THE DYNAMICS OF THE GENERALIZED MANIPULATOR

by

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ABSTRACT

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The concept of "generalized manipulator" is introduced, and the closed form and recursive form dynamical models of the generalized manipulator are presented in Newton-Euler formulation. The physical meaning of each term in the dynamical model is explained.

The dynamical models formulated by the Newton-Euler method and the Lagrangian-Euler method are proved equivalent. The dynamical model of the generalized manipulator is reduced to ordinary manipulators. The reduced dynamical model is shown identical to existing models. Futhermore, the reduced dynamical model of the generalized manipulator can be used to compute forces and torques components along any direction.

Application of the model to problems of mobile robots and flexible manipulators is shown.

CONTENT

TABLE

FIGURES

Notation List

A lower case letter represents a scalar.

Underlined of a lower case letter means a vector.

x, y, z are the three components of a Cartesian vector.

T represents the homogeneous transformation operation.

A represents the rotation of a coordinate frame.

$\underline{d}$ represents the translation of a coordinate frame.

Superscript at the upper right corner of an operator:
The coordinate frame that is referred to before operation.

Subscript at the lower left corner of an operator:
The coordinate frame that is referred to after operation.

The two subscripts at the lower right corner of an operator separated by a comma: The first subscript refers to the component of the subscript coordinate frame, and the second subscript indicates the component of the superscript coordinate frame.

$\underline{u}$ is the translational velocity.

$\underline{a}$ is the translational acceleration.

$\underline{\alpha}$ is the rotational velocity.

$\underline{w}$ is the rotational acceleration.

$\underline{v}$ is the sum of translational and rotational velocity.

Ω is the rotational velocity in matrix form.

$\underline{f}$ is the force of a mass element.

$\underline{n}$ is the torque of a mass element.

$\underline{F}$ is the force of a link.

$\underline{N}$ is the torque of a link.

$\underline{\delta}$ is the generalized angle of rotational vector.

$\underline{\mu}$ is the generalized translational vector.

$\bar{n}$ is the generalized velocity vector.

$\bar{N}$ is the generalized acceleration vector.

Γ is the generalized force vector.

$\underline{s}$ is the Cartesian component selection vector.

$\underline{r}$ is the position vector of a mass element.

I is the inertial matrix of a link.

Superscript ,j at the upper right corner of $\underline{r}$ or I indicates that mass element j is being referred.

CHAPTER 1

INTRODUCTION

In this thesis, the dynamical model of a "generalized manipulator" is derived and some applications of such model are shown.

A generalized manipulator is an hypothetical model whose main use is in facilitating certain analysis aspects of robotics. A generalized manipulator can have arbitrary number of links connected as an open chain by "generalized joints"(fig.1). A generalized joint has three translational and three rotational degrees of freedom which is the upper limit for any physical joint.

The closed form and recursive form dynamical models of the generalized manipulator are derived in chapters 3 and 5. Newton-Euler formulation is used. The equivalence to Lagrange formulation is shown in chapter 4. The dynamical model of the generalized manipulator can be specialized to the one-degree-of-freedom-per-link rigid manipulator. The reduction technique is also discussed in chapter 4.

The dynamical model of the generalized manipulator can be combined with other mechanical models to solve some nontrival problems, such as the trajectory computation of a

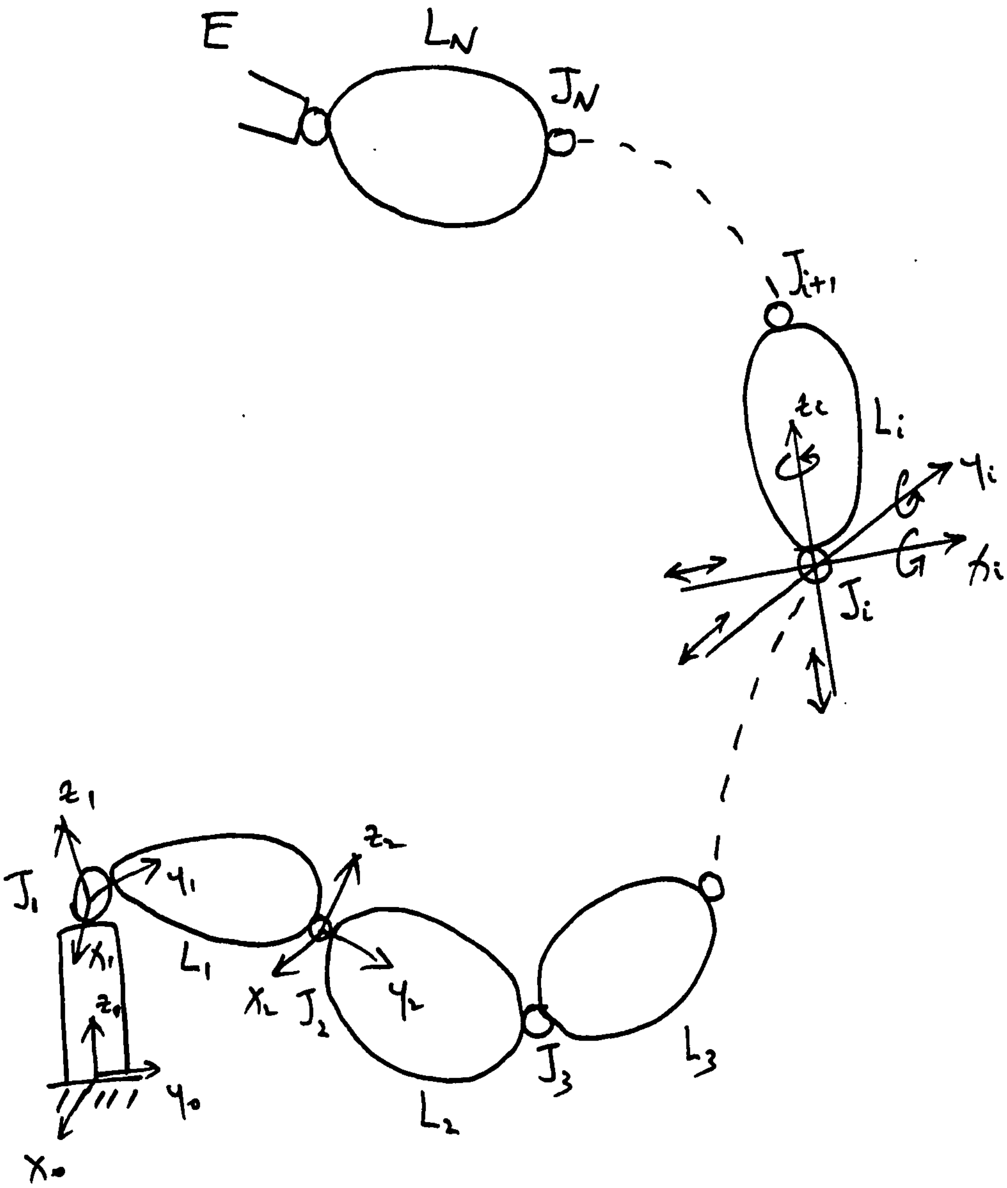


Figure 1 : The N-link Generalized Manipulator

movable robot(chapter 6), and the static deflection of a flexible manipulator(chapter 7).

Next let us outline in more detail the contents of each chapter highlighting the original contributions.

Three mathematical tools are required to derive the dynamical model of a generalized manipulator(chapter 2). These are the homogeneous transformation matrix, the kinematic model, and the propagation equations of static forces and torques. The homogeneous transformation matrix transfers the reference from one coordinate frame to the other coordinate frame (section 2.1). The kinematic model of manipulators provide the relationship between the set of joint variables and the set of global coordinates of the end effector(section 2.2). The propagation equations of static forces and torques are shown in section 2.3.

The distributive property of orientation matrices (equation 2-46) is shown to be valid only for orthogonal coordinate frames. This important feature is useful in simplifying the derivation of the dynamical model. In this thesis, Cartesian coordinate frames are chosen for every joint.

In chapter 3, the derivation of the closed form dynamical model of the generalized manipulator is done in a straight forward manner. The physical meaning of each term in the model is explained. In the closed form dynamical model, each term can be found independently of the others. A step-by-step procedure of writing the closed form dynamical model for the generalized manipulator is presented(section 3.3).

For deriving the dynamical model, the open kinematic chain concept and the open dynamic chain concept(section 3.3) are employed. The entire concept of "generalized manipulator" is new. The derivation of the model follows similar derivations that have been done by others for ordinary manipulators. Interestingly, the generalization is conceptually simple and has the same level of complexity as the derivation for ordinary manipulators.

Although the Newton-Euler method and the Lagrangian method are proved equivalent[Gold59], the dynamical models derived by these two methods do not have the same appearance. In chapter 4, the term by term equivalence of these two formulations is shown(section 4.2).

The dynamical model of the generalized manipulator can be specialized to the one-degree-of-freedom-per-link rigid manipulator. The reduced model is shown to be identical to existing models[Paul82]. Most importantly, the derivation in chapter 4 shows that the reduced dynamical model may be used to compute forces and torques components in any direction(not necessarily along the principal axes of motion). This property has never been shown before.

The recursive form dynamical model of the generalized manipulator is derived in section 5.2. The reduced recursive dynamical model in section 5.3 is equivalent to existing models[Luh80].

In this thesis, two possible applications of generalized manipulator theory are introduced. Both utilize the important feature of generalized manipulator, namely - the ability to compute forces and torques along any direction. The analysis of movable robots(chapter 6) and the analysis of static deflection of manipulators(chapter 7), applying the generalized manipulator dynamics, is original.

CHAPTER II

MATHEMATICAL BACKGROUND

The study of the dynamics of the robot manipulator has three purposes. It is for helping us to discover some important insights of the dynamical properties of the manipulator. It is used for simulation and it can be applied in on-line control.

Dynamical model can be written in various forms employing different formulations. The closed form is suitable for analysis and the recursive form is good for computational aspects, such as on-line control.

Lagrangian formulation dominated the field in the last decade. Only recently, Newton-Euler formulation start to become popular. Newton-Euler formulation is proved to be more efficient computationally. It also eases the task of visualizing the physical meaning of each term in the equations[Luh80].

In this thesis, general algorithms for writing the dynamical model in closed form and recursive form by the Newton-Euler formulation are derived. This dynamical model is even more general than the existing ones because it considers every force and torque components of each joint.

This is important in some applications, such as modelling the movable robot(chapter 6) and flexible manipulator (chapter 7).

For deriving the dynamical model, we need to develop several mathematical tools. These are the transformation of the reference to different coordinate frames(homogeneous transformation), the relationship between the movement of the end effector and the joints of the manipulator(kinematic model), and the changes of force and torque with reference to another coordinate frame(statics).

2.1 Homogeneous Transformations

We need to develope a method for transferring the reference from one coordinate frame to another. The coordinate frames can be arbitrarily defined and different sets of coordinate systems can be used within the same manipulator.

The notation which is used in this thesis is summarized in the notation table. For examples(fig.2):

$\underline{q}$ represents a generalized coordinate and $\underline{d}$ represents a displacement. They have three components α, β, τ and are referred to the coordinate frames i, j, k.

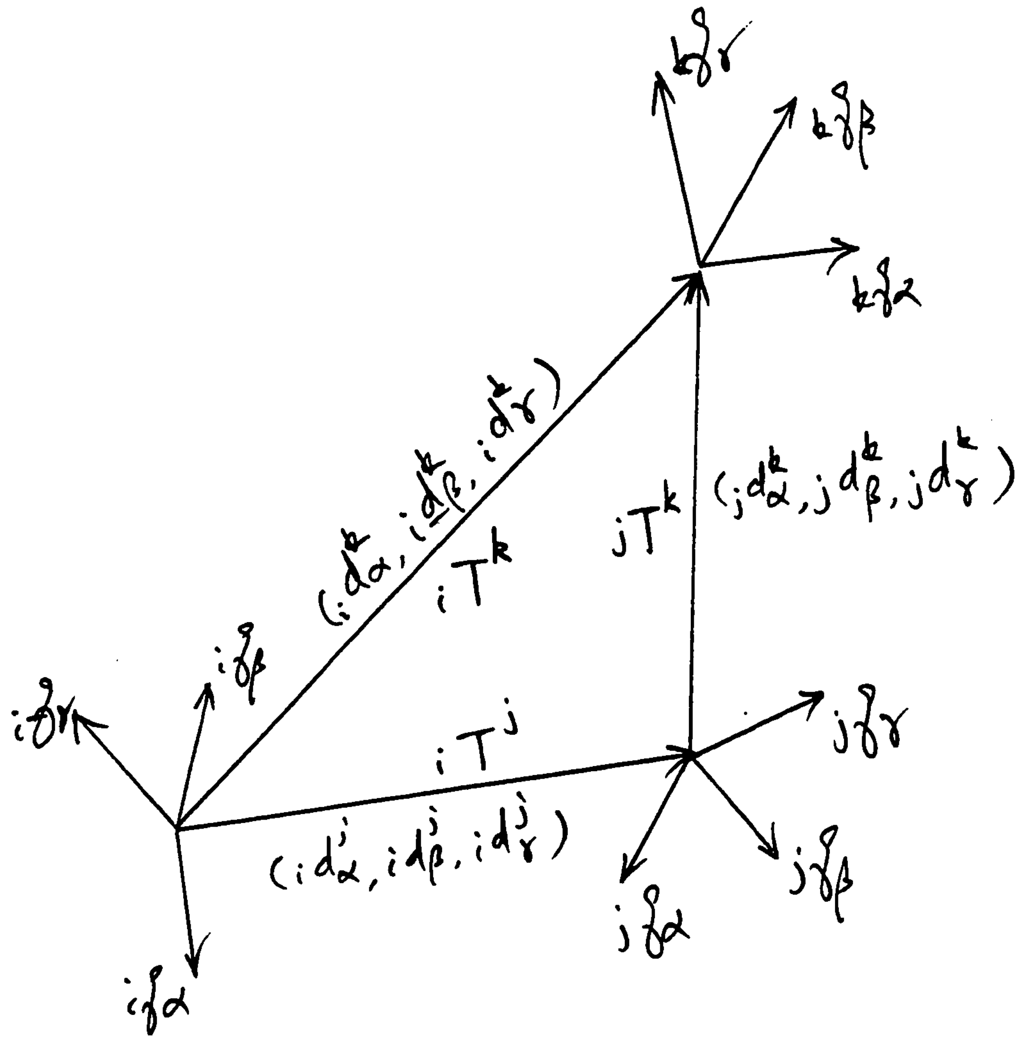


Figure 2 : Homogeneous Transformation

T represents a homogeneous transformation. The transformation is from the superscript coordinate to the subscript coordinate.

${}_{i}t^{j}{}_{x,y}$ (not shown in the figure) is the projection of the unity vector of the y-axis in coordinate j on the x-axis of coordinate i, and the projection is independent of the translation of the coordinates.

For simplicity, any obvious or unimportant superscript or subscript will not be written. For instance, we may write

$${}_{i}t^{j}{}_{x,y} = t_{x,y}$$

if we have indicated in the text that coordinates i and j are referred, or the relation is true for any coordinates.

2.1.1 Homogeneous Transformation Matrix

For a rigid body, the transformation from one coordinate to another coordinate should consist of the following scalars.

$$
{}_{i}T^{j} = \{ t_{x,x}, t_{x,y}, t_{x,z}, t_{y,x}, t_{y,y}, t_{y,z}, t_{z,x}, t_{z,y}, t_{z,z}, d_{x}, d_{y}, d_{z} \}
$$

We need to organize the above scalars such that the transformation can be cascaded as the following:

$${}_{i}T^{j}\,{}_{j}T^{k} = {}_{i}T^{k}$$

The projection of a component in coordinate k to any axis of coordinate i via the coordinate j goes through three different routes, and the following nine equations are obtained.

$${}_{i}t^{k}{}_{x,x} = {}_{i}t^{j}{}_{x,x}\,{}_{j}t^{k}{}_{x,x} + {}_{i}t^{j}{}_{x,y}\,{}_{j}t^{k}{}_{y,x} + {}_{i}t^{j}{}_{x,z}\,{}_{j}t^{k}{}_{z,x} \qquad (2\text{-}1)$$

$${}_{i}t^{k}{}_{x,y} = {}_{i}t^{j}{}_{x,x}\,{}_{j}t^{k}{}_{x,y} + {}_{i}t^{j}{}_{x,y}\,{}_{j}t^{k}{}_{y,y} + {}_{i}t^{j}{}_{x,z}\,{}_{j}t^{k}{}_{z,y} \qquad (2\text{-}2)$$

$${}_{i}t^{k}{}_{x,z} = {}_{i}t^{j}{}_{x,x}\,{}_{j}t^{k}{}_{x,z} + {}_{i}t^{j}{}_{x,y}\,{}_{j}t^{k}{}_{y,z} + {}_{i}t^{j}{}_{x,z}\,{}_{j}t^{k}{}_{z,z} \qquad (2\text{-}3)$$

$${}_{i}t^{k}{}_{y,x} = {}_{i}t^{j}{}_{y,x}\,{}_{j}t^{k}{}_{x,x} + {}_{i}t^{j}{}_{y,y}\,{}_{j}t^{k}{}_{y,x} + {}_{i}t^{j}{}_{y,z}\,{}_{j}t^{k}{}_{z,x} \qquad (2\text{-}4)$$

$${}_{i}t^{k}{}_{y,y} = {}_{i}t^{j}{}_{y,x}\,{}_{j}t^{k}{}_{x,y} + {}_{i}t^{j}{}_{y,y}\,{}_{j}t^{k}{}_{y,y} + {}_{i}t^{j}{}_{y,z}\,{}_{j}t^{k}{}_{z,y} \qquad (2\text{-}5)$$

$${}_{i}t^{k}{}_{y,z} = {}_{i}t^{j}{}_{y,x}\,{}_{j}t^{k}{}_{x,z} + {}_{i}t^{j}{}_{y,y}\,{}_{j}t^{k}{}_{y,z} + {}_{i}t^{j}{}_{y,z}\,{}_{j}t^{k}{}_{z,z} \qquad (2\text{-}6)$$

$${}_{i}t^{k}{}_{z,x} = {}_{i}t^{j}{}_{z,x}\,{}_{j}t^{k}{}_{x,x} + {}_{i}t^{j}{}_{z,y}\,{}_{j}t^{k}{}_{y,x} + {}_{i}t^{j}{}_{z,z}\,{}_{j}t^{k}{}_{z,x} \qquad (2\text{-}7)$$

$$ {}_it^k{}_{z,y} = {}_it^j{}_{z,x}\,{}_jt^k{}_{x,y} + {}_it^j{}_{z,y}\,{}_jt^k{}_{y,y} + {}_it^j{}_{z,z}\,{}_jt^k{}_{z,y} \quad (2\text{-}8) $$

$$ {}_it^k{}_{z,z} = {}_it^j{}_{z,x}\,{}_jt^k{}_{x,z} + {}_it^j{}_{z,y}\,{}_jt^k{}_{y,z} + {}_it^j{}_{z,z}\,{}_jt^k{}_{z,z} \quad (2\text{-}9) $$

The displacement components of the origin from coordinate k to coordinate i via coordinate j is not the sum of their respective displacement coordinates because their reference coordinates have different orientation. They are calculated by the equations.

$$ {}_id^k{}_x = {}_id^j{}_x + {}_it^j{}_{x,x}\,{}_jd^k{}_x + {}_it^j{}_{x,y}\,{}_jd^k{}_y + {}_it^j{}_{x,z}\,{}_jd^k{}_z \quad (2\text{-}10) $$

$$ {}_id^k{}_y = {}_id^j{}_y + {}_it^j{}_{y,x}\,{}_jd^k{}_x + {}_it^j{}_{y,y}\,{}_jd^k{}_y + {}_it^j{}_{y,z}\,{}_jd^k{}_z \quad (2\text{-}11) $$

$$ {}_id^k{}_z = {}_id^j{}_z + {}_it^j{}_{z,x}\,{}_jd^k{}_x + {}_it^j{}_{z,y}\,{}_jd^k{}_y + {}_it^j{}_{z,z}\,{}_jd^k{}_z \quad (2\text{-}12) $$

Combining equations 2-1:2-12 yields the following matrix :

$$ T = \begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} & d_x \\ t_{y,x} & t_{y,y} & t_{y,z} & d_y \\ t_{z,x} & t_{z,y} & t_{z,z} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and the cascade of transformations is equal to the matrix multiplication[Paul82].

$${}_{i}T^{j}\ {}_{j}T^{k} = {}_{i}T^{k} \qquad (2\text{-}13)$$

or:

$${}_{i}\begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} & d_x \\ t_{y,x} & t_{y,y} & t_{y,z} & d_y \\ t_{z,x} & t_{z,y} & t_{z,z} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{j} {}_{j}\begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} & d_x \\ t_{y,x} & t_{y,y} & t_{y,z} & d_y \\ t_{z,x} & t_{z,y} & t_{z,z} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{k}$$

$$= {}_{i}\begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} & d_x \\ t_{y,x} & t_{y,y} & t_{y,z} & d_y \\ t_{z,x} & t_{z,y} & t_{z,z} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{k} \qquad (2\text{-}14)$$

This matrix is called the <u>Homogeneous Transformation Matrix</u>.

The 3X3 matrix $\begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} \\ t_{y,x} & t_{y,y} & t_{y,z} \\ t_{z,x} & t_{z,y} & t_{z,z} \end{bmatrix}$ is called the <u>Orientation Matrix</u>. This matrix is independent of the displacement between coordinate frames (This matrix is sometimes referred to as the "matrix of directional cosines").

The 3X1 vector $[\ d_x\ \ d_y\ \ d_z\ ]^T$ is called the

Displacement Vector[Paul82].

In order to reduce the complexity of the notation, we define:

$$ {}_{i}A^{j} == \begin{bmatrix} t_{x,x} & t_{x,y} & t_{x,z} \\ t_{y,x} & t_{y,y} & t_{y,z} \\ t_{z,x} & t_{z,y} & t_{z,z} \end{bmatrix} == \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \qquad (2\text{-}15) $$

$$ {}_{i}\underline{d}^{j} == \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \qquad (2\text{-}16) $$

Thus

$$ {}_{i}T^{j} == \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}17) $$

2.1.2 Inverse Homogeneous Transformation Matrix

The inverse homogeneous transformation transfers the reference back to the original coordinate. Hence

$$ {}_{j}T^{i} = (\ {}_{i}T^{j}\ )^{-1} \qquad (2\text{-}18) $$

Due to the fact that the transformation is reversible, the following equalities must hold.

$$ {}_j t^i_{x,x} = {}_i t^j_{x,x} \quad (2\text{-}19)$$
$$ {}_j t^i_{x,y} = {}_i t^j_{y,x} \quad (2\text{-}20)$$
$$ {}_j t^i_{x,z} = {}_i t^j_{z,x} \quad (2\text{-}21)$$
$$ {}_j t^i_{y,x} = {}_i t^j_{x,y} \quad (2\text{-}22)$$
$$ {}_j t^i_{y,y} = {}_i t^j_{y,y} \quad (2\text{-}23)$$
$$ {}_j t^i_{y,z} = {}_i t^j_{z,y} \quad (2\text{-}24)$$
$$ {}_j t^i_{z,x} = {}_i t^j_{x,z} \quad (2\text{-}25)$$
$$ {}_j t^i_{z,y} = {}_i t^j_{y,z} \quad (2\text{-}26)$$
$$ {}_j t^i_{z,z} = {}_i t^j_{z,z} \quad (2\text{-}27)$$

Having used the above nine equations, the displacement referred to coordinate j in terms of coordinate i are:

$$ {}_j d^i_x = - {}_i d^j_x \, {}_i t^j_{x,x} - {}_i d^j_y \, {}_i t^j_{y,x} - {}_i d^j_z \, {}_i t^j_{z,x} \quad (2\text{-}28)$$
$$ {}_j d^i_y = - {}_i d^j_x \, {}_i t^j_{x,y} - {}_i d^j_y \, {}_i t^j_{y,y} - {}_i d^j_z \, {}_i t^j_{z,y} \quad (2\text{-}29)$$
$$ {}_j d^i_z = - {}_i d^j_x \, {}_i t^j_{x,z} - {}_i d^j_y \, {}_i t^j_{y,z} - {}_i d^j_z \, {}_i t^j_{z,z} \quad (2\text{-}30)$$

where the minus signs indicate that the direction of the displacement vector is reversed in the inverse homogeneous transformation.

Equations 2-18:2-30 & 2-15 identify the structure of the inverse homogeneous transformation matrix as:

$$T^{-1} == \begin{bmatrix} a_x & a_y & a_z & -\underline{a}\bullet\underline{d} \\ b_x & b_y & b_z & -\underline{b}\bullet\underline{d} \\ c_x & c_y & c_z & -\underline{c}\bullet\underline{d} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-31}$$

where $\underline{a}\bullet\underline{d} = a_x d_x + a_y d_y + a_z d_z$ is the scalar product of the vectors $\underline{a}$ & $\underline{d}$; and it is the same for $\underline{b}\bullet\underline{d}$ and $\underline{c}\bullet\underline{d}$. Also

$$A^{-1} == \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} \tag{2-32}$$

Refer to [Paul82].

2.1.3 Object Representation

An object is represented by a set of points. Each point is treated as a vector $\underline{p}$ in a reference coordinate, say j. The transformation of a vector to another coordinate, say i, is:

$${}_iT^j \; {}_j\underline{p} = {}_i\underline{p} \tag{2-33}$$

$$_i\begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^j {}_j\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = {}_i\begin{bmatrix} a_x p_x + b_x p_y + c_x p_z + d_x \\ a_y p_x + b_y p_y + c_y p_z + d_y \\ a_z p_x + b_z p_y + c_z p_z + d_z \\ 1 \end{bmatrix}$$

or

$$ {}_i\underline{d}^j + {}_iA^j \; {}_j\underline{p} = {}_i\underline{p} \qquad (2\text{-}34) $$

Equation 2-34 indicates that a homogeneous transformation of a vector can be decomposed into the translational part, ${}_i\underline{d}^j$, and the rotational part, ${}_iA^j{}_j\underline{p}$.

The transformation of a set of points(an object) is :

$$ {}_i\begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^j {}_j\begin{bmatrix} p_x & q_x & \cdots & r_x \\ p_y & q_y & \cdots & r_y \\ p_z & q_z & \cdots & r_z \\ 1 & 1 & \cdots & 1 \end{bmatrix} = {}_i\begin{bmatrix} p_x & q_x & \cdots & r_x \\ p_y & q_y & \cdots & r_y \\ p_z & q_z & \cdots & r_z \\ 1 & 1 & \cdots & 1 \end{bmatrix} $$

For a rigid object, a coordinate frame(four points) at the object is adquate for the representation[Paul82].

2.1.4 Properties of the Orientation Matrix

Since $A^{-1}A = I$, we substitute equations 2-32 & 2-15 to 2-1:2-9 and obtain

$$ a_x a_x + a_y a_y + a_z a_z = \underline{a} \bullet \underline{a} = 1 \qquad (2\text{-}35) $$

$$ a_x b_x + a_y b_y + a_z b_z = \underline{a} \bullet \underline{b} = 0 \qquad (2\text{-}36) $$

$$ a_x c_x + a_y c_y + a_z c_z = \underline{a} \bullet \underline{c} = 0 \qquad (2\text{-}37) $$

$$ b_x a_x + b_y a_y + b_z a_z = \underline{b} \bullet \underline{a} = 0 \qquad (2\text{-}38) $$

$$ b_x b_x + b_y b_y + b_z b_z = \underline{b} \bullet \underline{b} = 1 \qquad (2\text{-}39) $$

$$ b_x c_x + b_y c_y + b_z c_z = \underline{b} \bullet \underline{c} = 0 \qquad (2\text{-}40) $$

$$c_x a_x + c_y a_y + c_z a_z = \underline{c} \bullet \underline{a} = 0 \qquad (2\text{-}41)$$

$$c_x b_x + c_y b_y + c_z b_z = \underline{c} \bullet \underline{b} = 0 \qquad (2\text{-}42)$$

$$c_x c_x + c_y c_y + c_z c_z = \underline{c} \bullet \underline{c} = 1 \qquad (2\text{-}43)$$

These nine equations indicate that the three axes of the coordinate frame are mutually orthogonal.

By using the equations 2-35:2-43, we can prove the following equality.

$$_i A^j\, _j\underline{p} \bullet {}_i A^j\, _j\underline{q} = {}_j\underline{p} \bullet {}_j\underline{q} \qquad (2\text{-}44)$$

This equation means that the scalar product of any two vectors is independent of the orientation of the coordinate frame.

Defining the vector product of any two vectors as:

$$\underline{p} \times \underline{q} = \begin{bmatrix} p_y q_z - p_z q_y \\ p_z q_x - p_x q_z \\ p_x q_y - p_y q_x \end{bmatrix} \qquad (2\text{-}45)$$

Since the cross product often appears in computing the torque(discussed later), it is desired that the magnitude of the cross product will be invariant under rotation of the coordinate frame. In order words, we want

$$_i A^j\, _j\underline{p} \times {}_i A^j\, _j\underline{q} = {}_i A^j ({}_j\underline{p} \times {}_j\underline{q}) \qquad (2\text{-}46)$$

By direct substitution and comparison between corresponding terms, equation 2-46 holds when the following three equations are valid.

$$\underline{a} = \underline{b} \times \underline{c} \tag{2-47}$$

$$\underline{b} = \underline{c} \times \underline{a} \tag{2-48}$$

$$\underline{c} = \underline{a} \times \underline{b} \tag{2-49}$$

This means that the three axes of the coordinates must be perpendicular to each other. From now on, we restrict ourselves to this kind of coordinate systems.

The Cartesian coordinate system fulfilles the requirement of equations 2-35:2-43 & 2-47:2-49, and it is suitable for modelling the dynamics of robot manipulators. Now, we are ready to compute the orientation matrix in the Cartesian coordinate system.

Suppose the coordinate j rotates an angle θ in the direction of x-axis and becomes the coordinate i(fig.3). By definition, we get

$$t_{x,x} = 1$$

$$t_{x,y} = 0$$

$$t_{x,z} = 0$$

for the unity vector on x-axis; and

$$t_{y,x} = 0$$

$$t_{y,y} = \cos\theta$$

$$t_{y,z} = -\sin\theta$$

for the unity vector on y-axis; and

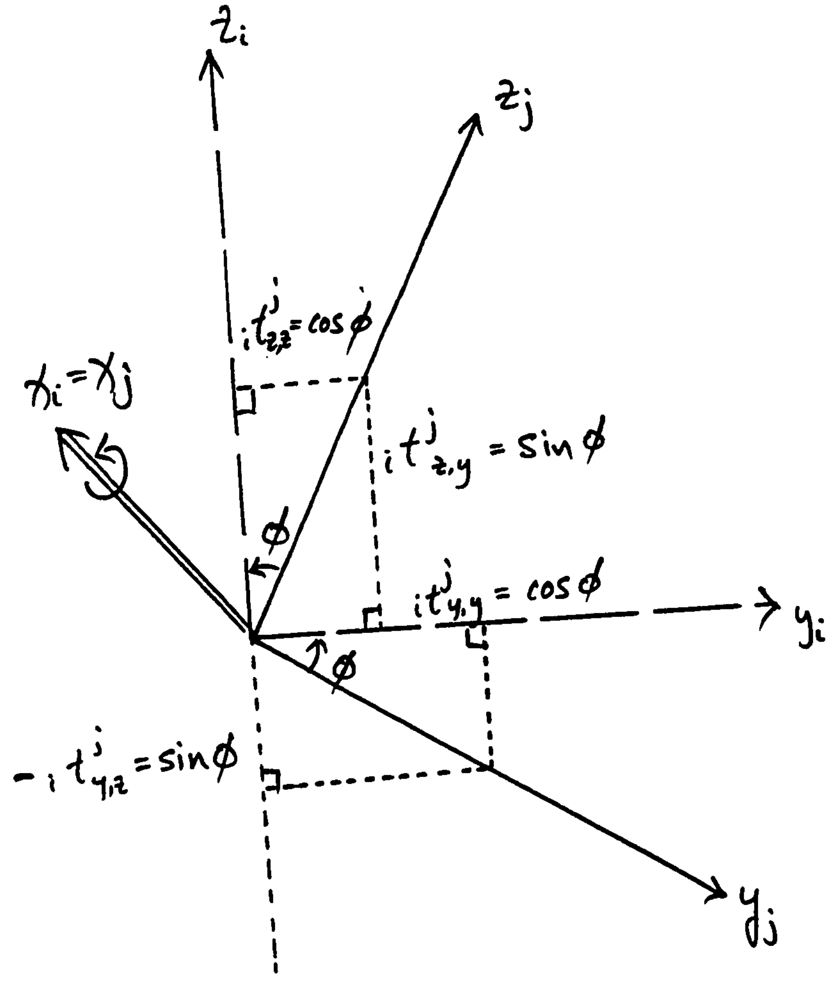


Figure 3 : Orientation Matrix

- Rotate an Angle Ø Around the x-axis.

