurement noise?

Answering those questions through simulation by no means implies any definite rule concerning the calibration of a general six degrees of freedom manipulator. It is hoped to obtain from this almost trivial example a feel and some intuition towards the Kalman filter behavior. Any observation is expected intuitively to carry through to the more general manipulator and should be the subject of a deeper investigation and more rigorous proof.

The simulations were run for three types of norms, i.e., the one norm, the infinity norm and euclidien norm (see Appendix E). All three types of norms gave very close numerical results. Hence, all the figures apply to all of the above mentioned norms.

Since the one-link manipulator system is always observable or determined previously, an angle of 45° was chosen to run the simulation because it maximizes the norm of the matrix $\underline{\underline{H}}$ and, thus, minimizes the number of measurements needed to achieve a predetermined accruacy as discussed in section 4.4. Those results as presented in figures 13:20. The number of measurements taken in order to estimate

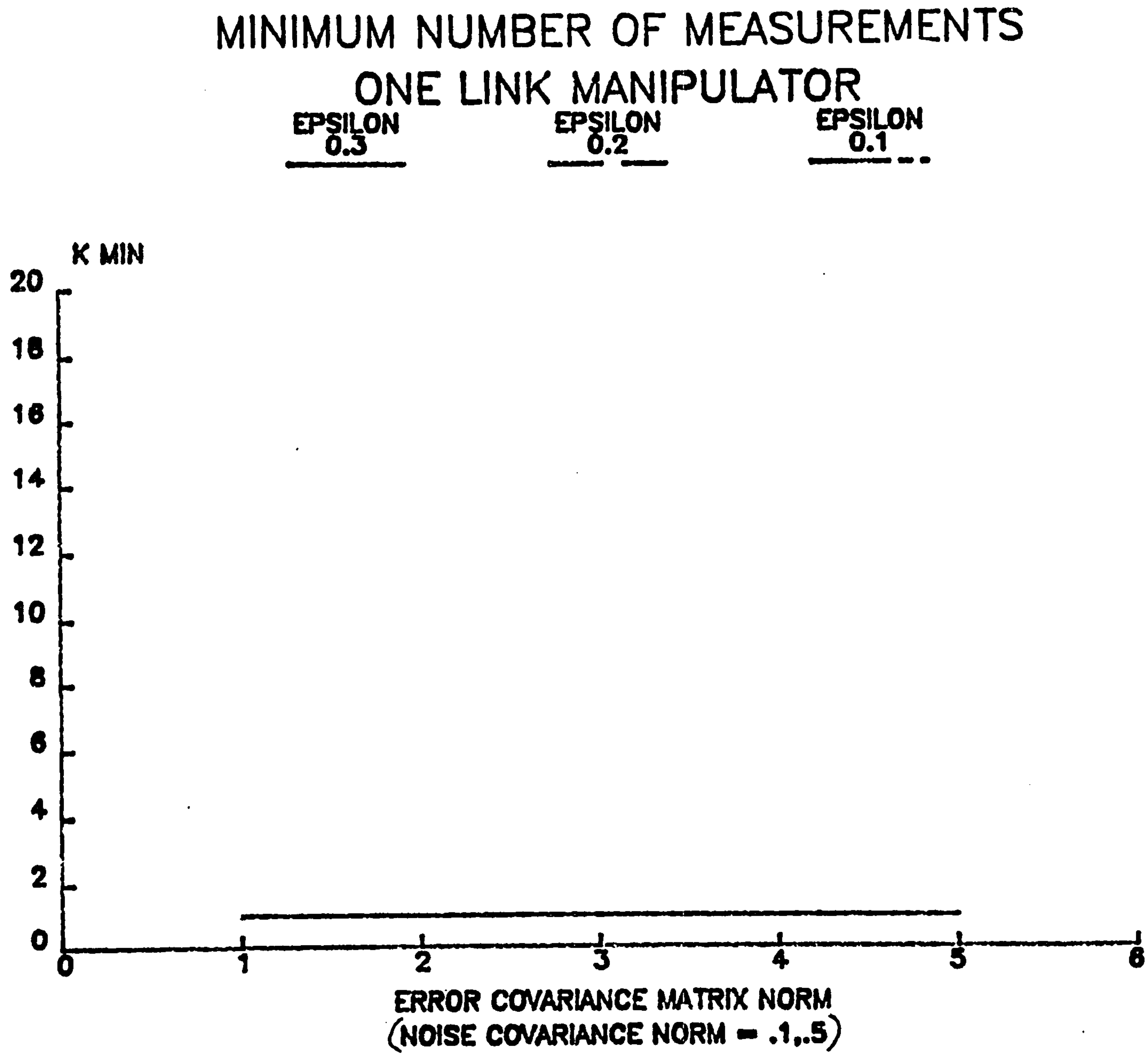


Fig. 13 Minimum Number of Measurements vs. Error Covariance Matrix Norm (|| R|| = .1 and .5)

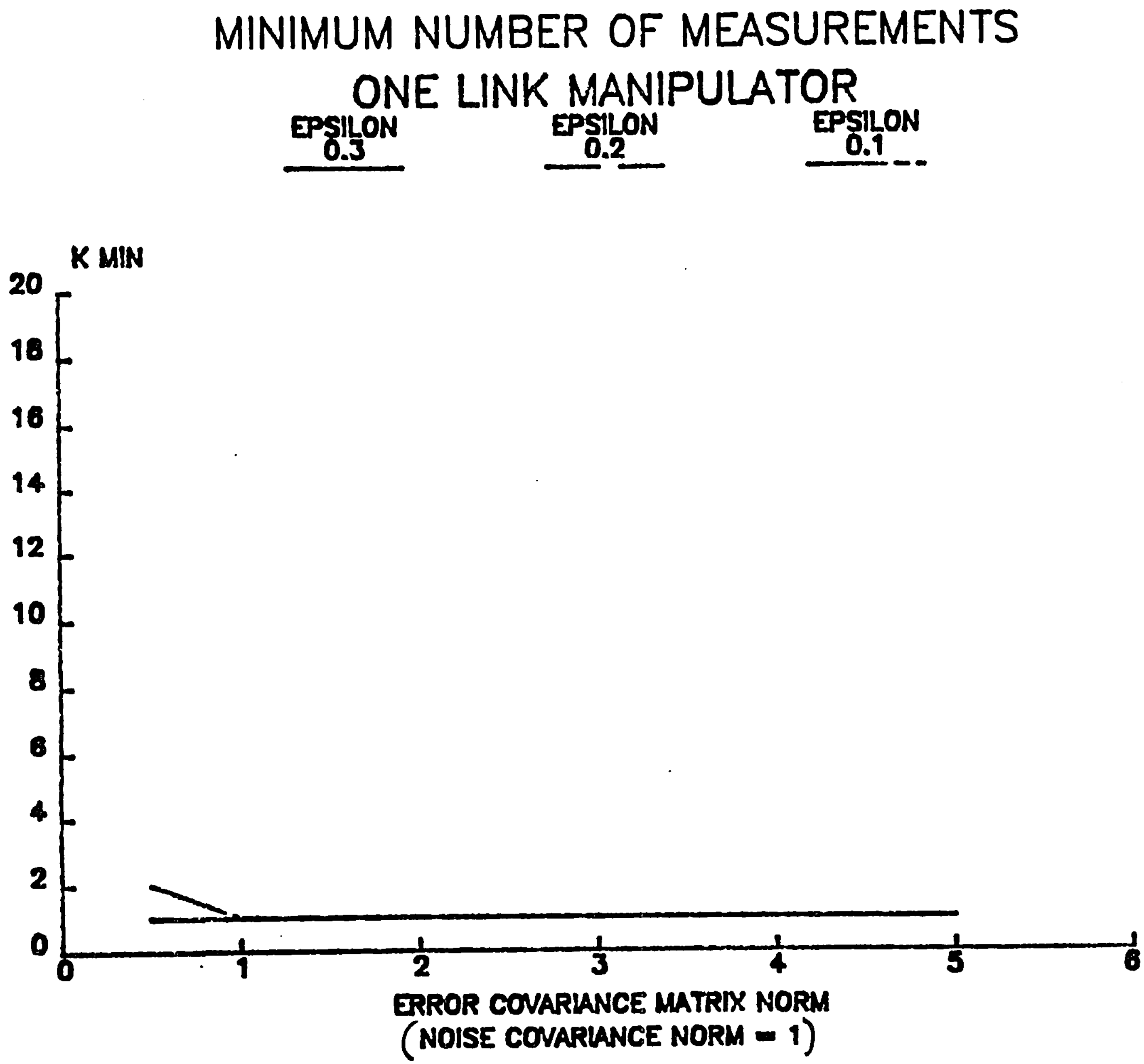


Fig. 14 Minimum Number of Measurements vs. Error Covariance Matrix Norm (||R|| = 1)

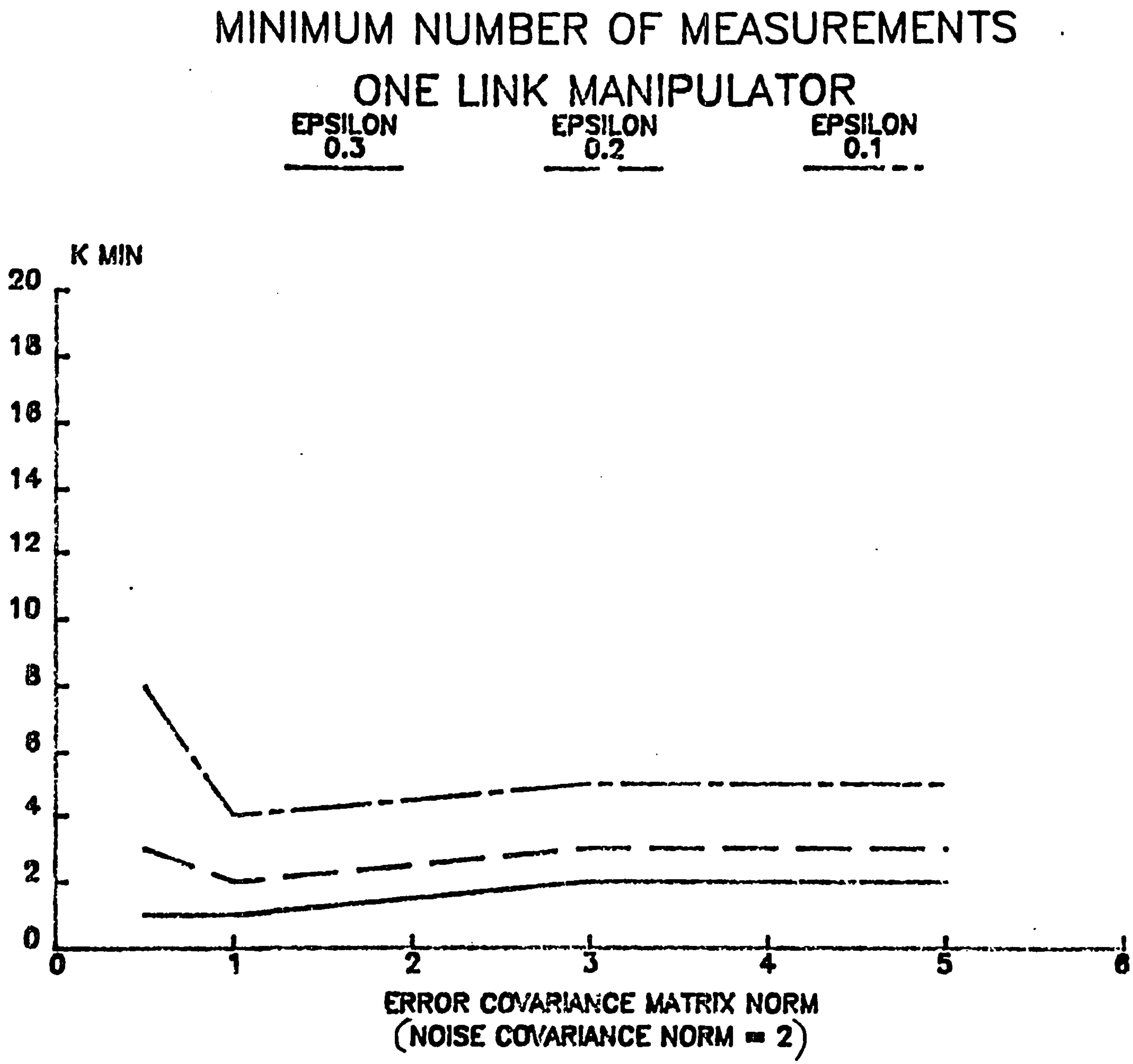


Fig. 15 Minimum Number of Measurements vs. Error Covariance Matrix Norm (||R|| = 2)

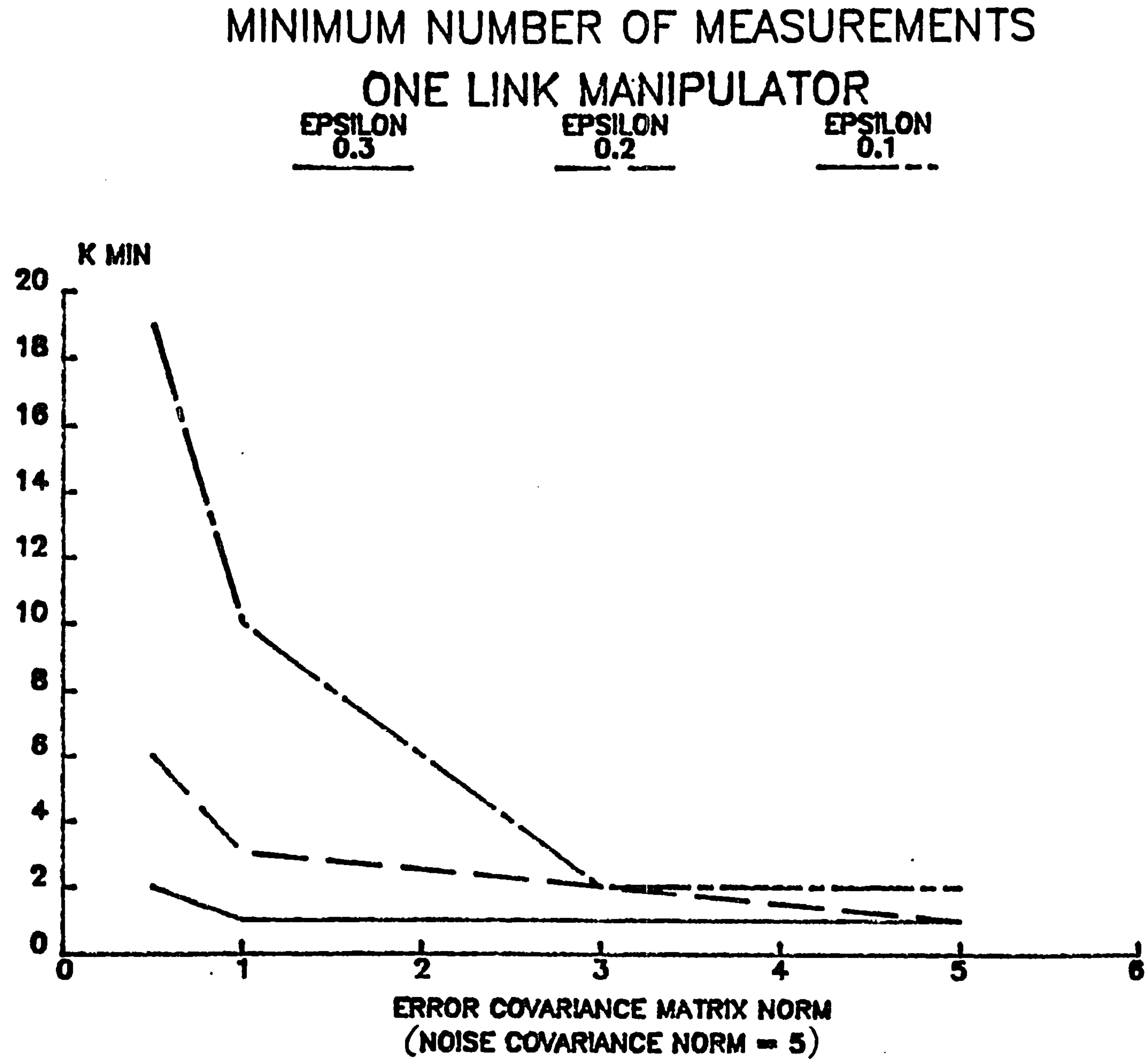


Fig. 16 Minimum Number of Measurements vs. Error Covariance Matrix Norm (|| R|| = 5)