$t_{z,x} = 0$

$t_{z,y} = \sin\phi$

$t_{z,z} = \cos\phi$

for the unity vector on z-axis. Hence

$$Rot(x;\phi) = {}_i\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}^j \quad (2\text{-}50)$$

Similarly,

$$Rot(y;\phi) = {}_i\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}^j \quad (2\text{-}51)$$

and

$$Rot(z;\phi) = {}_i\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^j \quad (2\text{-}52)$$

Note that rotation operations do not commute[Paul82].

Differentiation of the Orientation Matrix

The differentiation of a matrix is carried by differentiating every element. Therefore by the chain rule,

$$\frac{d}{dt} Rot(x;\phi) = \frac{d\phi}{dt}\Big|_{x\text{-axis}} {}_i\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix}^j$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w_x \\ 0 & w_x & 0 \end{bmatrix} {}_i\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\emptyset & -\sin\emptyset \\ 0 & \sin\emptyset & \cos\emptyset \end{bmatrix}^j$$

$$= \Omega_x \; {}_iA^j \qquad (2\text{-}56)$$

where

$$\left.\frac{d\emptyset}{dt}\right|_{x\text{-axis}} == w_x \qquad (2\text{-}57)$$

and

$$\Omega_x == \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w_x \\ 0 & w_x & 0 \end{bmatrix} \qquad (2\text{-}58)$$

are the angular velocity vector and matrix in x-direction [Paul82].

Similarly

$$\left.\frac{d\emptyset}{dt}\right|_{y\text{-axis}} == w_y \qquad (2\text{-}59)$$

and

$$\Omega_y == \begin{bmatrix} 0 & 0 & w_y \\ 0 & 0 & 0 \\ -w_y & 0 & 0 \end{bmatrix} \qquad (2\text{-}60)$$

are the angular velocity vector and matrix in y-direction.

Also,

$$\left.\frac{d\emptyset}{dt}\right|_{z\text{-axis}} == w_z \qquad (2\text{-}61)$$

and

$$\Omega_z == \begin{bmatrix} 0 & -w_z & 0 \\ w_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2\text{-}62)$$

are the angular velocity vector and matrix in z-direction.

It is easy to verify that the three angular velocity matrices mutually commute:

$$\Omega_x \Omega_y = \Omega_y \Omega_x \qquad (2\text{-}63)$$

$$\Omega_y \Omega_z = \Omega_z \Omega_y \qquad (2\text{-}64)$$

$$\Omega_z \Omega_x = \Omega_x \Omega_z \qquad (2\text{-}65)$$

Define

$$\Omega = \Omega_x \Omega_y \Omega_z = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \qquad (2\text{-}66)$$

or

$$\underline{w} = [w_x \; w_y \; w_z]^T \qquad (2\text{-}67)$$

are the angular velocity matrix and vector in any direction.

The time derivative of the orientation matrix can be shown to satisfy :

$$d_t(A) = \Omega A \qquad (2\text{-}68)$$

and $$d_{tt}(A) = d_t(\Omega)\ A + \Omega\ \Omega\ A \tag{2-69}$$

It can be written in a vector form as:

$$d_t({}_iA^j)\,{}_j\underline{r} = {}_i\underline{w} \times {}_iA^j\,{}_j\underline{r} \tag{2-70}$$

and

$$d_{tt}({}_iA^j)\,{}_j\underline{r} = d_t({}_i\underline{w}) \times {}_iA^j\,{}_j\underline{r} + {}_i\underline{w} \times ({}_i\underline{w} \times {}_iA^j\,{}_j\underline{r})$$
$$= {}_i\underline{\alpha} \times {}_iA^j\,{}_j\underline{r} + {}_i\underline{w} \times ({}_i\underline{w} \times {}_iA^j\,{}_j\underline{r}) \tag{2-71}$$

where

$${}_i\underline{\alpha} = d_t({}_i\underline{w}) \tag{2-72}$$

is the angular acceleration of coordinate i.

If the reference coordinate rotates an arbitrary angle, say ${}_iA^j$, then the angular velocity vector becomes

$${}_i\underline{w} = {}_iA^j\,{}_j\underline{w} \tag{2-73}$$

and the angular velocity matrix changes to

$${}_i\Omega = {}_iA^j\,{}_j\Omega\,{}_jA^i \tag{2-74}$$

2.1.5 The Displacement Vector

Let ${}_i\underline{d}^j$ be a displacement vector from coordinate j to coordinate i. To compute its velocity and acceleration, nonmoving coordinate frame is introduced denoted as coordinate 0. So

$${}_0\underline{d}^j = {}_0A^i\,{}_i\underline{d}^j \tag{2-75a}$$

$$d_t({}_0\underline{d}^j) = {}_0A^i d_t({}_i\underline{d}^j) + d_t({}_0A^i)\,{}_i\underline{d}^j$$
$$= {}_0A^i d_t({}_i\underline{d}^j) + {}_0\underline{w} \times {}_0A^i\,{}_i\underline{d}^j \tag{2-75b}$$

$$d_{tt}({}_0\underline{d}^j) = {}_0A^i d_{tt}({}_i\underline{d}^j) + d_t({}_0\underline{w}) \times {}_i\underline{d}^j$$
$$+ {}_0\underline{w} \times ({}_0\underline{w} \times {}_i\underline{d}^j) + 2\,{}_0\underline{w} \times {}_0A^i d_t({}_i\underline{d}^j) \tag{2-75c}$$

Denote

$$ {}_i\underline{u}^j = d_t({}_i\underline{d}^j) \qquad (2\text{-}76)$$

$$ {}_i\underline{a}^j = d_t({}_i\underline{u}^j) \qquad (2\text{-}77)$$

and $$ {}_i\underline{v}^j = {}_iA^0 d_t({}_0\underline{d}^j) \qquad (2\text{-}78)$$

as the translational velocity, acceleration, and the total velocity (translational and rotational) of the displacement vector respectively.

If we multiply equations 2-75b & 2-75c by ${}_iA^0$ and use equations 2-46, 2-72, 2-76:2-78, then we arrive at the following results:

$$ {}_i\underline{v}^j = {}_i\underline{u}^j + {}_i\underline{w} \times {}_i\underline{d}^j \qquad (2\text{-}79)$$

$$ d_t({}_i\underline{v}^j) = {}_i\underline{a}^j + {}_i\underline{\alpha} \times {}_i\underline{d}^j + {}_i\underline{w} \times ({}_i\underline{w} \times {}_i\underline{d}^j) + 2\,{}_i\underline{w} \times {}_i\underline{u}^j \qquad (2\text{-}80)$$

The total acceleration of the displacement vector consists of the translational acceleration, angular acceleration, centrifugal acceleration and the Coriolis acceleration(in the same order as in equation 2-80). Refer to [Luh80].

Finally, let us introduce some useful vector identities that will be used later in developing the dynamical model of the robot manipulator. They can be verified by direct substitution. (Refer also to [Spie71])

$$ \underline{p} \times (\underline{q} \times \underline{r}) = [\, tr(rp^T)I - rp^T \,]\underline{q} \qquad (2\text{-}81)$$

$$\underline{p} \times (\underline{q} \times \underline{r}) = (\underline{p} \bullet \underline{r})\underline{q} - (\underline{p} \bullet \underline{q})\underline{r} \qquad (2\text{-}82)$$

$$(\underline{p} \times \underline{q}) \times \underline{r} = (\underline{p} \bullet \underline{r})\underline{q} - (\underline{r} \bullet \underline{q})\underline{p} \qquad (2\text{-}83)$$

$$(\underline{p} \times \underline{q}) \times \underline{r} = \underline{p} \times (\underline{q} \times \underline{r}) + \underline{q} \times (\underline{r} \times \underline{q}) \qquad (2\text{-}84)$$

$$\underline{p} \bullet (\underline{q} \times \underline{r}) = \underline{q} \bullet (\underline{r} \times \underline{p}) = \underline{r} \bullet (\underline{p} \times \underline{q}) \qquad (2\text{-}85)$$

$$\underline{p} \times (\underline{q} \times (\underline{r} \times \underline{s})) = (tr(ps^T)I - ps^T - \tfrac{1}{2}tr(I)I)(\underline{r} \times \underline{q}) + (\underline{r} \times (tr(ps^T) - ps^T)\underline{q}) \qquad (2\text{-}86)$$

2.2 Kinematics

Kinematics relates the motion of joints to the movement of end effector. The configuration of the manipulator is given by the position of all the joints. Each joint(link) requires a coordinate frame to describe its position and a homogeneous transformation matrix relates the relationship between one frame and another. Every coordinate frame has six degrees of freedom which includes three translations and three rotations. Therefore, the maximum number of degrees of freedom for a rigid link is six, and the total number of degrees of freedom of the N-link manipulator is 6N.

The computation of the spatial position(translation and rotation) of the end effector from a given set of joint positions is known as the forward kinematics. Computing the joint variables from given end effector position and orientation is known as the inverse kinematics.

2.2.1 Coordinate System

The way of placing the coordinate frames of the manipulator links influences the effectiveness of analyzing the system. Here are some guidelines(fig.4):

1. Because most of the manipulators have their links either perpendicular or parallel to each other, the Cartesian coordinate frames provide an easy manageable reference.

2. Origin of the coordinate frames should be placed at some representative points. The intersection point of the axes of motion is a popular choice. The coordinate frame reflects the current position of that joint (after motion has been made). Finally, we pick a nonmoving point for the base coordinate as the global reference of the position of the joints and end effector.

3. Always align one of the coordinate axes to the axis of motion of that joint. For a revolute joint, the axis of rotation is chosen; for a prismatic joint, the axis of displacement is selected.

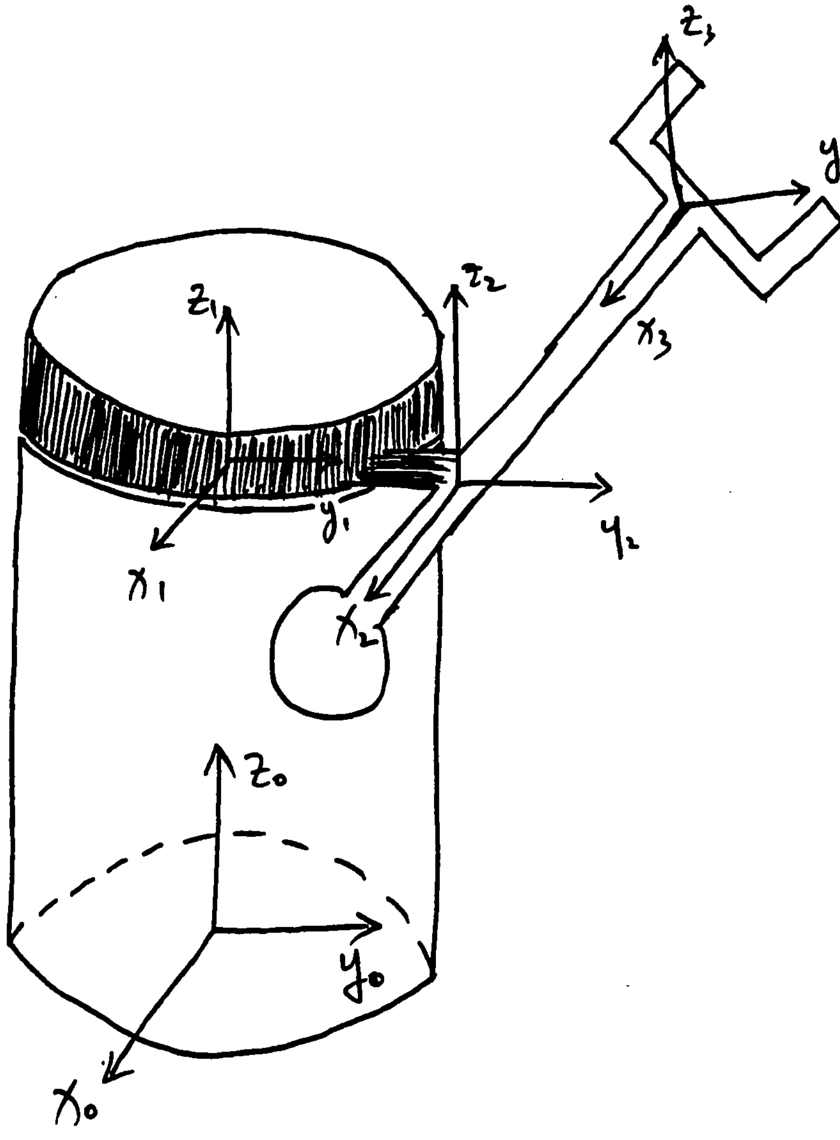


Figure 4 : The Cartesian Coordinate Frames of the Robot at the "Reference Position"

4. When the neighboring links are perpendicular to each other, align another coordinate axis in the same direction as the axis of motion of previous link. In more general situation, the axis is taken along the common normal to the two neighbouring axes of motion.

5. Given the right hand coordinate system, there are two possible choices of the third axis(Notice that we have not yet assign the positive sense to the direction of axes).

6. Finally, assign the z-axis of the coordinate frames. For each coordinate frame, it has three possible choices and there are two assignment strategies:

a Assign the axes in a unified way such that a general transformation can be applied[Paul82].

b Assign the axes arbitrary in order to reduce the complexities of the kinematic model of the manipulator(This thesis).

2.2.2 Kinematic Model

The kinematic model of the manipulator relates the position of the joints to the end effector position. The geometry of the manipulator is embedded within the kinematic model. Link is defined to be rigid, if the relative position between any two points in the link is fixed. Then the transformation from one link frame to the next consists of six elementary transformations.

Rotate an angle with respect to an axis of coordinate i.

$$\mathrm{Rot}(\alpha\ ;\ X_i) = \mathrm{Rot}(\alpha_i) \quad (2\text{-}87)$$

$$\mathrm{Rot}(\beta\ ;\ Y_i) = \mathrm{Rot}(\beta_i) \quad (2\text{-}88)$$

$$\mathrm{Rot}(\tau\ ;\ Z_i) = \mathrm{Rot}(\tau_i) \quad (2\text{-}89)$$

By these three rotations, the orientation of link frame i becomes the same as the orientation of the next link frame.

Three translations along the axes of coordinate frame i are required to align the origins of the two coordinate frames.

$$\mathrm{Tran}(a\ ;\ X_i) = \mathrm{Tran}(a_i) \quad (2\text{-}90)$$

$$\mathrm{Tran}(b\ ;\ Y_i) = \mathrm{Tran}(b_i) \quad (2\text{-}91)$$

$$\mathrm{Tran}(c\ ;\ Z_i) = \mathrm{Tran}(c_i) \quad (2\text{-}92)$$

In [Paul82], the transformation from one link frame to the next is done in terms of only two rotations and two translations(the Denavit-Hartenberg A_n matrices), if a certain convention in placing the link frames is followed. Equations 2-87:2-92 are a generaliztion of that method, since here link frames are assumed to be arbitrarily placed.

Once the coordinate frames have been assigned to the robot manipulator links, using the above six elementary transformations to move the reference from one frame to the other provides the homogeneous transformation between the two frames(Note, there may exist more than one way to move the reference of coordinate frame to align with the other frame). As an example to the development of the kinematic model, refer to figures 4 and 5 :

$$ {}_0T^1 = \mathrm{Tran}(d_0 \,;\, Z_0)\ \mathrm{Rot}(\emptyset_1 \,;\, Z_1 = Z_0) $$

$$ = \begin{bmatrix} \cos\emptyset_1 & -\sin\emptyset_1 & 0 & 0 \\ \sin\emptyset_1 & \cos\emptyset_1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}93) $$

$$ {}_1T^2 = \mathrm{Tran}(d_1 \,;\, Y_1)\ \mathrm{Rot}(\emptyset_2 \,;\, Y_2 = Y_1) $$

$$ = \begin{bmatrix} \cos\emptyset_2 & 0 & \sin\emptyset_2 & 0 \\ 0 & 1 & 0 & d_1 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}94) $$

$$ {}_2T^3 = \mathrm{Tran}(d_2 \,;\, -X_2) $$

$$ = \begin{bmatrix} 1 & 0 & 0 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}95) $$

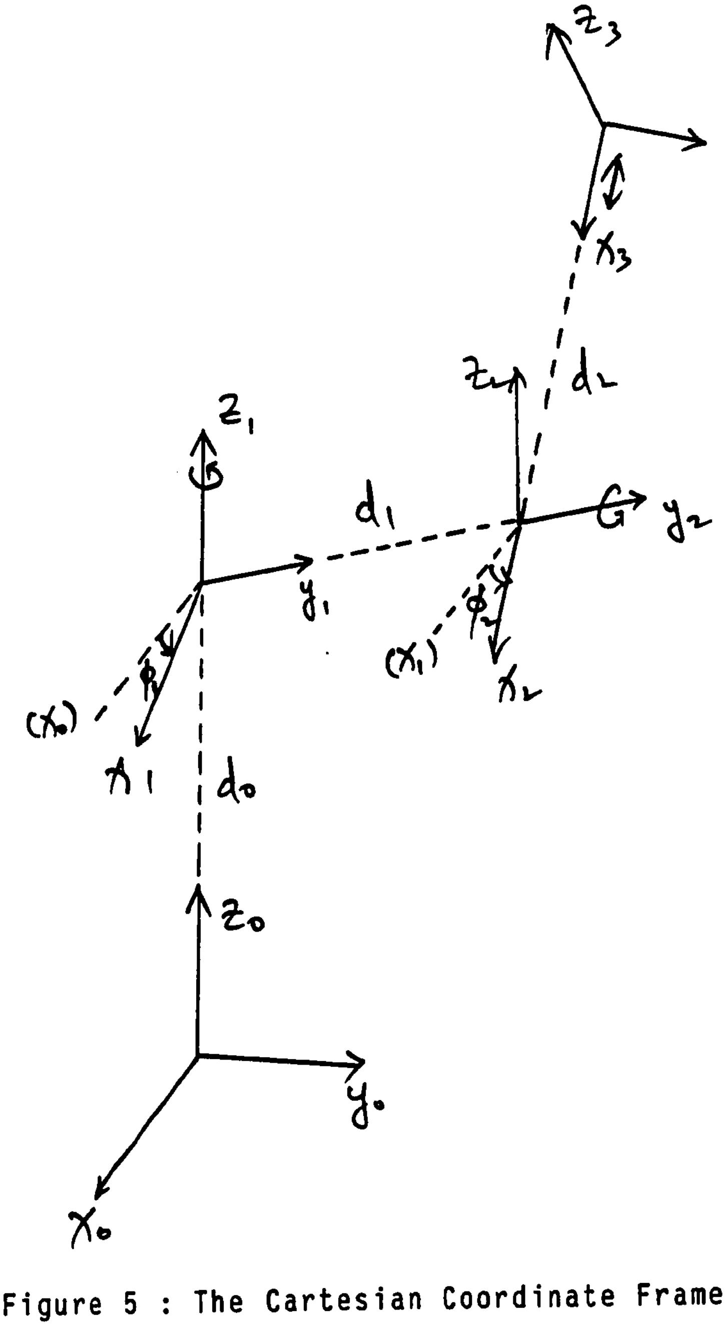


Figure 5 : The Cartesian Coordinate Frames of the Robot After Moved

This robot has three degrees of freedom, namely $\emptyset_1$, $\emptyset_2$ and d_2. The parameters d_0 and d_1 are the geometrical constant of the system.

Based on equations 2-93:2-95, we can construct the homogeneous transformation matrix for any pair of coordinate frames. Hence,

$$ {}_0T^2 = {}_0T^1 \ {}_1T^2 \tag{2-96}$$

$$= \begin{bmatrix} \cos\emptyset_1\cos\emptyset_2 & -\sin\emptyset_1 & \cos\emptyset_1\sin\emptyset_2 & -d_1\sin\emptyset_1 \\ \sin\emptyset_1\cos\emptyset_2 & \cos\emptyset_1 & \sin\emptyset_1\sin\emptyset_2 & d_1\cos\emptyset_1 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}_0T^3 = {}_0T^2 \ {}_2T^3 = \tag{2-97}$$

$$\begin{bmatrix} \cos\emptyset_1\cos\emptyset_2 & -\sin\emptyset_1 & \cos\emptyset_1\sin\emptyset_2 & -d_1\sin\emptyset_1-d_2\cos\emptyset_1\cos\emptyset_2 \\ \sin\emptyset_1\cos\emptyset_2 & \cos\emptyset_1 & \sin\emptyset_1\sin\emptyset_2 & d_1\cos\emptyset_1-d_2\sin\emptyset_1\cos\emptyset_2 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & d_0+d_2\sin\emptyset_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}_1T^3 = {}_1T^2 \ {}_2T^3 $$

$$= \begin{bmatrix} \cos\emptyset_2 & 0 & \sin\emptyset_2 & -d_2\cos\emptyset_2 \\ 0 & 1 & 0 & d_1 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & d_2\sin\emptyset_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-98}$$

Equations 2-93:2-98 form the complete kinematic model of the example robot manipulator. Similarly, for a N-link robot manipulator, its complete kinematic model is

$$ {}_{i}T^{j} \quad \text{where} \quad i,\ j = 0,1,2.......,N \tag{2-99}$$

2.3 Statics

By statics, we refer to the computation of the distribution of force and torque in an object at rest. For a rigid object at rest, the total sum of external forces and the total sum of external torques are zero at any point of the object(Statical balance). When the total sum of external forces or torques is not zero, the object is accelerated or revolved according to the Newton and Euler equations, and the system is no longer statically balanced. Nevertheless due to D'Alembert's principle[Beer62], one can replace the acceleration of the object by the equivalent reaction force and torque, to restore an "equivalent static balance" of the system.

2.3.1 Force

By Newton's second law, the force is defined as:

$$\underline{f} = d_t(\underline{p}) \tag{2-100}$$

where

$\underline{p} = m\,\underline{v}$ is the momemtum of the object. (2-101)

Suppose a force is acting on a point that is at ${}_j\underline{r}$ in coordinate j, or is ${}_i\underline{r}$ in coordinate i(fig.6). Recall equation 2-34, we have

$${}_i\underline{r} = {}_i\underline{d}^j + {}_iA^j\,{}_j\underline{r} \qquad (2\text{-}102)$$

then

$${}_i\underline{v} = {}_iA^j\,{}_j\underline{v} \qquad (2\text{-}103)$$

because the time differentiation of ${}_i\underline{d}^j$ and ${}_iA^j$ are zero provided that the coordinates i and j are not moving.

If the mass of the object is constant, then from equations 2-101 & 2-100, the transformation of the static force between coordinates is achieved as follows:

$${}_i\underline{p} = {}_iA^j\,{}_j\underline{p} \qquad (2\text{-}104)$$

and

$${}_i\underline{f} = {}_iA^j\,{}_j\underline{f} \qquad (2\text{-}105)$$

Equation 2-100 can be written as:

$$\underline{f} - d_t(\underline{p}) = 0 \qquad (2\text{-}106)$$

or

$$\underline{f} - \underline{f}_r = 0 \qquad (2\text{-}107)$$

where $\underline{f}_r$ is the reaction force and equation 2-107 provides the statical balance condition for a moving object (D'Alembert principle).

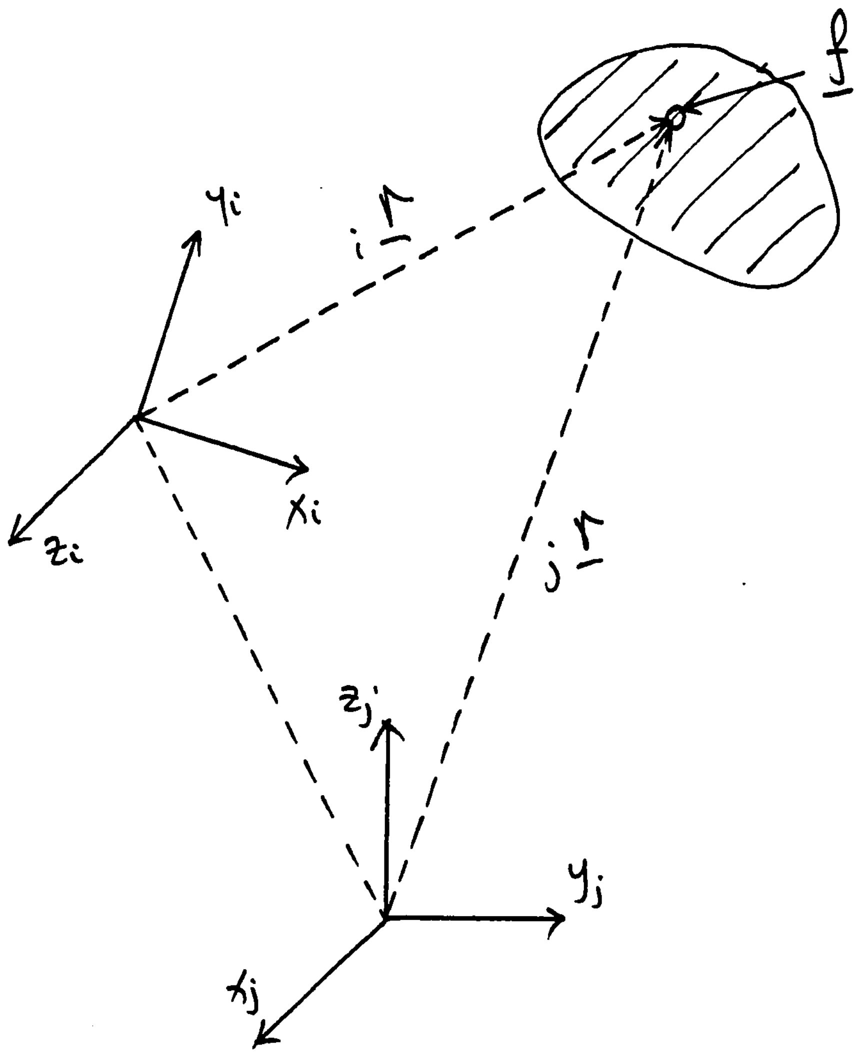


Figure 6 : The Transformation of Static Force and Torque Between Coordinate Frames

2.3.2 Torque

Torque is defined as

$$ {}_j\underline{n} = {}_j\underline{r} \times {}_j\underline{f} \tag{2-108} $$

in coordinate j. If the reference is transferred to coordinate i(fig.6),then

$$ {}_i\underline{n} = {}_i\underline{r} \times {}_i\underline{f} \tag{2-109} $$

Since

$$ {}_i\underline{r} = {}_i\underline{d}^j + {}_iA^j\,{}_j\underline{r}\ , $$

then

$$ {}_i\underline{n} = {}_i\underline{d}^j \times {}_iA^j\,{}_j\underline{f} + {}_iA^j\,{}_j\underline{r} \times {}_iA^j\,{}_j\underline{f} $$

$$ = {}_i\underline{d}^j \times {}_iA^j\,{}_j\underline{f} + {}_iA^j\,({}_j\underline{r} \times {}_j\underline{f}) $$

$$ = {}_i\underline{d}^j \times {}_i\underline{f} + {}_iA^j\,{}_j\underline{n} \tag{2-110} $$

after equations 2-46, 2-104 & 2-108 have been used.

Equation 2-110 describes the changes of the static torque between coordinate frames.

CHAPTER 3

CLOSED FORM NEWTON-EULER DYNAMICAL MODEL OF THE GENERALIZED MANIPULATOR

3.1 Introduction

The dynamical model of the manipulator describes the relationship between the force/torque and the translational/ rotational acceleration of the manipulator at a given position and velocity condition. The forward dynamical analysis computes the force and torque of each link by knowing the translational and rotational acceleration of the end effector. The inverse dynamical analysis provides the velocities and accelerations of each link, given the external forces and torques.

The dynamical model of the manipulator can be written in closed form or in recursive form. In the closed form dynamical model, each link is treated individually. In the recursive form dynamical model, each link is described with respect to the previous link.

The closed form dynamical model has a clear distinction of each individual effect and it is used for analysis and design purposes(table 1). The recursive form

Dynamical Model	Applications	Newton-Euler Method	Lagrangain Method
Closed Form	Analysis and Design	Intuitive but derivations require some innovations [Holl82] [This thesis]	Derivations are straight-forward but the physical meaning of the terms are not easy to understand [Paul82]
Recursive Form	Real-time Applications (on-line control)	More efficiency on computation [Luh80] [This thesis]	Less efficiency on computation [Holl82]

Table 1 : The Newton-Euler Method versus The Lagrangian Method

suitable for real time applications, such as on-line control of the robot manipulator.

For deriving the dynamical model of a manipulator, two methods are commonly used. The Lagrange formulation achieves the dynamical model in a straightforward manner, however the physical meaning of the terms is often hard to interprete. The Newton-Euler formulation is intuitive and more efficient, but its derivation requires a three dimensional view of the manipulator.

The general algorithm for constructing the dynamical model in closed form and recursive form by the Lagrangian formulation can be found in [Paul82] and [Holl80]. The general algorithm for writing the dynamical model in recursive form by the Newton-Euler formulation was derived by [Luh80]. The dynamical model in closed form by the Newton-Euler formulation can be found in[Holl82]. In both papers that deal with the Newton-Euler formulation, the center of mass of each link are used as origins for the respective coordinate frames.

There are two restrictions of the above dynamical models. One is that those models compute forces and torques only at the direction of motion but ignore the other directions. This restriction is acceptable if the robot is assumed to be rigid and stationary. The other restriction is that the coordinate frames have to be set in a specific way such that their models can be applied(In other word, the user must follow a fixed convention).

In this thesis, the derived dynamical models have the following characteristics:

a. The derivation follows the Newton-Euler formulation. The physical meaning of the terms will be explained and the derivation seems to be simpler than in the Lagrange method.

b. The dynamical model can be used for computing the force and torque in all directions for any joints. The way to do it is to assume that each joint can translate and rotate in all directions(fig.1). Such joints will be referred to as "generalized joints". A manipulator consists of N generalized joints is called the N-link generalized manipulator and it has 6N degrees of freedom. The derivation of the dynamical model for the generalized manipulator avoids the difficulties of identifying the axis-of-motion of each joint(link). The model contains all the possible information of any physical manipulator with the same number of links(section III.3). This generalized dynamical model can be reduced to (c).

c. the dynamical model for the one-degree-of-freedom-per-link manipulator. It is done easily by introducing the direction selection vectors. The reduced dynamical model has two advantages. The coordinate frames in it can be set arbitrarily and it can compute the force or torque at any desired direction. The method of reduction is discussed in detail in chapter 4.

d. The reduced dynamical model from Newton-Euler method will be shown to be identical to the dynamical model using Lagrangian method. It will be shown that the terms of the dynamical model from Newton-Euler method can be visualized but have each four different forms according to the four possible combinations of translational joints and rotational joints. The Lagrangian method however gives unique expressions for any combination of joints but the physical interpretation of the term is lost.

e. Futhermore, the unique expression for the reduced dynamical model can be used for computing the force and torque in any direction. This is proved using the Newton-Euler method but it cannot be shown using the Lagrangian method.

3.2 Derivation of the Closed Form Dynamical Model

The method of deriving the closed form dynamical model is, first to compute the force for accelerating a mass element of the manipulator. Second, we integrate the force over all mass elements from the desired joint to the end effector. This is the force demanded by that joint. The computation of the torques follow the same procedures. We first compute the torque of a mass element, then we integrate it to obtain the demanded torque from a joint. The method is simple, and it will be demonstrated to be efficient too.

The force of a mass element

The force for accelerating a mass element anywhere at link j of the generalized manipulator in an inertial coordinate frame is described by the Newton's second law.

$$ {}_0\underline{f} = m\, d_{tt}({}_0\underline{r}^{,j}) \tag{3-1}$$

The subscript 0 indicates that the base coordinate has been used and it is an inertial frame. The superscript ,j means that that mass element is in link j of the manipulator.