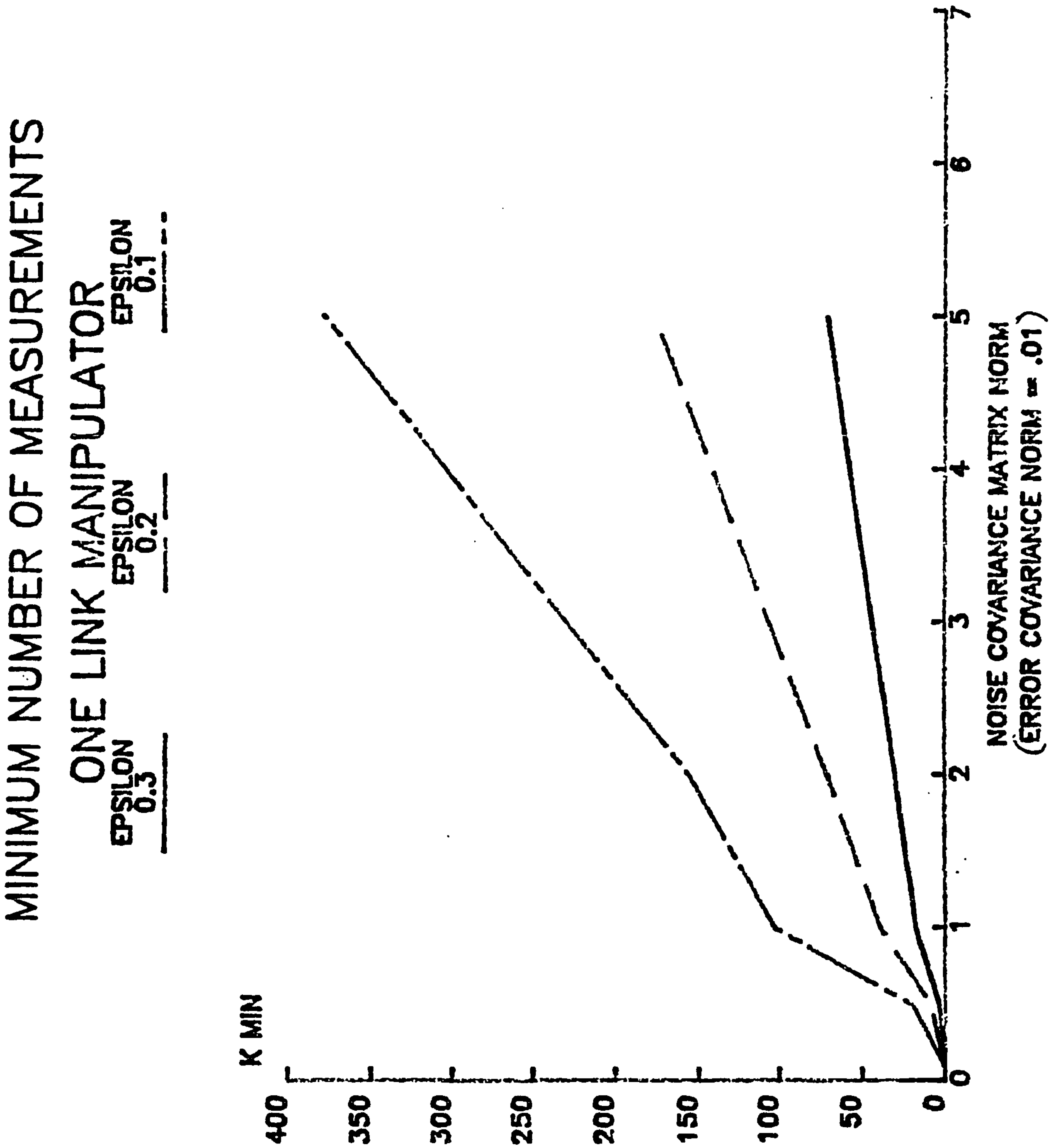


Fig. 17 Minimum Number of Measurments vs. Noise Covariance Matrix Norm ($||P_0||$ = .01)

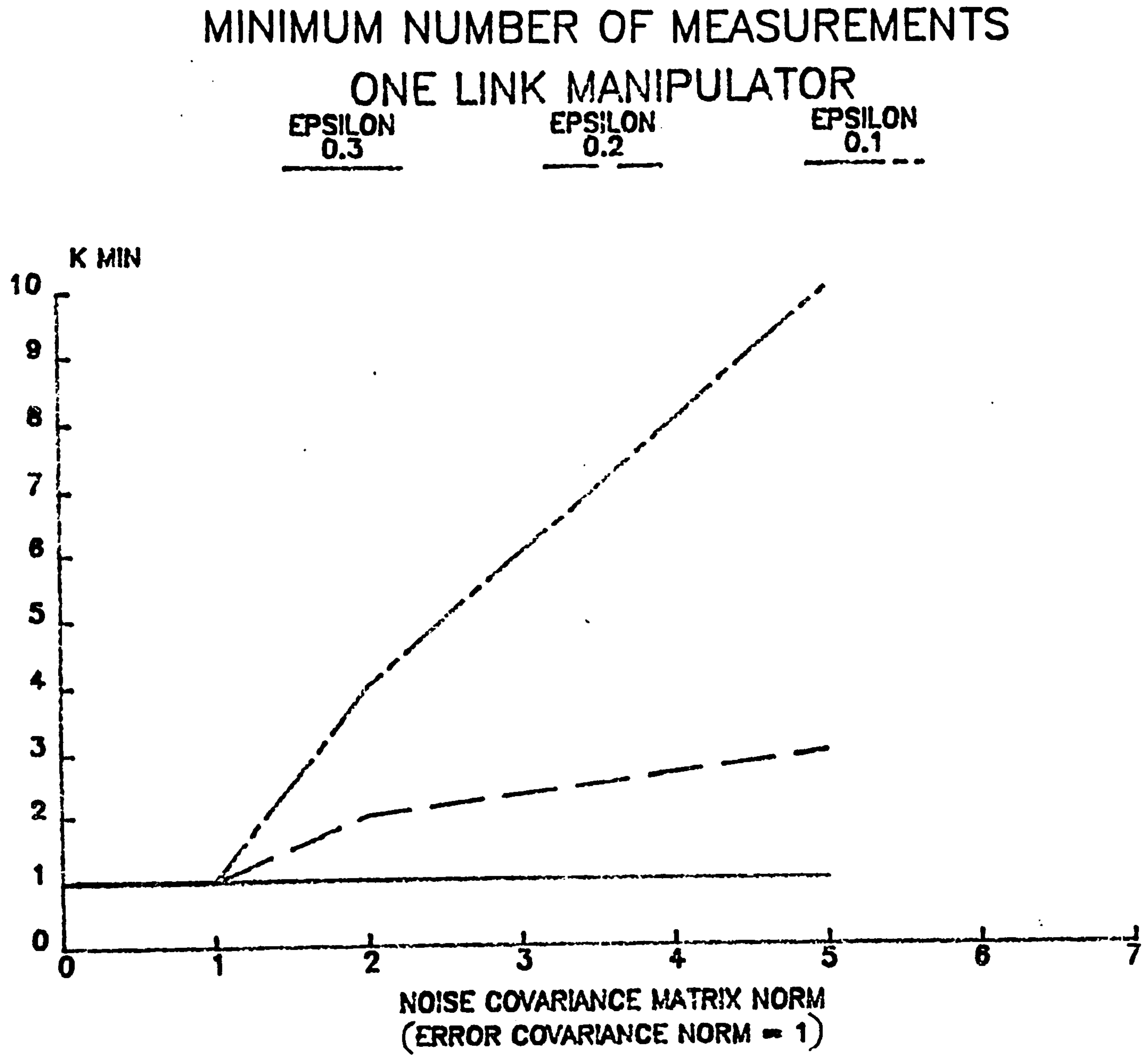


Fig. 18 Minimum Number of Measurements vs. Noise Covariance Matrix Norm ($||P_o|| = 1$)

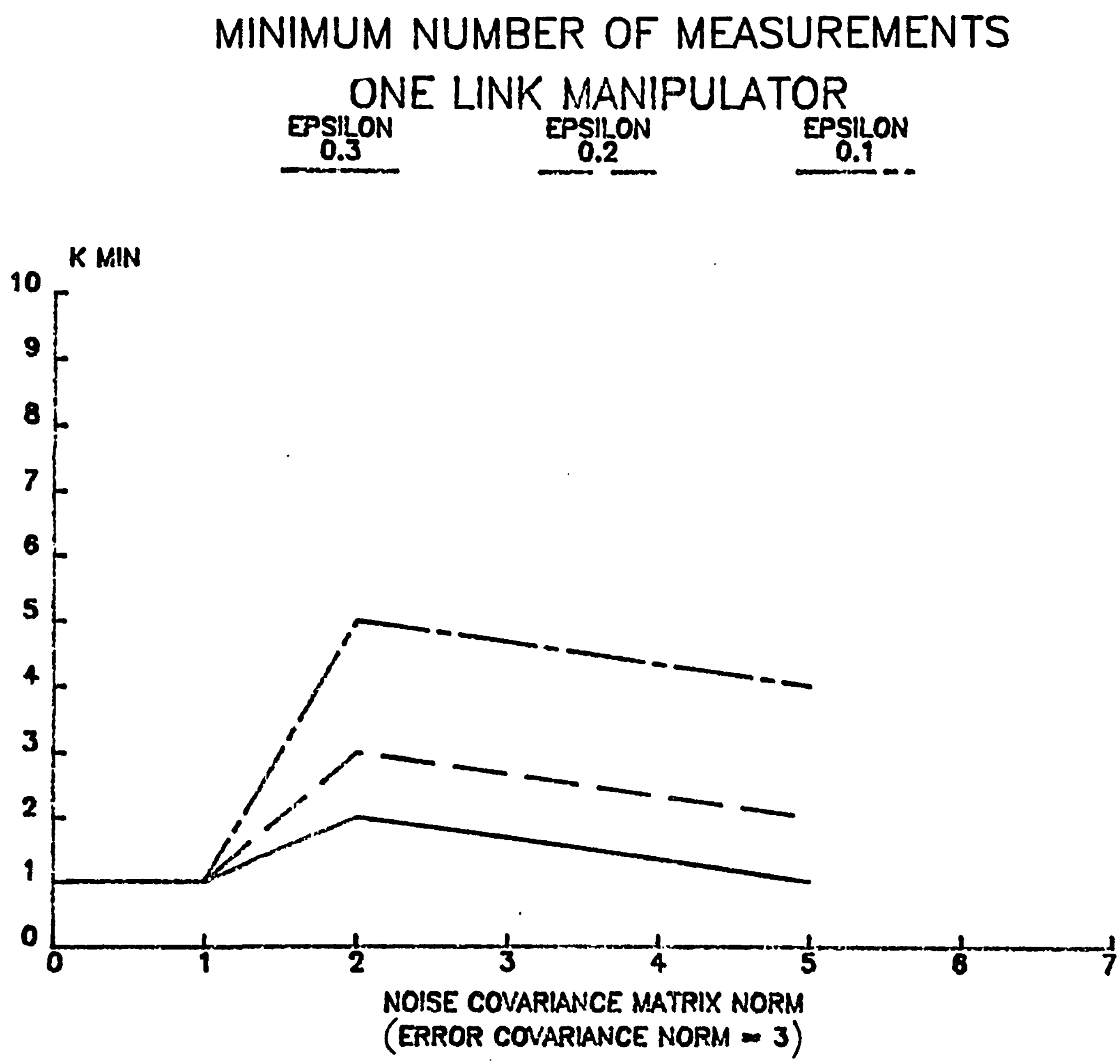


Fig. 19 Minimum Number of Measurements vs. Noise Covariance Matrix Norm ($||P_o|| = 3$)

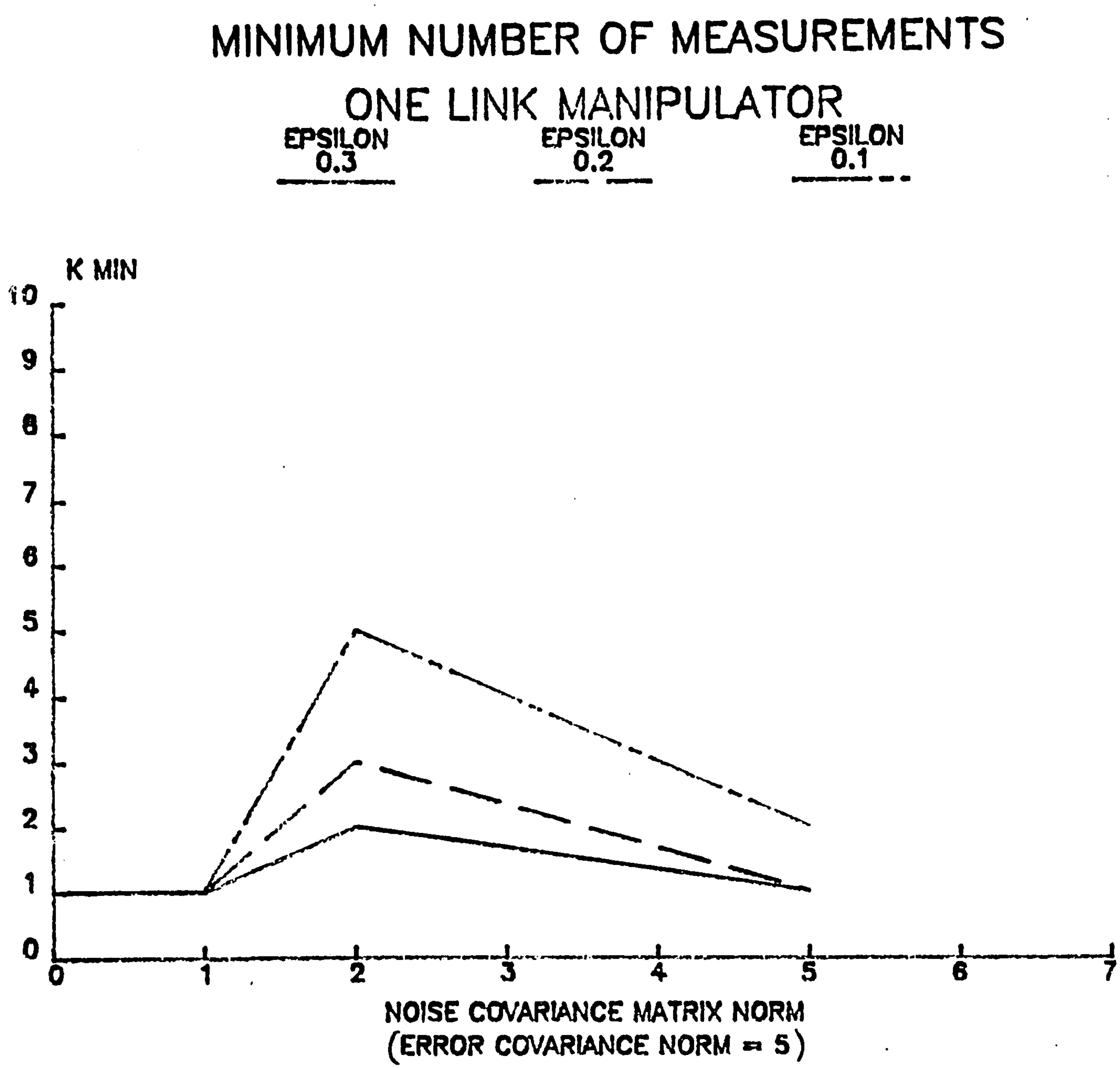


Fig. 20 Minimum Number of Measurements vs. Noise Covariance Matrix Norm ($||P_o|| = 5$)

uniquely the robot Kinematic mode error is the one for which the error convariance matrix norm reaches a certain acceptable fraction of the initial one, hence satisfying conditon 4-43.

A first glance at the results shows what seems to be a departure from what one might intuitively expect, as discussed in section 4.4, especially figure 10. The number of measurements is supposed to decrease as the initial uncertainty in the system's model decreases, but figures 13:16 seem to suggest the opposite. Also figures 19:20 show a decrease in the number of measurements as the norm of the noise covariance matrix increases above a certain value.

A closer look at the results and equation 5-23, helps in clearing up this discrepancy. Equation 5-23 suggests that K_{min} is proportional to the norm of the noise covariance matrix and inversely proportional to the norm of the initial system error covariance matrix. This being the case, it is more helpful to consider the evoluation of K_{min} as a function of a ratio of the error covariance to the noise covariance. This is shown in figure 21. The "state to noise ratio" (SNR) was taken to be the ratio of the true state vector norm to that of the measurement noise standard deviation. Then SNR is given by

$$SNR = \frac{||\underline{x}(0)||}{\sigma_v} \tag{5-27}$$

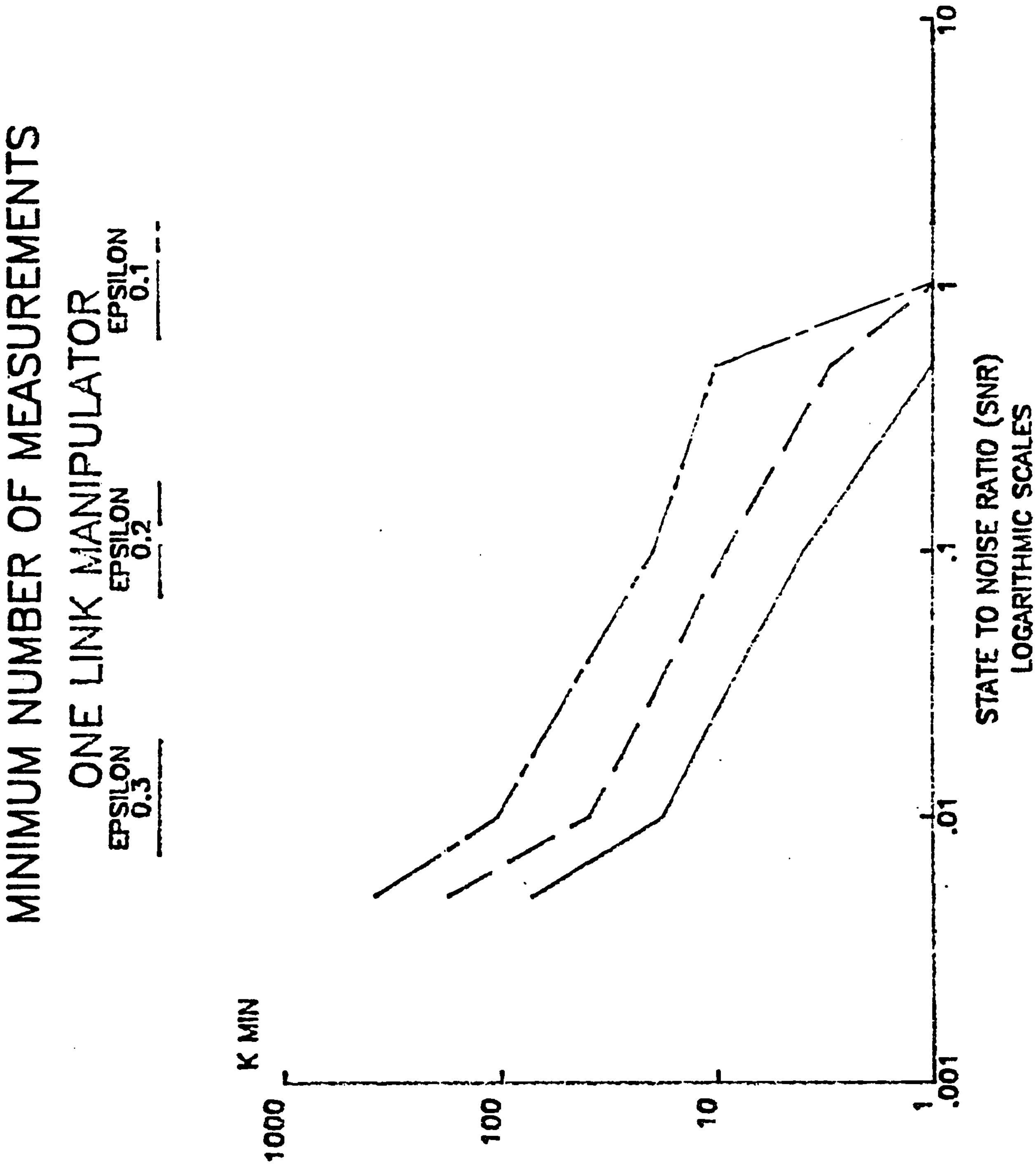


Fig. 21 Minimum Number of Measurements vs. State to Noise Ratio

Figure 21 shows the increase in the minimum number of measurements needed as the SNR decreases. In other words, more measurements are needed as the measurement noise increases (see section 4.5). Also as shown in figure 13:20, K_{min} is the same for all types of norms used.

Simulation also indicated that the minimum number of measurements needed to achieve a certain predetermined accuracy is the same for all the different types of matrix norms used (figure 11:20). In addition, the theoretical lower bound on the minimum number of measurements was found to be trivial (K_{min}=1) and of no practical use.

Moreover, the convergence of the filter is higher for larger SNR's as shown in figure 22:23.

Another result of the simulations indicates that the number of measurements needed decreases as the number of measurement configurations is increased (figure 24).

Figures 25:26 shows the result of the Kalman filter convergence towards an estimate of the state vector.

The reader should bear in mind that the figures are discrete in nature and should be interpreted as such.

5.2 The Planar Two Links Manipulator Example

The aim of this example is to determine the ability of the filter to generate an acceptable estimate for the errors in the kinematic parameter of a two-links manipulator.

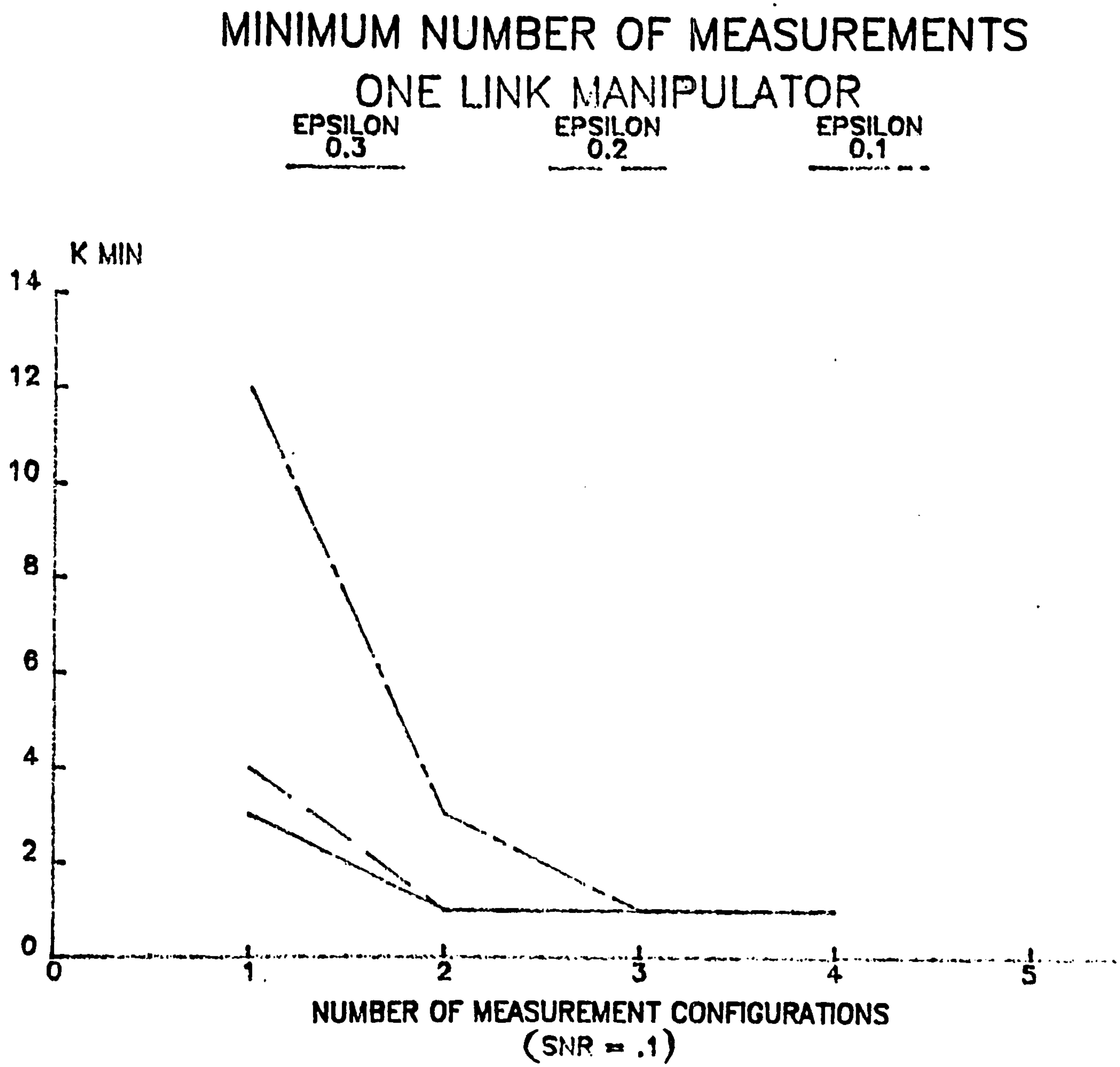


Fig. 22 Minimum Number of Measurements vs. Number of Measurement Configurations (SNR = .1)

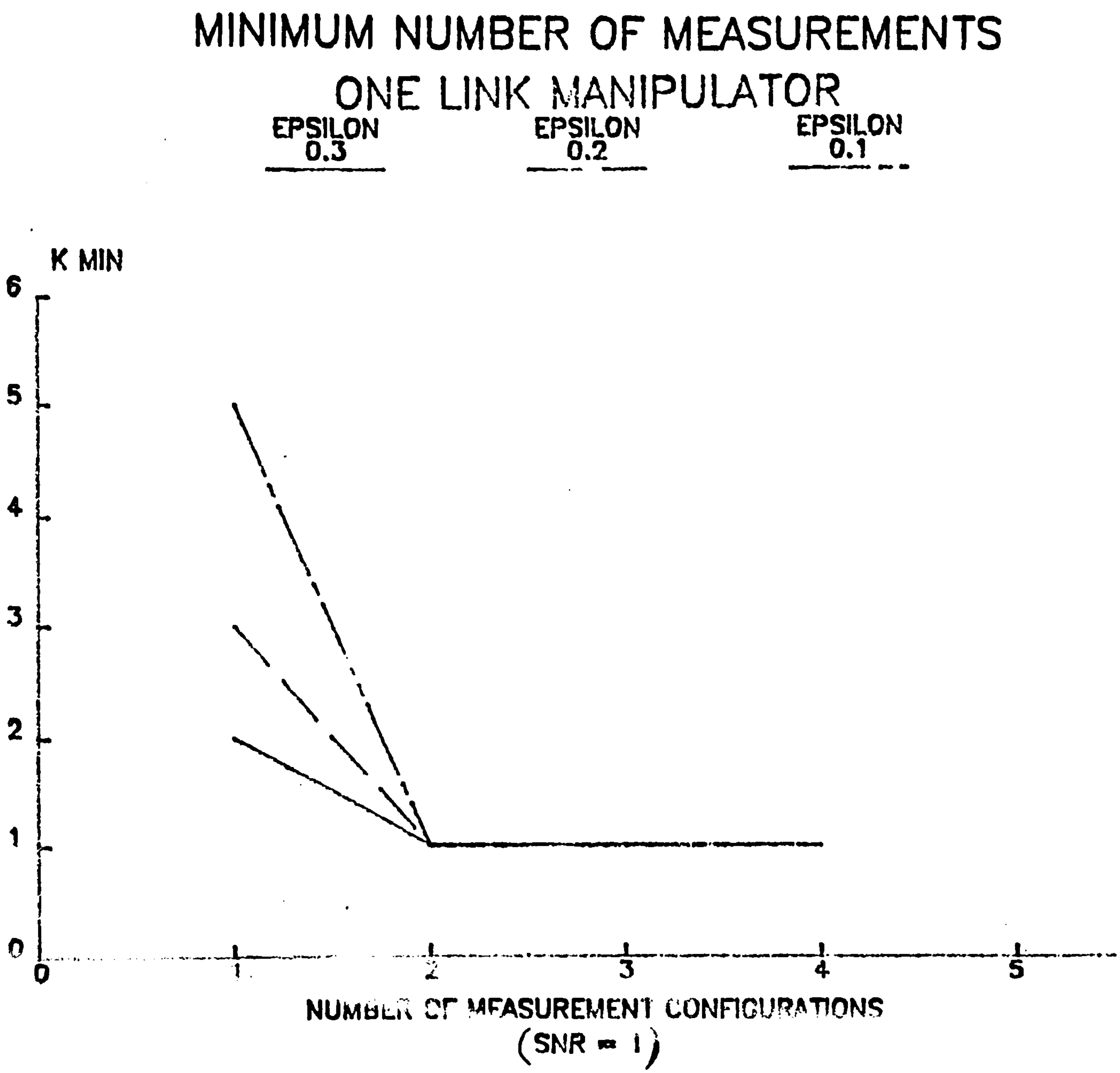


Fig. 23 Minimum Number of Measurements vs. Number of Measurement Configurations (SNR = 1)

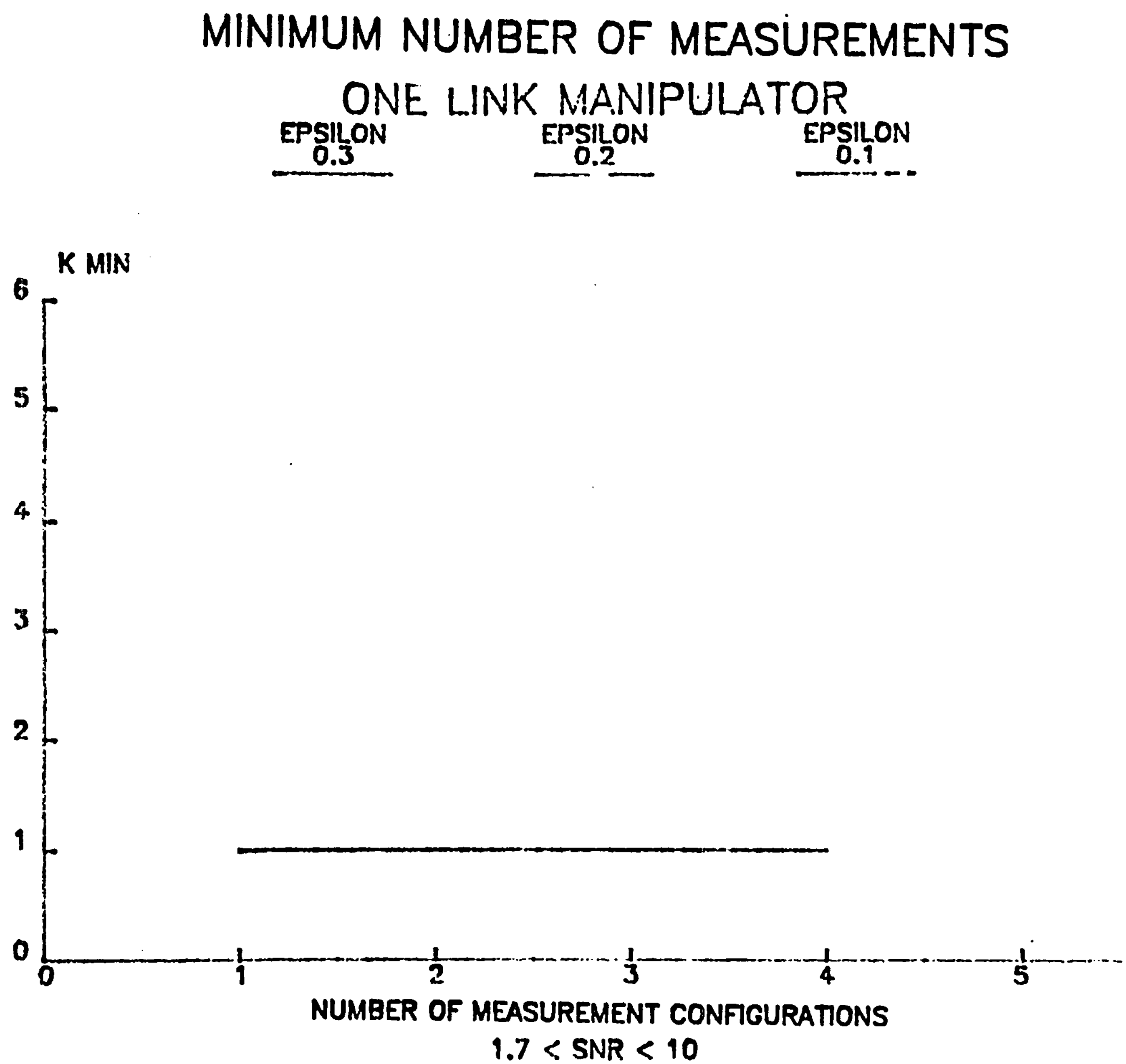


Fig. 24 Minimum Number of Measurements vs. Number of Measurement Configurations (1.7 <SNR< 10)

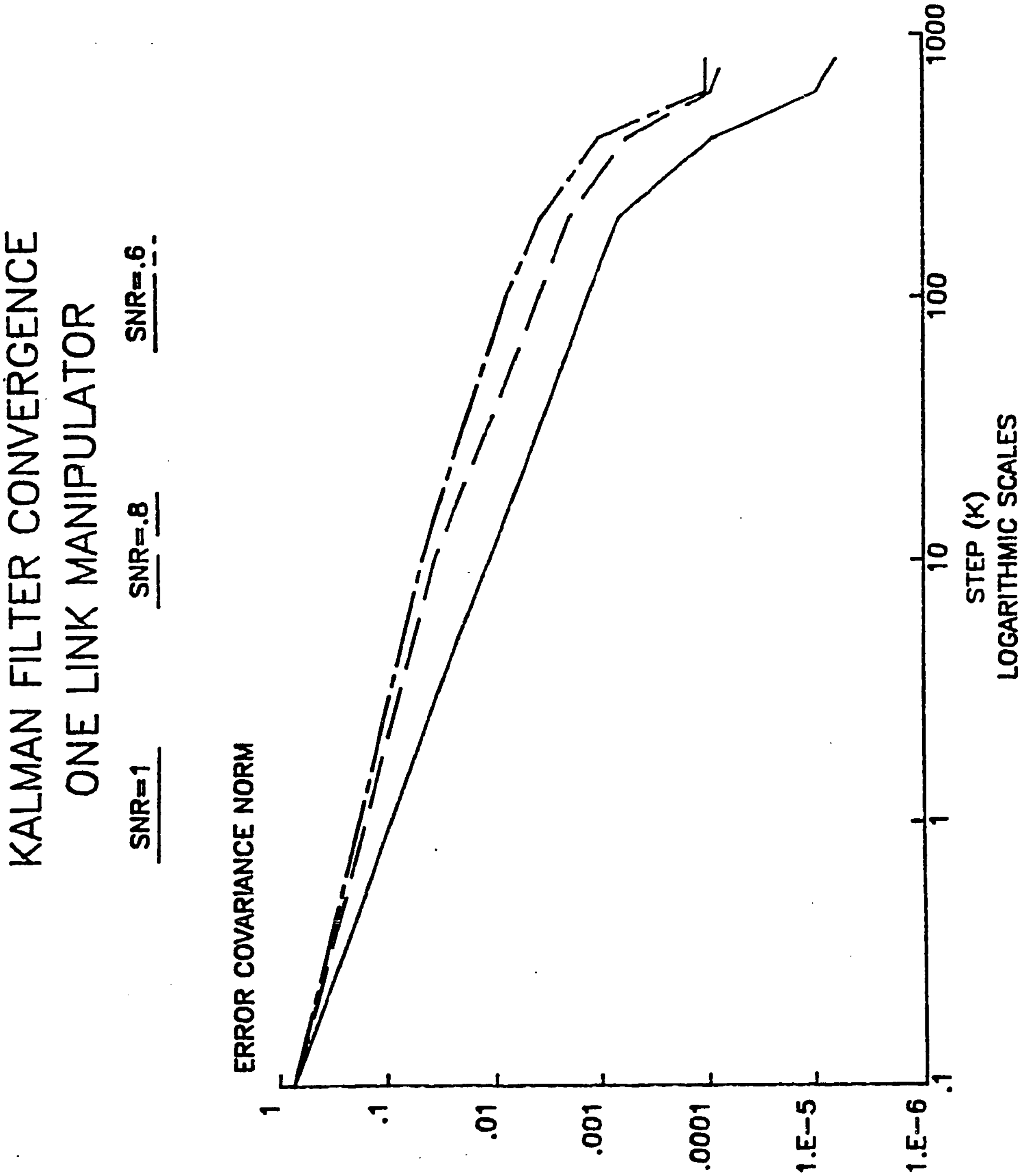


Fig. 25 One Link Manipulator Kalman Filter Convergence

XINIT(1,1)= 2.014406
XINIT(2,1)= -0.172029

STEP	XH(1,1)	XH(2,1)
1	0.199666512012E+01	-0.173712313175E+00
2	0.201473808289E+01	-0.173302292824E+00
3	0.201425170898E+01	-0.172875881195E+00
4	0.201425719261E+01	-0.172448635101E+00
5	0.201425623894E+01	-0.172035977244E+00
6	0.201425886154E+01	-0.172025092626E+00
7	0.201425981522E+01	-0.172022411443E+00
8	0.201426076889E+01	-0.172022074101E+00
9	0.201426124573E+01	-0.172022651303E+00
10	0.201426243782E+01	-0.172022616441E+00
11	0.201426339149E+01	-0.172022619234E+00
12	0.201426434517E+01	-0.172022646919E+00
13	0.201426529884E+01	-0.172022686961E+00
14	0.201426577568E+01	-0.172022681440E+00
15	0.201426696777E+01	-0.172022680989E+00
16	0.201426839828E+01	-0.172022689352E+00
17	0.201426959038E+01	-0.172022685503E+00
18	0.201427054405E+01	-0.172022667192E+00
19	0.201427173615E+01	-0.172022770338E+00
20	0.201427292824E+01	-0.172022801681E+00
21	0.201427388191E+01	-0.172022829314E+00
22	0.201427483559E+01	-0.172022829811E+00
23	0.201427531242E+01	-0.172022829765E+00
24	0.201427626610E+01	-0.172022831501E+00
25	0.201427721977E+01	-0.172022836776E+00