The computation of $d_{tt}({}_0\underline{r}^{,j})$ is as follows(fig.7):

$$ {}_0\underline{r}^{,j} = {}_0\underline{d}^{1} + {}_0A^{1}\,{}_1\underline{r}^{,j} \tag{3-2}$$

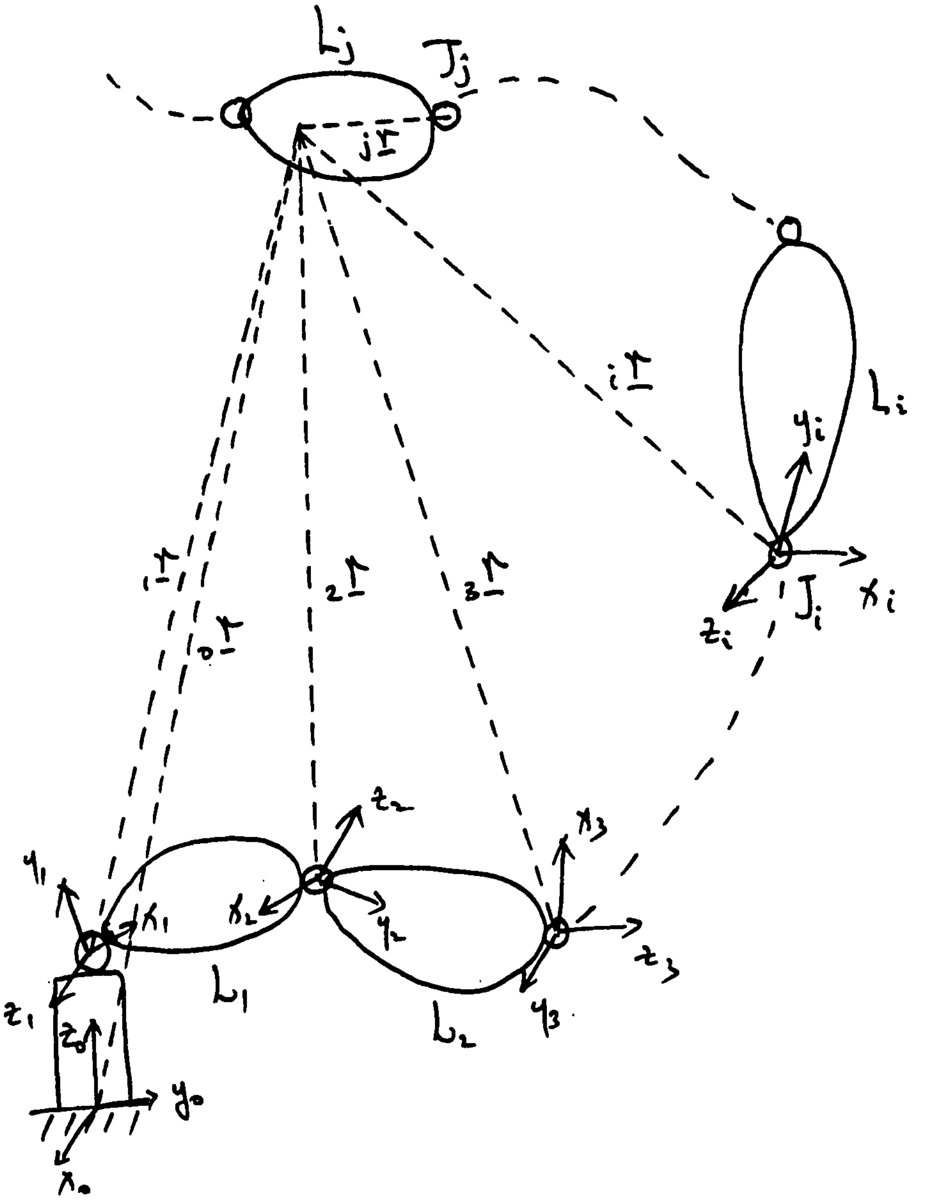


Figure 7 : The Force of a Mass Element in Link j.

$$d_t({}_0\underline{r}^{,j}) = d_t({}_0\underline{d}^1) + d_t({}_0A^1)\,{}_1\underline{r}^{,j} + {}_0A^1 d_t({}_1\underline{r}^{,j})$$

$$d_{tt}({}_0\underline{r}^{,j}) = d_{tt}({}_0\underline{d}^1) + d_{tt}({}_0A^1)\,{}_1\underline{r}^{,j}$$
$$+2d_t({}_0A^1)d_t({}_1\underline{r}^{,j}) + {}_0A^1 d_{tt}({}_1\underline{r}^{,j}) \qquad (3\text{-}4)$$

The superscript ,j will be supressed in the following derivations. Recall equations 2-75:2-80 & 2-70:2-71, we have

$$d_t\ ({}_0\underline{d}^1) = {}_0\underline{v}^1 = {}_0\underline{u}^1 + {}_0\underline{w}^1 \times {}_0\underline{d}^1 \qquad (3\text{-}5)$$

$$d_{tt}({}_0\underline{d}^1) = d_t({}_0\underline{v}^1) = {}_0\underline{a}^1 + {}_0\underline{\alpha}^1 \times {}_0\underline{d}^1$$
$$+ {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0\underline{d}^1) + 2\,{}_0\underline{w}^1 \times {}_0\underline{u}^1 \qquad (3\text{-}6)$$

$$d_t\ ({}_0A^1)\,{}_1\underline{r} = {}_0\underline{w}^1 \times {}_0A^1\,{}_1\underline{r} \qquad (3\text{-}7)$$

$$d_{tt}({}_0A^1)\,{}_1\underline{r} = {}_0\underline{\alpha}^1 \times {}_0A^1\,{}_1\underline{r}$$
$$+ {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0A^1\,{}_1\underline{r}) \qquad (3\text{-}8)$$

After substituting 3-5:3-8 into equations 3-3:3-4, we obtain

$$d_t\ ({}_0\underline{r}) = {}_0\underline{u}^1 + {}_0\underline{w}^1 \times {}_0\underline{d}^1 + {}_0\underline{w}^1 \times {}_0A^1\,{}_1\underline{r}$$
$$+{}_0A^1 d_t({}_1\underline{r}) \qquad (3\text{-}9)$$

$$d_{tt}({}_0\underline{r}) = {}_0\underline{a}^1 + {}_0\underline{\alpha}^1 \times {}_0\underline{d}^1 + {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0\underline{d}$$
$$+ {}_0\underline{\alpha}^1 \times {}_0A^1\,{}_1\underline{r} + {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0A^1\,{}_1\underline{r})$$
$$+2\,{}_0\underline{w}^1 \times {}_0\underline{u}^1$$
$$+2\,{}_0\underline{w}^1 \times {}_0A^1 d_t({}_1\underline{r}) + {}_0A^1 d_{tt}({}_1\underline{r}) \qquad (3\text{-}10)$$

sing equation 3-2, equations 3-9 & 3-10 become:

$$d_t\ ({}_0\underline{r}) = {}_0\underline{u}^1 + {}_0\underline{w}^1 \times {}_0\underline{r} + {}_0A^1 d_t({}_1\underline{r}) \qquad (3\text{-}11)$$

$$d_{tt}({}_0\underline{r}) = {}_0\underline{a}^1 + {}_0\underline{\alpha}^1 \times {}_0\underline{r} + {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0\underline{r})$$

$$+2{}_0\underline{w}^1 \times {}_0\underline{u}^1 + 2{}_0\underline{w}^1 \times {}_0A^1 d_t({}_1\underline{r})$$
$$+ {}_0A^1 d_{tt}({}_1\underline{r}) \qquad (3\text{-}12)$$

In equations 3-11 & 3-12, the variables ${}_0\underline{u}^1$, ${}_0\underline{a}^1$, ${}_0\underline{w}^1$ and ${}_0\underline{\alpha}^1$ are the translational and rotational velocities and accelerations of coordinate 1 with respect to 0. The variable ${}_0\underline{r}$ can be solved by the inverse kinematic method[Holl83]. The variables $d_t({}_1\underline{r})$ and $d_{tt}({}_1\underline{r})$ are unknown to us but they can be represented in terms of $d_t({}_2\underline{r})$ and $d_{tt}({}_2\underline{r})$:

$$d_t\ ({}_1\underline{r}) = {}_1\underline{u}^2 + {}_1\underline{w}^2 \times {}_1\underline{r} + {}_1A^2 d_t({}_2\underline{r}) \qquad (3\text{-}13)$$
$$d_{tt}({}_1\underline{r}) = {}_1\underline{a}^2 + {}_1\underline{\alpha}^2 \times {}_1\underline{r} + {}_1\underline{w}^2 \times ({}_1\underline{w}^2 \times {}_1\underline{r})$$
$$+ 2{}_1\underline{w}^2 \times {}_1\underline{u}^2 + 2{}_1\underline{w}^2 \times {}_1A^2 d_t({}_2\underline{r})$$
$$+ {}_1A^2 d_{tt}({}_2\underline{r}) \qquad (3\text{-}14)$$

Here, the variables ${}_1\underline{u}^2$, ${}_1\underline{a}^2$, ${}_1\underline{w}^2$ and ${}_1\underline{\alpha}^2$ are the translational and rotational velocities and accelerations of joint 1 that can be determined by the planned trajectory of the end effector using the inverse kinematic methods [Holl83].

Substituting equations 3-13 & 3-14 to equations 3-11 & 3-12, we get

$$d_t\ ({}_0\underline{r}) = {}_0\underline{u}^1 + {}_0\underline{w}^1 \times {}_0\underline{r}$$
$$+ {}_0A^1({}_1\underline{u}^2 + {}_1\underline{w}^2 \times d_t({}_1\underline{r}))$$

$$+{}_0A^2\ d_t({}_2\underline{r}) \tag{3-15}$$

$$\begin{aligned} d_{tt}({}_0\underline{r}) = \ & {}_0\underline{a}^1 + {}_0\underline{\alpha}^1 \times {}_0\underline{r} + {}_0\underline{w}^1 \times ({}_0\underline{w}^1 \times {}_0\underline{r}) \\ & +{}_0A^1({}_1\underline{a}^2 + {}_1\underline{\alpha}^2 \times {}_1\underline{r} + {}_1\underline{w}^2 \times ({}_1\underline{w}^2 \times {}_1\underline{r})) \\ & +{}_0A^2\ d_{tt}({}_2\underline{r}) \\ & +2\,{}_0\underline{w}^1 \times {}_0A^0({}_0\underline{u}^1) \\ & +2\,{}_0\underline{w}^1 \times {}_0A^1({}_1\underline{u}^2 + {}_1\underline{w}^2 \times {}_1\underline{r}) \\ & +2\,{}_0\underline{w}^1 \times {}_0A^2\ d_t({}_2\underline{r}) \\ & +2\,{}_0A^1({}_1\underline{w}^2 \times {}_1\underline{u}^2) \\ & +2\,{}_0A^1({}_1\underline{w}^2 \times {}_1\dot{A}^2 d_t({}_2\underline{r})) \end{aligned} \tag{3-16}$$

In equations 3-15:3-16, all the variables can be solved from the inverse kinematic methods except for $d_t({}_2\underline{r})$ and $d_{tt}({}_2\underline{r})$. We can expand $d_t({}_2\underline{r})$ and $d_{tt}({}_2\underline{r})$ in the similar manner. The expansion will be terminated at $d_t({}_j\underline{r})$ and $d_{tt}({}_j\underline{r})$ because we have assumed that the mass element is in link j of the manipulator. The equations for $d_t({}_j\underline{r})$ and $d_{tt}({}_j\underline{r})$ are :

$$d_t({}_j\underline{r}) = {}_j\underline{u}^{j+1} + {}_j\underline{w}^{j+1} \times {}_j\underline{r} \tag{3-17}$$

$$\begin{aligned} d_{tt}({}_j\underline{r}) = \ & {}_j\underline{a}^{j+1} + {}_j\underline{\alpha}^{j+1} \times {}_j\underline{r} + {}_j\underline{w}^{j+1} \times ({}_j\underline{w}^{j+1} \times {}_j\underline{r}) \\ & +2\,{}_j\underline{w}^{j+1} \times {}_j\underline{u}^{j+1} \end{aligned} \tag{3-18}$$

The physical interpretation of equations 3-17:3-18 is that the movement(velocity and acceleration) of the latter joints(joint j+1 to end effector) will not affect the movement of the mass element in the preceeding joints. This property characterizes open kinematic chains(One may take it as a definition of open kinematic chains).

By combining equations 3-15:3-18, one gets

$$d_t\ ({}_0\underline{r}) = \sum_{k=1}^{j} {}_0A^k({}_k\underline{u}^{k+1} + {}_k\underline{w}^{k+1} \times {}_k\underline{r}) \tag{3-19}$$

$$d_{tt}({}_0\underline{r}) = \sum_{k=1}^{j} {}_0A^k(\ {}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r} + {}_k\underline{w}^{k+1}\times({}_k\underline{w}^{k+1}\times$$

$$+2\,{}_j\underline{w}^{j+1} \times {}_j\underline{u}^{j+1})$$

$$+2 \sum_{l\geq k}^{j} \sum_{k=1}^{j} {}_0A^k({}_k\underline{w}^{k+1} \times {}_kA^l({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times \delta_{kl}\,{}_l\underline{r}) \tag{3-20}$$

where δ_{kl} equals to zero when k=l; otherwise it equals one.

From equations 3-1 & 3-20, the force of the mass element on link j of the generalized manipulator is:

$$_0\underline{f}^{,j} = \sum_{k=1}^{j} {}_0\underline{f}^{k,j} + \sum_{k=1}^{j} {}_0\underline{f}^{kk,j} + \sum_{l\geq k}^{j} \sum_{k=1}^{j} {}_0\underline{f}^{kl,j} \tag{3-21}$$

where $$_0\underline{f}^{k,j} = m\ {}_0A^k({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r}) \tag{3-22}$$

$$_0\underline{f}^{kk,j} = m\ {}_0A^k({}_k\underline{w}^{k+1} \times({}_k\underline{w}^{k+1} \times {}_k\underline{r})) \tag{3-23}$$

$$_0\underline{f}^{kl,j} = 2m\ {}_0A^k({}_k\underline{w}^{k+1} \times {}_kA^l({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1}\times\delta_{kl}\,{}_l\underline{r}) \tag{3-24}$$

are the inertial force, centrifugal force and Coriolis force of the mass element on link j of the generalized manipulator respectively.

The orientaion matrices are there to align the direction of the joint coordinates to the base coordinate.

If we define

$$ {}_0({}_k\underline{r}) == {}_0A^k\,{}_k\underline{r} \qquad (3\text{-}25)$$

as the vector in coordinate k that is referred to the direction of coordinate 0, then the expressions will be more readable. For example, equations 3-11:3-24 can be written as:

$$ {}_0\underline{f}^{k,j} = m\ {}_0({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r}) \qquad (3\text{-}26)$$

$$ {}_0\underline{f}^{kk,j} = m\ {}_0({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r})) \qquad (3\text{-}27)$$

$$ {}_0\underline{f}^{kl,j} = 2m\ {}_0({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times {}_l\underline{r}) \qquad (3\text{-}28)$$

The subscripts indicate the destination coordinates. They cascade from the inner parentheses to the outer parentheses, and all the subscripts within the same parentheses must be the same.

The total force of a joint

For the open dynamic chains, the force of a joint is the integration of forces of all the mass elements from that joint(denoted as J_i) to the end effector(denoted as E) plus the payload(One may use it as the definition of open dynamic chain). By integrating equation 3-21, we obtain the demanded force of joint i.

$$ {}_0({}_i\underline{F}) = \sum_{k=1}^{N} {}_0({}_i\underline{F}^k) + \sum_{k=1}^{N} {}_0({}_i\underline{F}^{kk}) + \sum_{l\geq k}^{N} \sum_{k=1}^{N} {}_0({}_i\underline{F}^{kl}) \qquad (3\text{-}29)$$

$$ {}_0({}_i\underline{F}^k) = \int_{Ji}^{E} {}_0({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r})\ dm \qquad (3\text{-}30)$$

$$ {}_0({}_i\underline{F}^{kk}) = \int_{Ji}^{E} {}_0({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r}))\, dm \qquad (3\text{-}31) $$

$$ {}_0({}_i\underline{F}^{kl}) = 2\int_{Ji}^{E} {}_0({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times {}_l\underline{r})\, dm \qquad (3\text{-}32) $$

These are respectively the inertial force, the centrifugal force and the Coriolis force of link j of the generalized manipulator. The direction of each force is referred to the base coordinate. If we want each force to be referred to its own coordinate, we simply change the subscript 0 to i that is equivalent to multiply the whole equation by ${}_iA^0$. Hence,

$$ {}_i\underline{F} = \sum_{k=1}^{N} {}_i\underline{F}^k + \sum_{k=1}^{N} {}_i\underline{F}^{kk} + \sum_{l\geq k}^{N} \sum_{k=1}^{N} {}_i\underline{F}^{kl} \qquad (3\text{-}33) $$

where :

$$ {}_i\underline{F}^k = \int_{Ji}^{E} {}_i({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r})dm \qquad (3\text{-}34) $$

$$ {}_i\underline{F}^{kk} = \int_{Ji}^{E} {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r}))dm \qquad (3\text{-}35) $$

$$ {}_i\underline{F}^{kl} = 2\int_{Ji}^{E} {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times \delta_{kl}\,{}_l\underline{r}))dm \qquad (3\text{-}36) $$

Define the mass and center of mass of link j in coordinate k as:

$$M^{,j} = \int_{\text{link } j} dm \tag{3-37}$$

$$_k\underline{p}^j = \int_{\text{link } j} {_k\underline{r}}^{,j}\, dm \,/\, M^{,j} = {_k\underline{d}}^j + {_kA}^j\, {_j\underline{p}} \tag{3-38}$$

where $_j\underline{p}$ is the center of mass of link j in its own coordinate frame.

Then equations 3-34:3-36 can be written as:

$$_i\underline{F}^k = \sum_{j=\max\{i,k\}}^{N} {_i\underline{F}}^{k,j} \tag{3-39}$$

$$_i\underline{F}^{kk} = \sum_{j=\max\{i,k\}}^{N} {_i\underline{F}}^{kk,j} \tag{3-40}$$

$$_i\underline{F}^{kl} = \sum_{j=\max\{i,k,l\}}^{N} {_i\underline{F}}^{kl,j} \tag{3-41}$$

where

$$_i\underline{F}^{k,j} = M^{,j}{}_i({_k\underline{a}}^{k+1} + {_k\underline{\alpha}}^{k+1} \times {_k\underline{p}}^{,j}) \tag{3-42}$$

$$_i\underline{F}^{kk,j} = M^{,j}{}_i({_k\underline{w}}^{k+1} \times ({_k\underline{w}}^{k+1} \times {_k\underline{p}}^{,j})) \tag{3-43}$$

$$_i\underline{F}^{kl,j} = 2M^{,j}{}_i({_k\underline{w}}^{k+1} \times {_k}({_l\underline{u}}^{l+1} + {_l\underline{w}}^{l+1} \times \delta_{kl}\, {_l\underline{p}}^{,j}))$$

are the inertial force, centrifugal force and Coriolis force of link j acting on joint i due to the movements of links k and l.

When k=i or l=i, the force of joint i is due to the movement of joint i itself. Conversely, the force of joint i is due to the movements of other links, and this is the coupling force.

The torque of a mass element

The torque of a mass element in link j of the generalized manipulator is computed by equation 2-109.

$${}_i\underline{n} = {}_i\underline{r} \times {}_i\underline{f}$$

where ${}_i\underline{f}$ is according to equations 3-21 & 3-26:3-28. Changing the subscript from 0 to i for referring the forces to the local coordinate, one gets

$${}_i\underline{f}^{,j} = \sum_{k=1}^{j} {}_i\underline{f}^{k,j} + \sum_{k=1}^{j} {}_i\underline{f}^{kk,j} + \sum_{l \geq k}^{j} \sum_{k=1}^{j} {}_i\underline{f}^{kl,j} \qquad (3\text{-}45)$$

where

$${}_i\underline{f}^{k,j} = m \; {}_i({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r}) \qquad (3\text{-}46)$$

$${}_i\underline{f}^{kk,j} = m \; {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r})) \qquad (3\text{-}47)$$

$${}_i\underline{f}^{kl,j} = 2m \; {}_i({}_k\underline{w}^{k+1} \times {}_kA^l({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times \delta_{kll}\underline{r}) \qquad (3\text{-}48)$$

Then the torques on a mass element in link j of the generalized manipulator are:

$${}_i\underline{n}^{,j} = \sum_{k=1}^{j} {}_i\underline{n}^{k,j} + \sum_{k=1}^{j} {}_i\underline{n}^{kk,j} + \sum_{l \geq k}^{j} \sum_{k=1}^{j} {}_i\underline{n}^{kl,j} \qquad (3\text{-}49)$$

where

$${}_i\underline{n}^{k,j} = m \; {}_i\underline{r} \times {}_i({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r}) \qquad (3\text{-}50)$$

$${}_i\underline{n}^{kk,j} = m \; {}_i\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r})) \qquad (3\text{-}51)$$

$${}_i\underline{n}^{kl,j} = 2m \; {}_i\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times {}_kA^l({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times \delta_{kll}\underline{r}) \qquad (3\text{-}52)$$

are the inertial torque, centrifugal torque and Coriolis torque of the mass element on link j of the generalized manipulator acting on joint i due to the movements of links k and l.

When k=i or l=i, the torque of joint i is due to the movement of joint i itself. Conversely, the torque of joint i is due to the movements of other links, and this is the coupling torque.

The torque of a joint

The torque of a joint is the integral of the torques of all the mass elements from that joint to the end effector plus the payload. By integrating equation 3-41:43, we obtain the demanded torque of joint i.

$$ {}_i\underline{N} = \sum_{k=1}^{N} {}_i\underline{N}^{k} + \sum_{k=1}^{N} {}_i\underline{N}^{kk} + \sum_{l\geq k}^{N} \sum_{k=1}^{N} {}_i\underline{N}^{kl} \qquad (3\text{-}53) $$

where

$$ {}_i\underline{N}^{k} = \sum_{j=\max\{i,k\}}^{N} {}_i\underline{N}^{k,j} \qquad (3\text{-}54) $$

$$ {}_i\underline{N}^{kk} = \sum_{j=\max\{i,k\}}^{N} {}_i\underline{N}^{kk,j} \qquad (3\text{-}55) $$

$$ {}_i\underline{N}^{kl} = \sum_{j=\max\{i,k,l\}}^{N} {}_i\underline{N}^{kl,j} \qquad (3\text{-}56) $$

$$ {}_{i}\underline{N}^{k,j} = \int_{\text{link } j} {}_{i}\underline{r} \times {}_{i}({}_{k}\underline{a}^{k+1} + {}_{k}\underline{\alpha}^{k+1} \times {}_{k}\underline{r})dm \tag{3-57} $$

$$ {}_{i}\underline{N}^{kk,j} = \int_{\text{link } j} {}_{i}\underline{r} \times {}_{i}({}_{k}\underline{w}^{k+1} \times ({}_{k}\underline{w}^{k+1} \times {}_{k}\underline{r}))dm \tag{3-58} $$

$$ {}_{i}\underline{N}^{kl,j} = 2\int_{\text{link } j} {}_{i}\underline{r} \times {}_{i}({}_{k}\underline{w}^{k+1} \times {}_{k}({}_{l}\underline{u}^{l+1} + {}_{l}\underline{w}^{l+1} \times \delta_{kl\,l}\underline{r})dm \tag{3-59} $$

Equations 3-54:3-56 are the inertial torque, the centrifugal torque and the Coriolis torque of link j of the generalized manipulator acting on joint i due to the movements of links k and l respectively.

a. the inertial torque

The integrand in equation 3-57 has two terms, the first term is

$$ \int_{\text{link } j} {}_{i}\underline{r} \times {}_{i}({}_{k}\underline{a}^{k+1})\, dm = M^{,j}({}_{i}\underline{p}^{,j} \times {}_{i}({}_{k}\underline{a}^{k+1})) \tag{3-60} $$

after using equations 3-37 & 3-38. Since ${}_{k}\underline{a}^{k+1}$ is the translational acceleration of link k, so $M^{,j}{}_{i}({}_{k}\underline{a}^{k+1})$ is the force to achieve that translational acceleration of link j in coordinate i, and equation 3-60 is the torque that is observed at the origin of coordinate i.

The second term is

$$ \int_{\text{link } j} {}_{i}\underline{r} \times {}_{i}({}_{k}\underline{\alpha}^{k+1} \times {}_{k}\underline{r})\, dm $$

$$= \int_{\text{link } j} ({}_i\underline{d}^j + {}_iA^j{}_j\underline{r}) \times {}_i({}_k\underline{\alpha}^{k+1} \times ({}_k\underline{d}^j + {}_kA^j{}_j\underline{r}))\, dm \qquad [2\text{-}34]$$

$$= \int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j)\, dm \qquad (3\text{-}61)$$

$$+ \int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j{}_j\underline{r})\, dm$$

$$+ \int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j)\, dm$$

$$+ \int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j{}_j\underline{r})\, dm$$

where [2-34] means "refer to equation 2-34".

The first term of equation 3-61 is

$$\int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j)\, dm \qquad [3\text{-}37]$$

$$= M'^j\, {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j) \qquad (3\text{-}62)$$

where the torque depends on the distances between coordinates j, i & k(fig.8). This is a version of the "parallel axis theorem"[Beer62].

The second term is

$$\int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j{}_j\underline{r})\, dm \qquad [3\text{-}37]$$

$$= M'^j\, {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j{}_j\underline{p}'^j) \qquad (3\text{-}63)$$

and the third term is

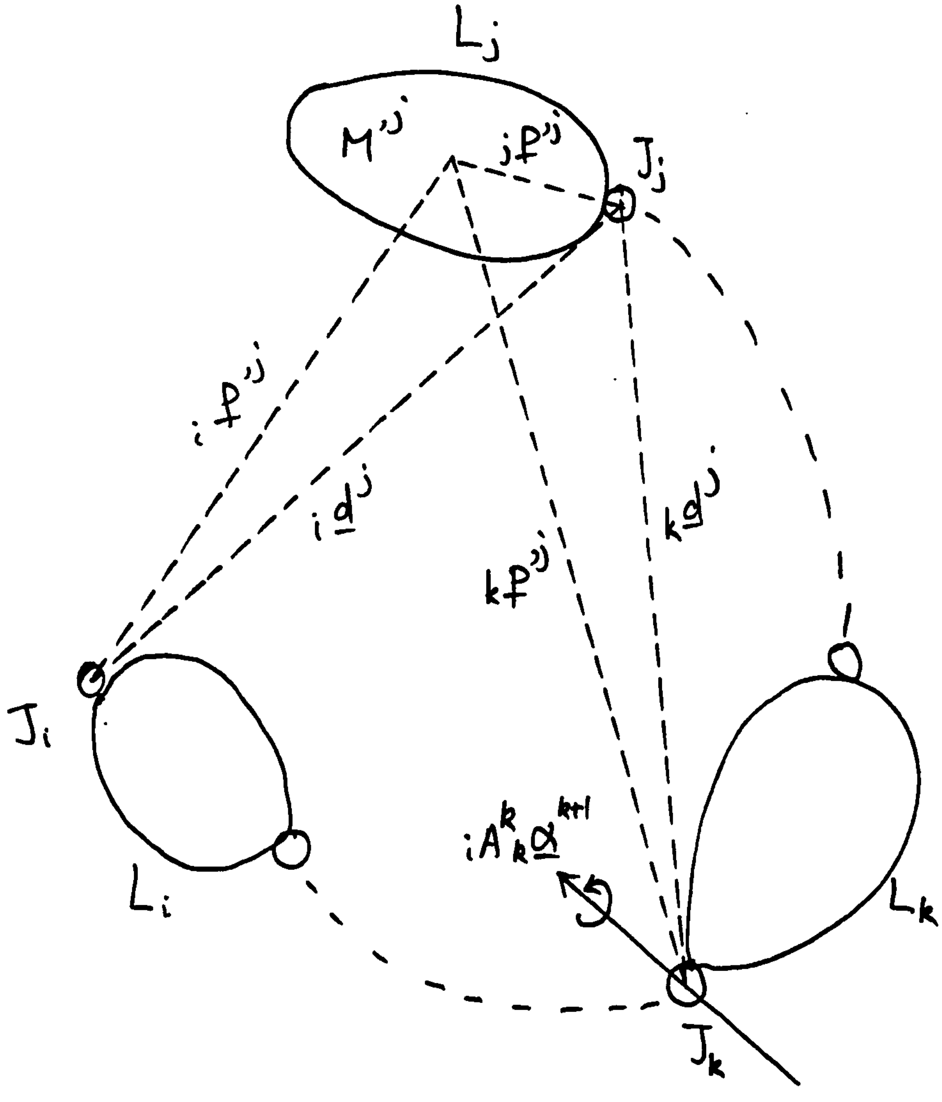


Figure 8 : The Inertial Torque

$$\int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j)\ dm \qquad [3\text{-}38]$$

$$= M^{,j}\ {}_iA^j{}_j\underline{p}^{,j} \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j) \qquad (3\text{-}64)$$

where the torques in equations 3-63 & 3-64 are due to the fact that the center of mass of link j is not at the origin of either coordinate frame i or k [fig.7].

The fourth term is

$$\int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j{}_j\underline{r})\ dm$$

$$= \int_{\text{link } j} {}_iA^j{}_j\underline{r} \times ({}_iA^k{}_k\underline{\alpha}^{k+1} \times {}_iA^j{}_j\underline{r})\ dm \qquad [2\text{-}46]$$

$$= \int_{\text{link } j} [tr(({}_iA^j{}_j\underline{r})({}_iA^j{}_j\underline{r})^T)I - ({}_iA^j{}_j\underline{r})({}_iA^j{}_j\underline{r})^T]({}_iA^k{}_k \qquad [2\text{-}81]$$

$$= {}_iA^j\{\int_{\text{link } j}[tr({}_j\underline{r}{}_j\underline{r}^T)I - ({}_j\underline{r}{}_j\underline{r}^T)]dm\}({}_iA^j)^T\ ({}_iA^k{}_k\underline{\alpha}^{k+1})$$

$$= {}_iA^jI^{,j}({}_iA^j)^T\ ({}_iA^k{}_k\underline{\alpha}^{k+1})$$

where

$$I^{,j} = \int_{\text{link } j} [tr({}_j\underline{r}{}_j\underline{r}^T)I - ({}_j\underline{r}{}_j\underline{r}^T)]\ dm \qquad (3\text{-}65)$$

is called the inertial matrix of link j measured at the origin of the local coordinate frame(here the inertia matrix of the link includes the inertia of the actuators). ${}_iA^jI^{,j}({}_iA^j)^T$ is the inertial matrix of link j under rotation of ${}_iA^j$(fig.8). ${}_iA^k{}_k\underline{\alpha}^{k+1}$ is the angular acceleration of joint k measured in the direction of coordinate i. So the fourth term is the torque to rotate a link at an arbitrary angle.