The vector XINIT is the vector of link error parameters. The filter was initiated with $\hat{\underline{x}}(0)=\underline{0}$

Fig. 26 One Link Manipulator, Kalman Filter Covergence towards Parameters true Value

5.2.1 Measurement Equation

In this section, the measurement equation for a planar two links manipulator will be developed. To that end, it will be assumed that both nominal and actual models are given by (figure 27)

Actual Model

$$P_x=(r_1+dr)\cdot\cos(\theta_1+d\theta_1)+(r_2+dr_2)\cdot\cos(\theta_1+d\theta_1\theta_2+d\theta_2) \qquad (5\text{-}28)$$

$$P_y=(r_1+dr_1)\cdot\sin(\theta_1+d\theta_1)+(r_2+dr_2)\cdot\sin(\theta_1+d\theta_1+\theta_2+d\theta_2) \qquad (5\text{-}29)$$

$$P_z = 0 \qquad (5\text{-}30)$$

Nominal Model

$$P_x = r_1\cos\theta_1+r_2\cos(\theta_1+\theta_2) \qquad (5\text{-}31)$$

$$P_y = r_1\sin\theta_1+r_2\sin(\theta_1+\theta_2) \qquad (5\text{-}32)$$

$$P_z = 0 \qquad (5\text{-}33)$$

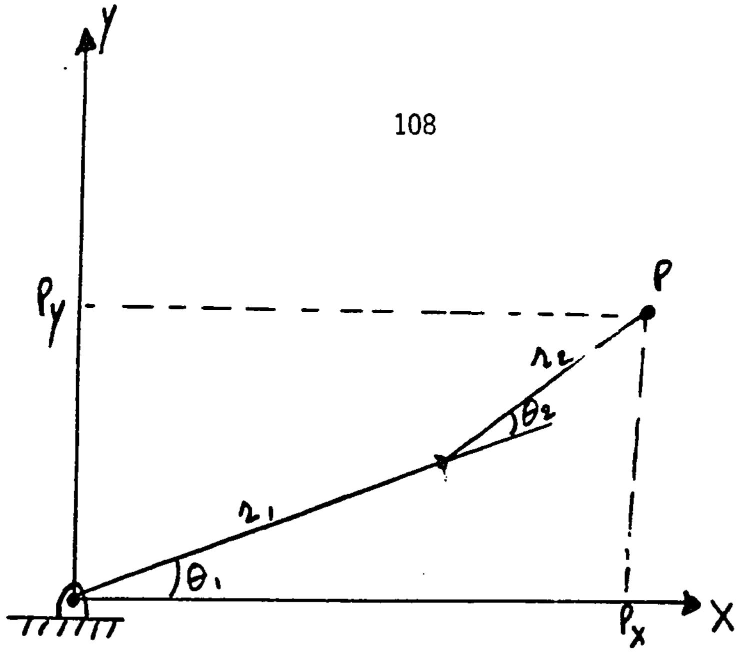
In addition, the following assumptions are taken

A.5: $\lim_{\theta_i \to 0} \cos(d\theta_i) = 1$ $\quad i=1,2$

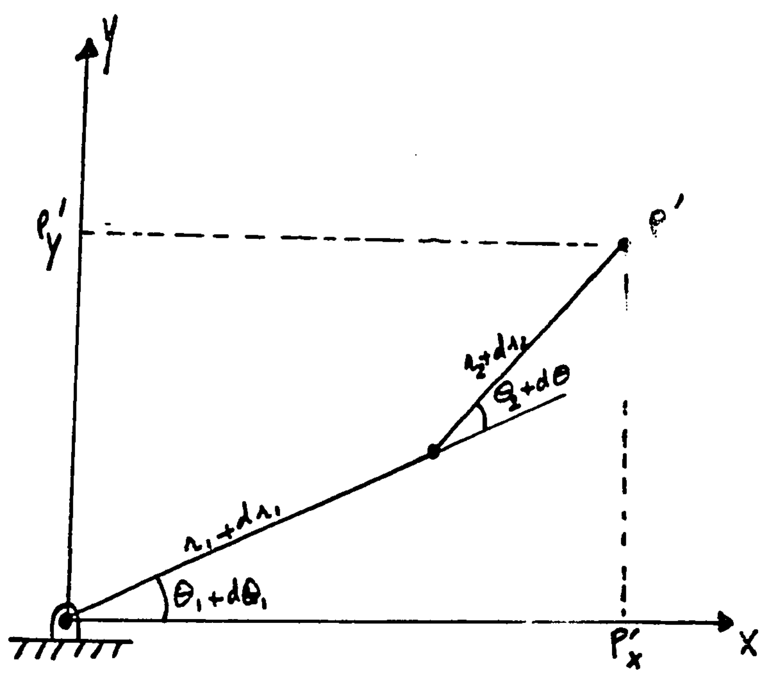
A.6: $\lim_{\theta_i \to 0} \sin(d\theta_i)=d\theta_i$ $\quad i-1,2$

A.7: Double differential error terms are neglible.

A.8: A method for measuring the difference between the actual and nominal position of the end point P is available.



a) Nominal Model



b) Actual Model

Fig. 27 Planar Two Links Manipulator Model

Using the above equations (5-28:5-33) and assumptions (A.5:A.8), it can be shown that

$$\underline{e}(k) = \begin{bmatrix} e_x \\ e_y \end{bmatrix}_K = \begin{bmatrix} P'_x - P_x \\ P'_y - P_y \end{bmatrix}_k$$

$$= \begin{bmatrix} -r_1 s\theta_1 - r_2(s\theta_1 \cdot c\theta_2 + c\theta_1 \cdot s\theta_2) & -r_2(s\theta_1 \cdot c\theta_2 + c\theta_1 \cdot s\theta_2) & c\theta_1 & c\theta_1 \cdot c\theta_2 - s\theta_1 s\theta_2 \\ -r_1 c\theta_1 - r_2(c\theta_1 \cdot c\theta_2 - s\theta_1 \cdot s\theta_2) & r_2(c\theta_1 \cdot c\theta_2 - s\theta_1 \cdot s\theta_2) & s\theta_1 & s\theta_1 \cdot c\theta_2 + c\theta_1 \cdot s\theta_2 \end{bmatrix}_K \cdot \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dr_1 \\ dr_2 \end{bmatrix}_K \quad (5\text{-}34)$$

Then, the measurements are given by the above difference vector and some additive white Gaussian noise,

$$\underline{z}(k) = \underline{e}(k) + \underline{v}(k) \qquad K = 1,2,... \qquad (5\text{-}35)$$

or

$$\underline{z}(k) = \underline{\underline{H}}(k) \cdot \underline{x}(k) + \underline{v}(k) \qquad K = 1,2,... \qquad (5\text{-}36)$$

where $\underline{v}(k)$ is a white Gaussian sequence with zero mean and postive definite covariance matrix $\underline{\underline{R}}(k)$, ie, $\underline{v}(k) \sim N(0,\underline{\underline{R}}(k))$, the matrix $\underline{\underline{H}}(k)$ is given by

$$\underline{\underline{H}}(k) = \begin{bmatrix} -r_1 s\theta_1 - r_2(s\theta_1 \cdot c\theta_2 + c\theta_1 \cdot s\theta_2) & -r_2(s\theta_1 \cdot c\theta_2 - c\theta_1 \cdot s\theta_2) & c\theta_1 & c\theta_1 \cdot c\theta_2 - s\theta_1 \cdot s\theta_2 \\ -r_1 c\theta_1 - r_2(c\theta_1 \cdot c\theta_2 - s\theta_1 \cdot s\theta_2) & r_2 (c\theta_1 \cdot c\theta_2 - s\theta_1 \cdot s\theta_2) & s\theta_1 & s\theta_1 \cdot c\theta_2 + c\theta_1 \cdot s\theta_2 \end{bmatrix}_K \qquad (5\text{-}37)$$

and where the state vector $\underline{x}(k)$ is

$$\underline{x}(k) = (d\theta_1, d\theta_2, dr_1, dr_2)^T_K \tag{5-38}$$

The identification problem formualtion and the Kalman filter equation setup is exactly the same as for the one link manipulator (section 5.1.2) and will not be repeated here.

5.2.2 Observability

As seen in the previous example, the observability matrix is given by equation (5-25) that is

$$\underline{\underline{\Omega}} = [\underline{\underline{H}}^T, \ldots, \underline{\underline{H}}^T]^T$$

where $\underline{\underline{\Omega}}$ is an ℓ x n matrix, ℓ is the number of measurements and n is the dimension of the state vector.

Hence, according to theorems E1 and E2 and corollary I, the system is observalbe if the rank of $\underline{\underline{\Omega}}$ is equal to 4 which is the dimension of the state vector $\underline{x} = (d\theta_1, d\theta_2, dr_1, dr_2)^T$.

It follows that the system is unobservable, if only one measurement is taken since rank of $\underline{\underline{\Omega}}$ will be equal to 2. Then, at least 2 measurements are needed for a proper identification by the Kalman filter.

One also expects the system to become unobservable if only two measurements are taken and one or both configurations is singular. The planar 2-link manipulator has two singular configurations, $\theta_2 = 0°$ and

$\theta_2 = 180°$. In addition, the system is expected to become unobservable if either or both the first link angle, i.e., θ_1 and the second link angle, i.e., θ_2, remains unchanged from one measurement configuration to the other. To overcome this problem more measurements can be taken and joints angles changed for each one such that some of the rows of the observability matrix become redundant and rank of $\underline{\underline{\Omega}}$ =4.

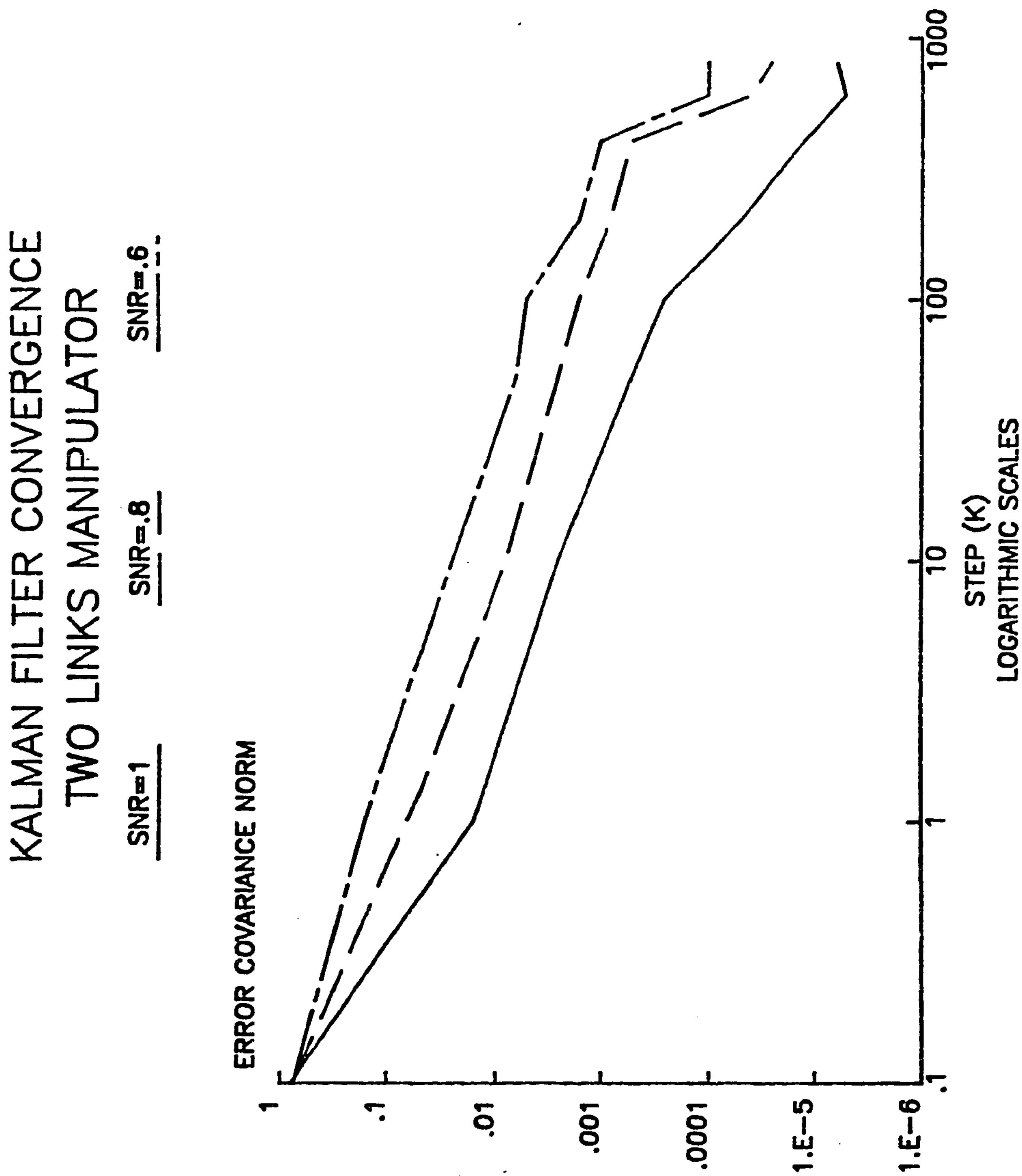


Fig. 28 Two Links Manipulator, Kalman Filter Convergence

XINIT(1,1)= 1.007233
XINIT(2,1)= -0.156770
XINIT(3,1)= -0.403607
XINIT(4,1)= 0.162232

STEP	XH(1,1)	XH(2,1)	XH(3,1)	XH(4,1)
1	0.100614571571E+01	-0.164994344115E+00	-0.101209688187E+01	0.818956553936E+00
2	0.100715136528E+01	-0.159073046679E+00	-0.621129631996E+00	0.380484014750E+00
3	0.100756795502E+01	-0.157059803605E+00	-0.520463466644E+00	0.367384999990E+00
4	0.100638600540E+01	-0.157142832375E+00	-0.426826864481E+00	0.260279327631E+00
5	0.100653898716E+01	-0.155882701278E+00	-0.393168359995E+00	0.250331789255E+00
6	0.100699543953E+01	-0.155652451921E+00	-0.349322706461E+00	0.240943593383E+00
7	0.100725567341E+01	-0.156714111567E+00	-0.388283669949E+00	0.239113041162E+00
8	0.100727868080E+01	-0.156709209085E+00	-0.401664972305E+00	0.229678511739E+00
9	0.100713026524E+01	-0.156748384237E+00	-0.423944443464E+00	0.229345351458E+00
10	0.100720059872E+01	-0.157175123692E+00	-0.438753843307E+00	0.220884859562E+00
11	0.100722277164E+01	-0.157489702106E+00	-0.453043073416E+00	0.220452666283E+00
12	0.100703179836E+01	-0.156513533326E+00	-0.425792016745E+00	0.206817716360E+00
13	0.100696551900E+01	-0.156391680241E+00	-0.446962445974E+00	0.204377587438E+00
14	0.100694532135E+01	-0.155601412058E+00	-0.449495404959E+00	0.198380281687E+00
15	0.100696761398E+01	-0.155416931374E+00	-0.433971703053E+00	0.196698813634E+00
16	0.100718787956E+01	-0.155313074589E+00	-0.415359513594E+00	0.196597004294E+00
17	0.100710640015E+01	-0.155661910772E+00	-0.416343092918E+00	0.192332715988E+00
18	0.100718741932E+01	-0.155712321401E+00	-0.424038201571E+00	0.188441719294E+00
19	0.100718475399E+01	-0.155973456140E+00	-0.427057832479E+00	0.186760474086E+00
20	0.100719363480E+01	-0.155679017305E+00	-0.410455018282E+00	0.185401719928E+00
21	0.100721412220E+01	-0.155534163117E+00	-0.406039178371E+00	0.184085206389E+00
22	0.100721471325E+01	-0.155494764447E+00	-0.407660812140E+00	0.178735033154E+00
23	0.100725318966E+01	-0.155438259244E+00	-0.430248290300E+00	0.178502353430E+00
24	0.100726189194E+01	-0.155349701643E+00	-0.428686738014E+00	0.173993858099E+00
25	0.100720634041E+01	-0.155218139291E+00	-0.426073819399E+00	0.174065219760E+00

The vector XINIT is the vector of link error parameters. The filter was initiated with $\hat{\underline{x}}(0) = 0$

Fig. 29 Two Links Manipulator, Kalman Filter Convergence Towards Parameters True Value

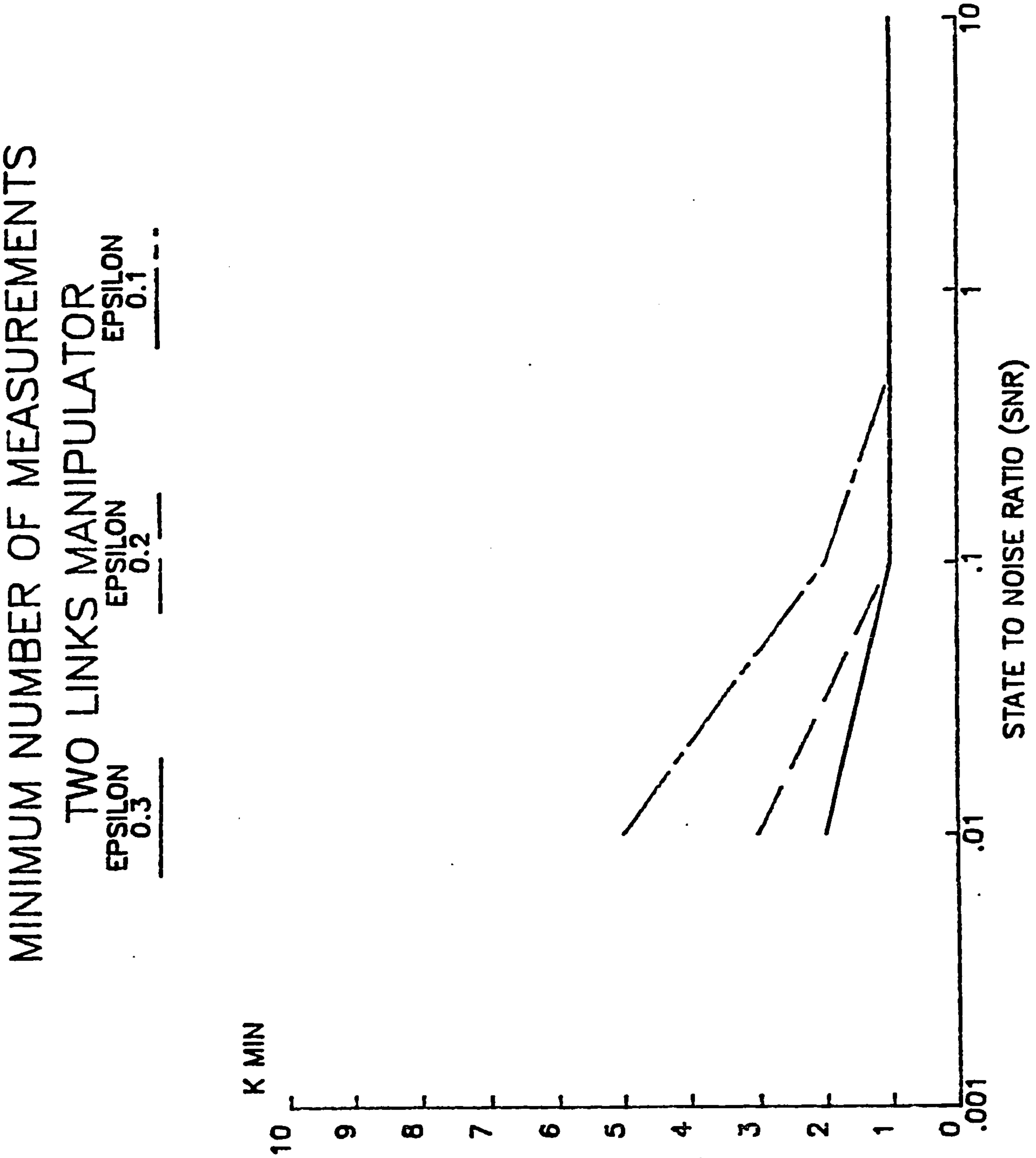


Fig. 30 Two Links Manipulator, Minimum Number of Measurements vs. State to Noise Ratio

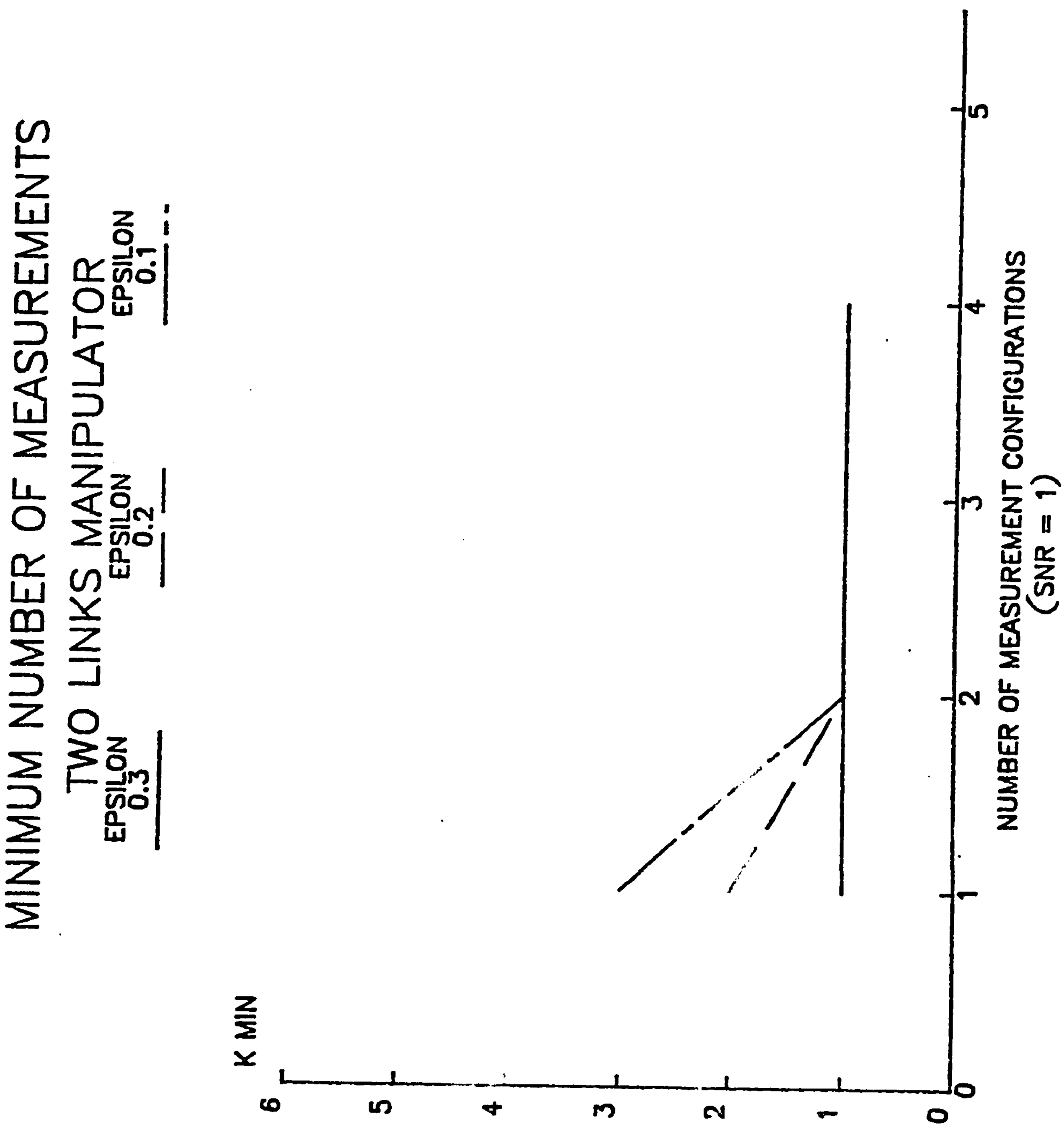


Fig. 31 Two Links Manipulator, Minimum Number of Measurements vs. Number of Measurement Configuraitons (SNR = 1)

XINIT(1,1)= 1.524829
XINIT(2,1)= -0.237331
XINIT(3,1)= -0.142697
XINIT(4,1)= 0.057358

STEP	XH(1,1)	XH(2,1)	XH(3,1)	XH(4,1)
1	0.152376961709E+01	-0.229559928179E+00	0.688300793990E-02	0.841876305640E-02
2	0.152138599110E+01	-0.225508123636E+00	0.683318637311E-02	0.837147422135E-02
3	0.151165711850E+01	-0.194501996040E+00	0.646407622844E-02	0.802663899958E-02
4	0.154432880979E+01	-0.298294216394E+00	0.769512308761E-02	0.917493831366E-02
5	0.153079295158E+01	-0.255959093571E+00	0.719250878319E-02	0.870583299547E-02
6	0.152802383900E+01	-0.246120199561E+00	0.707674259320E-02	0.859815813601E-02
7	0.152718079090E+01	-0.242734447122E+00	0.703727779910E-02	0.856139604002E-02
8	0.152665073640E+01	-0.240615604686E+00	0.701262662187E-02	0.853850133717E-02
9	0.152618131705E+01	-0.239444389939E+00	0.699843885377E-02	0.852510705590E-02
10	0.152549815178E+01	-0.237575918436E+00	0.697605032474E-02	0.850401539356E-02
11	0.152506029606E+01	-0.236443958516E+00	0.696246698499E-02	0.849118363112E-02
12	0.152475166321E+01	-0.235321998596E+00	0.694941775873E-02	0.847895164043E-02
13	0.152465963364E+01	-0.234707772732E+00	0.694254739210E-02	0.847251992673E-02
14	0.152474451065E+01	-0.234947293997E+00	0.694541074336E-02	0.847520958632E-02
15	0.152467393875E+01	-0.234687349379E+00	0.694237416610E-02	0.847241189331E-02
16	0.152494037151E+01	-0.235550701518E+00	0.695252744481E-02	0.848187506199E-02
17	0.152488708496E+01	-0.235423862934E+00	0.695097027346E-02	0.848040916026E-02
18	0.152487301826E+01	-0.235372126102E+00	0.695036072284E-02	0.847985409200E-02
19	0.152495360374E+01	-0.235700070358E+00	0.695417402312E-02	0.848333910108E-02
20	0.152467170696E+01	-0.235120475292E+00	0.694762729108E-02	0.847759470344E-02
21	0.152482271194E+01	-0.235204532743E+00	0.694822659716E-02	0.847807060927E-02
22	0.152481961250E+01	-0.235075354576E+00	0.694688083604E-02	0.847685616463E-02
23	0.152462456511E+01	-0.234662771225E+00	0.694174226373E-02	0.847184659051E-02
24	0.152454662323E+01	-0.234294459224E+00	0.693768123165E-02	0.846802257001E-02
25	0.152449357510E+01	-0.234240010381E+00	0.693686259910E-02	0.846721976995E-02

The filter is unable to estimate the value of the last two elements of the error parameters vector.

Fig. 32 Two Links Manipulator, Kalman Filter Divergence of an Unobservable System

5.2.3 Simulations Results

The goal of the two-links planar manipulator example is to investigate the behavior of the filter when the system is unobservable. All other conclusions or results of the one-link manipulator are suspected to carry through to the two-link one and are indeed true for this case.

Figures 28:29 show the convergence of the filter for an observable system made up of two nonsingular configurations.

Figure 30 shows the decrease in the number of measurements as the SNR increases.

Figure 31 shows the decrease in the number of measurements as the number of measurement configuraiton is increased.

Figure 32 shows the inability of the Kalman filter to estimate all of the state vector variables when the system is unobservable.

5.3 The Two Links Manipulator Example-(Thesis Set-Up)

This example will mainly deal with the set up of the two links manipulator Kinematic model and the Kalman filter measurement equation as described in chapter 3.

5.3.1 The Measurement Equation

Figure 33 shows the two links manipulator set-up that will be used for this example with coordinate frames assigned to the links.

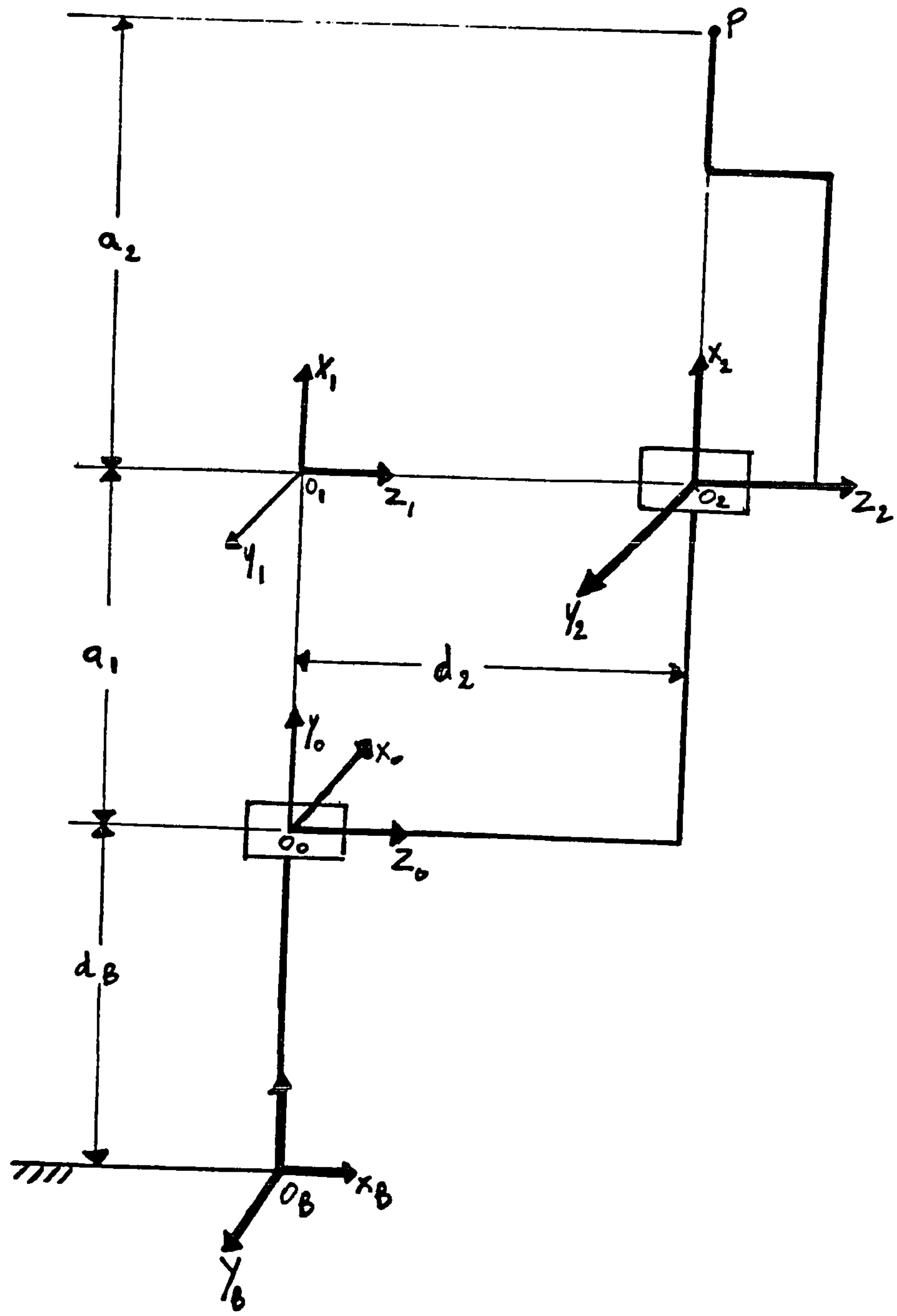


Fig. 33 Two Links Manipulator Model

Link parameters for the Two Links Manipulator

Link	Variable	α	a	d	β	μ
0	90°	-90°	0	d_B	N.A	0
1	θ_1	0°	a_1	0	0°	μ_1
2	θ_2	0°	a_2	d_2	N.A	μ_2

Assumptions:

- Counterclockwise rotations are positive rotations.
- Axes of rotation 1 and 2 are parallel revolute.