So the inertial torque represented in equation 3-57 is the combination of equations 3-60:3-65, hence

$$
\begin{aligned}
{}_i\underline{N}^{k,j} &= \int_{\text{link } j} \underline{r} \times {}_i({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{r} + 2{}_k\underline{w}^{k+1} \times {}_k\underline{u}^{k+1}) \\
&= M,^j({}_i\underline{p},^j \times {}_iA^k({}_k\underline{a}^{k+1})) \qquad (3\text{-}66) \\
&+ M,^j\ {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j) \\
&+ M,^j\ {}_i\underline{d}^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_kA^j\ {}_j\underline{p},^j) \\
&+ M,^j\ {}_iA^j\ {}_j\underline{p},^j \times {}_i({}_k\underline{\alpha}^{k+1} \times {}_k\underline{d}^j) \\
&+ {}_iA^jI,^j({}_iA^j)^T\ ({}_iA^k\ {}_k\underline{\alpha}^{k+1})
\end{aligned}
$$

where

$$
I,^j = \int_{\text{link } j} [tr({}_j\underline{r}\,{}_j\underline{r}^T)I - ({}_j\underline{r}\,{}_j\underline{r}^T)]\ dm
$$

$$
== \begin{bmatrix} \int_{\text{link } j}(y^2+z^2)dm & \int_{\text{link } j} xy\ dm & \int_{\text{link } j} xz\ dm \\ \int_{\text{link } j} yx\ dm & \int_{\text{link } j}(x^2+z^2)dm & \int_{\text{link } j} yz\ dm \\ \int_{\text{link } j} zx\ dm & \int_{\text{link } j} zy\ dm & \int_{\text{link } j}(x^2+y^2)dm \end{bmatrix} \qquad (3\text{-}67)
$$

$$== \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad (3\text{-}68)$$

b. the Coriolis torque

The Coriolis torque of link j for joint i due to the movements of joints k and l is computed by equation 3-58, where the operator δ_{kl} is suppressed in the derivations.

$$ {}_i\underline{N}^{kl,j} = 2\int_{\text{link } j} {}_i\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1} + {}_l\underline{w}^{l+1} \times {}_l\underline{r}))dm $$

$$ = 2\int_{\text{link } j} {}_i\underline{r}\, dm \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) $$

$$ +2\int_{\text{link } j} ({}_i\underline{d}^j + {}_iA^j{}_j\underline{r}) \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times ({}_l\underline{d}^j + {}_lA^j{}_j\underline{r})))\, dm \qquad [2\text{-}34] $$

$$ = 2M^{,j}\, {}_i\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \qquad [3\text{-}38] $$

$$ +2\int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^j))\, dm \qquad (3\text{-}69) $$

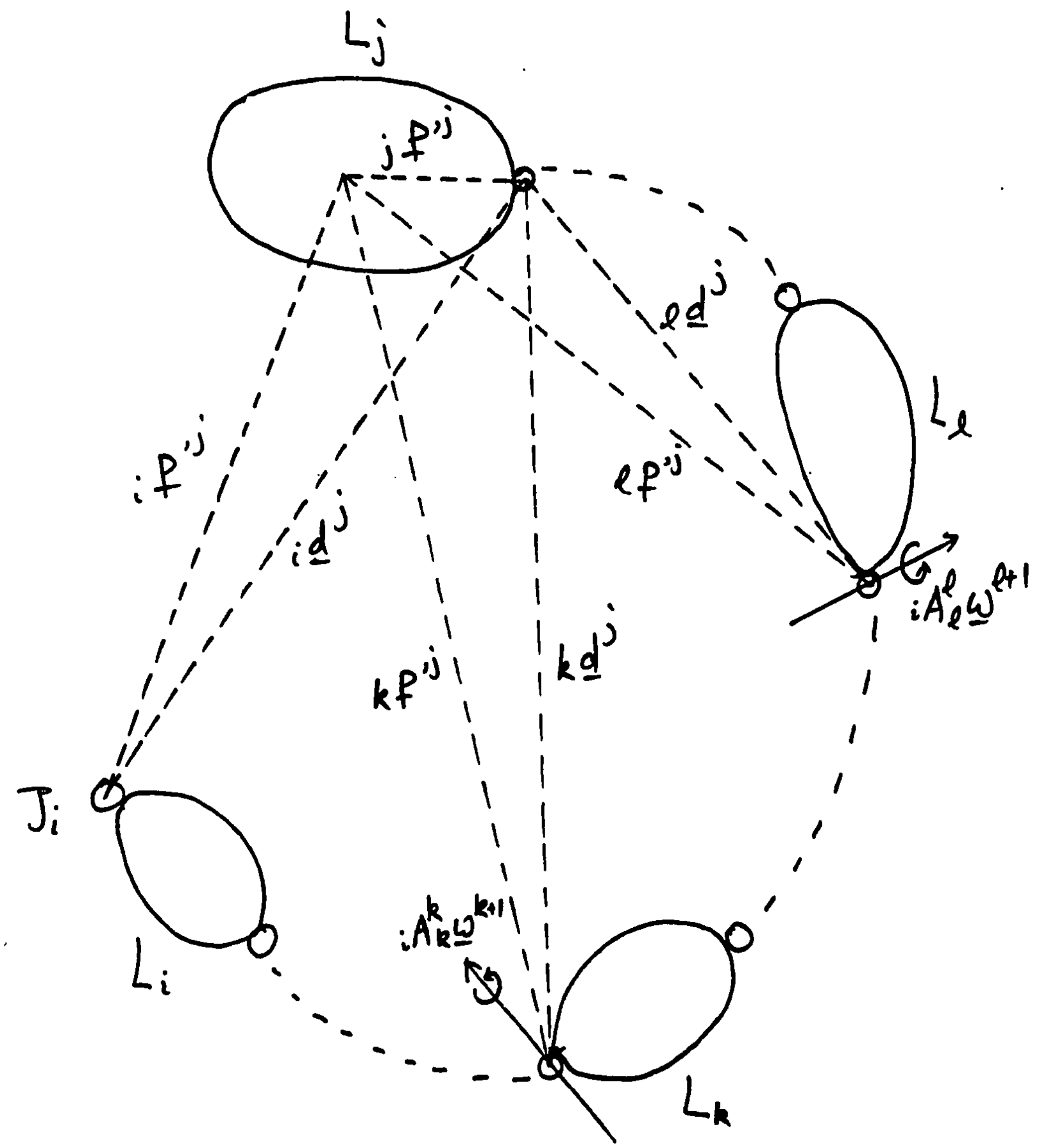
$$ +2\int_{\text{link } j} {}_i\underline{d}^j \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_lA^j{}_j\underline{r}))\, dm $$

$$ +2\int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^j))\, dm $$

$$ +2\int_{\text{link } j} {}_iA^j{}_j\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_lA^j{}_j\underline{r}))\, dm $$

$$
\begin{aligned}
&= 2M'^{j}\ {}_i\underline{p}'^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \\
&\quad +2M'^{j}\ {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^{j})) \qquad [3\text{-}37] \\
&\quad +2M'^{j}\ {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_lA^{j}\,{}_j\underline{p}'^{j}))[3\text{-}38] \\
&\quad +2M'^{j}\ {}_iA^{j}\,{}_j\underline{p}'^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^{j}))[3\text{-}38] \\
&\quad +2\int_{\text{link } j} {}_iA^{j}\,{}_j\underline{r} \times ({}_iA^{k}\,{}_k\underline{w}^{k+1} \times ({}_iA^{l}\,{}_l\underline{w}^{l+1} \times {}_iA^{j}\,{}_j\underline{r}))\,dm \qquad [2\text{-}46] \\
&= 2M'^{j}\ {}_i\underline{p}'^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \\
&\quad +2M'^{j}\ {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^{j})) \\
&\quad +2M'^{j}\ {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_lA^{j}\,{}_j\underline{p}'^{j})) \\
&\quad +2M'^{j}\ {}_iA^{j}\,{}_j\underline{p}'^{j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^{j})) \\
&\quad +2[{}_iA^{j}I'^{j}({}_iA^{j})^{T} - \tfrac{1}{2}tr(I'^{j})][({}_iA^{l}\,{}_l\underline{w}^{l+1} \times {}_iA^{k}\,{}_k\underline{w}^{k+1})] \\
&\quad +2({}_iA^{l}\,{}_l\underline{w}^{l+1}) \times \{[{}_iA^{j}I'^{j}({}_iA^{j})^{T}]({}_iA^{k}\,{}_k\underline{w}^{k+1})\} \qquad (3\text{-}70a)
\end{aligned}
$$

Equation 3-70 is the Coriolis torque of link j for joint i due to the movements of joints k & l. This equation has six terms all caused by the fact that joint k is revolute. If joint l is prismatic(fig.9), the Coriolis force is then $2M'^{j}{}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1}))$ at the center of mass of link j(3-44). Therefore, the arm of torque is ${}_i\underline{p}'^{j}$ in the first term. The next five terms exist only if joint l is revolute. The physical explanation is not obvious but the meaning is understood by following the derivations that is similar to the explanation for the inertial torque. Anyway, the geometrical interpretation of such variables are presented in fig.9.



Firgure 9 : The Coriolis Torque

If k=l, then δ_{kl} is zero and equation 3-70a becomes:

$$ {}_i\underline{N}^{kl,j} = 2M^{,j} \; {}_i\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \tag{3-70b} $$

c. the Centrifugal torque

The equation for the centrifugal torque is achieved by letting the coordinate frames k & l coincide, without considering the translational velocity and the factor 2 in (3-57:3-58). Hence

$$ \begin{aligned} {}_i\underline{N}^{kk,j} &= \int_{\text{link } j} {}_i\underline{r} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{r}))dm \\ &= M^{,j} \; {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{d}^{j})) \\ &+ M^{,j} \; {}_i\underline{d}^{j} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_kA^{j}\,{}_j\underline{p}^{,j})) \\ &+ M^{,j} \; {}_iA^{j}\,{}_j\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{d}^{j})) \\ &+ ({}_iA^{k}\,{}_k\underline{w}^{k+1}) \times \{[{}_iA^{j} I^{,j} ({}_iA^{j})^T]({}_iA^{k}\,{}_k\underline{w}^{k+1})\} \end{aligned} \tag{3-71} $$

The fourth term of equation 3-70 vanishes since the cross product of two identical vectors is zero. The last term is due to the generalized Euler equation. The rest of the terms treat the link as a point mass due to the variables of separation of coordinate frame origins and the position of center of mass.

The gravity force and torque

The gravity force of a mass element in link j of the generalized manipulator in the base coordinate is:

$$ {}_0\underline{f}_g = M^{,j}\ {}_0\underline{g} \tag{3-72}$$

where $\underline{g}$ is the gravity acceleration.

If one refers to a local coordinate, then

$$ {}_i\underline{f}_g = m^{,j}\ {}_i\underline{g} \tag{3-73}$$

For the open dynamic chains, the total gravity force acting on joint i is the sum of the gravity force of all the mass elements from that joint(J_i) to the end effector(E). Therefore,

$$ {}_i\underline{F}_g = \int_{Ji}^{E} {}_i\underline{g}\ dm = \sum_{j=i}^{N} {}_i\underline{F}_g{}^{,j} \tag{3-74}$$

or

$$ {}_i\underline{F}_g{}^{,j} = M^{,j}\ {}_i\underline{g} \tag{3-75}$$

according to equation 3-37.

The gravity torque of a mass element in link j of the generalized manipulator at the origin of coordinate frame i is:

$$ {}_i\underline{n}_g = {}_i\underline{r}\ X\ {}_i\underline{f}_g = m^{,j}\ {}_i\underline{r}\ X\ {}_i\underline{f} \tag{3-76}$$

The total gravity torque acting on joint i in the local coordinate is:

$$ {}_{i}\underline{N}_{g} = \int_{Ji}^{E} {}_{i}\underline{r}\ dm \times {}_{i}\underline{g} = \sum_{j=i}^{N} {}_{i}\underline{N}_{g}{}^{,j} \qquad (3\text{-}77) $$

where

$$ {}_{i}\underline{N}_{g}{}^{,j} = M^{,j}\ {}_{i}\underline{p}^{,j} \times {}_{i}\underline{g} $$
$$ = M^{,j}\ ({}_{i}\underline{d}^{j} + {}_{i}A^{j}\ {}_{j}\underline{p}^{,j}) \times {}_{i}\underline{g} \qquad (3\text{-}78) $$

3.3 Algorithm For Constructing The Closed Form Dynamical Model of the Generalized Manipulator

In the previous section, we have derived the closed form dynamical model for the generalized manipulator. The following step-by-step procedures summarize the derivation of the model:

A. Link Variables

Measure :

a. The mass of each link ($M^{,j}$).

b. The center of mass of each link(in terms of the link coordinate frame(${}_{j}\underline{p}^{,j}$)).

c. The inertia matrix of each link (in link's coordinate ($I^{,j}$)).

B. The Kinematic Model

Find the geometrical parameters of each link, then establish the kinematic model according to the procedures of section 2.2.

C. Joint Variables

Specify :

a. The translational velocity($_k\underline{u}^{k+1}$) & acceleration($_k\underline{a}^{k+1}$),

b. The angular velocity($_k\underline{w}^{k+1}$) and acceleration($_k\underline{\alpha}^{k+1}$) of joint k, k=1,2,......,N.

D. Construction of the dynamical model

The total force of joint i is the sum of equations 3-33, 3-74 & 3-79.

$$_i\underline{F} = \sum_{k=1}^{N} {}_i\underline{F}^k + \sum_{k=1}^{N} {}_i\underline{F}^{kk} + \sum_{l \geq k}^{N} \sum_{k=1}^{N} {}_i\underline{F}^{kl} + {}_i\underline{F}_g + {}_i\underline{F}_m \quad (3\text{-}81)$$

where

$$_i\underline{F}^k = \sum_{j=\max\{i,k\}}^{N} {}_i\underline{F}^{k,j} \quad (3\text{-}82)$$

$$_i\underline{F}^{kk} = \sum_{j=\max\{i,k\}}^{N} {}_i\underline{F}^{kk,j} \quad (3\text{-}83)$$

$$_i\underline{F}^{kl} = \sum_{j=\max\{i,k,l\}}^{N} {}_i\underline{F}^{kl,j} \quad (3\text{-}84)$$

$$_i\underline{F}_g = \sum_{j=i}^{N} {}_i\underline{F}_g{}^{,j} \quad (3\text{-}85)$$

and

$$_i\underline{F}^{k,j} = M^{,j}{}_i({}_k\underline{a}^{k+1} + {}_k\underline{\alpha}^{k+1} \times {}_k\underline{p}^{,j}) \quad (3\text{-}87)$$

$$_i\underline{F}^{kk,j} = M^{,j}{}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{p}^{,j})) \quad (3\text{-}88)$$

$$_i\underline{F}^{kl,j} = 2M^{,j}\,_i(_k\underline{w}^{k+1} \times \,_k(_l\underline{u}^{l+1} + \,_l\underline{w}^{l+1} \times \delta_{kl}\,_l\underline{p}^{,j}))$$

$$_i\underline{F}_g{}^{,j} = M^{,j}\,_i\underline{g}$$

The total torque of joint i is the sum of equations 3-52, 3-77 & 3-80.

$$_i\underline{N} = \sum_{k=1}^{N} {_i\underline{N}^k} + \sum_{k=1}^{N} {_i\underline{N}^{kk}} + \sum_{l \geq k}^{N} \sum_{k=1}^{N} {_i\underline{N}^{kl}} + {_i\underline{N}_g} + {_i\underline{N}_m} \quad (3\text{-}90)$$

where

$$_i\underline{N}^k = \sum_{j=\max\{i,k\}}^{N} {_i\underline{N}^{k,j}} \quad (3\text{-}91)$$

$$_i\underline{N}^{kk} = \sum_{j=\max\{i,k\}}^{N} {_i\underline{N}^{kk,j}} \quad (3\text{-}92)$$

$$_i\underline{N}^{kl} = \sum_{j=\max\{i,k,l\}}^{N} {_i\underline{N}^{kl,j}} \quad (3\text{-}93)$$

$$_i\underline{N}_g = \sum_{j=i}^{N} {_i\underline{N}_g{}^{,j}} \quad (3\text{-}94)$$

and

$$\begin{aligned} _i\underline{N}^{k,j} = {} & M^{,j}(_i\underline{p}^{,j} \times \,_iA^k(_k\underline{a}^{k+1})) \\ & + M^{,j}\,_i\underline{d}^j \times \,_i(_k\underline{\alpha}^{k+1} \times \,_k\underline{d}^j) \\ & + M^{,j}\,_i\underline{d}^j \times \,_i(_k\underline{\alpha}^{k+1} \times \,_kA^j\,_j\underline{p}^{,j}) \\ & + M^{,j}\,_iA^j\,_j\underline{p}^{,j} \times \,_i(_k\underline{\alpha}^{k+1} \times \,_k\underline{d}^j) \\ & + \,_iA^jI^{,j}(_iA^j)^T\,(_iA^k\,_k\underline{\alpha}^{k+1}) \end{aligned} \quad (3\text{-}96)$$

$$\begin{aligned} _i\underline{N}^{kk,j} = {} & M^{,j}\,_i\underline{d}^j \times \,_i(_k\underline{w}^{k+1} \times (_k\underline{w}^{k+1} \times \,_k\underline{d}^j)) \\ & + M^{,j}\,_i\underline{d}^j \times \,_i(_k\underline{w}^{k+1} \times (_k\underline{w}^{k+1} \times \,_kA^j\,_j\underline{p}^{,j})) \end{aligned} \quad (3\text{-}97)$$

$$+ M^{,j}\ {}_iA^j\ {}_j\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times ({}_k\underline{w}^{k+1} \times {}_k\underline{d}^j))$$
$$+ ({}_iA^k\ {}_k\underline{w}^{k+1}) \times \{[{}_iA^j I^{,j} ({}_iA^j)^T]({}_iA^k\ {}_k\underline{w}^{k+1})\}$$

If $l \geq k$, then

$$
\begin{aligned}
{}_i\underline{N}^{kl,j} = {} & 2M^{,j}\ {}_i\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \\
& +2M^{,j}\ {}_i\underline{d}^j \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^j)) \\
& +2M^{,j}\ {}_i\underline{d}^j \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_lA^j\ {}_j\underline{p}^{,j})) \\
& +2M^{,j}\ {}_iA^j\ {}_j\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{w}^{l+1} \times {}_l\underline{d}^j)) \\
& +2[{}_iA^j I^{,j} ({}_iA^j)^T - \tfrac{1}{2}tr(I^{,j})][({}_iA^l\ {}_l\underline{w}^{l+1} \times {}_iA^k\ {}_k\underline{w}^{k+1})] \\
& +2({}_iA^l\ {}_l\underline{w}^{l+1}) \times \{[{}_iA^j I^{,j} ({}_iA^j)^T]({}_iA^k\ {}_k\underline{w}^{k+1})\}
\end{aligned}
\tag{3-98a}
$$

If $l = k$, then

$${}_i\underline{N}^{kl,j} = 2M^{,j}\ {}_i\underline{p}^{,j} \times {}_i({}_k\underline{w}^{k+1} \times {}_k({}_l\underline{u}^{l+1})) \tag{3-98b}$$

$${}_i\underline{N}_g^{,j} = M^{,j}\ {}_i\underline{p}^{,j} \times {}_i\underline{g} \tag{3-99}$$

where $I^{,j}$ is the inertial matrix of link j measured at the coordinate origin of joint j.

The derivation of the dynamical model is based on four concepts:

a. <u>The Rigid Body Assumption</u>

The relative position of any two points in the link does not change while the link moves.

b. <u>The Open Kinematic Chain Concept</u>

The kinematic variables of a mass element do not depend on the kinematics of those mass elements which are closer to the open end of the chain.

c. The Open Dynamical Chain Concept

The overall required force(torque) at joint i is the sum of all forces(torques) that act on all mass elements from link i to the end effector.

d. The Generalized Manipulator Concept

The generalized mainpulator is constructed by the generalized joints which can be translated and rotated in any direction. In other words, their direction of motion are arbitrary.

Example

The robot in figures 4-5 is used as an example. In order to demonstrate the multiple degree-of-freedom link, joint 2 is thought of having two degrees of freedoms. It can rotate w.r.t. Y_2-axis and can translate along $-X_2$-axis too. Thus it is a two link robot that has three degrees of freedom.

The derivation of the dynamical model consists of the following steps :

Step A. Specifying all the link variables

The mass of link 1 is M_1 and its center of mass is located at the frame origin($\underline{p}_1$ = 0). The inertial matrix of link 1 is I_1 and it can be derived by equation 3-67. Since link 1 is a flat cylindical mass, the inertial matrix of link 1 is:

$$I_{1xx} = M_1 \, d_1^2 / 4 \tag{3-100}$$

$$I_{1yy} = M_1 \, d_1^2 / 2 \tag{3-101}$$

$$I_{1zz} = M_1 \, d_1^2 / 4 \tag{3-102}$$

$$I_{1xy} = I_{1yx} = I_{1yz} = I_{1zy} = I_{1xz} = I_{1zx} = 0 \tag{3-103}$$

where M_1 is the mass of link 1 and d_1 is the radius of the cylindrical mass.

Let d_0 be the thickness of the top.

The mass of link 2 is M_2 and its center of mass is situated at the middle of the link($\underline{p}_{2x} = -d_2/2$ = varying). Since link 2 is a slender rod, the inertial matrix of link 2(I_2) is:

$$I_{2xx} = 0 \tag{3-104}$$

$$I_{2yy} = M_2 \, d_2^2 / 12 \tag{3-105}$$

$$I_{2zz} = M_2 \, d_2^2 / 12 \tag{3-106}$$

$$I_{2xy} = I_{2yx} = I_{2yz} = I_{2zy} = I_{2xz} = I_{2zx} = 0 \tag{3-107}$$

Step B. Writing the kinematic model

The open kinematic chain concept implies that the kinematic model is established from the base to the end effector.

$${}_0T_1 = \mathrm{Tran}(d_0 \; ; \; Z_0) \; \mathrm{Rot}(\emptyset_1 \; ; \; Z_1 = Z_0)$$

$$= \begin{bmatrix} \cos\emptyset_1 & -\sin\emptyset_1 & 0 & 0 \\ \sin\emptyset_1 & \cos\emptyset_1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3\text{-}108)$$

$${}_1T_2 = \text{Tran}(d_1 \ ; \ Y_1) \ \text{Rot}(\emptyset_2 \ ; \ Y_2{=}Y_1) \ \text{Tran}(d_2 \ ;-X_2)$$

$$= \begin{bmatrix} \cos\emptyset_2 & 0 & \sin\emptyset_2 & -d_2\cos\emptyset_2 \\ 0 & 1 & 0 & d_1 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & d_2\sin\emptyset_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3\text{-}109)$$

Thus:

$${}_0T_2 = \text{Tran}(d_0 \ ; \ Z_0) \ \text{Rot}(\emptyset_1 \ ; \ Z_1) \ \text{Tran}(d_1 \ ; \ Y_1)$$
$$\text{Rot}(\emptyset_2 \ ; \ Y_2)\text{Tran}(d_2 \ ;-X_2) = \quad (3\text{-}110)$$

$$\begin{bmatrix} \cos\emptyset_1\cos\emptyset_2 & -\sin\emptyset_1 & \cos\emptyset_1\sin\emptyset_2 & -d_1\sin\emptyset_1-d_2\cos\emptyset_1\cos\emptyset_2 \\ \sin\emptyset_1\sin\emptyset_2 & \cos\emptyset_1 & \sin\emptyset_1\sin\emptyset_2 & d_1\cos\emptyset_1-d_2\sin\emptyset_1\cos\emptyset_2 \\ -\sin\emptyset_2 & 0 & \cos\emptyset_2 & d_0 + d_2\sin\emptyset_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Step C. Specifying the dynamical variables of each joint</u>

The angular velocity and acceleration of link 1 are indicated by the vectors $[0, 0, \underline{w}_1]^T$ & $[0, 0, \underline{\alpha}_1]^T$.

The angular velcoity and acceleration of link 2 are represented by the vectors $[0, \underline{w}_2, 0]^T$, $[0, \underline{\alpha}_2, 0]^T$, and the translational velocity and acceleration of link 2 are specified by the vectors $[-\underline{u}_2, 0, 0]^T$ and $[-\underline{a}_2, 0, 0]^T$.

Step D. Constructing the dynamical model

Because of the open dynamic chain concept, the dynamical model is constructed from the end effector to the base.

The gravity terms

The gravity vector in the local coordinate is:

$$ {}_0\underline{g} = [0, 0, -g]^T \qquad (3\text{-}111)$$

$$ {}_1\underline{g} = ({}_0A^1)^T {}_0\underline{g} = [0, 0, -g]^T \qquad (3\text{-}112)$$

$$ {}_2\underline{g} = ({}_0A^2)^T {}_0\underline{g} = g[\sin\theta_2, 0, -\cos\theta_2]^T \qquad (3\text{-}113)$$

The gravity force of joint 2 is:

$$ {}_2\underline{F}_g = {}_2\underline{F}'^2_g = M'^2 {}_2\underline{g} = M_2 g[\sin\theta_2, 0, -\cos\theta_2]^T \qquad (3\text{-}114)$$

The gravity force of joint 1 is:

$$ {}_1\underline{F}'^2_g = M'^2 {}_1\underline{g} = -M_2 g[0, 0, 1]^T $$

$$ {}_1\underline{F}'^1_g = M'^1 {}_1\underline{g} = -M_1 g[0, 0, 1]^T $$

$$ {}_1\underline{F}_g = {}_1\underline{F}'^2_g + {}_1\underline{F}'^1_g = -(M_1+M_2) g[0, 0, 1]^T \qquad (3\text{-}115)$$

The coordinates of the centers of mass are:

$$ {}_2\underline{p}'^2 = [-\tfrac{1}{2}d_2, 0, 0]^T \qquad (3\text{-}116)$$

$$ {}_1\underline{p}'^2 = [-\tfrac{1}{2}d_2\cos\theta_2, d_1, \tfrac{1}{2}d_2\sin\theta_2]^T \qquad (3\text{-}117)$$

$${}_{1}\underline{p}'^{1} = [\ 0,\ 0,\ 0]^T \qquad (3\text{-}118)$$

The gravity torque of joint 2 is:

$${}_{1}\underline{N}_g = {}_{2}\underline{N}'^{2}{}_g = M'^{2}\ {}_{2}\underline{p}'^{2} \times {}_{2}\underline{g}$$
$$= -\tfrac{1}{2}M^2 g d_2[0,\ \cos\emptyset_2,\ 0]^T \qquad (3\text{-}119)$$

The gravity torque of joint 1 is:

$${}_{1}\underline{N}'^{2}{}_g = M'^{2}\ {}_{1}\underline{p}'^{2} \times {}_{1}\underline{g}$$
$$= -\tfrac{1}{2}M_2 g[d_1,\ d_2\cos\emptyset_2,\ 0]^T$$
$${}_{1}\underline{N}'^{1}{}_g = M'^{1}\ {}_{1}\underline{p}'^{1} \times {}_{1}\underline{g} = [0,\ 0,\ 0]^T$$
$${}_{1}\underline{N}_g = {}_{1}\underline{N}'^{2}{}_g + {}_{1}\underline{N}'^{1}{}_g = -\tfrac{1}{2}M_2 g[d_1,\ d_2\cos\emptyset_2,\ 0]^T \qquad (3\text{-}120)$$

<u>The inertial terms</u>

Let:

$${}_{2}\underline{F}^{2,2} = M_2(\underline{a}_2 + \underline{\alpha}_2 \times \underline{p}_2)$$
$$= M_2 \begin{bmatrix} -a_2 \\ 0 \\ \tfrac{1}{2}\alpha_2 d_2 \end{bmatrix}$$

$${}_{2}\underline{F}^{1,2} = M_2\ {}_{2}A^1(\underline{\alpha}_1 \times {}_{1}\underline{p}'^{2})$$
$$= M_2 \begin{bmatrix} -\ \alpha_1 d_1 \cos\emptyset_2 \\ -\tfrac{1}{2}\alpha_1 d_2 \sin\emptyset_2 \\ -\ \alpha_1 d_1 \sin\emptyset_2 \end{bmatrix}$$

Then the inertial force of joint 2 is:

$$ {}_2\underline{F}_I = {}_2\underline{F}^{2,2} + {}_2\underline{F}^{1,2} $$

$$ = M_2 \begin{bmatrix} -a_2-\alpha_1 d_1 \cos\emptyset_2 \\ -\frac{1}{2}\alpha_1 d_2 \sin\emptyset_2 \\ \frac{1}{2}\alpha_2 d_2-\alpha_1 d_1 \sin\emptyset_2 \end{bmatrix} \qquad (3\text{-}121) $$

Similarly:

$$ {}_1\underline{F}^{2,2} = M_2 \; {}_1A^2(\underline{a}_2 + \underline{\alpha}_2 \times \underline{p}_2) $$

$$ = M_2 \begin{bmatrix} -a_2\cos\emptyset_2+\frac{1}{2}d_2\alpha_2\sin\emptyset_2 \\ 0 \\ a_2\sin\emptyset_2+\frac{1}{2}d_2\alpha_2\cos\emptyset_2 \end{bmatrix} $$

$$ {}_1\underline{F}^{1,2} = M_2 \; \underline{\alpha}_1 \times {}_1\underline{p}_{,2} $$

$$ = M_2 \begin{bmatrix} -d_1\alpha_1 \\ -\frac{1}{2}d_2\sin\emptyset_2\alpha_1 \\ 0 \end{bmatrix} $$

$$ {}_1\underline{F}^{1,1} = M_1 \; \underline{w}_1 \times \underline{p}_1 = \underline{0} $$

Then the inertial torque of joint 1 is:

$$ {}_1\underline{F}_I = {}_1\underline{F}^{2,2} + {}_1\underline{F}^{1,2} + {}_1\underline{F}^{1,1} $$

$$ = M_2 \begin{bmatrix} -a_2\cos\emptyset_2+\frac{1}{2}d_2\alpha_2\sin\emptyset_2-d_1\alpha_1 \\ -\frac{1}{2}d_2\sin\emptyset_2\alpha_1 \\ a_2\sin\emptyset_2+\frac{1}{2}d_2\alpha_2\cos\emptyset_2 \end{bmatrix} \qquad (3\text{-}122) $$

The inertial torque of joint 2 is calculated as follows:

$$_2\underline{N}^{2,2} = M_2(\underline{p}_2 \times \underline{a}_2) + I'^2\alpha_2$$

$$= \begin{bmatrix} 0 \\ I_{2yy}\alpha_2 \\ 0 \end{bmatrix}$$

$$_2\underline{N}^{1,2} = M_2(\underline{p}_2 \times {}_2A^1(\underline{\alpha}_1 \times {}_1\underline{d}^2)) + I'^2{}_2A_1\alpha_1$$

$$= \alpha_1 \begin{bmatrix} \tfrac{1}{2}M_2d_1d_2\cos\emptyset_2 - I_{2xx}\sin\emptyset_2 \\ \tfrac{1}{2}M_2d_1d_2\sin\emptyset_2 \\ \tfrac{1}{2}M_2(d_2)^2\cos\emptyset_2 + I_{2zz}\cos\emptyset_2 \end{bmatrix}$$

$$_2\underline{N}_I = {}_1\underline{N}^{2,2} + {}_1\underline{N}^{1,2}$$

$$= \begin{bmatrix} (\tfrac{1}{2}M_2d_1d_2\cos\emptyset_2 - I_{2xx}\sin\emptyset_2)\alpha_1 \\ I_{2yy}\alpha_2 + \tfrac{1}{2}M_2d_1d_2\sin\emptyset_2\alpha_1 \\ (\tfrac{1}{2}M_2d_1d_2\sin\emptyset_2 + I_{2zz}\cos\emptyset_2)\alpha_1 \end{bmatrix} \quad (3\text{-}123)$$

Similarly, the inertial torque of joint 1:

$$_1\underline{N}^{2,2} = M_2\ ({}_1\underline{p}_{,2} \times {}_1A^2\underline{a}_2)$$
$$+ M_2\ {}_1\underline{d}^2 \times {}_1A^2(\underline{\alpha}_2 \times \underline{p}_2)$$
$$+ {}_1A^2 I'^2{}_2A^1({}_1A^2\underline{\alpha}_2)$$

$$= \begin{bmatrix} M_2 d_1(a_2 \sin\emptyset_2 + \frac{1}{2} d_2 \alpha_2 \cos\emptyset_2) \\ -\frac{1}{2} M_2 (d_2)^2 \alpha_2 + I_{2yy}\alpha_2 \\ M_2 d_1 (a_2 \cos\emptyset_2 - \frac{1}{2} d_2 \alpha_2 \sin\emptyset_2) \end{bmatrix}$$

$$_1\underline{N}^{1,2} = M_2({}_1\underline{d}^2 \times (\underline{\alpha}_1 \times {}_1\underline{p}'^{,2}))$$
$$+M_2({}_1A^2\underline{p}_2 \times (\underline{\alpha}_1 \times {}_1\underline{d}^2))$$
$$+{}_1A^2 I'^{,2}\, {}_2A^1 \underline{\alpha}_1$$

$$= \alpha_1 \begin{bmatrix} (M_2(d_2)^2 + (I_{2zz} - I_{2xx}))\cos\emptyset_2 \sin\emptyset_2 \\ -3/2 M_2 d_1 d_2 \sin\emptyset_2 \\ M_2(d_1)^2 + I_{2xx}(\sin\emptyset_2)^2 + I_{2zz}(\cos\emptyset_2)^2 \end{bmatrix}$$

$$_1\underline{N}^{1,1} = I'^{,1}\alpha_1$$

$$= \begin{bmatrix} 0 \\ 0 \\ I_{1zz}\alpha_1 \end{bmatrix}$$

$$_1\underline{N}_I = {}_1\underline{N}^{2,2} + {}_1\underline{N}^{1,2} + {}_1\underline{N}^{1,1} \qquad (3\text{-}124)$$

$$= \begin{bmatrix} M_2 d_1(a_2 \sin\emptyset_2 + \frac{1}{2} d_2\alpha_2 \cos\emptyset_2) \\ \quad + \alpha_1\{(M_2(d_2)^2 + (I_{2zz} - I_{2xx}))\cos\emptyset_2 \sin\emptyset_2 \\ -\frac{1}{2} M_2(d_2)^2\alpha_2 + I_{2yy}\alpha_2 - 3/2 M_2 d_1 d_2 \sin\emptyset_2 \alpha_1 \\ M_2 d_1(a_2\cos\emptyset_2 - \frac{1}{2} d_2 \alpha_2 \sin\emptyset_2) + \{M_2(d_1)^2 \\ \quad + I_{2xx}(\sin\emptyset_2)^2 + I_{2zz}(\cos\emptyset_2)^2 + I_{1zz}\}\alpha_1 \end{bmatrix}$$

The centrifugal and Coriolis terms

The centrifugal and Coriolis terms can be computed in a similar manner. In this example, we assume that the robot is moving in slow speed, thus these two effects are insignificant.