According to equation 3-38, the measurement equation is given by

$$\underline{z}_M = \underline{\underline{H}}\cdot\underline{x} + \underline{\underline{L}}\cdot\underline{v} \qquad (5\text{-}38)$$

where

$$z_M = [d_{x_M}, d_{y_M}, d_{z_M}, \delta_{x_M}, \delta_{y_M}, \delta_{z_M}]^T \qquad (5\text{-}39)$$

and where $\underline{\underline{H}}$, $\underline{x}$, $\underline{\underline{L}}$, and $\underline{v}$ were given by equations 3-40:3-43.

i) Link 0

Using equation 3-8 and link 0 parameters the Jacobian matrix can be obtained

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -d_B \\ 0 & 0 & -1 & 0 & d_B & 0 \end{bmatrix}$$

$$\underline{\underline{^{B}J^{0}}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{5-41}$$

For a revolute joint, the $\underline{\underline{G}}$ matrix is given by equation 3-14

$$\underline{\underline{G}}_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5-42}$$

Hence,

$$\underline{\underline{^{B}J^{0}}} \cdot \underline{\underline{G}}_0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -d_B & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5-43}$$

Also,

$$\underline{b}_0 = (0,\ a_0,\ c\alpha_0,\ -a_0,\ s\alpha_0,\ 0,\ s\alpha_0,\ c\alpha_0)^T \tag{5-44}$$

$$= (0,\ 0,\ 0,\ 0,\ -1,\ 0)^T$$

and,

$$^{B}\underline{\underline{J}}^{0} \cdot \underline{b}_0 = (0,\ -d_B,\ 0,\ -1,\ 0,\ 0)^T \tag{5-45}$$

ii) Link 1

Since the axes of rotation 1 and 2 are parallel, the convention given in Appendix B should be used. Then using link 1 parameters, equation B-2 becomes

$$\underline{\underline{G}}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5-46}$$

and, the Jacobian matrix is

$$
{}^{B}\underline{\underline{J}}^{1} = {}^{B}\underline{\underline{J}}^{0} \cdot {}^{0}\underline{\underline{J}}^{1} = \begin{bmatrix} -s\theta_1 & c\theta_1 & 0 & 0 & 0 & -d_B - a_1 c\theta_1 \\ 0 & 0 & -1 & 2a_1 s\theta_1 c\theta_1 - d_B s\theta_1 & 0 & 0 \\ -c\theta_1 & -s\theta_1 & 0 & 0 & 0 & -a_1 s\theta_1 \\ 0 & 0 & 0 & -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -c\theta_1 & -s\theta_1 & 0 \end{bmatrix} \tag{5-47}
$$

Hence,

$$
{}^{B}\underline{\underline{J}}^{1} \cdot \underline{\underline{G}}_1 = \begin{bmatrix} a_1 c\theta_1 & -s\theta_1 & 0 & 0 \\ 0 & 0 & 2a_1 s\theta_1 c\theta_1 - d_B s\theta_1 & 0 \\ -as\theta_1 & -c\theta_1 & 0 & 0 \\ 0 & 0 & -s\theta_1 & -c\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c\theta_1 & s\theta_1 \end{bmatrix} \tag{5-48}
$$

Also,

$$
\underline{b}_1 = (0,\ a_1,\ c\alpha_1,\ -a_1,\ s\alpha_1,\ 0,\ s\alpha_1,\ c\alpha_1)^T \tag{5-49}
$$

$$
= (0,\ a_1,\ 0,\ 0,\ 0,\ 1)^T
$$

and

$$
{}^{B}\underline{\underline{J}}^{1} \cdot \underline{b}_1 = (-d_B,\ 0,\ -2a_1,\ s\theta_1,\ 0,\ -1,\ 0)^T \tag{5-50}
$$

iii) Link 2

Using Equation 3-23 and link 2 parameters, the Jacobian matrix becomes

$$
{}^{B}\underline{\underline{J}}^{2} = {}^{B}\underline{\underline{J}}^{0} \cdot {}^{0}\underline{\underline{J}}^{1} \cdot {}^{1}\underline{\underline{J}}^{2} =
\begin{bmatrix}
-s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & 0 & d_2(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) & d_2(c\theta_1 s\theta_2 + s\theta_1 c\theta_2) & -a_2(s\theta_1 s\theta_2 + c\theta_1 c\theta_2 (-d_B - a_1 c\theta_1 \\
0 & 0 & -1 & 2a_2 s\theta_2 c\theta_2 + 2a_1 c\theta_1 s\theta_1 c\theta_2 - d_B s\theta_1 c\theta_2 & 2a_1 c\theta_1 s\theta_1 s\theta_2 - d_B s\theta_1 s\theta_2 & 0 \\
s\theta_1 s\theta_2 - c\theta_1 c\theta_2 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & 0 & -d_2(c\theta_1 s\theta_2 + c\theta_2 s\theta_1) & d_2(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) & a_2(s\theta_1 c\theta_2 - s\theta_2 c\theta_1) - a_1 s\theta_1 \\
0 & 0 & 0 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & s\theta_1 s\theta_2 - c\theta_1 c\theta_2 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & 0
\end{bmatrix}
\quad (5\text{-}51)
$$

And, the $\underline{\underline{G}}_2$ matrix is obtained from equation B-2

$$\underline{\underline{G}}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{5-52}$$

Hence,

$$^{B}\underline{\underline{J}}^{2} \cdot \underline{\underline{G}}_2 = \begin{bmatrix} -d_B - a_1 c\theta_1 - 2a_2 s\theta_1 s\theta_2 & 0 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & d_2(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) \\ 0 & -1 & 0 & 2a_2 s\theta_2 c\theta_2 + 2a_1 c\theta_1 s\theta_1 c\theta_2 - d_B s\theta_1 c\theta_2 \\ -a_1 s\theta_1 - 2a_2 c\theta_1 s\theta_2 & 0 & s\theta_1 s\theta_2 - c\theta_1 c\theta_2 & -d_2(c\theta_1 s\theta_2 + c\theta_2 s\theta_2) \\ 0 & 0 & 0 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s\theta_1 s\theta_2 - c\theta_1 c\theta_2 \end{bmatrix} \tag{5-53}$$

Also,

$$\underline{b}_2 = (0,\ a_2,\ c\alpha_2,\ -a_2,\ s\alpha_2,\ 0,\ s\alpha_2,\ c\alpha_2)^T$$

$$= (0,\ a_2,\ 0,\ 0,\ 0,\ 1)^T \qquad (5\text{-}54)$$

and

$$^{B}\underline{\underline{J}}^{2}\cdot\underline{b}_2 = (-a_1c\theta_1-2a_2s\theta_1s\theta_2-d_B\ ,\ 0,\ -a_1s\theta_1-2a_2c\theta_1s\theta_2,\ 0,\ -1,\ 0)^T \qquad (5\text{-}55)$$

Using equations 5-43, 5-48, and 5-53, the $\underline{\underline{H}}$ matrix becomes

$$\underline{\underline{H}}(k) = [{}^{B}\underline{\underline{J}}^{0} \cdot \underline{\underline{G}}_{0},\ {}^{B}\underline{\underline{J}}^{1} \cdot \underline{\underline{G}}_{1},\ {}^{B}\underline{\underline{J}}^{2} \cdot \underline{\underline{G}}_{2}]$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 & -a_1 s\theta_1 & -s\theta_1 \\ -d_B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -a_1 s\theta_1 & -c\theta_1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & -d_B - a_1 c\theta_1 2a_2 s\theta_1 s\theta_2 \\ 2a_1 s\theta_1 c\theta_1 - d_B s\theta_1 & 0 & 0 \\ 0 & 0 & -a_1 s\theta_1 - 2a_2 c\theta_1 s\theta_2 \\ -s\theta_1 & -c\theta_1 & 0 \\ 0 & 0 & -1 \\ -c\theta_1 & s\theta_1 & 0 \end{matrix}$$

$$\left.\begin{matrix} 0 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 & d_2(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) \\ -1 & 0 & 2a_2 s\theta_2 c\theta_2 + 2a_1 c\theta_1 c\theta_2 - d_B s\theta_1 c\theta_2 \\ 0 & s\theta_1 s\theta_2 - c\theta_1 c\theta_2 & -d_2(c\theta_1 s\theta_2 + c\theta_2 s\theta_2) \\ 0 & 0 & -s\theta_1 c\theta_2 - s\theta_2 c\theta_1 \\ 0 & 0 & 0 \\ 0 & 0 & s\theta_1 s\theta_2 - c\theta_1 c\theta_2 \end{matrix}\right] \quad (5\text{-}56)$$

and $\underline{x} = [\delta\theta_0,\ \delta d_0,\ \delta a_0,\ \delta\alpha_0,\ \delta\theta_1,\ \delta a_1,\ \delta\alpha_1\ \delta\beta_1,\ \delta\theta_2 \delta d_2,\ \delta a_2, \delta\alpha_2]^T$ (5-57)

Concatenating equations 5-45, 5-50, and 5-55, the $\underline{\underline{L}}$ matrix is obtained:

$$\underline{\underline{L}}(k) = [{}^B\underline{\underline{J}}^0 \cdot \underline{b}_0,\ {}^B\underline{\underline{J}}^1 \cdot \underline{b}_1,\ {}^B\underline{\underline{J}}^2 \cdot \underline{b}_2,\ -\underline{\underline{I}}]$$

$$= \begin{bmatrix} 0 & d_B & -\alpha_1 c\theta_1 - 2a_2 s\theta_1 s\theta_2 - d_B & -1 & 0 & 0 & 0 & 0 & 0 \\ -d_B & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2a_1 s\theta_1 & -a_1 s\theta_1 - 2a_2 c\theta_1 s\theta_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(5-58)

Then, for the time variant case, the measurement equation is

$$\underline{z}_M(k) = \underline{\underline{H}}(k)\cdot\underline{x}(k) + \underline{\underline{L}}(k)\cdot\underline{v}(k)$$

where the $\underline{\underline{H}}(k)$ and $\underline{\underline{L}}(k)$ matrices are given by equations 5-56 and 5-58 respectively, $\underline{z}_M(k)$ is given by equation 5-39 and where

$$\underline{x}(k)=(\delta\theta_0,\delta d_0,\delta a_0,\delta\alpha_0,\delta\theta_1,\delta a_1,\delta\theta_1,\delta\alpha_1,\delta\beta_1,\delta\theta_2,\delta d_2,\delta a_2,\ \delta\alpha_2)^T$$

$$\underline{v}(k) = (0,\ \mu_1,\ \mu_2,\ z_{v_1},\ z_{v_2},\ z_{v_3})^T \qquad (5\text{-}59)$$

The above equations can then be reformulated to represent a time-invariant system measurements as described in section 4.3.

CHAPTER 6

SOME FINAL REMARKS

Many questions posed previously, namely in Chapter 4, still need to be answered.

In the robot model developed earlier many possible error sources were left unmodeled (e.g., gears backlash....) and the process was assumed to be a constant random one. Also, repeatability effect were left out. The choice of a proper model, a suitable process dynamics, $\underline{\underline{\phi}}(k)$, and process noise covariance, $\underline{\underline{Q}}(k)$, is still an open and challenging research problem.

The relationship between calibration errors and the accruacy of the calibration sensors needs also to be defined.

Additionally, the Kalman filter algorithm provides for a probabilistic characterization of the measurement noise and unknown robot parameters and a suitable theoretical vehicle for answering questions pertinent to the robot calibration problems, an important question needs to be answered; from a practical and computational point of view, does this probabilistic characterization and computational effort pay off? In other words, does the Kalman filter formulation introduce a significant improvement to the error parameters identification as to warrant the added effort? For that purpose, a comparative study between other identification methods, such as the least square estimation method, and the Kalman filter algorithm is needed to determine the practical advantages, if any, of the Kalman filter.

Furthermore, the system observability problem and the proper measurement configurations was investigated. The simulations confirmed the presence of a relationship between singular configurations of the robot and the unobservability of the system, and hence, the inability of the Kalman filter to estimate the robot Kinematic error parameters. Further research is still needed to develop a general theoretical link between unobservability and robot singularities.

Finally, a theoretical lower bound on the number of measurements needed to achieve a predetermined calibration accuracy was developed. This theoretical bound proved to be trivial and inconclusive. Further investigation is still needed to reach a more reliable and useful one.

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APPENDIX A

Derivation of the error Jacobian

From equation 3-3 one have,

$$ {}^{(i-1)A}\underline{\underline{dA}}^{iN}_{(i-1)} = {}^{(i-1)A}\underline{\underline{A}}^{iN} \cdot {}^{iA}\underline{\underline{dA}}^{iN}_{(i)} \cdot {}^{iN}\underline{\underline{A}}^{(i-1)A} \qquad \text{(A-1)} $$

From equation 2-15 yields

$$ {}^{i}\underline{\underline{A}}^{i-1} = \left[\begin{array}{c|c} {}^{i}\underline{\underline{R}}^{i-1} & \underline{P}_{i,i-1} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \qquad \text{(A-2)} $$

Using equation 2-41 becomes

$$ {}^{Ai}\underline{\underline{dA}}^{iN}_{i} = \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_i\Delta & -\underline{d}_i \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \qquad \text{(A-3)} $$

Equation A-1 can be expanded to obtain

$$ \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_i\Delta & -\underline{d}_i \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} {}^{(i-1)A}\underline{\underline{R}}^{iN} & \underline{P}_{i,i-1} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_i\Delta & -\underline{d}_i \\ \hline 0\ 0\ 0 & 1 \end{array}\right] $$

$$ \cdot \left[\begin{array}{c|c} {}^{iN}\underline{\underline{R}}^{(i-1)A} & \underline{P}_{i-1,i} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \underline{\underline{I}} - {}^{(i-1)A}\underline{\underline{R}}^{iN} \cdot (\underline{\delta}_i\ \Delta)\, {}^{iN}\underline{\underline{R}}^{(i-1)A} & -{}^{(i-1)A}\underline{\underline{R}}^{iN}(\underline{\delta}_i\Delta)\, \underline{P}_{i-1,i} - {}^{(i-1)}\underline{\underline{R}}^{iN} \cdot \underline{d}_i \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right] $$

By equating the corresponding sub matrices one gets

$$ (\underline{\delta}_i\Delta)_{i-1} = {}^{(i-1)A}\underline{\underline{R}}^{iN}(\underline{\delta}_i\ \Delta) \cdot {}^{iN}\underline{\underline{R}}^{(i-1)A} = ({}^{iN}\underline{\underline{R}}^{(i-1)A})^{-1} \cdot (\underline{\delta}_i\ \Delta) \cdot {}^{iN}\underline{\underline{R}}^{(i-1)A} \qquad \text{(A-4)} $$

and

$$ (\underline{d}_i)_{i-1} = {}^{(i-1)A}\underline{\underline{R}}^{iN}(\underline{\delta}_i\ \Delta) \cdot \underline{P}_{i,i-1} + {}^{(i-1)A}\underline{R}^{iN} \cdot \underline{d}_i \qquad \text{(A-5)} $$

Now utilizing the following general matrix equalities [35]

$$\underline{\underline{R}}\cdot(\underline{\delta}\Delta)\cdot\underline{\underline{R}}^{-1} = (\underline{\underline{R}}\underline{\delta})\Delta \quad \text{(A-6)}$$

and

$$(\underline{\delta}\Delta)\cdot f = -(f\Delta)\cdot\underline{\delta} \quad \text{(A-7)}$$

Equations A-4 and A-5 are reduced to

$$(\underline{\delta}_i\Delta)_{i-1} = ({}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN}\cdot\underline{\delta}_i)\Delta \quad \text{(A-8)}$$

and

$$(\underline{d}_i)_{i-1} = {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN}(\underline{P}_{i,i-1}\Delta)\cdot\underline{\delta}_i + {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN}\cdot\underline{d}_i \quad \text{(A-9)}$$

Note that equation (A-8) implies

$$(\underline{\delta}_i)_{i-1} = {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN}\cdot(\underline{\delta}_i)_i \quad \text{(A-10)}$$

Using the definition of $\underline{z}$ (equation 3-4) i.e., $\underline{z}=[d^T,\delta^T]^T$, equation A-9 and A-10 can be combined in a matrix form to yield

$$(\underline{z}_i)_{i-1} = \left[\begin{array}{c|c} {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN} & {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN}\ (\underline{P}_{i,i-1}\Delta) \\ \hline 0\ \ 0\ \ 0 & {}^{(i-1)}A_{R}{}^{iN} \end{array}\right]\cdot(z_i)_i \quad \text{(A-11)}$$

$$= \left[\begin{array}{c|c} {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN} & +\ (\underline{P}_{i-1,i}\Delta)\cdot{}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN} \\ \hline 0 & {}^{(i-1)}\underline{\underline{A}}_{R}{}^{iN} \end{array}\right]\cdot(z_i)_i \quad \text{(A-12)}$$

or

$$(\underline{z}_i)_{i-1} = {}^{i-1}\underline{\underline{J}}^{i}(\underline{z}_i)_i \quad \text{(A-13)}$$

Then, referring to section 2.1.3 and 2.1.6 Jacobian matrix ${}^{i-1}\underline{\underline{J}}^{i}$ can be determined.

Thus, from equation 2-15 and 2-33, the Jacobian matrix is

$$
{}^{i-1}\underline{\underline{J}}^{i} = \begin{bmatrix}
c\theta_i & +s\theta_i & 0 & d_i s\theta_i c\alpha_i + a_i s^2\theta_i s\alpha & -d_i c\theta_i c\alpha_i - a_i s\theta_i c\theta_i s\alpha_i & -d_i s\alpha_i + a_i s\theta_i c\alpha_i \\
-s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & d_i c\theta_i - a_i s\theta_i c\theta_i s\alpha_i & d_i s\theta_i + a_i c^2\theta_i s\alpha_i & -a_i c\theta_i c\alpha_i \\
s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -a_i s\theta_i c\theta_i [1+c\alpha_i] & -a_i [s^2\theta_i - c^2\theta_i c\alpha_i] & a_i c\theta_i s\alpha_i \\
0 & 0 & 0 & c\theta_i & c\theta_i c\alpha_i & s\alpha_i \\
0 & 0 & 0 & -s\theta_i c\alpha_i & -c\theta_i s\alpha_i & c\alpha_i \\
0 & 0 & 0 & s\theta_i s\alpha_i & &
\end{bmatrix}
\quad \text{(A-14)}
$$

APPENDIX B

I. Revolute Joints

The following convention for obtaining a coordinate frame for revolute link i whe the joint axes i and i+1 are parallel or near parallel will be established. The frame i+1 is obtained by performing the following four steps: (figure B.1)

i) Pass a plane perpendicular to the z_{i-1} axis (see fig. B.1). The origin of frame i , 0_i, will be defined as the intersection of this plane with the joint axes i+1.

ii) Rotate frame i-1 about z_{i-1} to align x_{i-1} with the line connecting the points 0_{i-1} and 0_i.

iii) Translate the origin of the last frame to 0_i.

iv) Perform two rotation about the resulting frame's x and y axes to align to z axis with that of joint i.

Mathematically, the above steps can be described by

$$ {}^{i}\underline{\underline{A}}^{i-1} = Rot(y''',\beta)\cdot Rot(x'',\alpha)\ Transl\ (x'd)\cdot R(z,\theta) \qquad (B.1) $$

Where the above homogeneous transformation are given in section 2.1.