CHAPTER 4

EQUIVALENCE BETWEEN THE NEWTON-EULER AND THE LAGRANIAN METHODS

In the previous section, the closed form dynamical model for the generalized rigid robot manipulator has been derived. In this section, we limit ourself to rigid manipulators that have only one degree of freedom per link. Joints that have more than one degree of freedom are decomposed into multiple of one degree of freedom joints, and each "subjoint" is described by one coordinate frame. This is done for the sake of properly identifying the generalized variables of the Lagrange formulation.

If the joint has only one degree of freedom, then only the component along the axis of motion is of interest. Thus the complexity of the dynamical model is reduced significantly. Also, since the links are rigid, we have either force or torque along the axis of motion of that link. That makes the dynamical model even simpler. If the link has a prismatic joint, the force component is of interest. If it has a revolute joint, the torque component is considered.

4.1 The Generalized Variables

Before reducing the closed form dynamical model of the previous chapter to the single degree of freedom per link case, let us define the generalized variables(needed for Lagrange formulation).

The selection vectors are chosen as follows:

$\underline{s} == [1, 0, 0]^T$ if x-component of the joint (4-1)

$\underline{s} == [0, 1, 0]^T$ if y-component of the joint (4-2)

$\underline{s} == [0, 0, 1]^T$ if z-component of the joint (4-3)

is selected.

The generalized angle of rotation is(fig.10):

$\underline{\delta} == \underline{0}$ if the joint/link is prismatic. (4-4)

$\underline{\delta} == A^T\underline{s}$ if the joint/link is revolute. (4-5)

The selection vector $\underline{s}$ specifies the axis of rotation of the original coodinate, and the vector $\underline{\delta}$ is the direction of the axis of rotation in the destination coordinate where A is the orientation matrix between the two coordinates.

The generalized displacement vector is(fig.11):

$\underline{\mu} == A^T\underline{s}$ if the joint/link is prismatic. (4-6)

$\underline{\mu} == A^T(\underline{s}X\underline{d})$ if the joint/link is revolute. (4-7)

If the joint is prismatic, then the selection vector $\underline{s}$ specifies the axis of translation of the original coodinate,

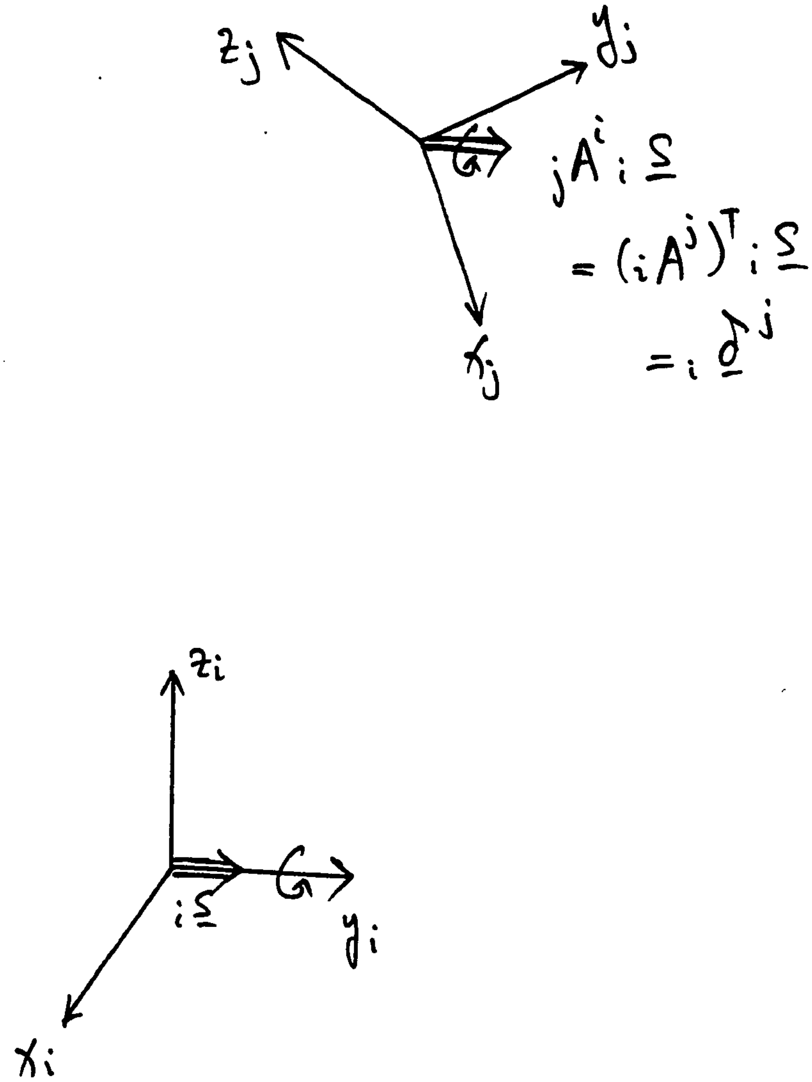


Figure 10 : The Generalized Angle of Rotation

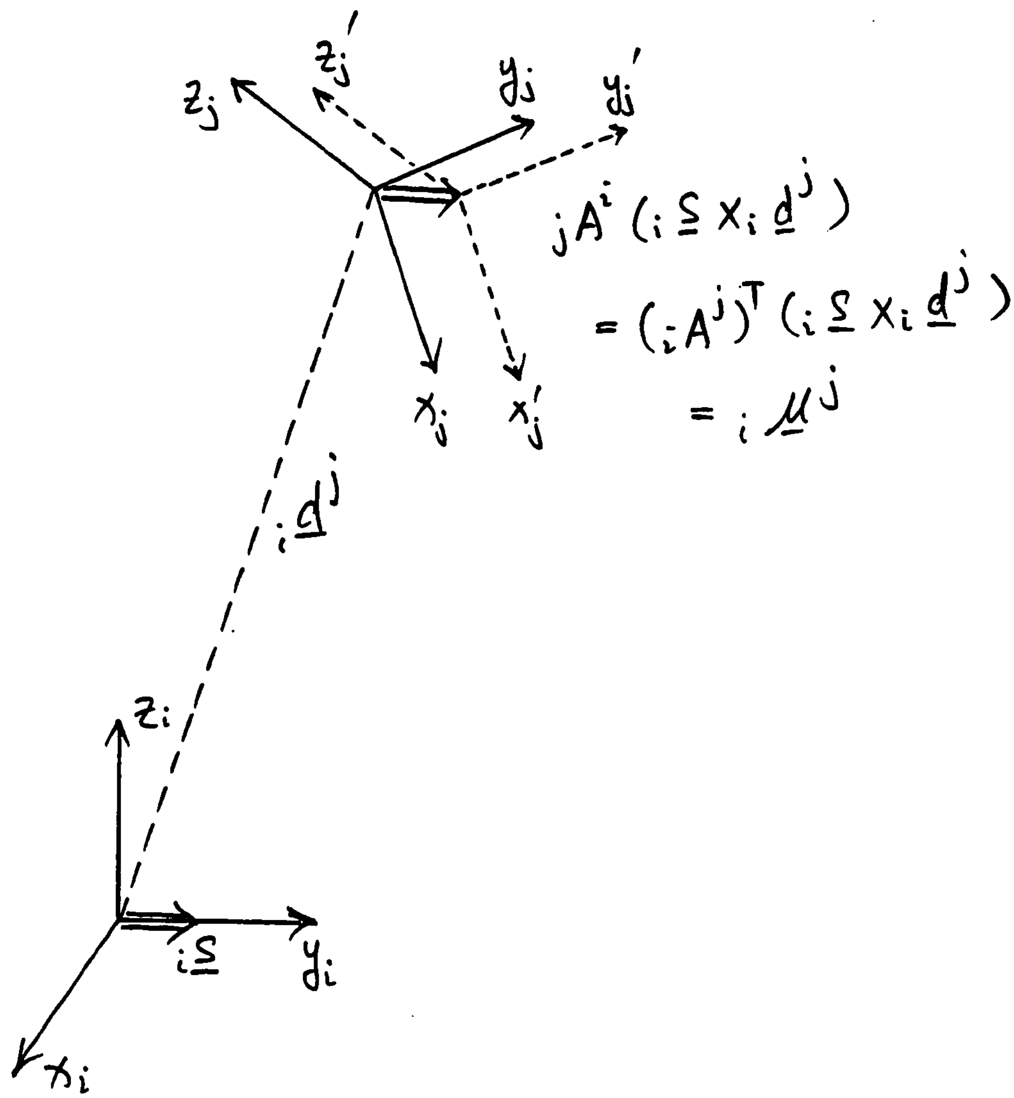


Figure 11 : The Generalized Displacement Vector

and the vector $\underline{\mu}$ is the direction of the axis of translation in the destination coordinate where A is the orientation matrix between the two coordinates. If the joint is revolute, then the selection vector $\underline{s}$ determines the axis of rotation, and the displacement equals to $\underline{s}$ X $\underline{d}$ where $\underline{d}$ is the displacement vector and A is the orientation matrix between the two coordinates.

The generalized velocity is:

$\tilde{n}$ == u$\underline{s}$ if the joint is prismatic. (4-8)

$\bar{n}$ == w$\underline{s}$ if the joint is revolute. (4-9)

The generalized velocity is defined as the component along the direction of the axis-of-motion.

The generalized acceleration is:

$\tilde{N}$ == a$\underline{s}$ if the joint is prismatic. (4-10)

$\tilde{N}$ == $\alpha\underline{s}$ if the joint is revolute. (4-11)

The generalized acceleration is defined as the component along the direction of the axis-of-motion.

The generalized force is:

Γ == F$\underline{s}$ if the joint is prismatic. (4-12)

Γ == N$\underline{s}$ if the joint is revolute. (4-13)

The generalized force is defined as the component along the direction of the axis-of-motion.

For the dynamical model derived by the Newton-Euler method, different mathematical expressions result when using different ordering of prismatic and revolute joints(when representing a multi-degree of freedom joint by a sequence of single degree of freedom joints), and each mathematical expression can be interpreted physically. However, when generalized coordinates(equations 4-1:4-13) are used, all combinations of prismatic and revolute joints lead to the same mathematical equation. In this "generalized" equation, the physical interpretation of the various terms is implicit and sometimes cannot be visualized.

4.2 Equivalence Between the Newton-Euler and the Lagrangian Methods

Lagrange formulation yields a unique dynmaical model. In this section, we shall prove that the dynamical model derived from the Newton-Euler method can be transformed into the same unique dynamical model. Hence, the Newton-Euler and Lagrangian methods are equivalent[Holl82].

In the following derivations, the superscripts of the prismatic and revolute velocity and acceleration will be mitted since they are always larger by one than the subscripts and the subscripts indicate the joints which are referred. For example, ${}_k\underline{W}^{k+1} = {}_k\underline{W}$, etc.

The inertial terms

If we assume that both joints k and i are revolute, then from equations 4-1:4-7, we obtain

$$ {}_i\underline{\delta}^j = ({}_iA^j)^T {}_i\underline{s} \qquad (4\text{-}14)$$

$$ {}_k\underline{\delta}^j = ({}_kA^j)^T {}_k\underline{s} \qquad (4\text{-}15)$$

$$ {}_i\underline{\mu}^j = ({}_iA^j)^T ({}_i\underline{s} \times {}_i\underline{d}^j) \qquad (4\text{-}16)$$

$$ {}_k\underline{\mu}^j = ({}_kA^j)^T ({}_k\underline{s} \times {}_k\underline{d}^j) \qquad (4\text{-}17)$$

In equation 3-96, both joints k and i are revolute, then the inertial torque is

$$ {}_i\underline{s}\ {}_iN^{k,j} = M^{,j} \{ {}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_k\underline{d}^j) + {}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^j{}_j\underline{p}^{,j}) + {}_iA^j{}_j\underline{p}^{,j} \times {}_iA^k({}_k\underline{s} \times {}_k\underline{d}^j) + {}_iA^jK^{,j}{}_jA^k{}_k\underline{s} \} {}_k\alpha \qquad (4\text{-}18)$$

where $k^{,j} == I^{,j} / M^{,j}$ (4-19)

is the unit mass inertial matrix of link j measured at the coordinate origin of joint j.

Using the definition of generalized force and acceleration, equation 4-19 can be written as:

$$ {}_i\Gamma^{k,j} = D_{ik}{}^{,j}\ {}_k\ddot{N} \qquad (4\text{-}20)$$

where $D_{ik}{}^{,j} = M^{,j} \{ {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_k\underline{d}^j)) + {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^j{}_j\underline{p}^{,j}))$ (4-21)

$$+{}_i\underline{s} \bullet ({}_iA^j\,{}_j\underline{p}^{,j} \times {}_iA^k({}_k\underline{s} \times {}_k\underline{d}^j))$$
$$+{}_i\underline{s}^T\,{}_iA^j K^{,j}\,{}_jA^k\,{}_k\underline{s} \qquad \}$$

The fourth term of equation 4-21 is:

$${}_i\underline{s}^T\,{}_iA^j K^{,j}\,{}_jA^k\,{}_k\underline{s}$$
$$= (({}_iA^j)^T\,{}_i\underline{s})^T\; K^{,j}\; (({}_kA^j)^T\,{}_k\underline{s})$$
$$= ({}_i\underline{\delta}^j)^T\; K^{,j}\; ({}_k\underline{\delta}^j) \qquad (4\text{-}22)$$

The first term of equation 4-21 is:

$${}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s}\;\; {}_k\underline{d}^j))$$
$$= {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^j({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j))$$
$$= {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^j\;{}_k\underline{\mu}^j)$$
$$= ({}_iA^j\,{}_k\underline{\mu}^j) \bullet ({}_i\underline{s} \times {}_i\underline{d}^j)$$
$$= ({}_k\underline{\mu}^j)^T({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}^j)$$
$$= ({}_k\underline{\mu}^j)^T({}_i\underline{\mu}^j)$$
$$= {}_k\underline{\mu}^j \bullet {}_i\underline{\mu}^j \qquad (4\text{-}23)$$

where equations 2-44, 2-46 and 2-85 have been used and they will be used very often in the following derivations as well:

The second term of the equation 4-21 is:

$$= {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^j\,{}_j\underline{p}^{,j}))$$
$$= {}_iA^k({}_k\underline{s} \times {}_kA^j\,{}_j\underline{p}^{,j}) \bullet ({}_i\underline{s} \times {}_i\underline{d}^j)$$
$$= {}_iA^j({}_kA^j)^T({}_k\underline{s} \times {}_kA^j\,{}_j\underline{p}^{,j}) \bullet {}_iA^j({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}$$
$$= ({}_kA^j)^T({}_k\underline{s} \times {}_kA^j\,{}_j\underline{p}^{,j}) \bullet ({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}^j)$$

$$= (({}_kA^j)^T {}_k\underline{s} \times {}_j\underline{p}'^{,j}) \bullet ({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}^j)$$
$$= ({}_k\underline{\delta}^j \times {}_j\underline{p}'^{,j}) \bullet {}_i\underline{\mu}^j$$
$$= {}_j\underline{p}'^{,j} \bullet ({}_i\underline{\mu}^j \times {}_k\underline{\delta}^j) \qquad (4\text{-}24)$$

The third term of equation 4-21 is:

$$= {}_i\underline{s} \bullet ({}_iA^j {}_j\underline{p}'^{,j} \times {}_iA^k({}_k\underline{s} \times {}_k\underline{d}^j))$$
$$= {}_iA^j({}_iA^j)^T {}_i\underline{s} \bullet ({}_iA^j {}_j\underline{p}'^{,j} \times {}_iA^j({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j))$$
$$= ({}_iA^j)^T {}_i\underline{s} \bullet ({}_j\underline{p}'^{,j} \times ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j))$$
$$= {}_i\underline{\delta}^j \bullet ({}_j\underline{p}'^{,j} \times {}_k\underline{\mu}^j)$$
$$= {}_j\underline{p}'^{,j} \bullet ({}_k\underline{\mu}^j \times {}_i\underline{\delta}^j) \qquad (4\text{-}25)$$

After combining equations 4-22:4-25, equation 4-21 becomes

$$D_{ik}{}'^{,j} = M'^{,j} \{ ({}_i\underline{\delta}^j)^T K'^{,j} ({}_k\underline{\delta}^j) + {}_k\underline{\mu}^j \bullet {}_i\underline{\mu}^j + {}_j\underline{p}'^{,j} \bullet ({}_i\underline{\mu}^j \times {}_k\underline{\delta}^j + {}_k\underline{\mu}^j \times {}_i\underline{\delta}^j) \} \qquad (4\text{-}26)$$

Assume that both joints k & i are prismatic, then from equations 4-1:4-7, we obtain

$${}_i\underline{\delta}^j = \underline{0} \qquad (4\text{-}27)$$
$${}_k\underline{\delta}^j = \underline{0} \qquad (4\text{-}28)$$
$${}_i\underline{\mu}^j = ({}_iA^j)^T {}_i\underline{s} \qquad (4\text{-}29)$$
$${}_k\underline{\mu}^j = ({}_kA^j)^T {}_k\underline{s} \qquad (4\text{-}30)$$

In equation 3-87, if both joints k & i are prismatic, then the inertial force is:

$${}_i\underline{s}\ {}_iF^{k,j} = M'^{,j}\ {}_iA^k\ {}_k\underline{s}\ {}_ka$$
$$\Rightarrow \quad {}_i\Gamma^{k,j} = D_{ik}{}'^{,j}\ {}_k\ddot{N} \qquad (4\text{-}31)$$

where $D_{ik}{}^{,j} = M^{,j}\ \{({}_i\underline{s})^T)\ {}_iA^k\ {}_k\underline{s}\}$

$= M^{,j}\ \{[({}_iA^j)^T\ {}_i\underline{s}]^T[({}_kA^j)^T\ {}_k\underline{s}]\}$

$= M^{,j}\ \{({}_i\underline{\mu}^j)^T\ {}_k\underline{\mu}^j\}$

$= M^{,j}\ \{{}_i\underline{\mu}^j \bullet {}_k\underline{\mu}^j\}$ (4-32)

Using the definitions of 4-27:4-30, equations 4-32 & 4-32 equal to equations 4-20 & 4-26 respectively but the definition of the terms is different.

Assume that joint k is revolute and joint i is prismatic, then from equations 4-1:4-7, we obtain

$${}_i\underline{\delta}^j = \underline{0} \quad (4\text{-}33)$$

$${}_k\underline{\delta}^j = ({}_kA^j)^T\ {}_k\underline{s} \quad (4\text{-}34)$$

$${}_i\underline{\mu}^j = ({}_iA^j)^T\ {}_i\underline{s} \quad (4\text{-}35)$$

$${}_k\underline{\mu}^j = ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j) \quad (4\text{-}36)$$

In equation 3-87, if joints k is revolute and joint i is prismatic, then the inertial force is:

$${}_i\underline{s}\ {}_iF^{k,j} = M^{,j}\ \{{}_iA^k({}_k\underline{s} \times {}_k\underline{p}^{,j})\}\ {}_k\alpha$$

=> $${}_i\Gamma^{k,j} = D_{ik}{}^{,j}\ {}_k\ddot{N} \quad (4\text{-}37)$$

where $D_{ik}{}^{,j} = M^{,j}\ \{\ {}_i\underline{s} \bullet {}_iA^k({}_k\underline{s} \times {}_k\underline{p}^{,j})\}$

$= M^{,j}\ \{{}_i\underline{s}\bullet{}_iA^k({}_k\underline{s}\times{}_k\underline{d}^j) + {}_i\underline{s}\bullet{}_iA^k({}_k\underline{s}\times{}_kA^j\ {}_j\underline{p}^{,j})\}$

$= M^{,j}\ \{\ ({}_iA^j)^T\ {}_i\underline{s} \bullet ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j)$

$+({}_iA^j)^T\ {}_i\underline{s} \bullet(({}_kA^j)^T\ {}_k\underline{s} \times {}_j\underline{p}^{,j})\ \}$

$= M^{,j}\ \{\ {}_i\underline{\mu}^j \bullet {}_k\underline{\mu}^j + {}_j\underline{p}^{,j} \bullet ({}_i\underline{\mu}^j \times {}_k\underline{\delta}^j)\ \}$ (4-38)

With the defintions of 4-33:4-36, equations 4-37 & 4-38 equal to equations 4-20 & 4-26 respectively.

Assume that joint k is prismatic and joint i is revolute, then from equations 4-1:4-7, we obtain

$${}_i\underline{\delta}^j = ({}_iA^j)^T {}_i\underline{s} \tag{4-39}$$

$${}_k\underline{\delta}^j = \underline{0} \tag{4-40}$$

$${}_i\underline{\mu}^j = ({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}^j) \tag{4-41}$$

$${}_k\underline{\mu}^j = ({}_kA^j)^T {}_k\underline{s} \tag{4-42}$$

In equation 3-96, if joints k is prismatic and joint i is revolute, then the inertial torque is:

$${}_i\underline{s}\ {}_iN^{k,j} = M^{,j}\ \{\ {}_i\underline{p}^{,j} \times {}_iA^k{}_k\underline{s}\ \}\ {}_ka$$

$$\Rightarrow \quad {}_i\Gamma^{k,j} = D_{ik}{}^{,j}\ {}_k\ddot{q} \tag{4-43}$$

where
$$\begin{aligned} D_{ik}{}^{,j} &= M^{,j}\ \{\ {}_i\underline{s} \bullet ({}_i\underline{p}^{,j} \times {}_iA^k{}_k\underline{s})\ \} \\ &= M^{,j}\ \{{}_i\underline{s}\bullet({}_i\underline{d}^j \times {}_iA^k{}_k\underline{s}) + {}_i\underline{s}\bullet({}_iA^j{}_j\underline{p}^{,j} \times {}_iA^k{}_k\underline{s})\} \\ &= M^{,j}\ \{{}_iA^k{}_k\underline{s}\bullet({}_i\underline{s} \times {}_i\underline{d}^j) + ({}_iA^j)^T{}_i\underline{s}\bullet({}_j\underline{p}^{,j} \times ({}_kA^j)^T{}_k\underline{s})\} \\ &= M^{,j}\ \{({}_kA^j)^T{}_k\underline{s}\bullet({}_iA^j)^T({}_i\underline{s} \times {}_i\underline{d}^j) + {}_i\underline{\delta}^j\bullet({}_j\underline{p}^{,j} \times {}_k\underline{\mu}^j)\} \\ &= M^{,j}\ \{\ {}_k\underline{\mu}^j \bullet {}_i\underline{\mu}^j + {}_j\underline{p}^{,j} \bullet ({}_k\underline{\mu}^j \times {}_i\underline{\delta}^j)\ \} \end{aligned} \tag{4-44}$$

With the definitions of 4-39:4-42, equations 4-43 & 4-44 equal to equations 4-20 & 4-26 respectively.

In conclusion, the following equations work for any combination of joint k and i, providing the generalized terms(4-1:4-7) are used accordingly.

$$ {}_{i}\Gamma^{k,j} = D_{ik}{}^{,j}\ {}_{k}\ddot{N} \qquad (4\text{-}45)$$

$$ D_{ik}{}^{,j} = M^{,j}\ \{\ ({}_{i}\underline{\delta}^{j})^{T}\ K^{,j}\ ({}_{k}\underline{\delta}^{j}) + {}_{k}\underline{\mu}^{j} \bullet {}_{i}\underline{\mu}^{j} $$
$$ + {}_{j}\underline{p}^{,j} \bullet ({}_{i}\underline{\mu}^{j} \times {}_{k}\underline{\delta}^{j} + {}_{k}\underline{\mu}^{j} \times {}_{i}\underline{\delta}^{j})\ \} \qquad (4\text{-}46)$$

where $K^{,j} = [tr({}_{j}\underline{r}^{,j}({}_{j}\underline{r}^{,j})^{T} - {}_{j}\underline{r}^{,j}({}_{j}\underline{r}^{,j})^{T}] \ /\ M^{,j}$ (4-47)

In equation 4-45, if both joints k & i are revolute, then the first term is due to the arbitrary angle of link j; the second term is due to the "parallel axes theorem"; the last two terms are due to the fact that the center of mass of link j does not coincide with the coordinate origin of joint j. If joints k & i are in different combinations, the explanation of the terms is not obvious since the definitions of the variables are altered.

<u>The Coriolis terms</u>

In computing the Coriolis force and torque(equations 2-82:2-83), joint k is always taken to be revolute. Let joints l & i be both revolute, then according to equations 4-1:4-7, we obtain:

$$ {}_{i}\underline{\delta}^{j} = ({}_{i}A^{j})^{T}\ {}_{i}\underline{s} \qquad (4\text{-}48)$$

$$ {}_{k}\underline{\delta}^{j} = ({}_{k}A^{j})^{T}\ {}_{k}\underline{s} \qquad (4\text{-}49)$$

$$ {}_{l}\underline{\delta}^{j} = ({}_{l}A^{j})^{T}\ {}_{l}\underline{s} \qquad (4\text{-}50)$$

$$ {}_{i}\underline{\mu}^{j} = ({}_{i}A^{j})^{T}({}_{i}\underline{s} \times {}_{i}\underline{d}^{j}) \qquad (4\text{-}51)$$

$${}_k\underline{\mu}^j = ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j) \qquad (4\text{-}52)$$

$${}_l\underline{\mu}^j = ({}_lA^j)^T({}_l\underline{s} \times {}_l\underline{d}^j) \qquad (4\text{-}53)$$

If both joints l & i are revolute, according to equation 3-98a, the Coriolis torque is:

$$\begin{aligned}
{}_i\underline{s}\ {}_iN^{kl,j} = 2M'^j \{ & \\
& {}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j)) \\
& + {}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_lA^j{}_j\underline{p}'^j)) \\
& + {}_iA^j{}_j\underline{p}'^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j)) \\
& + [{}_iA^jK'^j({}_iA^j)^T - \tfrac{1}{2}tr(K'^j)]({}_iA^l{}_l\underline{s} \times {}_iA^k{}_k\underline{s}) \\
& + ({}_iA^l{}_l\underline{s}) \times \{[{}_iA^jK'^j({}_iA^j)^T]({}_iA^k{}_k\underline{s}) \quad \}\ {}_kw\ {}_lw
\end{aligned}$$

hence $\quad {}_i\Gamma^{kl,j} = 2\ D_{ikl}{}'^j\ {}_k\bar{n}\ {}_l\bar{n}$ (4-54)

where $D_{ikl}{}'^j = M'^j \{$ (4-55)

$$\begin{aligned}
& {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j))) \\
& + {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_lA^j{}_j\underline{p}'^j))) \\
& + {}_i\underline{s} \bullet ({}_iA^j{}_j\underline{p}'^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j)) \\
& + {}_i\underline{s} \bullet ([{}_iA^jK'^j({}_iA^j)^T - \tfrac{1}{2}tr(K'^j)]({}_iA^l{}_l\underline{s} \times {}_iA^k{}_k\underline{s})) \\
& + {}_i\underline{s} \bullet (({}_iA^l{}_l\underline{s}) \times ([{}_iA^jK'^j({}_iA^j)^T]({}_iA^k{}_k\underline{s})) \qquad \}
\end{aligned}$$

In equation 4-55, the last term is:

$$\begin{aligned}
& {}_i\underline{s} \bullet (({}_iA^l{}_l\underline{s}) \times \{[{}_iA^jK'^j({}_iA^j)^T]({}_iA^k{}_k\underline{s})) \\
= {} & {}_iA^j({}_iA^j)^T{}_i\underline{s} \bullet [({}_iA^j({}_lA^j)^T{}_l\underline{s}) \times [{}_iA^jK'^j({}_iA^j)^T]({}_iA \\
= {} & ({}_iA^j)^T{}_i\underline{s} \bullet [({}_lA^j)^T{}_l\underline{s}) \times K'^j(({}_kA^j)^T{}_k\underline{s})] \\
= {} & {}_i\underline{\delta}^j \bullet ({}_l\underline{\delta}^j \times K'^j{}_k\underline{\delta}^j) \qquad (4\text{-}56)
\end{aligned}$$

The fourth term of equation 4-55 is:

$$
\begin{aligned}
& {}_i\underline{s} \bullet ([{}_iA^j K^{,j}({}_iA^j)^T - \tfrac{1}{2}tr(K^{,j})]({}_iA^l{}_l\underline{s} \times {}_iA^k{}_k\underline{s})) \\
&= ({}_iA^j)^T{}_i\underline{s} \bullet [K^{,j} - \tfrac{1}{2}tr(K^{,j})](({}_lA^j)^T{}_l\underline{s} \times ({}_kA^j)^T{}_k\underline{s}) \\
&= {}_i\underline{\delta}^j \bullet [K^{,j} - \tfrac{1}{2}tr(K^{,j})]({}_l\underline{\delta}^j \times {}_k\underline{\delta}^j) \qquad (4\text{-}57)
\end{aligned}
$$

The first term of equation 4-55 is:

$$
\begin{aligned}
& {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j))) \\
&= [{}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j))] \bullet ({}_i\underline{s} \times {}_i\underline{d}^j) \\
&= [({}_kA^j)^T{}_k\underline{s} \times ({}_lA^j)^T({}_l\underline{s} \times {}_l\underline{d}^j)] \bullet ({}_iA^j)^T({}_i\underline{s} \times {}_i\dot{\underline{d}}^j) \\
&= ({}_k\underline{\delta}^j \times {}_l\underline{\mu}^j) \bullet {}_i\underline{\mu}^j \\
&= {}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j) \qquad (4\text{-}58)
\end{aligned}
$$

The second term of equation 4-55 is:

$$
\begin{aligned}
& {}_i\underline{s} \bullet ({}_i\underline{d}^j \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_lA^j{}_j\underline{p}^{,j}))) \\
&= [{}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_lA^j{}_j\underline{p}^{,j}))] \bullet ({}_i\underline{s} \times {}_i\underline{d}^j) \\
&= [({}_kA^j)^T{}_k\underline{s} \times (({}_lA^j)^T{}_l\underline{s} \times {}_j\underline{p}^{,j})] \bullet ({}_iA^j)^T({}_i\underline{s} \times {}_i\dot{\underline{d}}^j) \\
&= ({}_k\underline{\delta}^j \times ({}_l\underline{\delta}^j \times {}_j\underline{p}^{,j})) \bullet {}_i\underline{\mu}^j \\
&= {}_j\underline{p}^{,j} \bullet ({}_l\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j)) \qquad (4\text{-}59)
\end{aligned}
$$

The third term of equation 4-55 is:

$$
\begin{aligned}
& {}_i\underline{s} \bullet ({}_iA^j{}_j\underline{p}^{,j} \times {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{d}^j)) \\
&= ({}_iA^j)^T{}_i\underline{s} \bullet ({}_j\underline{p}^{,j} \times (({}_kA^j)^T{}_k\underline{s} \times ({}_lA^j)^T({}_l\underline{s} \times {}_l\dot{\underline{d}}^j))) \\
&= {}_i\underline{\delta}^j \bullet ({}_j\underline{p}^{,j} \times ({}_k\underline{\delta}^j \times {}_l\underline{\mu}^j)) \\
&= {}_j\underline{p}^{,j} \bullet ({}_i\underline{\delta}^j \times ({}_l\underline{\mu}^j \times {}_k\underline{\delta}^j)) \qquad (4\text{-}60)
\end{aligned}
$$

After substituting equations 4-56:4-60 into equation

4-55, we obtain

$$D_{ikl}{}^{,j} = M^{,j} \{ {}_i\underline{\delta}^j \bullet ({}_l\underline{\delta}^j \times K^{,j}\,{}_k\underline{\delta}^j) \qquad (4\text{-}61)$$
$$+{}_i\underline{\delta}^j \bullet [K^{,j} - \tfrac{1}{2}tr(K^{,j})]({}_l\underline{\delta}^j \times {}_k\underline{\delta}^j)$$
$$+{}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j)$$
$$+{}_j\underline{p}^{,j} \bullet ({}_l\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j))$$
$$+{}_j\underline{p}^{,j} \bullet ({}_i\underline{\delta}^j \times ({}_l\underline{\mu}^j \times {}_k\underline{\delta}^j)) \quad \}$$

Now assume that both joints l & i are prismatic and joint k is revolute, then

$${}_i\underline{\delta}^j = \underline{0} \qquad (4\text{-}62)$$
$${}_k\underline{\delta}^j = ({}_kA^j)^T\,{}_k\underline{s} \qquad (4\text{-}63)$$
$${}_l\underline{\delta}^j = \underline{0} \qquad (4\text{-}64)$$
$${}_i\underline{\mu}^j = ({}_iA^j)^T\,{}_i\underline{s} \qquad (4\text{-}65)$$
$${}_k\underline{\mu}^j = ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j) \qquad (4\text{-}66)$$
$${}_l\underline{\mu}^j = ({}_lA^j)^T\,{}_l\underline{s} \qquad (4\text{-}67)$$

If both joints l & i are prismatic and joint k is revolute, then from equation 3-89, we get:

$${}_i\underline{s} \bullet {}_iF^{kl,j} = 2M^{,j}\,{}_iA^k({}_k\underline{s} \times {}_kA^l\,{}_l\underline{s})\ {}_kw\ {}_lu$$

$$\Rightarrow \quad {}_i\Gamma^{kl,j} = 2\,D_{ijk}{}^{,j}\ {}_k\bar{n}\ {}_l\bar{n} \qquad (4\text{-}68)$$

where
$$D_{ikl}{}^{,j} = M^{,j}\{{}_i\underline{s} \bullet {}_iA^k({}_k\underline{s} \times {}_kA^l\,{}_l\underline{s})\}$$
$$= M^{,j}\{({}_iA^j)^T\,{}_i\underline{s} \bullet (({}_kA^j)^T\,{}_k\underline{s} \times ({}_lA^j)^T\,{}_l\underline{s}\}$$
$$= M^{,j}\{\ {}_i\underline{\mu}^j \bullet ({}_k\underline{\delta}^j \times {}_l\underline{\mu}^j)\ \}$$
$$= M^{,j}\{\ {}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j)\ \} \qquad (4\text{-}69)$$

Under definitions 4-62:4-67, equations 4-68 & 4-69 equal to equations 4-54 & 4-61 respectively.

Let joint i be prismatic and joints k & l be revolute, then

$$ {}_i\underline{\delta}^j = \underline{0} \tag{4-70}$$
$$ {}_k\underline{\delta}^j = ({}_kA^j)^T {}_k\underline{s} \tag{4-71}$$
$$ {}_l\underline{\delta}^j = ({}_lA^j)^T {}_l\underline{s} \tag{4-72}$$
$$ {}_i\underline{\mu}^j = ({}_iA^j)^T {}_i\underline{s} \tag{4-73}$$
$$ {}_k\underline{\mu}^j = ({}_kA^j)^T({}_k\underline{s} \times {}_k\underline{d}^j) \tag{4-74}$$
$$ {}_l\underline{\mu}^j = ({}_lA^j)^T({}_l\underline{s} \times {}_l\underline{d}^j) \tag{4-75}$$

If joint i is prismatic and both joints k & l are revolute, then from equation 3-89, the Coriolis force is:

$$ {}_i\underline{s}\ {}_iF^{kl,j} = 2M^{,j}\ {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{p}^{,j}))\ {}_kw\ {}_lw $$

$$ \Rightarrow \quad {}_i\Gamma^{kl,j} = 2\, D_{ikl}{}^{,j}\ {}_k\bar{n}\ {}_l\bar{n} \tag{4-76}$$

where
$$ \begin{aligned} D_{ikl}{}^{,j} &= M^{,j}\ \{{}_i\underline{s} \bullet {}_iA^k({}_k\underline{s} \times {}_kA^l({}_l\underline{s} \times {}_l\underline{p}^{,j}))\} \\ &= M^{,j}\ \{({}_iA^j)^T {}_i\underline{s} \bullet (({}_kA^j)^T {}_k\underline{s} \times ({}_lA^j)^T({}_l\underline{s} \times {}_l\underline{d}^j)) \\ &\qquad + ({}_iA^j)^T {}_i\underline{s} \bullet (({}_kA^j)^T {}_k\underline{s} \times (({}_lA^j)^T {}_l\underline{s} \times {}_j\underline{p}^{,j}))\} \\ &= M^{,j}\ \{\ {}_i\underline{\mu}^j \bullet ({}_k\underline{\delta}^j \times {}_l\underline{\mu}^j) \\ &\qquad + {}_i\underline{\mu}^j \bullet ({}_k\underline{\delta}^j \times ({}_l\underline{\delta}^j \times {}_j\underline{p}^{,j}))\ \} \\ &= M^{,j}\ \{\ {}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j) \\ &\qquad + {}_j\underline{p}^{,j} \bullet ({}_l\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j))\ \} \end{aligned} \tag{4-77}$$

Under definitions 4-70:4-75, equations 4-76 & 4-77 equal to equations 4-54 & 4-61 respectively.

Finally, assume that joint l is prismatic and joints k & i are revolute, then

$${}_{i}\underline{\delta}^{j} = ({}_{i}A^{j})^{T}\,{}_{i}\underline{s} \qquad (4\text{-}78)$$

$${}_{k}\underline{\delta}^{j} = ({}_{k}A^{j})^{T}\,{}_{k}\underline{s} \qquad (4\text{-}79)$$

$${}_{l}\underline{\delta}^{j} = \underline{0} \qquad (4\text{-}80)$$

$${}_{i}\underline{\mu}^{j} = ({}_{i}A^{j})^{T}({}_{i}\underline{s} \times {}_{i}\underline{d}^{j}) \qquad (4\text{-}81)$$

$${}_{k}\underline{\mu}^{j} = ({}_{k}A^{j})^{T}({}_{k}\underline{s} \times {}_{k}\underline{d}^{j}) \qquad (4\text{-}82)$$

$${}_{l}\underline{\mu}^{j} = ({}_{l}A^{j})^{T}\,{}_{l}\underline{s} \qquad (4\text{-}83)$$

Since joint l is prismatic and both joints k & i are revolute, then from equation 3-98, the Coriolis torque is:

$${}_{i}\underline{s}\;{}_{i}N^{kl,j} = 2M^{,j}\;({}_{i}\underline{p}^{,j} \times {}_{i}A^{k}({}_{k}\underline{s} \times {}_{k}A^{l}\,{}_{l}\underline{s}))\;{}_{k}w\;{}_{l}u$$

$$\Rightarrow \quad {}_{i}\Gamma^{kl,j} = 2\,D_{ikl}{}^{,j}\;{}_{k}\bar{n}\;{}_{l}\bar{n} \qquad (4\text{-}84)$$

$$\begin{aligned}
\text{where } D_{ikl}{}^{,j} &= M^{,j}\,\{{}_{i}\underline{s} \bullet ({}_{i}\underline{p}^{,j} \times {}_{i}A^{k}({}_{k}\underline{s} \times {}_{k}A^{l}\,{}_{l}\underline{s}))\} \\
&= M^{,j}\,\{[{}_{i}A^{k}({}_{k}\underline{s} \times {}_{k}A^{l}\,{}_{l}\underline{s})] \bullet ({}_{i}\underline{s} \times ({}_{i}\underline{d}^{j} + {}_{i}A^{j}\,{}_{j}\underline{p}^{,j}))\} \\
&= M^{,j}\,\{[({}_{k}A^{j})^{T}\,{}_{k}\underline{s} \times ({}_{l}A^{j})^{T}\,{}_{l}\underline{s}] \bullet [({}_{i}A^{j})^{T}({}_{i}\underline{s} \times {}_{i}\underline{d}^{j})] \\
&\qquad + ({}_{k}A^{j})^{T}\,{}_{k}\underline{s} \times ({}_{l}A^{j})^{T}\,{}_{l}\underline{s}] \bullet (({}_{i}A^{j})^{T}\,{}_{i}\underline{s} \times {}_{j}\underline{p}^{,j})\} \\
&= M^{,j}\,\{\, ({}_{k}\underline{\delta}^{j} \times {}_{l}\underline{\mu}^{j}) \bullet {}_{i}\underline{\mu}^{j} \\
&\qquad + ({}_{k}\underline{\delta}^{j} \times {}_{l}\underline{\mu}^{j}) \bullet ({}_{i}\underline{\delta}^{j} \times {}_{j}\underline{p}^{,j})\,\} \\
&= M^{,j}\,\{\, {}_{k}\underline{\delta}^{j} \bullet ({}_{l}\underline{\mu}^{j} \times {}_{i}\underline{\mu}^{j}) \qquad (4\text{-}85) \\
&\qquad + {}_{j}\underline{p}^{,j} \bullet ({}_{i}\underline{\delta}^{j} \times ({}_{l}\underline{\mu}^{j} \times {}_{k}\underline{\delta}^{j}))\,\}
\end{aligned}$$

Under defintions 4-78:4-83, equations 4-84 & 4-85 equal to equations 4-54 & 4-61 respectively.

So no matter what kind of combination of joint l & i, we still get the same equation for computing the Coriolis force or torque if the generalized variables(4-1:4-7) are applied correctly. The equation is:

$$ {}_i\Gamma^{kl,j} = 2\, D_{ikl}{}^{,j}\ {}_k\bar{n}\ {}_l\bar{n} \tag{4-86}$$

where

$$\begin{aligned} D_{ikl}{}^{,j} = M^{,j}\ \{\ & {}_i\underline{\delta}^j \bullet ({}_l\underline{\delta}^j \times K^{,j}\,{}_k\underline{\delta}^j) \\ & + {}_i\underline{\delta}^j \bullet [K^{,j} - \tfrac{1}{2}tr(K^{,j})]({}_l\underline{\delta}^j \times {}_k\underline{\delta}^j) \\ & + {}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j) \\ & + {}_j\underline{p}^{,j} \bullet ({}_l\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j)) \\ & + {}_j\underline{p}^{,j} \bullet ({}_i\underline{\delta}^j \times ({}_l\underline{\mu}^j \times {}_k\underline{\delta}^j)) \quad \} \end{aligned} \tag{4-87}$$

<u>The centrifugal terms</u>

The centrifugal terms are a special case of the Coriolis terms. If we let k=l and divide the equation by 2 of the Coriolis terms, we get the expression for the centrifugal terms.

$$ {}_i\Gamma^{kl,j} = D_{ikk}{}^{,j}\ {}_k\bar{n}\ {}_k\bar{n} \tag{4-88}$$

where

$$\begin{aligned} D_{ikk}{}^{,j} = M^{,j}\ \{\ & {}_i\underline{\delta}^j \bullet ({}_k\underline{\delta}^j \times K^{,j}\,{}_k\underline{\delta}^j) \\ & + {}_k\underline{\delta}^j \bullet ({}_k\underline{\mu}^j \times {}_i\underline{\mu}^j) \\ & + {}_j\underline{p}^{,j} \bullet ({}_k\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j)) \\ & + {}_j\underline{p}^{,j} \bullet ({}_i\underline{\delta}^j \times ({}_k\underline{\mu}^j \times {}_k\underline{\delta}^j)) \quad \} \end{aligned} \tag{4-89}$$

and the variables are defined according to equations 4-1:4-7.

<u>The gravity term</u>

If joint i is revolute, then

$$ {}_{i}\underline{\delta}^{j} = ({}_{i}A^{j})^{T}\,{}_{i}\underline{s} \tag{4-90}$$

$$ {}_{i}\underline{\mu}^{j} = ({}_{i}A^{j})^{T}({}_{i}\underline{s} \times {}_{i}\underline{d}^{j}) \tag{4-91}$$

Referring to equation 3-99, the gravity torque is:

$$ {}_{i}\underline{s}\;{}_{i}N'^{j}_{g} = M'^{j}\;{}_{i}\underline{p}'^{j} \times {}_{i}\underline{g} $$

$$ \Rightarrow \quad {}_{i}\Gamma'^{j}_{g} = D_{i}'^{j} \tag{4-92}$$

$$\begin{aligned}
\text{where } D_{i}'^{j} &= M'^{j}\;{}_{i}\underline{s} \bullet ({}_{i}\underline{p}'^{j} \times {}_{i}\underline{g}) \\
&= M'^{j}\;{}_{i}\underline{g} \bullet ({}_{i}\underline{s} \times ({}_{i}\underline{d}^{j} + {}_{i}A^{j}\,{}_{j}\underline{p}'^{j})) \\
&= M'^{j}\;{}_{i}\underline{g} \bullet {}_{i}A^{j}[({}_{i}A^{j})^{T}({}_{i}\underline{s} \times {}_{i}\underline{d}^{j}) \\
&\qquad + ({}_{i}A^{j})^{T}\,{}_{i}\underline{s} \times {}_{j}\underline{p}'^{j})] \\
&= M'^{j}\;{}_{i}\underline{g} \bullet {}_{i}A^{j}({}_{i}\underline{\mu}^{j} + {}_{i}\underline{\delta}^{j} \times {}_{j}\underline{p}'^{j})
\end{aligned} \tag{4-93}$$

If joint i is prismatic, then

$$ {}_{i}\underline{\delta}^{j} = \underline{0} \tag{4-94}$$

$$ {}_{i}\underline{\mu}^{j} = ({}_{i}A^{j})^{T}\,{}_{i}\underline{s} \tag{4-95}$$

According to equation 3-89, the gravity force is:

$$ {}_{i}\underline{s}\;{}_{i}F'^{j}_{g} = M'^{j}\;{}_{i}\underline{g} $$

$$ \Rightarrow \quad {}_{i}\Gamma'^{j}_{g} = D_{i}'^{j} \tag{4-96}$$

$$\begin{aligned}
\text{where } D_{i}'^{j} &= M'^{j}\;{}_{i}\underline{s} \bullet {}_{i}\underline{g} \\
&= M'^{j}\;{}_{i}\underline{g} \bullet ({}_{i}A^{j})[({}_{i}A^{j})^{T}\,{}_{i}\underline{s}] \\
&= M'^{j}\;{}_{i}\underline{g} \bullet {}_{i}A^{j}\;{}_{i}\underline{\mu}^{j}
\end{aligned} \tag{4-97}$$

Under defintions 4-94:4-95, equations 4-96 & 4-97 equal to equations 4-92 & 4-93 respectively.

4.3 The Closed Form Dynamical Model For The One-Degree-of-Freedom-Per-Link Rigid Manipulator

Based on definitions 4-1:4-7, we have a unique closed form dynamical model for the one-degree-of-freedom-per-link rigid manipulator, and it can be written as:

$$ {}_i\Gamma = D_i + \sum_{k=1}^{N} D_{ik}\, {}_k\ddot{n} + \sum_{l=1}^{N} \sum_{k=1}^{N} D_{ikl}\, {}_k\dot{n}\, {}_l\dot{n} + Ia_i\, {}_i\ddot{n} \quad (4\text{-}98) $$

where

$$ D_i = \sum_{j=i}^{N} D_i{}^{,j} \quad (4\text{-}99) $$

$$ D_{ik} = \sum_{j=\max(i,k)}^{N} D_{ik}{}^{,j} \quad (4\text{-}100) $$

$$ D_{ikl} = \sum_{j=\max\{i,k,l\}}^{N} D_{ikl}{}^{,j} \quad (4\text{-}101) $$

and

$$ D_i{}^{,j} = M^{,j}\, {}_i\underline{g} \bullet {}_iA^j({}_i\underline{\mu}^j + {}_i\underline{\delta}^j \times {}_j\underline{p}^{,j}) \quad (4\text{-}102) $$

$$ D_{ik}{}^{,j} = M^{,j} \{ ({}_i\underline{\delta}^j)^T K^{,j} ({}_k\underline{\delta}^j) + {}_k\underline{\mu}^j \bullet {}_i\underline{\mu}^j + {}_j\underline{p}^{,j} \bullet ({}_i\underline{\mu}^j \times {}_k\underline{\delta}^j + {}_k\underline{\mu}^j \times {}_i\underline{\delta}^j) \} \quad (4\text{-}103) $$

If $l \geq k$, then

$$ D_{ikl}{}^{,j} = M^{,j} \{ {}_i\underline{\delta}^j \bullet ({}_l\underline{\delta}^j \times K^{,j}{}_k\underline{\delta}^j) + {}_i\underline{\delta}^j \bullet [K^{,j} - \tfrac{1}{2}tr(K^{,j})]({}_l\underline{\delta}^j \times {}_k\underline{\delta}^j) \quad (4\text{-}104) $$

$$+{}_k\underline{\delta}^j \bullet ({}_l\underline{\mu}^j \times {}_i\underline{\mu}^j)$$
$$+{}_j\underline{p}'^{,j} \bullet ({}_l\underline{\delta}^j \times ({}_k\underline{\delta}^j \times {}_i\underline{\mu}^j))$$
$$+{}_j\underline{p}'^{,j} \bullet ({}_i\underline{\delta}^j \times ({}_l\underline{\mu}^j \times {}_k\underline{\delta}^j)) \quad \}$$

If $l < k$, then

$$D_{ikl}{}^{,j} = D_{ilk}{}^{,j} \tag{4-105}$$

Notice that equation 4-104 will be different if we exchange the indices k and l and it works only when $l \geq k$. If $k > l$, equation 4-105 is used. Under this arrangement, the index l in equation 4-98 runs from l=1 to l=N instead of from l>k to l=N. When l=k, it is the case of generalized centrifugal force.

Equations 4-98:4-105 form the dynamical model for the one-degree-of-freedom-per-link rigid manipulator. This same dynamical model has also been derived by the Lagrangian method [Paul82]. Hence, we demonstrated that the Newton-Euler and the Lagrangian methods are equivalent.

Example

The robot in figures 4 is again used as an example. The coordinate frames are set according to fig.5. The procedures to derive the dynamical model are the same as the example in chapter 3, but with less amount of computation because it is a one-degree-of-freedom-per-link rigid manipulator.

Step A. Specification of all link parameters

The mass of link 1 is M_1 and its center of mass locates right at the origin($\underline{p}_1$ = 0). The inertial matrix of link 1 is I_1 and it can be derived by equation 3-67. Since link 1 is a flat cylindical mass, thus the inertial matrix of link 1 is:

$$I_{1xx} = M_1 \ d_1^2 \ / \ 4 \tag{4-106}$$

$$I_{1yy} = M_1 \ d_1^2 \ / \ 2 \tag{4-107}$$

$$I_{1zz} = M_1 \ d_1^2 \ / \ 4 \tag{4-108}$$

$$I_{1xy} = I_{1yx} = I_{1yz} = I_{1zy} = I_{1xz} = I_{1zx} = 0 \tag{4-109}$$

The mass of link 2 is zero(M_2=0) because there is no mass from joint 2 to joint 3.

The mass of link 3 is M_3 and its center of mass is situated at the middle of the link($\underline{p}_{3x}$ =$-d_3/2$ = varying). Since the link 3 is a slender rod, the inertial matrix of link 3(I_3) is:

$$I_{3xx} = 0 \tag{4-110}$$

$$I_{3yy} = M_3 \ d_3^2 \ / \ 12 \tag{4-111}$$

$$I_{3zz} = M_3 \ d_3^2 \ / \ 12 \tag{4-112}$$

$$I_{3xy} = I_{3yx} = I_{3yz} = I_{3zy} = I_{3xz} = I_{3zx} = 0 \tag{4-113}$$

Step B. Establishing the kinematic model

The kinematic model of the robot of figures 4 & 5 is given by equations 2-92:2-98.

Step C. Specifies of the dynamical variables of each joint

The variables of link 1 are $[0, 0, \underline{w}_1]^T$ & $[0, 0, \underline{\alpha}_1]^T$.
The variables of link 2 are $[0, \underline{w}_2, 0]^T$ & $[0, \underline{\alpha}_2, 0]^T$.
The variables of link 3 are $[-\underline{u}_2, 0,0]^T$ & $[-\underline{a}_2, 0,0]^T$.

Step D. Constructing the dynamical model

The more convenient way to construct the dynamical model of a robot is from the end effector to the base.

The vectors of the center of mass are:

$${}_3\underline{p}'^{,3} = [\tfrac{1}{2}d_3, 0, 0]^T \quad (4\text{-}114)$$

$${}_2\underline{p}'^{,3} = [\tfrac{1}{2}d_3\cos\theta_2, 0 , -\tfrac{1}{2}d_3\sin\theta_2]^T \quad (4\text{-}115)$$

$${}_1\underline{p}'^{,3} = [\tfrac{1}{2}d_3\cos\theta_2, d_1, -\tfrac{1}{2}d_3\sin\theta_2]^T \quad (4\text{-}116)$$

$${}_2\underline{p}'^{,2} = [0, 0, 0]^T \quad (4\text{-}117)$$

$${}_1\underline{p}'^{,2} = [0, 0, 0]^T \quad (4\text{-}118)$$

$${}_1\underline{p}'^{,1} = [0, 0, 0]^T \quad (4\text{-}119)$$

The gravity terms

The gravity vector in the local coordinate is:

$${}_0\underline{g} = [0, 0, -g]^T \quad (4\text{-}120)$$

$${}_1\underline{g} = ({}_0A^1)^T\,{}_0\underline{g} = [0, 0, -g]^T \quad (4\text{-}121)$$

$$_2\underline{g} = ({_0A^2})^T {_0\underline{g}} = g[\sin\emptyset_2,\ 0,\ -\cos\emptyset_2]^T \qquad (4\text{-}122)$$

$$_3\underline{g} = ({_0A^3})^T {_0\underline{g}} = g[\sin\emptyset_2,\ 0,\ -\cos\emptyset_2]^T \qquad (4\text{-}123)$$

Since joint 3 is prismatic along the X_3-axis, then

$$_3\underline{s} = [1,\ 0,\ 0]^T$$

$$_3\underline{\delta}^3 = [0,\ 0,\ 0]^T$$

$$_3\underline{\mu}^3 = ({_3A^3})^T {_3\underline{s}} = [1,\ 0,\ 0]^T$$

So $D_3{}^{,3} = M^{,3}\ {_3\underline{g}} \bullet {_3A^3}\ {_3\underline{\mu}^3} = M_3 g \sin\emptyset_2$

Hence $_3F_g = D_3{}^{,3} = M_3 g \sin\emptyset_2$ (4-124)

Joint 2 is revolute(having Y_2-axis as the axis of rotation), therefore

$$_2\underline{s} = [0,\ 1,\ 0]^T$$

$$_2\underline{\delta}^3 = ({_2A^3})^T {_2\underline{s}} = [0,\ 1,\ 0]^T$$

$$_2\underline{\mu}^3 = ({_2A^3})^T ({_2\underline{s}} \times {_2\underline{d}^3}) = [0,\ 0,\ d_3]^T$$

So $D_2{}^{,3} = M^{,3}\ {_2\underline{g}} \bullet {_2A^3}({_2\underline{\mu}^3} + {_2\underline{\delta}^3} \times {_3\underline{p}^{,3}}) = -\tfrac{1}{2} M_3 g d_3 \cos\emptyset_2$

Since $D_2{}^{,2} = 0$

Hence $_2\Gamma_g = D_2{}^{,3} + D_2{}^{,2} = -\tfrac{1}{2} M_3 d_3 \cos\emptyset_2$ (4-125)

Joint 1 is revolute(having the Z_1-axis as the axis of rotation), therefore

$$_1\underline{s} = [0,\ 0,\ 1]^T$$

$$_1\underline{\delta}^3 = ({_1A^3})^T {_1\underline{s}} = [-\sin\emptyset_2,\ 0,\ \cos\emptyset_2]^T$$

$$_1\underline{\mu}^3 = ({_1A^3})^T ({_1\underline{s}} \times {_1\underline{d}^3}) = -[d_1\cos\emptyset_2,\ d_2\cos\emptyset_2,\ d_1\sin\emptyset_2]^T$$

So $D_1{}^{,3} = M^{,3}\ {_1\underline{g}} \bullet {_1A^3}({_1\underline{\mu}^3} + {_1\underline{\delta}^3} \times {_3\underline{p}^{,3}}) = 0$

Since $D_1{}^{,2} = 0$

and $D_1{}^{,1} = 0$

Hence ${}_1\Gamma_g = 0$ (4-126)

<u>The inertial terms</u>

Under the above :

$${}_3\underline{s} = [1, 0, 0]^T$$
$${}_3\underline{\delta}^3 = [0, 0, 0]^T$$
$${}_3\underline{\mu}^3 = ({}_3A^3)^T {}_3\underline{s} = [1, 0, 0]^T$$
$${}_2\underline{s} = [0, 1, 0]^T$$
$${}_2\underline{\delta}^3 = ({}_2A^3)^T {}_2\underline{s} = [0, 1, 0]^T$$
$${}_2\underline{\mu}^3 = ({}_2A^3)^T ({}_2\underline{s} \times {}_2\underline{d}^3) = [0, 0, d_3]^T$$
$${}_1\underline{s} = [0, 0, 1]^T$$
$${}_1\underline{\delta}^3 = ({}_1A^3)^T {}_1\underline{s} = [-\sin\theta_2, 0, \cos\theta_2]^T$$
$${}_1\underline{\mu}^3 = ({}_1A3)T({}_1\underline{s} \times {}_1\underline{d}3) = -[d_1\cos\theta_2, d_2\cos\theta_2, d_1\sin\theta_2]^T$$

So
$$D_{33}{}^{,3} = M^{,3} \{ ({}_3\underline{\delta}^3)^T K^{,3} ({}_3\underline{\delta}^3) + {}_3\underline{\mu}^3 \bullet {}_3\underline{\mu}^3 + {}_3\underline{p}^{,3} \bullet ({}_3\underline{\mu}^3 \times {}_3\underline{\delta}^3 + {}_3\underline{\mu}^3 \times {}_3\underline{\delta}^3) \}$$
$$= M_3$$

and $D_{33} = D_{33}{}^{,3} = M_3$

So
$$D_{32}{}^{,3} = M^{,3} \{ ({}_3\underline{\delta}^3)^T K^{,3} ({}_2\underline{\delta}^3) + {}_2\underline{\mu}^3 \bullet {}_3\underline{\mu}^3 + {}_3\underline{p}^{,3} \bullet ({}_3\underline{\mu}^3 \times {}_2\underline{\delta}^3 + {}_2\underline{\mu}^3 \times {}_3\underline{\delta}^3) \}$$
$$= [0, 0, 0]^T$$

and $D_{32} = D_{33}{}^{,3} = 0$

So $D_{31}{,}^{3} = M{,}^{3} \{ ({}_3\underline{\delta}^3)^T K{,}^3 ({}_1\underline{\delta}^3) + {}_1\underline{\mu}^3 \bullet {}_3\underline{\mu}^3$

$+ {}_3\underline{p}{,}^3 \bullet ({}_3\underline{\mu}^3 \times {}_1\underline{\delta}^3 + {}_1\underline{\mu}^3 \times {}_3\underline{\delta}^3) \}$

$= -M_3 d_1 \cos\emptyset_2$

and $D_{31} = D_{31}{,}^3 = -M_3 d_1 \cos\emptyset_2$

Hence the total inertial force for joint 3 is:

$$\begin{aligned} {}_3F_I &= D_{33}\, \text{-}a_3 + D_{32}\,\alpha_2 + D_{31}\,\alpha_1 \\ &= -M_3(a_1 + \alpha_1 d_1 \cos\emptyset_1) \end{aligned} \qquad (4\text{-}127)$$

So $D_{23}{,}^3 = 0$

and $D_{23} = D_{23}{,}^3 \doteq 0$

So $D_{22}{,}^3 = I_{3yy} + M_3(d_3)^2$

$D_{22}{,}^2 = 0$

and $D_{22} = D_{22}{,}^3 + D_{22}{,}^2 = I_{3yy} + (d_3)^2$

So $D_{21}{,}^3 = M_3\ (\tfrac{1}{2} d_1 d_3 \sin\emptyset_2)$

$D_{21}{,}^2 = 0$

and $D_{21} = D_{21}{,}^3 + D_{21}{,}^2 = \tfrac{1}{2} M_3 d_1 d_3 \sin\emptyset_2$

Hence the total inertial torque of joint 2 is:

$$\begin{aligned} {}_2N_I &= D_{23}\, \text{-}a_3 + D_{22}\,\alpha_2 + D_{21}\,\alpha_1 \\ &= [I_{3yy} + M_3(d_3)^2]\alpha_2 + \tfrac{1}{2} M_3 d_1 d_3 \sin\emptyset_2 \alpha_1 \end{aligned} \qquad (4\text{-}128)$$

So $D_{13}{,}^3 = M_3\ (-d_1 \cos\emptyset_2)$

and $D_{13} = D_{13}{,}^3 = -M_3 d_1 \cos\emptyset_1$

So $D_{12},^{3} = M_3\ (-\frac{1}{2}d_1 d_3 \sin\emptyset_2)$

$D_{12},^{2} = 0$

and $D_{12} = D_{12},^{3} + D_{12},^{2} = -\frac{1}{2}M_3 d_1 d_3 \sin\emptyset_2$

so $D_{11},^{3} = I_{3xx}(\sin\emptyset_2)^2 + I_{3zz}(\cos\emptyset_2)^2$
$+M_3(d_1)^2 + M_3(d_3)^2(\cos\emptyset_2)^2$

$D_{11},^{2} = 0$

$D_{11},^{1} = I_{1zz}$

and $D_{11} = D_{11},^{3} + D_{11},^{2} + D_{11},^{1}$
$= M_3(d_1)^2 + I_{3xx}(\sin\emptyset_2)^2 + [I_{3zz}+(d_2)^2](\cos\emptyset_2)^2$
$+I_{1zz}$

Hence the total inertial torque of joint 2 is:

$$
\begin{aligned}
{}_1N_I &= D_{13}\ {-a_3} + D_{12}\ \alpha_2 + D_{11}\ \alpha_1 \\
&= M_3 d_1 \cos\emptyset_1 a_3 - \tfrac{1}{2}M_3 d_1 d_3 \sin\emptyset_2 \alpha_2 \\
&\quad +\{ M_3(d_1)^2 + I_{3xx}(\sin\emptyset_2)^2 + [I_{3zz}+(d_3)^2](\cos\emptyset_2)^2 \\
&\quad +I_{1zz} \}\alpha_1 \qquad (4\text{-}129)
\end{aligned}
$$

The centrifugal and Coriolis terms

The centrifugal and Coriolis terms can be computed in the similar manner. In this example, we assume that the robot is moving in slow speed, thus these two effects are insignificant.

<u>Conclusion</u>

The result of the last example is identical to the result from the generalized dynamical model(chapter 3). In the example of last chapter, link 2 of the robot was assumed to have two degrees of freedom. In the example of this chapter, it was assumed to be two separate links, link 2 & 3, and the two links are separated by a distance d_3 in the $(-X_3)$-axis direction. For the sake of comparison, the following terms of equations 4-124:4-129 are identified:

M_1	by	M_1	
d_1	by	d_1	
M_3	by	M_2	
d_3	by	d_2	
I_{3xx}	by	I_{2xx}	
$I_{3yy} + M_3(d_3)^2$	by	I_{2yy}	(parallel axis theorem)
$I_{3zz} + M_3(d_3)^2$	by	I_{2zz}	(parallel axis theorem)

then equation 4-124 equals to the x-component of equation 3-114, and so on.

One important remark is that equations 4-98:4-105 may be used to compute the component of the generalized force in any direction since these are a version of the general dynamical model of the previous chapter. If the equations 4-98:105 are derived using the Lagrangian method, they are valid only along the principal axes of motion.

CHAPTER 5

RECURSIVE FORM NEWTON-EULER DYNAMICAL MODEL OF THE GENERALIZED MANIPULATOR

5.1 Introduction

The recursive form dynamical model for an open chain robot manipulator computes all the dynamics values of a link in terms of the neighbouring link dynamic values. Usually, it computes the translational and rotational displacement, velocity and acceleration from the base coordinate to the end effector's coordinate; then it computes the force and torque at each coordinate origin backwards from the end effector to the base[Luh80][Holl80]. The computational complexity of the recursive form dynamical model can be shown to increase linearly with the number of links. As we have seen in the previous two chapters, the computational complexity of the closed form dynamical model increases in a much faster way. Hence, the recursive form dynamical model is used for problems which require intensive computation, such as in on-line control of the manipulator.

There are two common ways to derive the recursive form the Lagrangian formulation[Luh80]. The reason that Newton-Euler is more efficient is that the rotational velocity and acceleration are represented as vectors in the Newton-Euler formulation, but are represented in matrix form in the Lagrangian formulation[Holl82]. Both methods as appear in the above cited references use the center of mass of the link as the reference for balancing the force and the torque. Both deal only with the one-degree-of-freedom-per-link rigid manipulator case.

In this chapter, the recursive form dynamical model is derived for the generalized manipulator, and the coordinate origin of the link is used for balancing the forces and the torques and can be at arbitrary location. The generalized recursive form dynamical model will be specialized to the one-degree-of-freedom-per-link rigid manipulator, and will be shown to be equivalent to the model where the centers of mass of the link are used as the reference points.

5.2 Derivation of the Recursive Form Dynamical Model

The key idea in obtaining the recursive form dynamical model is to align all the local coordinate frames to have the same orientation as the base coordinate frame(fig.12). The reason for that will become evident later.