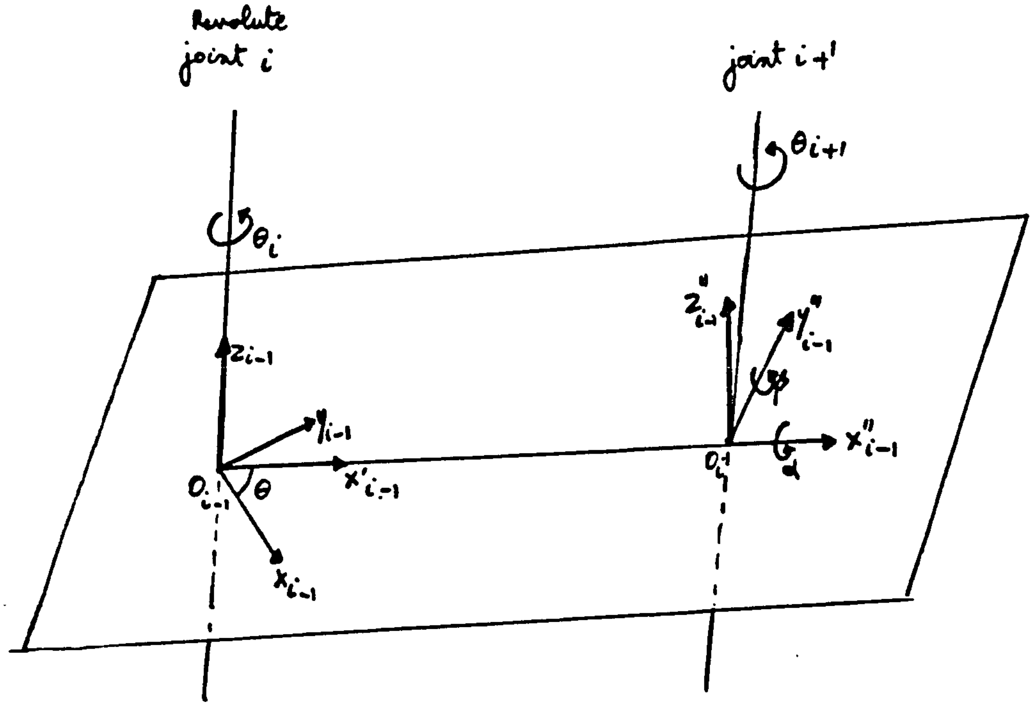

Fig. B.1: Revolute joint Link Parameters for Parallel or near Parallel joint Axes

Now following the same error mapping techniques outlined in section 3.1, one has, for near parallel revolute joints,

$$\underline{\underline{G}}_i = \begin{bmatrix} -d_i s\alpha_i s\beta_i & c\beta_i & 0 & 0 \\ d_i c\alpha_i & 0 & 0 & 0 \\ d_i s\alpha_i c\beta_i & -s\beta_i & 0 & 0 \\ c\alpha_i s\beta_i & 0 & s\beta_i & 0 \\ s\alpha_i & 0 & 0 & -1 \\ c\alpha_i c\beta_i & 0 & -s\beta_i & 0 \end{bmatrix} \qquad \text{(B-2)}$$

and

$$\underline{g}_i = (\delta\theta_i', \ \delta d_i, \ \delta\alpha_i, \ \delta\alpha_i, \delta\beta_i)^T \qquad \text{(B-3)}$$

where

$$\delta\theta_i' = \delta\theta_i + \mu_{\theta i} \qquad \text{(B-5)}$$

This transformation has the property that small joint location and orientation variation may be modeled by small parameters variations, ie, $\delta\beta$, $\delta\alpha$, δd and $\delta\theta$. It should be noted also that this property is lost when the consecutive joint axes are perpendiuclar or nearly perpendicular.

II. Prismatic Joints

Similar equations are obtained when a prismatic joint axes is parallel or near parallel to the next joint axes. It should be noted that it is assumed that the coordinate frame i can be obtained by using the following convention: (figure B.2)

i) Locate a reference point on the prismatic joint axes, i.e., 0_{i-1}. Pass a plane through this point perpendicular to the axis. Denote the intersection of this plane with that of joint i+1 as 0_i.

ii) Translate the origin of frame i-1 along z_i to point 0_{i-1} (Fig. B.1).

iii) Rotate the resulting frame about its z axis to align the x-axis with the line 0_{i-1}, 0_i.

iv) Translate the resulting frame to the point 0_i.

v) Perform two rotations about the last frame's x and y axis to align its z axis with that of joint i+1.

Mathematically, this is expressed as

$$ {}^{i}\underline{\underline{A}}^{i-1} = \mathrm{Rot}(y'''',\beta)\cdot\mathrm{Rot}(x''',\alpha)\cdot\mathrm{Transl}(x'',a)\cdot\mathrm{Rot}(z',\theta)\cdot\mathrm{Transl}(z,d) $$

Then, for parallel or near parallel prismatic joints, one have.

$$ \underline{\underline{G}}_i = \begin{bmatrix} C\alpha_i C\beta_i & -a_i S\alpha_i S\beta_i & C\beta_i & 0 & 0 \\ S\alpha_i & a_i C\alpha_i & 0 & 0 & 0 \\ C\alpha_i & -a_i S\alpha_i C\beta_i & -S\beta_i & 0 & 0 \\ 0 & C\alpha_i S\beta_i & 0 & C\beta_i & 0 \\ 0 & S\alpha_i & 0 & 0 & -1 \\ 0 & C\alpha_i C\beta_i & 0 & -S\beta_i & 0 \end{bmatrix} \qquad \text{(B-5)} $$

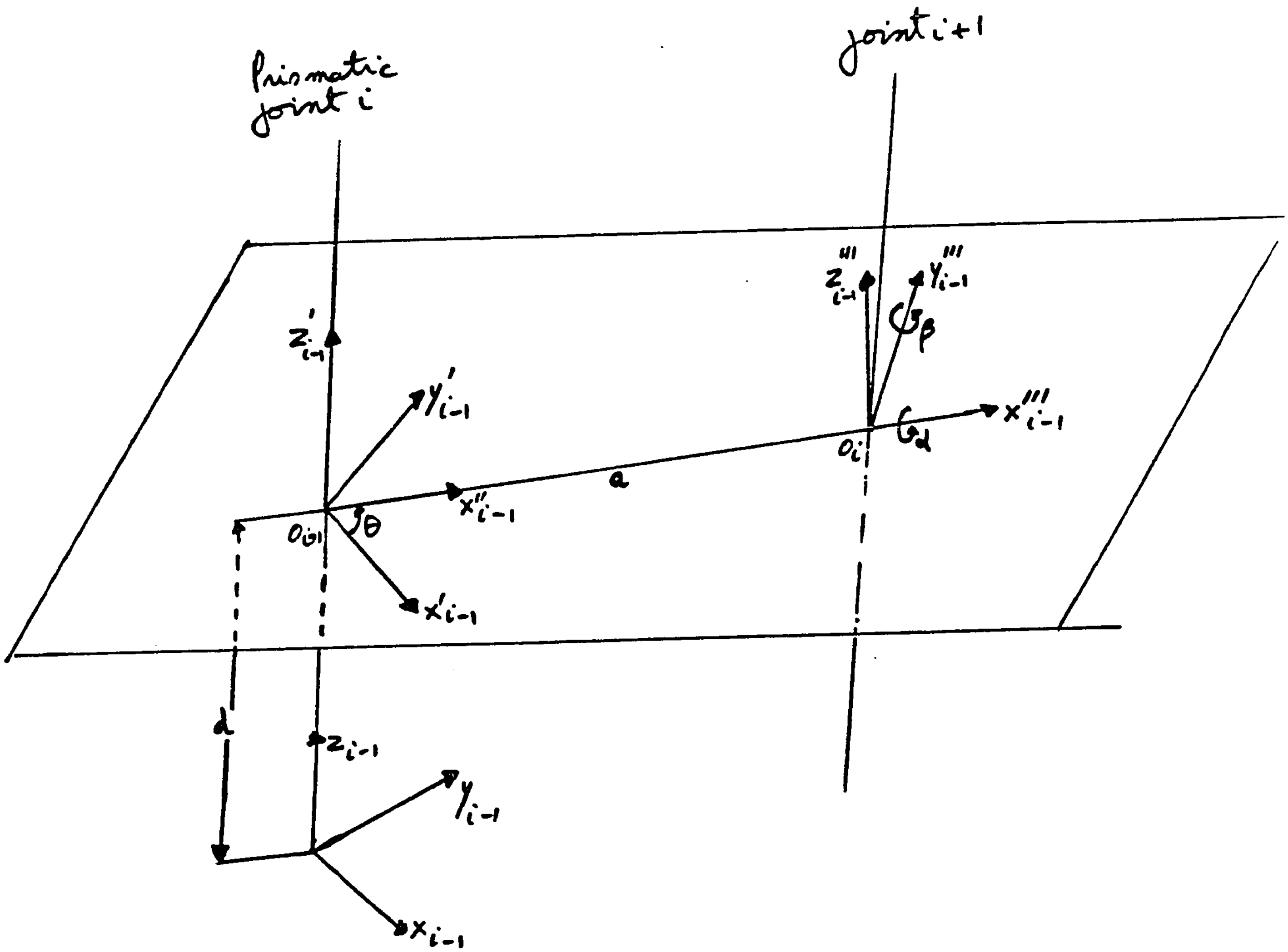

Fig. B.2: Prismatic Joint Link Parameters for Parallel or Near Parallel Joint Axes and

$$\underline{g}_i = (\delta d_i', \ \delta\theta_i, \ \delta a_i, \ \delta\alpha_i, \ \delta\beta_i)^T \tag{B-6}$$

where

$$\delta d_i' = \delta d_i + \mu_{di} \tag{B-7}$$

APPENDIX C

Derivation of Equaiton 3-19

The transformation between a base coordinate frame and an ith link coordinate frame of a manipulator can be expressed as

$$ {}^{B}\underline{\underline{T}}^{i} = {}^{B}\underline{\underline{A}}^{0} \cdot {}^{0}\underline{\underline{A}}^{1} \cdots\cdots {}^{i-1}\underline{\underline{A}}^{i} \qquad \text{(C-1)} $$

Using the result of equation 2-15 to compute the right hand side of equation C-1 one gets

$$ {}^{B}\underline{\underline{T}}^{i} = \left[\begin{array}{c|c} {}^{B}\underline{\underline{R}}^{0} \cdot {}^{0}\underline{\underline{R}}^{1} \cdots {}^{i-1}\underline{\underline{R}}^{i} & \underline{P}_{B,0} + {}^{B}\underline{\underline{R}}^{0} \cdot \underline{P}_{0,1} + \cdots + {}^{B}\underline{\underline{R}}^{i-1} \cdot \underline{P}_{i-1,i} \\ \hline \underline{\underline{0}} & 1 \end{array} \right] \qquad \text{(C-2)} $$

Now, the Jacobian of ${}^{B}\underline{\underline{T}}^{i}$ is computed, from equation 3-4, as

$$ {}^{B}\underline{\underline{J}}^{i} = \left[\begin{array}{c|c} {}^{B}\underline{\underline{R}}^{i} & (\underline{P}_{B,0}\Delta) \cdot {}^{B}\underline{\underline{R}}^{i} + ({}^{B}\underline{\underline{R}}^{0} \cdot \underline{P}_{0,1})\Delta {}^{B}\underline{\underline{R}}^{1} + \cdots\cdots + ({}^{B}\underline{\underline{R}}^{i-1} \cdot \underline{P}_{i-1,i})\Delta \cdot {}^{B}\underline{\underline{R}}^{i} \\ \hline 0\ 0\ 0 & {}^{B}\underline{\underline{R}}^{i} \end{array} \right] \qquad \text{(C-3)} $$

On the other hand, by successive multiplication of the error Jacobian, one have

$$ {}^{B}\underline{\underline{J}}^{0} \cdot {}^{0}\underline{\underline{J}}^{1} \cdots {}^{i-1}\underline{\underline{J}}^{i} = \left[\begin{array}{c|c} {}^{B}\underline{\underline{R}}^{i} & (\underline{P}_{B,0}\Delta)\, {}^{B}\underline{\underline{R}}^{i} + {}^{B}\underline{\underline{R}}^{0} (\underline{P}_{0,1}\Delta)\, {}^{0}\underline{\underline{R}}^{i} + \cdots + {}^{B}\underline{\underline{R}}^{i-1} (\underline{P}_{i-1,i}\Delta)\, {}^{i-1}\underline{\underline{R}}^{i} \\ \hline 0\ 0\ 0 & {}^{B}\underline{\underline{R}}^{i} \end{array} \right] \qquad \text{(C-4)} $$

Equations C-3 and C-4 are equal in view of the equality A-5. Hence,

$$ {}^{B}\underline{\underline{J}}^{i} = {}^{B}\underline{\underline{J}}^{0} \cdot {}^{0}\underline{\underline{J}}^{1} \cdots\cdots {}^{i-1}\underline{\underline{J}}^{i} \qquad \text{(C-5)} $$

APPENDIX D

Derivation of equations 3-23 and 3-24:

From equations 3-20 and 3-21, the $\underline{\underline{dA}}_B$ matrix is given by

$$\underline{\underline{dA}}_B = {}^{B}\underline{\underline{T}}^{mN}\; {}^{n}\underline{\underline{T}}^{B}_{m} \cdot \underline{\underline{dA}}_v = \underline{\underline{T}} \cdot \underline{\underline{dA}}_v \qquad \text{(D-1)}$$

Using equation 2-41.

$$\underline{\underline{dA}}_i = \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_i\Delta & -\underline{d}_i \\ \hline 0 & 1 \end{array}\right] \qquad \text{(D-2)}$$

Expanding equation D.1, the following result is obtained

$$\left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_b\Delta & -\underline{d}_B \\ \hline 0\;\;0\;\;0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \underline{\underline{R}} & -\underline{p} \\ \hline 0\;0\;0 & 1 \end{array}\right] \left[\begin{array}{c|c} \underline{\underline{I}} - \underline{\delta}_v\Delta & -\underline{d}_v \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underline{\underline{R}}(\underline{\underline{I}} - \underline{\delta}_v\Delta) & -\underline{\underline{R}}\underline{d}_v - \underline{p} \\ \hline 0\;\;0\;\;0 & 1 \end{array}\right] \qquad \text{(D-3)}$$

By equating corresponding submatrices, one gets

$$\underline{\underline{I}} - \underline{\delta}_b\Delta = \underline{\underline{R}}\cdot(\underline{\underline{I}} - \underline{\delta}_v\Delta) \qquad \text{(D-4)}$$

or

$$\underline{\delta}_B\Delta = \underline{\underline{I}} - \underline{\underline{R}}\cdot(\underline{\underline{I}} - \underline{\delta}_v\Delta) \qquad \text{(D-5)}$$

and

$$\underline{d}_B = \underline{\underline{R}}\cdot\underline{d}_v - \underline{p} \qquad \text{(D-6)}$$

APPENDIX E

E.1 Matrix Norms:

In ordinary 2 dimensional or 3 dimensional space there is an obvious and natural sense in which one would use the term "length of a vector". Thus, if

$\underline{x} = (x_1, x_2, x_3)$

Then one would normally understand the length of $\underline{x}$ to be number

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Conceptually, this just the same as the "distance form the origin" of the point whose coordinates are x_1, x_2 and x_3. As a result, one goes on to speak of the distance between 2 points say (x_1, x_2, x_3) and (y_1, y_2, y_3) as the length of the vector $\underline{x} - \underline{y}$:

$$d(x,y) = ||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)}$$

The concept of distance allows us to talk meaningfully about convergence and limiting processes in space, since there will be a specific sense in which one can speak of a sequence of points (or vectors) approaching more and more closely to some particular limit point.

Thus, one wishes to generalize these ideas to n-dimensional space (and, in fact, to any kind of vector spaces) and to do so it is necessary to isolate the essential formal properties of "length" and of "distance".

All the concepts in this section are applicable to any linear space over the field of complex numbers or over the field of real numbers or kind of vector spaces. However, for convenience, the discussion will be to the real vector space (R^n, R).

The concept of the norm of a vector $\underline{x}$ in (R^n, R) is a generalization of the idea of length. Any real valued function of $\underline{x}$, denoted by $||\underline{x}||$, can be defined as a norm if it has the properties that for any $\underline{x}$ in (R^n, R) and any α in R

1. $||\underline{x}|| \geq 0$ and $||\underline{x}|| = 0$ iff $\underline{x} = \underline{0}$
2. $||\alpha\underline{x}|| = |\alpha| \cdot ||\underline{x}||$ where $|\cdot|$ denote the abosolute value
3. $||x_1 + x_2|| \leq ||x_1|| + ||x_2||$

The last inequality is call triangular inequality. The distance $d(\underline{x}, \underline{y})$ between any two members $\underline{x}$ and $\underline{y}$ of R^n is then well defined as

$$d(\underline{x}, \underline{y}) = ||\underline{x} - \underline{y}||$$

and the real valued function $d(\underline{x},\underline{y})$ is often called "metric" on R^n. Finally, a sequence of vectors $\{\underline{x}_k\}$ in R^n is said to converge to $\underline{x}$ in R^n as its limit iff

$$\lim_{k\to\infty} d(x,x_k) = \lim_{k\to\infty} ||\underline{x} - \underline{x}_k|| = 0$$

Convergence in this node is described as convergence "in the sense of the norm $||\underline{x}||$ on R^n". (or "convergence in norm"). A useful sequence of possible norms in (R^n,R) can be obtained by writing

$$||\underline{x}||_P = \left[\sum_{i=1}^{n} |x_i|^P \right]^{1/P} \qquad P = 1,2,3,\ldots$$

Moreover, if we allow P to tend to ∞ it can be shown that

$$||\underline{x}||_\infty = \lim_{P\to\infty} \left[\sum_{i=1}^{n} |x_i|^P \right]^{1/P} = \max_{1\leq i\leq n} |x_i|$$

It is easily confirmed that the limiting value satisfies the conditions 1 through 3. It should be noted that $||\underline{x}||$ is finite iff all components of $\underline{x}$ are finite.

Once a specific norm has been chosen for (R^n, R), there will be a corresponding concept of "distance".

$$d_P(x,y) = ||\underline{x} - \underline{y}||_P = \left[\sum_{i=1}^{n} |x_i - y_i|^P \right]^{1/P}$$

in particular; $d_\infty(x,y) = \max \left(|x_i - y_i|\right)$; $1\leq i\leq n$

The sequence $\{\underline{x}_k\}$, in R^n converges to the limit $\underline{x}$ in the sense of the norm $||\underline{x}||$ in R^n iff

$$\lim_{k\to\infty} ||\underline{x} - \underline{x}_k||_p = 0$$

E. 2 Norm of a bounded transformation

The concept of norm can be extended to linear operators that map R^n into itself, or equivalently, to square matrices with real coefficients.