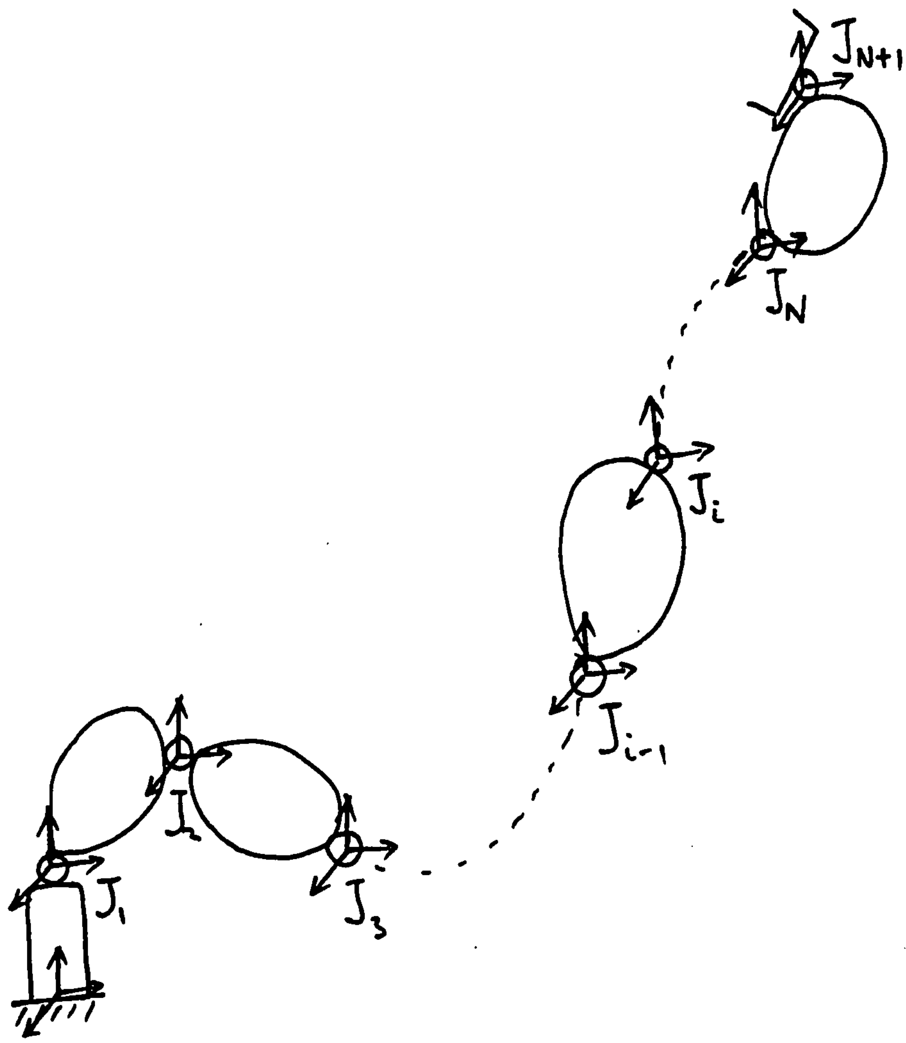


Figure 12 : The Coordinate System for Deriving the Dynamical Model for the Generalized Manipulator

The recursive formula of rotational displacement, velocity and acceleration

Since $\quad {}_0A^{i+1} = {}_0A^{i}\,{}_iA^{i+1}$ (5-1)

then $\quad d_t\,({}_0A^{i+1}) = d_t({}_0A^{i}) + {}_0A^{i} d_t({}_iA^{i+1})$ (5-2)

and $\quad d_{tt}({}_0A^{i+1}) = d_{tt}({}_0A^{i}) + 2d_t({}_0A^{i})d_t({}_iA^{i+1})$
$\qquad + {}_0A^{i} d_{tt}({}_iA^{i+1})$ (5-3)

The recursive formula of the displacement, velocity and acceleration of the coordinate origin

Since $\quad {}_0\underline{d}^{i+1} = {}_0\underline{d}^{i} + {}_0A^{i}\,{}_i\underline{d}^{i+1}$ (5-4)

then $\quad d_t({}_0\underline{d}^{i+1}) = d_t({}_0\underline{d}^{i}) + d_t({}_0A^{i})\,{}_i\underline{d}^{i+1} + {}_0A^{i} d_t({}_i\underline{d}^{i+1})$

and $\quad d_{tt}({}_0\underline{d}^{i+1}) = d_{tt}({}_0\underline{d}^{i}) + d_{tt}({}_0A^{i})\,{}_i\underline{d}^{i+1}$
$\qquad + 2d_t({}_0A^{i})d_t({}_i\underline{d}^{i+1}) + {}_0A^{i} d_{tt}({}_i\underline{d}^{i+1})$ (5-6)

The recursive formula of the displacement, velocity & acceleration of a position in link j w.r.t. the base coordinate

Since $\quad {}_0\underline{r} = {}_0\underline{d}^{i} + {}_0A^{i}\,{}_i\underline{r}$ (5-7)

then $\quad d_t({}_0\underline{r}) = d_t({}_0\underline{d}^{i}) + d_t({}_0A^{i})\,{}_i\underline{r} + {}_0A^{i} d_t({}_i\underline{r})$ (5-8)

and $\quad d_{tt}({}_0\underline{r}) = d_{tt}({}_0\underline{d}^{i}) + d_{tt}({}_0A^{i})\,{}_i\underline{r} + 2d_t({}_0A^{i})d_t({}_i\underline{r})$
$\qquad + {}_0A^{i} d_{tt}({}_i\underline{r})$ (5-9)

where the superscript ,j of the position vector $\underline{r}$ has been suppressed.

The recursive formula of the displacement, velocity & acceleration of a position in link j w.r.t. the local coordinate

Since $\quad {}_i\underline{r} = {}_i\underline{d}^{i+1} + {}_iA^{i+1}\,{}_{i+1}\underline{r}$ (5-10) then $d_t({}_i\underline{r})=d_t({}_i\underline{d}^{i+1}) + d_t({}_iA^{i+1})_{i+1}\underline{r} + {}_iA^{i+1}d_t({}_{i+1}\underline{r})$ (5-11)

and $d_{tt}({}_i\underline{r})=d_{tt}({}_i\underline{d}^{i+1}) + d_{tt}({}_iA^{i+1})_{i+1}\underline{r}$

$$+2d_t({}_iA^{i+1})d_t({}_{i+1}\underline{r}) + {}_iA^{i+1}d_{tt}({}_{i+1}\underline{r}) \quad (5\text{-}12)$$

where the superscript ,j of the position vector $\underline{r}$ has been suppressed.

Recursive formula of the force

Let's define

$$({}_0A^i\underline{F}_{i-1,i}) == \int_{J_i}^{E} d_{tt}({}_0\underline{r}^{,j})\ dm \quad (5\text{-}13)$$

as the force of link i acting on link i-1, that is the total force accumulating from joint i to the end effector. The orientation matrix ${}_0A^i$ adjusts the direction from the local coordinate i to the base coordinate 0.

Similarly, the force of link i+1 acting on link i is:

$$({}_0A^{i+1}\underline{F}_{i,i+1}) == \int_{J_{i+1}}^{E} d_{tt}({}_0\underline{r}^{,j})\ dm \quad (5\text{-}14)$$

By sustituting equations 5-7:5-9 to equation 5-13, and suppress the superscript ,j of the position vector $\underline{r}$ in the following derivations, we obtain

$$({}_0A^i\underline{F}_{i-1,i})$$

$$= \int_{J_i}^{E} \{\ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)_i\underline{r} + 2d_t({}_0A^i)d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r})\ \}dm \quad [5\text{-}9]$$

$$= \int_{J_i}^{J_{i+1}} \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r}^{,i} + 2d_t({}_0A^i)d_t({}_i\underline{r}^{,i}) + {}_0A^i d_{tt}({}_i\underline{r}^{,i}) \}dm$$

$$+ \int_{J_{i+1}}^{E} \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r} + 2d_t({}_0A^i)d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r}) \}dm \qquad (5\text{-}15)$$

Recall equation 3-38:

$${}_i\underline{p}^{,i} = \int_{\text{link } i} {}_i\underline{r}^{,i}\, dm \,/M^{,i}$$

then the first term of equation 5-15 is:

$$\int_{J_i}^{J_{i+1}} \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r}^{,i} + 2d_t({}_0A^i)d_t({}_i\underline{r}^{,i}) + {}_0A^i d_{tt}({}_i\underline{r}^{,i}) \}dm$$

$$= M^{,i}\{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{p}^{,i} + 2d_t({}_0A^i)d_t({}_i\underline{p}^{,i}) + {}_0A^i d_{tt}({}_i\underline{p}^{,i})\} \qquad (5\text{-}16)$$

By using equations 5-1:5-12 & 5-14, and suppressing the superscript ,j, the second term of equation 5-15 can be rewritten as follows:

$$\int_{J_{i+1}}^{E} \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r} + 2d_t({}_0A^i)d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r}) \}dm$$

$$= \int_{J_{i+1}}^{E} \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{d}^{i+1} + d_{tt}({}_0A^i)\,{}_iA^{i+1} + 2d_t({}_0A^i)d_t({}_i\underline{d}^{i+1}) + 2d_t({}_0A^i)d_t({}_iA^{i+1})\,{}_{i+1}\underline{r} + 2d_t({}_0A^i)\,{}_iA^{i+1}d_t({}_{i+1}\underline{r}) + {}_0A^i d_{tt}({}_i\underline{d}^{i+1}) + {}_0A^i d_{tt}({}_iA^{i+1})\,{}_{i+1}\underline{r} + 2\,{}_0A^i d_t({}_iA^{i+1})d_t({}_{i+1}\underline{r}) + {}_0A^i\,{}_iA^{i+1}d_{tt}({}_{i+1}\underline{r}) \}\,dm$$

$$= \int_{J_{i+1}}^{E} \{ d_{tt}({}_0\underline{d}^{i+1}) + d_{tt}({}_0A^{i+1})\,{}_{i+1}\underline{r} + 2d_t({}_0A^{i+1})d_t({}_{i+1}\underline{r}) + {}_0A^{i+1}\,{}_{i+1}\underline{r} \}\,dm$$

$$= \int_{J_{i+1}}^{E} d_{tt}({}_0\underline{r})\,dm$$

$$= ({}_0A^{i+1}\underline{F}_{i,i+1}) \qquad (5\text{-}17)$$

After substituting equations 5-16:5-17 into equation 5-15, we obtain the following recursive equation for the force.

$$({}_0A^i\underline{F}_{i-1,i})$$

$$= M'^{i}\{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{p}'^{i} + 2d_t({}_0A^i)d_t({}_i\underline{p}'^{i}) + {}_0A^i d_{tt}({}_i\underline{p}'^{i}) \}$$

$$+ ({}_0A^{i+1}\underline{F}_{i,i+1}) \qquad (5\text{-}18)$$

Recursive formula of the torque

Define

$$({}_0A^i \underline{N}_{i-1,i}) == \int_{J_i}^{E} ({}_0A^i\, {}_i\underline{r}^{,j}) \times d_{tt}({}_0\underline{r}^{,j})\, dm \qquad (5\text{-}19)$$

as the torque of link i acting on link i-1, that is the total force accumulating from joint i to the end effector. The orientation matrix ${}_0A^i$ adjusts the direction from the local coordinate i to the base coordinate 0.

Similarly, the torque of link i+1 acting on link i is:

$$({}_0A^{i+1} \underline{N}_{i,i+1}) == \int_{J_{i+1}}^{E} ({}_0A^{i+1}\, {}_{i+1}\underline{r}^{,j}) \times d_{tt}({}_0\underline{r}^{,j})\, dm \qquad (5\text{-}20)$$

By substituting equation 5-9 into equation 5-19, and suppressing the superscript ,j, we get

$$({}_0A^i \underline{N}_{i-1,i})$$

$$= \int_{J_i}^{E} ({}_0A^i\, {}_i\underline{r}) \times \{\, d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\, {}_i\underline{r} + 2d_t({}_0A^i) d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r}) \,\}\, dm$$

$$= \int_{J_i}^{J_{i+1}} ({}_0A^i\, {}_i\underline{r}^{,i}) \times \{\, d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\, {}_i\underline{r}^{,i} + 2d_t({}_0A^i) d_t({}_i\underline{r}^{,i}) + {}_0A^i d_{tt}({}_i\underline{r}^{,i}) \,\}\, dm$$

$$+\int_{J_{i+1}}^{E} ({}_0A^i\,{}_i\underline{d}^{i+1}) \times \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r} + 2d_t({}_0A^i)d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r}) \}\, dm$$

$$+\int_{J_i}^{E} ({}_0A^i\,{}_iA^{i+1}\,{}_{i+1}\underline{r}) \times \{ d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r} + 2d_t({}_0A^i)d_t({}_i\underline{r}) + {}_0A^i d_{tt}({}_i\underline{r}) \}\, dm \qquad (5\text{-}21)$$

Recall equation 3-38, then

$$\int_{J_i}^{J_{i+1}} ({}_0A^i\,{}_i\underline{r}'^i) \times d_{tt}({}_0\underline{d}^i)\, dm = M'^i\{{}_0A^i\,{}_i\underline{p}'^i \times d_{tt}({}_0\underline{d}^i)\} \qquad (5\text{-}22)$$

Recall equations 2-71, 2-81, 2-86 & 3-65, then

$$\int_{J_i}^{J_{i+1}} ({}_0A^i\,{}_i\underline{r}'^i) \times d_{tt}({}_0A^i)\,{}_i\underline{r}'^i\, dm$$

$$= \int_{\text{link } i} ({}_0A^i\,{}_i\underline{r}'^i) \times \{{}_0\underline{\alpha}^i \times {}_0A^i\,{}_i\underline{r}'^i) + {}_0\underline{w}^i \times ({}_0\underline{w}^i \times {}_0A^i\,{}_i\underline{r}'^i)\}\, dm$$

$$= {}_0A^i I'^i ({}_0A^i)^T\,{}_0\underline{\alpha}^i + {}_0\underline{w}^i \times [{}_0A^i I'^i ({}_0A^i)^T]\,{}_0\underline{w}^i$$

where I'^i is the inertial matrix of link i measured at the origin of coordinate frame i.

Since link i is rigid, then the translational and rotational velocity and acceleration of all the points in the link i are the same(i.e the rigid link). Then, by equation 2-77, we have:

$$d_t\,({}_i\underline{r}'^{\,i}) = d_t\,({}_i\underline{d}^{\,i+1}) = {}_i\underline{u} \tag{5-24}$$

$$d_{tt}({}_i\underline{r}'^{\,i}) = d_{tt}({}_i\underline{d}^{\,i+1}) = {}_i\underline{a} \tag{5-25}$$

then

$$\int_{J_i}^{J_{i+1}} ({}_0A^i\,{}_i\underline{r}'^{\,i}) \times (2d_t({}_0A^i)d_t({}_i\underline{r}))\,dm$$

$$= 2\int_{\text{link } i} ({}_0A^i\,{}_i\underline{r}'^{\,i}) \times d_t({}_0A^i)\,{}_i\underline{u}\,dm$$

$$= 2M'^{\,i}\{{}_0A^i\,{}_i\underline{p}'^{\,i} \times d_t({}_0A^i)\,{}_i\underline{u}\} \tag{5-26}$$

and

$$\int_{J_i}^{J_{i+1}} ({}_0A^i\,{}_i\underline{r}'^{\,i}) \times {}_0A^i d_{tt}({}_i\underline{r})\,dm$$

$$= \int_{\text{link } i} ({}_0A^i\,{}_i\underline{r}'^{\,i}) \times {}_0A^i\,{}_i\underline{a}$$

$$= M'^{\,i}\{{}_0A^i\,{}_i\underline{p}'^{\,i} \times {}_0A^i\,{}_i\underline{a}\} \tag{5-27}$$

Combining 5-22:5-23 & 5-26:5-27, the first term on the right hand side of equation 5-21 is:

$$\int_{J_i}^{J_{i+1}} ({}_0A^i\,{}_i\underline{r}'^{\,i}) \times \{\, d_{tt}({}_0\underline{d}^i) + d_{tt}({}_0A^i)\,{}_i\underline{r}'^{\,i} + 2d_t({}_0A^i)d_t({}_i\underline{r}'^{\,i}) + {}_0A^i d_{tt}({}_i\underline{r}'^{\,i}) \,\}\,dm$$

$$= M'^{i}\{{}_{0}A^{i}\,{}_{i}\underline{p}'^{i} \times d_{tt}({}_{0}\underline{d}^{i})\}$$
$$+M'^{i}\{{}_{0}A^{i}\,{}_{i}\underline{p}'^{i} \times {}_{0}A^{i}\,{}_{i}\underline{a}\}$$
$$+2M'^{i}\{{}_{0}A^{i}\,{}_{i}\underline{p}'^{i} \times d_{t}({}_{0}A^{i})\,{}_{i}\underline{u}\}$$
$$+{}_{0}A^{i}I'^{i}({}_{0}A^{i})^{T}\,{}_{0}\underline{\alpha}^{i} + {}_{0}\underline{w}^{i} \times [{}_{0}A^{i}I'^{i}({}_{0}A^{i})^{T}]\,{}_{0}\underline{w}^{i}$$

where ${}_{0}\underline{\alpha}^{i}$ and ${}_{0}\underline{w}^{i}$ is the vector form of $d_{t}({}_{0}A^{i})$ and $d_{tt}({}_{0}A^{i})$ respectively.

Based on equations 5-9, 5-14 & 5-17, the second term on the right hand side of equation 5-21 is:

$$\int_{J_{i+1}}^{E} ({}_{0}A^{i}\,{}_{i}\underline{d}^{i+1}) \times \{ d_{tt}({}_{0}\underline{d}^{i}) + d_{tt}({}_{0}A^{i})\,{}_{i}\underline{r} + 2d_{t}({}_{0}A^{i})d_{t}({}_{i}\underline{r}) + {}_{0}A^{i}d_{tt}({}_{i}\underline{r}) \}\, dm$$
$$= ({}_{0}A^{i}\,{}_{i}\underline{d}^{i+1}) \times ({}_{0}A^{i+1}\underline{F}_{i,i+1}) \qquad (5\text{-}29)$$

Based on equations 5-17 and 5-20, the third term on the right hand side of equation 5-21 is:

$$\int_{J_{i}}^{E} ({}_{0}A^{i+1}\,{}_{i+1}\underline{r}) \times \{ d_{tt}({}_{0}\underline{d}^{i}) + d_{tt}({}_{0}A^{i})\,{}_{i}\underline{r} + 2d_{t}({}_{0}A^{i})d_{t}({}_{i}\underline{r}) + {}_{0}A^{i}d_{tt}({}_{i}\underline{r}) \}\, dm$$
$$= ({}_{0}A^{i+1}\underline{N}_{i,i+1}) \qquad (5\text{-}30)$$

After substituting equations 5-28:5-30 to equation 5-21, we achieve the following recursive equation for computing the torque.

$$
\begin{aligned}
&({}_0A^i\underline{N}_{i-1,i}) \\
&= M'^i\{{}_0A^i{}_i\underline{p}'^i \times d_{tt}({}_0\underline{d}^i)\} \\
&+M'^i\{{}_0A^i{}_i\underline{p}'^i \times {}_0A^i{}_i\underline{a}\} \\
&+2M'^i\{{}_0A^i{}_i\underline{p}'^i \times d_t({}_0A^i){}_i\underline{u}\} \\
&+{}_0A^iI'^i({}_0A^i)^T{}_0\underline{\alpha}^i + {}_0\underline{w}^i \times [{}_0A^iI'^i({}_0A^i)^T]{}_0\underline{w}^i \\
&+({}_0A^i{}_i\underline{d}^{i+1}) \times ({}_0A^{i+1}\underline{F}_{i,i+1}) \\
&+({}_0A^{i+1}\underline{N}_{i,i+1})
\end{aligned}
\tag{5-31}
$$

Recursive dynamical model of the generalized manipulator

Equations 5-1:5-6, 5-18 and 5-31 establish the recursive dynamical model of the generalized manipulator. The rotational displacement/velocity/acceleration, then the total displacement/velocity/acceleration of the link are computed from the base coordinate to the end effector; The force and the torque are computed backwards from the end effector to the base. The boundary conditions of the model are:

$$ {}_0\underline{d}^0 = \underline{0} \tag{5-32} $$

$$ d_t({}_0\underline{d}^0) = \underline{0} \tag{5-33} $$

$$ d_{tt}({}_0\underline{d}^0) = \underline{g} \tag{5-34} $$

$$ {}_0A^0 = I \tag{5-35} $$

$$ d_t({}_0A^0) = 0 \tag{5-36} $$

$$ d_{tt}({}_0A^0) = 0 \tag{5-37} $$

${}_0A^{N+1}\underline{F}_{N,N+1}$ is the external force (5-38)

${}_0A^{N+1}\underline{N}_{N,N+1}$ is the external torque (5-39)

acting on the end effector.

The gravity effect is taken into account in the boundary condition 5-34, the gravity acceleration is thought of as an external acceleration acting on the base, and it will propagate throughout the whole manipulator via the recursive formulas 5-1:5-6.

5.3 The Recursive Form Dynamical Model of the One-Degree-of-Freedom-Per-Link Rigid Manipulator

The recursive form dynamical model of the last section is not very efficient in computation, and it can be simplified by the following two modifications:

a. In last section, the rotational velocity and acceleration are represented in matrix form that has a lot of redundancy. In fact, the rotational velocity and acceleration can be represented by vectors, then the efficiency is three times larger[Holl82].

b. In last section, we have assumed that every joint can be rotated and translated in any direction. In pratical applications, the joint is either prismatic or revolute in a specfic direction. Hence the dynamical model can be simplified.

Recursive formula of rotational velocity and acceleration in vector form

The rotational displacement can not be represented in the vector form(section 1.4), but the rotational velocity and acceleration can.

From equation 2-68,

$d_t(A) = \Omega A$

then $d_t({}_0A^{i+1}) = {}_0\Omega^{i+1}\ {}_0A^{i+1}$ (5-40)

and $d_t({}_0A^{i}) = {}_0\Omega^{i}\ {}_0A^{i}$ (5-41)

and $d_t({}_iA^{i+1}) = {}_i\Omega^{i+1}\ {}_iA^{i+1}$ (5-42)

hence equation 5-2 becomes:

$${}_0\Omega^{i+1}\,{}_0A^{i+1} = {}_0\Omega^{i}\,{}_0A^{i}\,{}_iA^{i+1} + {}_0A^{i}\,{}_i\Omega^{i+1}\,{}_iA^{i+1} \quad (5\text{-}43)$$

after substituting equation 5-1 to equation 5-43, it becomes

$${}_0\Omega^{i+1} = {}_0\Omega^{i} + {}_0A^{i}\,{}_i\Omega^{i+1}\,{}_iA^{0} \quad (5\text{-}44)$$

then by equations 2-66, 2-67, 2-72 & 2-73, we obtain

$${}_0\underline{w}^{i+1} = {}_0\underline{w}^{i} + {}_0A^{i}\,{}_i\underline{w}^{i+1} \quad (5\text{-}45)$$

By differentiating equation 5-45, we obtain

$$d_t({}_0\underline{w}^{i+1}) = d_t({}_0\underline{w}^{i}) + d_t({}_0A^{i})\,{}_i\underline{w}^{i+1} + {}_0A^{i}d_t({}_i\underline{w}^{i+1})$$

hence $${}_0\underline{\alpha}^{i+1} = {}_0\underline{\alpha}^{i} + {}_0\underline{w}^{i} \times ({}_0A^{i}\,{}_i\underline{w}^{i+1}) + {}_0A^{i}\,{}_i\underline{\alpha}^{i+1} \quad (5\text{-}46)$$

Equations 5-54 and 5-46 are the recursive formula of rotational velocity and acceleration in vector form.

Note that the presence of the orientation matrix ${}_0A^i$ is to align the local coordinate frame to the same direction as the base coordinate frame. Recall that ${}_i\underline{w}^{i+1} == {}_i\underline{w}$ and ${}_i\underline{\alpha}^{i+1} == {}_i\underline{\alpha}$ are the revolute velocity and acceleration of link i in local coordinate frame.

Since joint i can be either prismatic or revolute in the direction ${}_i\underline{s}$, therefore,

$$ {}_i\underline{w} = \underline{0} \qquad \text{if joint i is prismatic} \qquad (5\text{-}47) $$

$$ {}_i\underline{\alpha} = \underline{0} \qquad (5\text{-}48) $$

and

$$ {}_i\underline{w} = {}_i\underline{s}\ {}_i\dot{n} \qquad \text{if joint i is revolute} \qquad (5\text{-}49) $$

$$ {}_i\underline{\alpha} = {}_i\underline{s}\ {}_i\ddot{N} \qquad (5\text{-}50) $$

Substituting equations 5-47:5-50 into equations 5-45:5-46, the recursive formulas of the revolute velocity and acceleration are achieved as follows:

$$ {}_0\underline{w}^{i+1} = {}_0\underline{w}^i \qquad \text{if joint i is prismatic} \qquad (5\text{-}51) $$

$$ {}_0\underline{\alpha}^{i+1} = {}_0\underline{\alpha}^i \qquad (5\text{-}52) $$

and

$$ {}_0\underline{w}^{i+1} = {}_0\underline{w}^i + ({}_0A^i\ {}_i\underline{s})\,{}_i\dot{n} \quad \text{if joint i is revolute} \qquad (5\text{-}53) $$

$$ {}_0\underline{\alpha}^{i+1} = {}_0\underline{\alpha}^i + {}_0\underline{w}^i \times ({}_0A^i\ {}_i\underline{s})\,{}_i\dot{n} + ({}_0A^i\ {}_i\underline{s})\,{}_i\ddot{N} \qquad (5\text{-}54) $$

Equations 5-51:5-54 are the recursive formulas of the revolute velocity and acceleration for the one-degree-of-freedom-per-link rigid manipulator in vector form.

Recursive formula of the displacement/velocity/acceleration

Since the revolute displacement cannot be simplified, the recursive formula of displacement remains unchanged(equation 5-4).

Define the total velocity and acceleration of the origin of coordinate frame i w.r.t. to base coordinate as:

$$d_t\ ({}_0\underline{d}^i) == {}_0\underline{v}^i \tag{5-55}$$

$$d_{tt}({}_0\underline{d}^i) == {}_0\underline{\epsilon}^i \tag{5-56}$$

Using equations 2-70:2-71, equations 5-5:5-6 become:

$${}_0\underline{v}^{i+1} = {}_0\underline{v}^i + {}_0\underline{w}^i \text{ X } {}_0A^i\,{}_i\underline{d}^{i+1} + {}_0A^i d_t({}_i\underline{d}^{i+1})$$

$${}_0\underline{\epsilon}^{i+1} = {}_0\underline{\epsilon}^i + {}_0\underline{\alpha}^i \text{ X } {}_0A^i\,{}_i\underline{d}^{i+1} + {}_0\underline{w}^i \text{ X } ({}_0\underline{w}^i \text{ X }$$
$$+2{}_0\underline{w}^i \text{ X } {}_0A^i d_t({}_i\underline{d}^{i+1}) + {}_0A^i d_{tt}({}_i\underline{d}^{i+1}) \tag{5-58}$$

which are the recursive formula of the total velocity and acceleration of the origins of local coordinate frames.

Recall equations 2-79:2-80, the following equations are derived:

$$d_t\ ({}_i\underline{d}^{i+1}) = {}_i\underline{u} \qquad \text{if joint i is prismatic} \tag{5-59}$$

$$d_{tt}({}_i\underline{d}^{i+1}) = {}_i\underline{a} \tag{5-60}$$

and $$d_t\ ({}_i\underline{d}^{i+1}) = {}_i\underline{w} \text{ X } {}_i\underline{d}^{i+1} \qquad \text{if joint i is revolute} \tag{5-61}$$

$$d_{tt}({}_i\underline{d}^{i+1}) = {}_i\underline{\alpha} \text{ X } {}_i\underline{d}^{i+1} + {}_i\underline{w} \text{ X } ({}_i\underline{w} \text{ X } {}_i\underline{d}^{i+1}) \tag{5-62}$$

Now, suppose that the joint has only one degree of freedom, then

$${}_i\underline{u} = {}_i\underline{s}\ {}_i\tilde{n} \qquad \text{if joint i is prismatic} \tag{5-63}$$

$${}_i\underline{a} = {}_i\underline{s}\ {}_i\tilde{N} \tag{5-64}$$

and $_{i}\underline{u} = \underline{0}$ if joint i is revolute (5-65)

$$_{i}\underline{a} = \underline{0} \quad (5\text{-}66)$$

By using equations 5-59:5-66 and 5-51:5-54, equations 5-57:5-58 become:

$$_{0}\underline{v}^{i+1} = {}_{0}\underline{v}^{i} + {}_{0}\underline{w}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{d}^{i+1} + ({}_{0}A^{i}\,{}_{i}\underline{s})_{i}\dot{n} \quad (5\text{-}67)$$

$$_{0}\underline{\epsilon}^{i+1} = {}_{0}\underline{\epsilon}^{i} + {}_{0}\underline{\alpha}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{d}^{i+1} + {}_{0}\underline{w}^{i+1} \times ({}_{0}\underline{w}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{d}^{i+1})$$
$$+ 2({}_{0}\underline{w}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{s})_{i}\dot{n} + ({}_{0}A^{i}\,{}_{i}\underline{s})_{i}\ddot{N} \quad (5\text{-}68)$$

if joint i is prismatic.

$$_{0}\underline{v}^{i+1} = {}_{0}\underline{v}^{i} + {}_{0}\underline{w}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{d}^{i+1} \quad (5\text{-}69)$$

$$_{0}\underline{\epsilon}^{i+1} = {}_{0}\underline{\epsilon}^{i} + {}_{0}\underline{\alpha}^{i+1} \times {}_{0}A^{i}\,{}_{i}\underline{d}^{i+1} + {}_{0}\underline{w}^{i+1} \times ({}_{0}\underline{w}^{i+1} \times$$

if joint i is revolute. (5-70)

Equations 5-67:5-70 are the recursive formulas of the the total velocity and acceleration of the origins of local coordinates for the one-degree-of-freedom-per-link rigid manipulator in vector form.

Recursive formula of force

Equation 5-18 is the recursive formula of force, where $_{i}\underline{p}'^{i}$ is the position vector of the center of mass of link i in local coordinate frame.

$$({}_{0}A^{i}\underline{F}_{i-1,i})$$
$$= M'^{i}\{ d_{tt}({}_{0}\underline{d}^{i}) + d_{tt}({}_{0}A^{i})\,{}_{i}\underline{p}'^{i} + 2d_{t}({}_{0}A^{i})\,d_{t}({}_{i}\underline{p}'^{i})$$
$$+ {}_{0}A^{i} d_{tt}({}_{i}\underline{p}'^{i}) \}$$
$$+ ({}_{0}A^{i+1}\underline{F}_{i,i+1}) \quad [5\text{-}18]$$

If we assume that joint i is prismatic and it consists of a single piece of mass, then the velocity and acceleration of the center of mass of link i is:

$$d_t\ ({}_i\underline{p}^{i+1}) = {}_i\underline{s}\ {}_i\tilde{n} \qquad (5\text{-}71)$$

$$d_{tt}({}_i\underline{p}^{i+1}) = {}_i\underline{s}\ {}_i\tilde{N} \qquad (5\text{-}72)$$

Because of equations 5-51:5-52, therefore

$$d_{tt}({}_0A^i)\,{}_i\underline{p}'^{\,i} = {}_0\underline{\alpha}^{i+1} X\, {}_0A^i\, {}_i\underline{p}'^{\,i} + {}_0\underline{w}^{i+1} X({}_0\underline{w}^{i+1} X\, {}_0A^i\, {}_i\underline{p}'^{\,i})$$

$$dt({}_0A^i)\,d_t({}_i\underline{p}'^{\,i}) = ({}_0\underline{w}^{i+1}\ X\ {}_0A^i\, {}_i\underline{s})\,{}_i\tilde{n} \qquad (5\text{-}74)$$

Substituting equations 5-71:5-74 into equation 5-18, and using the equation 5-68, one gets

$$({}_0A^i\underline{F}_{i-1,i})$$
$$= M'^{\,i}\{{}_0\underline{\epsilon}^{i+1} + {}_0\underline{\alpha}^{i+1} X\, {}_0A^i\, {}_{i+1}\underline{p}'^{\,i} + {}_0\underline{w}^{i+1} X({}_0\underline{w}^{i+1} X\, {}_0A^i\, {}_{i+1}\underline{p}'^{\,i})\}$$
$$+({}_0A^{i+1}\underline{F}_{i,i+1}) \qquad (5\text{-}77)$$

where $_{i+1}\underline{p}'^{\,i} == ({}_iA^{i+1})^T({}_i\underline{p}'^{\,i} - {}_i\underline{d}^{i+1})$ (5-78)

is the position vector of the center of mass of link i but it is refered to the i+1 coordinate frame.