Let $\underline{\underline{A}}$ be a linear transformation mapping the elements of the normed linear space, R^n into itself. The, $\underline{\underline{A}}$ is said to be continuous with respect to the norms R^n if the following situation occurs.

Let $\{\underline{x}_k\}$ be any sequence of vectors in (R^n) which converges to the limit $\underline{x}$ in the sense of the norm on R^n; the, the corresponding sequence $\{\underline{\underline{A}}\cdot\underline{x}_k\}$ of the image vectors in R^n coverges to the limit of $\underline{\underline{A}}\cdot\underline{x}$ in the sense of the norm on F.

A is a continuous linear tranformation from R^n into R^n iff it is bounded in the sense that there exists a finite positive number M such athat,

$$||\underline{\underline{A}}\cdot\underline{x}|| \leq M\ ||\underline{x}|| \quad \text{for every } \underline{x} \text{ in } R^n$$

The smallest number M such tht this relation is true for all $\underline{x}$ in R^n is :alled "norm of the bounded linear transformation A", and is written as $||\underline{\underline{A}}||$. It can be defined explicitly by:

$$||\underline{\underline{A}}|| \triangleq \sup_{x\neq 0} \frac{||\underline{\underline{A}}\cdot\underline{x}||}{||\underline{x}||} = \sup_{||\underline{x}|| = 1} ||\underline{\underline{A}}\cdot\underline{x}||$$

where "sup" stands for supremum, the largest possible number of $||\underline{\underline{A}}\cdot\underline{x}||$ or the least upper bound of $||\underline{\underline{A}}\cdot\underline{x}||$.

It is easily verifiable that $||\underline{\underline{A}}||$, as defined above, satisfies properties 1 through 3, which characterize a norm on R^n.

Note that the fundamental property of a bounded linear transformation can be expressed in the form

$$||\underline{\underline{A}}\cdot\underline{x}|| \leq ||\underline{\underline{A}}||\cdot||\underline{x}|| \quad \text{for } \underline{x} \text{ in } R^n$$

E.3 Matrix Norms

Since the norm of $\underline{\underline{A}}$ is defined through the norm of $\underline{x}$; hence it is called "induced norm". Furthermore it should possess the following properties:

M1. $||\underline{\underline{A}}|| \geq 0$ always and $||\underline{\underline{A}}|| = 0$ iff $\underline{\underline{A}} = \underline{\underline{0}}$

M2. $||\underline{\underline{A}} + \underline{\underline{B}}|| \leq ||\underline{\underline{A}}|| + ||\underline{\underline{B}}||$

M3. $||\alpha A|| = |\alpha|\cdot||A||$ for every scalar α.

A necessary condition for a matrix norm to be cinsistent with a given vector norm $||\underline{x}||$ is obtained when we consider matrix multiplication. Thus, suppose that the matrix norm $||\underline{\underline{A}}||$ is consistent with the vector norm $||\underline{x}||$, and consider, the proudct $\underline{\underline{C}} = \underline{\underline{A}}\cdot\underline{\underline{B}}$.

$$||\underline{\underline{C}}\cdot\underline{x}|| = ||\underline{\underline{A}}\cdot\underline{\underline{B}}\cdot\underline{x}|| \leq ||\underline{\underline{A}}|| \cdot||\underline{\underline{B}}\cdot\underline{x}|| \leq ||\underline{\underline{A}}||\cdot||\underline{\underline{B}}||\cdot||\underline{x}||$$

Then for arbitrary matrices A and B one must have

M4. $||\underline{\underline{A}}\cdot\underline{\underline{B}}|| \leq ||\underline{\underline{A}}||\cdot||\underline{\underline{B}}||$

E.4 Choices of Norms

Some possible choices of matrix norms are

i) $||\underline{\underline{A}}|| = \sum_{i=j}^{n} \sum_{j=1}^{m} |a_{ij}|$

(where the a_{ij}'s are the entries of A)

ii) $||\underline{\underline{A}}|| = \max_{i,j} |a_{ij}|$

iii) $||\underline{\underline{A}}|| = \max_{j} \{ \sum_{i=1}^{n} |a_{ij}|\}$ (called column-sum norm)

iv) $||\underline{\underline{A}}|| = \max_{i} \{ \sum_{j=1}^{n} |a_{ij}|\}$ (called row-sum norm)

It should be noted here that the fairly common error designating a matrix norm to be the result of finding the minimum of the standard matrix row be deleterious if the so called norm is used in an application where use of a valid norm is crucial.

Hence

$$||\underline{\underline{A}}|| \triangleq \min\{||\underline{\underline{A}}||_{iii,} ||\underline{\underline{A}}||_{iv}\}$$

in not a norm because it violates the trianglar inequality.

All the above choices of matrix norms are ad-hoc choices. Moreover, the preceding norms are not induced norm either.

E.5 Induced Norms

If the vector norm of R^n is $||\underline{x}||_\infty$ then, for the corresponding induced matrix norm $||\underline{\underline{A}}||_\infty$,

$$||\underline{\underline{A}}\cdot\underline{x}||_\infty = \max_i |\sum_{k=1}^{n} a_{i,k}\cdot x_k| \leq \max_i \sum_{k=1}^{n} |a_{ik}||x_k|$$

If $\max_i \sum_{k=1}^{n} |a_{ik}|$ is reached when $i=\alpha$, take for $\underline{x}$ the vector defined by

$$x_k = \frac{|a_{\alpha k}|}{a_{\alpha k}},$$ if $a_{\alpha k} \neq 0$ and $x_k=1$ otherwise. Then $||\underline{x}||_\infty=1$ and

$$||\underline{\underline{A}}||_\infty = \sup_{||x||_\infty=1} ||\underline{\underline{A}}\cdot\underline{x}||_\infty = \max_i \sum_{k=1}^{n} |a_{ik}|$$

Similarily, if the vector norm on R^n is $||x||$, then

$$||\underline{\underline{A}}\cdot\underline{x}||_1 = \sum_{i=1}^{n} |\sum_{k=1}^{n} a_{ik}x_k| \leq \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ik}| |x_k| < \sum_{k=1}^{n} |x_k|\cdot(\max_k \sum_{i=1}^{n} |a_{ik}|)$$

If $\max_k \sum_{i=1}^{n} |a_{ik}|$ is attained $k=\beta$, take for x the vector defined by

$$x_k= 0 \text{ if } k\neq\beta \text{ and } x_\beta=1$$

then $||\underline{x}||_1 = 1$ and

$$||\underline{\underline{A}}||_1 = \sup_{||\underline{x}||_1=1} ||\underline{\underline{A}}\cdot\underline{x}||_1 = \max_k \sum_{i=1}^{n} |a_{ik}|$$

Finally, let the vector norm on R^n be given by

$$||\underline{x}||_2 = \left[\sum_{k=1}^{n} |x_k|^2 \right]^{1/2}$$ (The vector euclidien norm)

Then $||\underline{x}||^2 = \underline{x}^*\cdot\underline{x}$ and $||\underline{\underline{A}}\cdot\underline{x}||_2^2 = (\underline{\underline{A}}\cdot\underline{x})^*(\underline{\underline{A}}\cdot\underline{x}) = \underline{x}^*\cdot\underline{\underline{A}}^*\cdot\underline{\underline{A}}\ \underline{x}$

where $()^*$denotes the simple transposer (or complex conjugate transpose if the field is complex).

Hence, the induced matrix norm $||\underline{\underline{A}}||_2$ is given by

$$||\underline{\underline{A}}||_2^2 = \max_{x \neq 0} \frac{||\underline{\underline{A}} \cdot \underline{x}||_2^2}{||x||_2^2} = \max \frac{\underline{x}^* \cdot \underline{\underline{A}}^* \cdot \underline{\underline{A}} \cdot \underline{x}}{x^* \cdot x}$$

Now the matrix $\underline{\underline{A}}^* \cdot \underline{\underline{A}}$ is symmetric and positive semi-definite (i.e. for all $\underline{x}$, $\underline{x}^* \cdot \underline{\underline{A}}^* \cdot \underline{\underline{A}} \cdot \underline{x} \geq 0$ should be true). It follows that the eigenvalues of $\underline{\underline{A}}^* \cdot \underline{\underline{A}}$ are all real and non negative. If we denote those eigenvalues by λ_k^2 for $1 \leq k \leq n$ where $\lambda_1^2 \geq \lambda_2^2 \ldots \geq \lambda_n^2$ and if the corresponding (orthorgonalized) real eigenvectors are $x_1, \ldots, x_n$ then for any non-zero vector $\underline{x}$

$$\underline{x} = \sum_{k=1}^{n} c_k x_k$$

and so,

$$\frac{||\underline{\underline{A}}\underline{x}||_2^2}{||\underline{x}||_2^2} = \frac{\underline{x}^* (\underline{\underline{A}}^* \cdot \underline{\underline{A}}_x) x}{\underline{x}^* \ \underline{x}} = \frac{\sum_{k=1}^{n} |c_k|^2 \lambda_k^2}{\sum_{k=1}^{n} |c_k|^2}$$

and hence $0 \leq \lambda_n^2 \leq \dfrac{||\underline{\underline{A}} \cdot \underline{x}||^2}{||\underline{x}||_2^2} \leq \lambda_1^2$

Furthermore, by choosing $\underline{x} = \underline{x}_n$, it is clear that the value λ_1^2 can be attained. That is to say

$$||\underline{\underline{A}}||_2 = \max_{x=0} \left[\frac{||\underline{\underline{A}}\cdot\underline{x}||_2^2}{||\underline{x}||_2^2} \right]^{1/2} = \lambda_1$$

or in the other form

$$||\underline{\underline{A}}||_2 = \left[\lambda_{max}(\underline{\underline{A}}^*\cdot\underline{\underline{A}}) \right]^{1/2}$$

This is often referred to as the spectral norm of the matrix $\underline{\underline{A}}$.

EX: Let $\underline{\underline{A}} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ Define $||\underline{\underline{A}}||_1$, $||\underline{\underline{A}}||_2$ and $||\underline{\underline{A}}||_\infty$.

1) note that $\max_k \sum_{i=1}^{2} |a_{ik}|$ is attained when k=1

then $||\underline{\underline{A}}||_1 = \max_1 \sum_{i=1}^{2} |a_{i1}| = |3| + |-1| = 4$

2) $\underline{\underline{A}}^*\cdot\underline{\underline{A}} = \underline{\underline{A}}^T\cdot\underline{\underline{A}} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$

$$[\underline{\underline{A}} - \lambda\underline{\underline{I}}] = \begin{bmatrix} 10-\lambda & 6 \\ 6 & 4-\lambda \end{bmatrix}$$

It is clear that λ is a solution of $\underline{\underline{A}} - \lambda\underline{\underline{I}}$ iff $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = (10-\lambda)(4-\lambda) - 36 = \lambda^2 - 14\lambda + 4 = 0$$

$$\text{or} \quad \lambda = 7 \pm 6.7 = \begin{cases} 13.7 \\ 0.3 \end{cases}$$

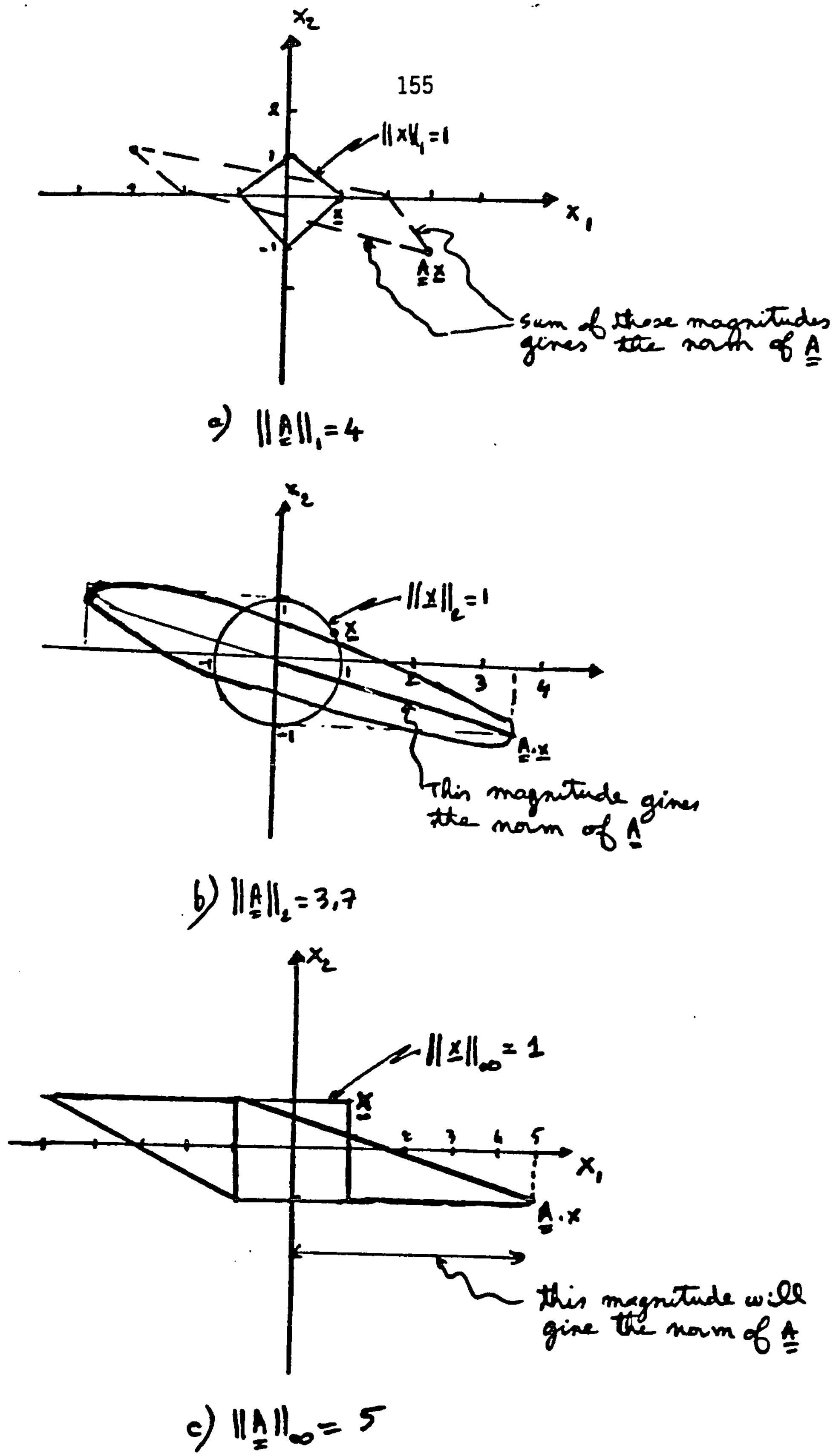

Fig. E.1: Results of different Norms

Then $||\underline{\underline{A}}||_2 = [\lambda_{max}(\underline{\underline{A}}^* \cdot \underline{\underline{A}})]^{1/2} = \sqrt{13.7} = 3.7$

3) The $\max_i \sum_{k=1}^{n} |a_{ik}|$ is reached when i=1

then $||\underline{\underline{A}}||_\infty = \max_1 \sum_{k=1}^{2} |a_{1k}| = |3| + |2| = 5$

As can be seen from E.1 all those norms are different.

E.6 Some Useful Theorems and Matrix Properties

Theorem E1 [3,5]

Let $\underline{\underline{A}}$ be an M by N matrix with coefficients in a field and Let $\underline{\underline{A}}^*$ be the complex conjugate transpose of $\underline{\underline{A}}$. Then

1) $\text{Rank}(\underline{\underline{A}}) = N$ iff $\text{Rank}(\underline{\underline{A}}^* \cdot \underline{\underline{A}}) = N$

   or equivalently,

   $\det(\underline{\underline{A}}^* \cdot \underline{\underline{A}}) \neq 0$

2) $\text{Rank}(\underline{\underline{A}}) = M$ iff $\text{Rank}(\underline{\underline{A}} \cdot \underline{\underline{A}}^*) = M$

   or equivalently

   $\det(\underline{\underline{A}} \cdot \underline{\underline{A}}^*) = M$

Note that $\underline{\underline{A}}^* \cdot \underline{\underline{A}}$ is an N x N matrix and $\underline{\underline{A}} \cdot \underline{\underline{A}}^*$ is an M x M matrix.

Theorem E2 [14,35]

The rank of a matris $\underline{\underline{A}}$ is equal to the size of the largest minor of $\underline{\underline{A}}$ that is nonsingular.

Corollary E1:

Let $\underline{\underline{A}}$ be an M x N matrix. Then,

$\text{Rank}(\underline{\underline{A}}) \leq \min(N,M)$

Matrix inversion lemma:

If $\underline{\underline{P}}$ and $\underline{\underline{R}}$ are nonsingular matrices of order N x N and M x M, respectively, and $\underline{\underline{H}}$ is a M x N matrix, then the following identitity holds

$$(\underline{\underline{H}}^T \cdot \underline{\underline{R}}^{-1} \cdot \underline{\underline{H}} + \underline{\underline{P}}^{-1}) = \underline{\underline{P}} - \underline{\underline{P}} \cdot \underline{\underline{H}}^T (\underline{\underline{H}} \cdot \underline{\underline{P}} \cdot \underline{\underline{H}}^T + \underline{\underline{R}})^{-1} \cdot \underline{\underline{H}} \cdot \underline{\underline{P}}$$

Schwartz Inequality

Let A and B be 2 marices with coefficients in a field

$$||\underline{\underline{A}} \cdot \underline{\underline{B}}|| \leq ||\underline{\underline{A}}|| \cdot ||\underline{\underline{B}}||$$

Partial Derivative Property

Let $\underline{\underline{A}}$ and $\underline{\underline{B}}$ be two matrices with coefficients in a field. Also, let $\underline{\underline{B}}$ be symmetric. Then,

$$\frac{\delta}{\delta A} [\text{trace } (\underline{\underline{A}} \cdot \underline{\underline{B}} \cdot \underline{\underline{A}}^T)] = 2\underline{\underline{A}} \cdot \underline{\underline{B}}$$