Assuming that joint i is revolute, then the velocity and acceleration of the center of mass of link i are:

$$d_t\ ({}_i\underline{p}^{i+1}) = {}_i\underline{w}\ X\ {}_i\underline{p}^{i+1} \qquad (5\text{-}79)$$

$$d_{tt}({}_i\underline{p}^{i+1}) = {}_i\underline{\alpha}\ X\ {}_i\underline{p}^{i+1} + {}_i\underline{w}\ X\ ({}_i\underline{w}\ X\ {}_i\underline{p}^{i+1}) \qquad (5\text{-}80)$$

and $d_{tt}({}_0A^i)\,{}_i\underline{p}'^{\,i} = {}_0\underline{\alpha}^i X\, {}_0A^i\, {}_i\underline{p}'^{\,i} + {}_0\underline{w}^i X({}_0\underline{w}^i X\, {}_0A$

$$d_t({}_0A^i)\,d_t({}_i\underline{p}'^{\,i}) = ({}_0\underline{w}^i\ X\ ({}_0A^i\, {}_i\underline{w}\ X\ {}_i\underline{p}'^{\,i})) \qquad (5\text{-}82)$$

If using equations 5-18, 5-53:5-54, 5-70, 5-79:5-82 and 2-84, then one can prove that equations 5-77 & 5-78 are indeed valid for revolute joint.

Recursive formula of torque

Substituting equations 5-56 and 2-70 into equation 5-31,

$$
\begin{aligned}
&({}_0A^i \underline{N}_{i-1,i}) \\
&= M'^i\{{}_0A^i{}_i\underline{p}'^i \times ({}_0\underline{\epsilon}^i + {}_0A^i{}_i\underline{a} + 2{}_0\underline{w}^i \times {}_0A^i{}_i\underline{v})\} \\
&+{}_0A^i I'^i({}_0A^i)^T{}_0\underline{\alpha}^i + {}_0\underline{w}^i \times [{}_0A^i I'^i({}_0A^i)^T]{}_0\underline{w}^i \\
&+({}_0A^i{}_i\underline{d}^{i+1}) \times ({}_0A^{i+1}\underline{F}_{i,i+1}) \\
&+({}_0A^{i+1}\underline{N}_{i,i+1})
\end{aligned}
\qquad (5\text{-}83)
$$

Since ${}_0\underline{\epsilon}^i + {}_0A^i{}_i\underline{a} + 2{}_0\underline{w}^i \times {}_0A^i{}_i\underline{v}$ is the acceleration of the center of mass of link i, therefore the term

$$M'^i\{{}_0A^i{}_i\underline{p}'^i \times ({}_0\underline{\epsilon}^i + {}_0A^i{}_i\underline{a} + 2{}_0\underline{w}^i \times {}_0A^i{}_i\underline{v})\}$$

is the torque at the origin of coordinate frame i due to the acceleration of the center of mass of link i. This term will not appear if we use the center of mass of the link as the reference to balance the torque[Luh80], however, some other terms will appear in the equation.

In general, the following equation is used when the reference point for balancing the torque is not at the center of mass.

$$\underline{N} = d_t(\underline{L}) + \underline{r}_c \times d_t(\underline{M}) \tag{5-84}$$

where $\underline{N}$ is the torque at the reference point, $\underline{L}$ is the total angular momentum, and $\underline{M}$ is the total linear momentum of the body. The vector $\underline{r}_c$ is measured from the reference point to the center of mass of the body.

5.4 Algorithm for Constructing the Recursive Dynamical Model

The results of this section suggest the following algorithm:

Given the following information:

a. the kinematic model of the manipulator.

b. the joint variables, ${}_i\underline{s}$ ${}_i\bar{n}$ & ${}_i\dot{\underline{s}}$ ${}_i\bar{N}$ for i=1,2....,N

c. the boundary conditions:

$${}_0\underline{w}^0 = \underline{0}$$

$${}_0\underline{\alpha}^0 = \underline{0}$$

$${}_0\underline{d}^0 = \underline{0}$$

$${}_0\underline{v}^0 = \underline{0}$$

$${}_0\underline{\epsilon}^0 = \underline{g}$$

${}_0A^{N+1}\underline{F}_{N,N+1}$ the external force acting on end effector

${}_0A^{N+1}\underline{N}_{N,N+1}$ the external torque acting on end effector

then the algorithm of construction the recursive dynamic model:

Step

0: Set i=1

1: Compute ${}_0\underline{w}^i$, ${}_0\underline{\alpha}^i$, ${}_0\underline{v}^i$ and ${}_0\underline{\epsilon}^i$ by equations 5-51:5-54 and 5-67:5-70.

2: if i=N, then continue to step 3;
Otherwise, set i=i+1 and return to step 1.

3: Compute ${}_0A^i\underline{F}_{i-1,i}$ and ${}_0A^i\underline{N}_{i-1,i}$ by equations 3-77, 3-78 and 3-83.

4: If i=1, then stop;
Otherwise, set i=i-1 and return to step 3.

CHAPTER 6

DYNAMICS OF MOVABLE ROBOT

In the last three chapters, we have derived the dynamical model for an arbitrary open chain manipulator. From the dynamical model one is able to compute the force and torque required of the actuactors at each joint. Above all, it is possible to compute forces and torques that are not necessarily in the direction of motion. These are the constraint forces and torques of each joint respectively. The constraint force and torque are normally not considered in a stationary rigid robot manipulator. However, if the robot is movable(i.e. has a nonstationary base. Note also that we deliberately use the word "movable" and not "mobile"), or the manipulator is flexible, then the constraint forces and torques become important. This thesis will present an example of each case to illustrate the significance of the constraint force and torques. No attempt has been made to thoroughly investigate these two problems. We just want to demonstrate possible applications of "generalized manipulators".

This chapter deals with the first application of the dynamical models, namely the analysis of movable robot.

6.1 The Problems in Modelling the Movable Robot

There are three problems in modelling a movable robot: a. Constraints on the wheels b. Coupling effects between the moving base and the other robot links. c. The changing configuration of the system.

These three problems will be discussed in the above order and the equations that describe the effects will be derived. These equations are very complicated and require numerical solution. The last section of this chapter presents an algorithm to solve a class of the problems related to movable robots.

6.1.1 The Constraints on The Wheels

The movable robot uses its wheels to travel on the ground through the frictional force. The frictional force(F) is proportional to the reaction force(N) from the ground to the wheel. The constant of proportionality μ is called the "frictional coefficient".

$$\mu = \frac{F}{N} \tag{6-1}$$

The wheels can either roll or slide on the ground as different frictional forces on the wheels are encountered.

Usually, the rolling fricitonal force is much smaller than the sliding frictional force. Since all the frictional forces are proportinal to the reaction force, the rolling fricitonal coefficient(μ_r) is much smaller than the sliding fricitional coefficeint(μ_s).

$$\mu_r << \mu_s \tag{6-2}$$

The robot will encounter stronger resistance to start moving than to maintain its speed(sticking effect). Thus the statical frictional force(denoted by μN) is larger than the dynamical frictional force(denoted by $\mu' N$). In terms of the frictional coefficients, the following inequality holds:

$$\mu' < \mu \tag{6-3}$$

The above frictional coefficients provide the maximum resistance force of that kind for which the ground can hold the wheels. If the applied force is not big enough, the magnitude of the frictional force is exactly the same as that of the applied force. This till the applied force overcomes the friction, then the frictional force will be constant according to equation 6-1. Mathematically, denote the applied force by F, then the frictional force(f) will be:

$$f = -F \qquad \text{if} \quad F < \mu N \tag{6-4}$$

$$f = \mu N \qquad \text{if} \quad F \geq \mu N \tag{6-5}$$

a. The Fixed Type Wheels

Fixed type wheels are restricted to roll in one direction, although sliding may occur in any direction.

Let v and a be the linear velocity and acceleration of the robot base respectively, and w be the angular velocity of the wheels. The the equations of motion of the wheels along the direction of rolling are:

$$\text{if } F < \mu_r N < \mu_s N \text{ , then } \quad v = 0 \tag{6-6}$$

$$\text{and} \quad a = 0 \tag{6-7}$$

$$\text{if } \mu_r N \leq F < \mu_s N \text{ , then } \quad v = R\,w \tag{6-8}$$

$$\text{and} \quad M\,a = F - \mu_r' N \tag{6-9}$$

$$\text{if } \mu_r N < \mu_s N \leq F \text{ , then } \quad M\,a = F - \mu_s' N \tag{6-11}$$

where M is the total mass of the system and R is the radius of the wheel.

Equation 6-6:6-7 imply that if the force is not large enough to move the system, no motion is expected. Equation 6-8:6-9 say that the force is sufficiently large to roll the wheels but not large enough to make the wheels slide. The last equation indicates that the force is large enough to roll as well as to slide the wheels.

The equations of motion of the wheels in the direction that is perpendicular to the direction of rolling (due to a perpendicular planar applied force $F_\perp$) are:

$$\text{if } F_\perp < \mu_s N \text{ , then } v_\perp = 0 \quad (6\text{-}12)$$

$$\text{and } a_\perp = 0 \quad (6\text{-}13)$$

$$\text{if } F_\perp \geq \mu_s N \text{ , then } M\, a_\perp = F - \mu_s{}' N \quad (6\text{-}15)$$

The rolling frictional coefficient does not enter the equation because rolling is not permitted in this direction.

The equations which relate the linear and angular velocities are the kinematical constraint equations; and the equations which relate the acceleration and the force are the dynamical constraint equations.

b. The Free Spinning Type Wheels

For the free spinning type wheels, the wheels always align themselves along the direction of the applied force.

Therefore, along the direction of the applied force, the constraint equations are :

$$\text{if } F < \mu_r N < \mu_s N \text{ , then } v = 0 \quad (6\text{-}16)$$

$$\text{and } a = 0 \quad (6\text{-}17)$$

$$\text{if } \mu_r N \leq F < \mu_s N \text{ , then } v = R\, w \quad (6\text{-}18)$$

$$\text{and } M\, a = F - \mu_r{}' N \quad (6\text{-}19)$$

$$\text{if } \mu_r N < \mu_s N \leq F \text{ , then } M\, a = F - \mu_s{}' N \quad (6\text{-}21)$$

where M is the total mass of the system and R is the radius of the wheel.

But, perpendicular to the direction of the force, the constraint equations are:

$$v_{\perp} = 0 \qquad (6\text{-}22)$$

$$a_{\perp} = 0 \qquad (6\text{-}23)$$

6.1.1.1 Dynamical Stability of The Robot

Consider an example. Let us assume that the two link with three degree of freedom robot(fig.4) is equipped with three wheels at its base(fig.13). One of the wheels is of the free spinning type and the other two are of the fixed type. The method of solving this particular movable robot indicates the direction of approach to this kind of problems. This thesis does not attempt to derive a general solution to the problem of mobile robots.

The first consideration of a movable robot is to investigate whether the robot is dynamically stable while it is moving(What we mean by that is - what are the conditions for the robot and the motion, such that the robot maintains an upright position while moving and does not topple down). In order to reduce the complexity of the equations, let us place the x-y coordinate plane on the ground and z-axis vertically upwards(fig.13).

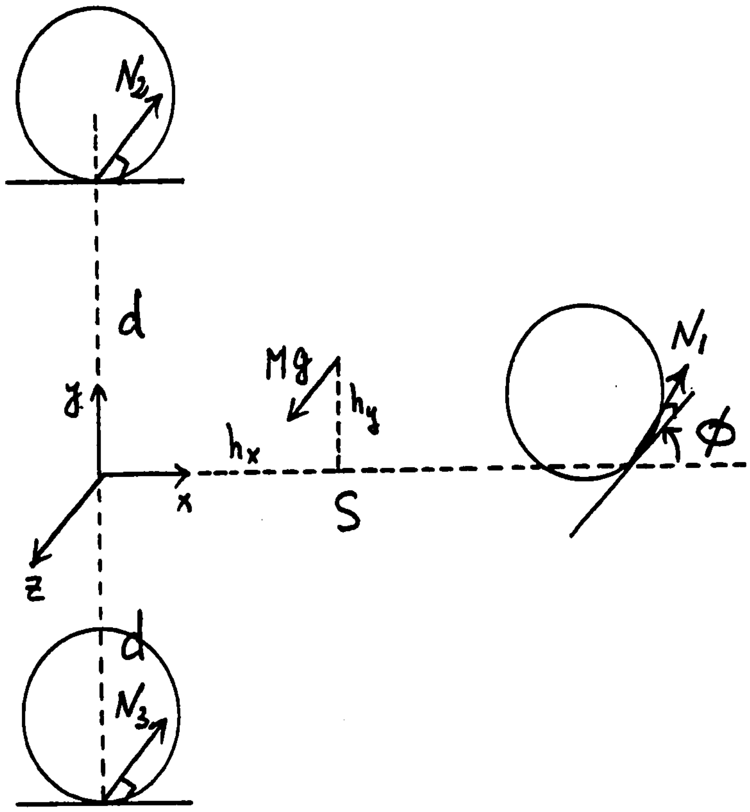


Figure 13 : Coordinate Frame of the Mobile Base of a Movable Robot

Note, none of the wheels is actuated. The base may move just as a reaction to motion of the robot links. Let N_1, N_2 & N_3 be the ground reaction force to wheel 1, 2, 3 respectively(fig.13), and h_x, h_y be the horizontal position of the center of mass of the robot in the base coordinate. Let s be the distance of the free spinning wheel from the origin of the base coordinate, and d is half the distance of the two fixed wheels. Assuming that the robot does not topple while moving, the following equations should hold:

$$F_z - M \underline{g} - N_1 - N_2 - N_3 = 0 \quad (6\text{-}31)$$

$$N_x - M \underline{g}\, h_y + N_2\, d - N_3\, d = 0 \quad (6\text{-}32)$$

$$N_y + M \underline{g} (s - h_x) - N_1\, s = 0 \quad (6\text{-}33)$$

By the stability assumption,

$$N_1 \geq 0 \quad (6\text{-}34)$$

$$N_2 \geq 0 \quad (6\text{-}35)$$

$$N_3 \geq 0 \quad (6\text{-}36)$$

Thus from equation 6-33,

$$N_1 = (N_y + M \underline{g}(s - h_x)) / s \quad (6\text{-}37)$$

which implies

$$h_x \leq s + N_y / (M \underline{g}) \quad (6\text{-}38)$$

From equations 6-31 & 6-37 and 6-32, we get

$$N_2 + N_3 = F_z + (M \underline{g}\, h_x) / s - N_y / s \quad (6\text{-}40)$$

$$N_2 - N_3 = (M \underline{g} h_y) / d - N_x / d \tag{6-41}$$

then $N_2 = \frac{1}{2}\{ F_z + M \underline{g} (h_x/s + h_y/d) - (N_y/s + N_x/d)\}$ (6-42)

and $N_3 = \frac{1}{2}\{ F_z + M \underline{g} (h_x/s - h_y/d) - (N_y/s - N_x/d)\}$ (6-43)

Using equations 6-35, 6-38 & 6-42, we obtain

$$h_y \geq N_x / (M \underline{g}) - (F_z d) / (M \underline{g}) - d \tag{6-44}$$

Equations 6-36, 6-38 & 6-43 will yield

$$h_x \leq N_x / (M \underline{g}) + (F_z d) / (M \underline{g}) + d \tag{6-45}$$

Equations 6-38, 6-44 & 6-45 define the dynamical stability region of the center of mass for a moving robot. If the system is stable, equations 6-37, 6-42 & 6-43 provide the reaction force of the three wheels respectively.

In the above inequalities, N_x, N_y and F_z are the external inputs.

6.1.1.2 The Spatial Axis of Rotation

If the robot is stable when it is moving, then it is restricted to travel on the horizontal plane only. Assuming that all the wheels are rolling without slipping, then the robot rotates about a certain common spatial axis. This spatial axis of rotation of the robot can be shown to be the vertical line coming out of the unique intersection point of all planar normals to the wheels directions(fig.14). The

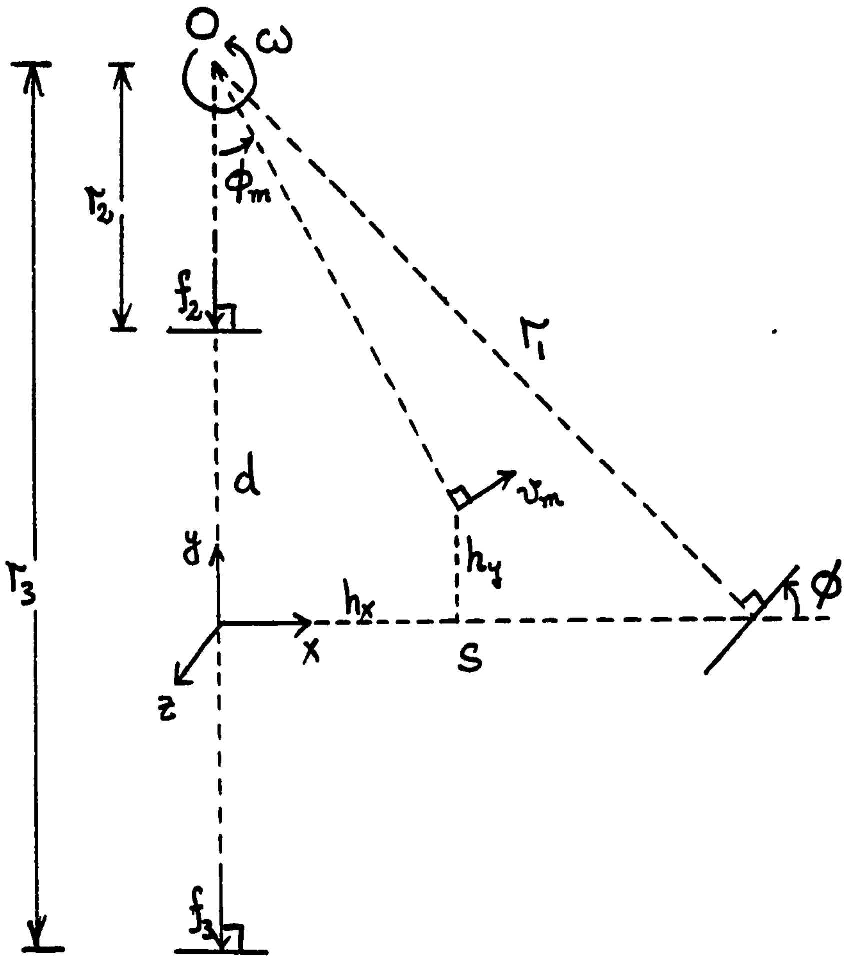


Figure 14 : Spatial Axis of Rotation of the Three Wheels Movable Robot

wheels travel along a circle whose center is the above intersection point since no rolling is allowed along the radial direction. Such common intersection point may not always exist. In that case, the robot's travel involves not just rolling but also slipping. Analysis of this problem is very involved. Here, the wheels of the robot are assumed to be well aligned such that a spatial axis of rotation indeed exists.

6.1.1.3 Kinematic Constraints of Rolling Robot

The robot in fig.14 has three wheels. Wheel 1 is the free spinning type and wheels 2 & 3 are the fixed type(fig.14). Assuming that the robot rolls only, then the position of the spatial axis of rotation is determined by the angle of wheel 1, $\emptyset$. The angular velocity and acceleration of the robot with respect to the spatial axis of rotation is w and α respectively. The radius of the wheels is denoted by R and they rolls at an angular speed β.

The distances that the wheels travel on the circles equal to the total length that the wheels have rolled respectively. Denote by r_1, r_2 & r_3 the distance between each wheel to the spatial axis of rotation. Then the following equations hold:

$$R_1 \ \beta_1 = r_1 \ w = s \ \csc\emptyset \ w \qquad (6\text{-}46)$$

$$R_2 \ \beta_2 = r_2 \ w = (s \ \cot\emptyset - d) \ w \qquad (6\text{-}47)$$

$$R_3 \ \beta_3 = r_3 \ w = (s \ \cot\emptyset + d) \ w \qquad (6\text{-}48)$$

These are the kinematic constraint equations of the wheels of the robot.

Since the Newton-Euler equations are referred to the center of mass, a kinematic constraint equation of the center of mass (denoted by subscript m) of the robot is needed.

Since $v_m = r_m \ w$ (6-49)

where $r_m = (h_x^2 + (s \ \cot\emptyset - h_y)^2)^{\frac{1}{2}}$ (6-60)

and $\emptyset_m = \text{arc cot} ((s \ \cot\emptyset - h_y) / h_x)$ (6-61)

$w = d_t(\emptyset_m) = (s \ \sin^2\emptyset_m \ d_t(\emptyset)) / (h_x \ \sin^2\emptyset)$ (6-62)

$\Omega == d_t(\emptyset)$ (6-63)

then $v_x == (v_m)_x = r_m \cos\emptyset_m \ w$ (6-64)

and $v_y == (v_m)_y = r_m \sin\emptyset_m \ w$ (6-65)

hence $a_x = d_t(v_x)$

$= d_t(r_m) \cos\emptyset_m \ w - r_m \sin\emptyset_m \ w + r_m \cos\emptyset_m \ \alpha$ (6-66)

and $a_y = d_t(v_y)$

$= d_t(r_m) \sin\emptyset_m \ w + r_m \cos\emptyset_m \ w + r_m \sin\emptyset_m \ \alpha$ (6-67)

By differentiating equation 6-50, one gets:

$d_t(r_m) = -(s \ \Omega (s \ \cot\emptyset - h_y)) / (r_m \sin^2\emptyset)$ (6-67)

After simplications, equations 6-66:6-67 become:

$$a_x = - (s / \sin^2\emptyset) \Omega w + (s \cot\emptyset - h_y) \alpha \qquad (6\text{-}68)$$

$$a_y = h_x \alpha \qquad (6\text{-}69)$$

Equations 6-68 & 6-69 describe the relationship between a_x, a_y and w, Ω. In forcing mode application, the wheels are actuated(thus w, Ω are inputs), then a_x, a_y are determined from the equations. In non-forcing mode application, the wheels are not actuated and are rolled by the reaction force/torque from the manipulator. Then w, Ω are computed from the dynamical model(6-62:6-63) and a_x, a_y are determined by equations 6-68:6-69.

6.1.1.4 Dynamical Constraints of the Non-slipping Robot

The robot in fig.14 has three wheels. The wheel 1 is the free-spinning type, therefore, there is no frictional force perpendicular to it. On the other hand, wheels 2 & 3 are of the fixed type, so that there exists a frictional force in a perpendicular direction. The perpendicular frictional force of the two fixed wheels keep the robot on the circumference of the circle(fig.14) and prevent the robot from sliding away from it. The magnitudes of these two frictional forces are described by the constraint equations of the non-slipping robot as follow:

$$f_2 \leq \mu_s N_2 \qquad (6\text{-}70)$$

or $$f_3 \leq \mu_s N_3 \qquad (6\text{-}71)$$

We may combine equations 6-70 & 6-71, since we exclude the possibility of one wheel rolling while the other wheel is slipping. In this case, then the constraint equation of the non-slipping robot becomes:

$$f = f_2 + f_3 \leq \mu_s (N_2 + N_3) \qquad (6\text{-}72)$$

6.1.1.5 Horizontal Movement of the Robot

If the robot in motion is stable(6.1.1.1) and there is no slipping, then the wheels of the robot are just rolling(6.1.1.3) around a vertical axis in space(6.1.1.2). In this example, assuming that there is no external force, so that the robot moves purely by the reaction to motions of the robot links.

The coordinate frame of the base of the robot is set at the horizontal plane on the ground. Under this convention, the forces and torques are separated into two functional groups where F_z, N_x, N_y determine the stability of the system and F_x, F_y, N_z control the horizontal movement of the robot.

Refer to fig.14. According to Newton's second law, the acceleration of the center of mass of the system is:

$$M\ a_x = F_x - \mu_r' N_2 - \mu_r' N_3 - \mu_r' N_1 \cos\emptyset \qquad (6\text{-}73)$$

$$M\ a_y = F_y - f_1 - f_2 - \mu_r' \sin\emptyset$$
$$= F_y - f - \mu_r' \sin\emptyset \qquad (6\text{-}74)$$

According to Euler's equation, the balance of torques with respect to the spatial axis at point O is:

$$M\ a_x\ (\ s \cot\emptyset - h_y\ ) + (\ M\ a_y\ h_x + I_{zz} + M\ s^2 \cot^2\emptyset\ )\alpha$$
$$= N_z + F_x\ s \cot\emptyset - \mu_r' N_2 (s \cot\emptyset - d) - \mu_r' N_3 (s \cot\emptyset + d)$$
$$- \mu_r'\ N_1\ s \csc\emptyset \qquad (6\text{-}75)$$

where I_{zz} is the inertial moment w.r.t. the z-axis at the coordinate origin. Notice that the cross inertial moments are assumed negligible.

After substituting equations 6-37, 6,42, 6-43, 6-68 & 6-69 into equations 6-73:6-75, we obtain:

$$C_{11}\ \alpha + \qquad C_{13} = -f \qquad (6\text{-}76)$$

$$C_{21}\ \alpha + C_{22}\ \Omega\ w + C_{23} = 0 \qquad (6\text{-}77)$$

$$C_{31}\ \alpha + C_{32}\ \Omega\ w + C_{33} = 0 \qquad (6\text{-}78)$$

where

$$C_{11} = M\ h_x \qquad (6\text{-}79)$$

$$C_{12} = 0 \qquad (6\text{-}80)$$

$$C_{13} = \mu_r'\ (\ N_y/s + M\ g\ (1 - h_x/s))\sin\emptyset - F_y \qquad (6\text{-}81)$$

$$C_{21} = M\ s\ (\ \cot\ \emptyset - h_y\ /\ S\ ) \qquad (6\text{-}82)$$

$$C_{22} = -\ M\ s\ /\ \sin^2\emptyset \qquad (6\text{-}83)$$

$$C_{23} = \mu_r'(F_z - N_y/s + M g h_x/s)$$
$$+\mu_r'(N_y/s + M g (1 - h_x/s))\cos\emptyset - F_x \quad (6\text{-}84)$$
$$C_{31} = I_{zz} + M h_x^2 + M(s \cot\emptyset - h_y)^2 + M s^2\cot^2\emptyset \quad (6\text{-}85)$$
$$C_{32} = - M s (s \cot\emptyset - h_y) / \sin^2\emptyset \quad (6\text{-}86)$$
$$C_{33} = \mu_r'[(F_z s - N_y + M g h_x) - F_x s]\cot\emptyset$$
$$+\mu_r'[N_y + M g (s - h_x)]\csc\emptyset$$
$$+\mu_r'(N_x + M g h_y) - N_z \quad (6\text{-}87)$$

In non-forcing mode applications, we are interested in estimating the react horizontal movement of the system due to the motions of the links so that control law for the wheels can be designed. In this case, $\underline{F}$, $\underline{N}$ are determined from the dynamical model, h_x, h_y are derived from the kinematical model, and M, s, d are the given system's parameters together with the physical constants μ_r' and g. Therefore, there remain three unknowns in equations 6-76:6-78, namely, Ø(t), w(t) & f. So solution to this set of coupled nonlinear differential equations is possible only numerically.

The solution of the equation set is easier to obtain if it the equations are solved iteratively. Multiplying equation 6-77 by $(s \cot\emptyset - h_y)$ and subtracting from equation 6-78, one obtains:

$$\alpha = \{ N_z + F_x h_y + \mu_r' (M g h_y - N_x) - \mu_r' (N_y + M g (s - h_x))(\sin\emptyset + h_y \sin\emptyset / s) \} / (I_{zz} + M h_x^2 + M s^2 \cot^2\emptyset) \quad (6\text{-}88)$$

Given the initial conditions $\emptyset(t=0)$ & $w(t=0)$, then equation(6-88) will provide the values of $\alpha(t=0)$ and $w(t=1)$. From equation 6-76, we obtain $\dot{f}$ to verify whether the no-slipping condition is met or not. If slipping does not occur, we can use equation 6-77 or 6-78 to estimate Ω and update $\emptyset$ for the next computation cycle.

One disadvantage of iterative solution is that error accumulates. The estimation may be supported by introducing a position sensor at the free-spinning wheel such that $\emptyset(t)$ is measured. The position control algorithm is implemented to hold the system on track. The above iterative algorithm provides the estimation of $\emptyset$ for feedforward control. The value of $\emptyset$ can be easily controlled within a certain required tolerance.

The following three difficulties must be resolved before solving the above set of equations. The first is that the links are actuated(i.e. external energy is provided to the robot). Thus the system cannot be considered to be a passive mechanical system as was assumed throughout. This is a problem for future studies.

The second difficulty is the coupling effect between the links and the base. The motion of the links cause the system to move, while motion of the base influences back the motion of the links. This interaction can be resolved by a simple algebraic manipulation of the dynamical equations as will be shown in section 6.1.2.

Finally, the center of mass of the system depends on the robot configuration. Thus it is time varying. The exact equation showing this phenomenon is presented in section 6.1.3.

6.1.2 Coupling Effect Between the Links and the Moving Base

In the dynamical models derivation of chapters 3-5, it has been assumed that the base of the robot is stationary. If the base of the robot is not firmly fixed, then the reaction force and torque(6.1.2) from the motions of the links will make the base move(6.1.1). The problem of movable robot is not completely solved by the above two sections since the motion of the base affects back the movements of the links. This coupling effect will be addressed next.

a. The Coupling Force

Consider a particle, dm, in one of the links. It has an acceleration, $\underline{a}_t$. The acceleration $\underline{a}_t$ of the particle consists of two components. The acceleration $\underline{a}_0$ if the base does not move, and the acceleration $\underline{a}_b$ of the base. According to the Newton 2nd law, the total force $\underline{F}_t$ on the base is:

$$\underline{F}_t = \int_{J_0}^{E} \underline{a}_t \, dm = \int_{J_0}^{E} \underline{a}_0 \, dm + \int_{J_0}^{E} \underline{a}_b \, dm \qquad (6\text{-}89)$$

So the reaction force to accelerate the base is:

$$M_b \, \underline{a}_b = -\underline{F}_t = -\int_{J_0}^{E} \underline{a}_0 \, dm - \int_{J_0}^{E} \underline{a}_b \, dm \qquad (6\text{-}90)$$

hence $\left(M_b + \int_{J_0}^{E} dm \right) \underline{a}_b = - \int_{J_0}^{E} \underline{a}_0 \, dm$

hence $M \, \underline{a}_b = - \underline{F}_0$ (6-91)

where M is the total mass of the system and $\underline{F}_0$ is the force of the joint at coordinate 0 where the minus sign indicates the reaction force.

b. The Coupling Torque

The total torque of the joint at the base is:

$$\underline{N}_t = \int_{J_0}^{E} {}_0\underline{r} \times \underline{a}_t \, dm = \int_{J_0}^{E} {}_0\underline{r} \times \underline{a}_0 \, dm + \int_{J_0}^{E} {}_0\underline{r} \times \underline{a}_b \, dm$$

$$= \underline{N}_0 + \int_{J_0}^{E} {}_0\underline{r}\ dm \times \underline{a}_b$$

$$= \underline{N}_0 + \sum_{i=0}^{N} M_i \ ({}_0\underline{p}^i \times \underline{a}_b) \tag{6-92}$$

According to Euler's equation to the origin of coordinate 0 that does not coincide with the center of mass of the base:

$$- \underline{N}_t = - \underline{N}_0 - \sum_{i=0}^{N} M_i \ ({}_0\underline{p}^i \times \underline{a}_b)$$

$$= d_t(I_b\ w) + M_b \ ({}_0\underline{p}^b \times \underline{a}_b) \tag{6-93}$$

$$\text{hence } - \underline{N}_0 = d_t(I_b\ w) + \sum_{i=b}^{N} M_i \ ({}_0\underline{p}^i \times \underline{a}_b) \tag{6-94}$$

$$= d_t(I_b\ w) + M\ {}_0\underline{h} \times \underline{a}_b \tag{6-95}$$

where $-N_0$ is the reaction torque from the base to the manipulator, M is the total mass of the system and ${}_0\underline{h}$ is the position of the center of mass of the system. I_b is the inertial matrix and w is the angular velocity of the base.

One final remark is: in equations 6-91 & 6-95, M is the total mass of the system and I_b is the inertial matrix of the base that has not been obvious to forsee that coupling occurs only with regard to mass but not with regard to inertia.

6.1.3 Changing Configuration of the System

As indicated in the last section, the center of mass of the system has been used to compute the reaction movement of the base. The center of mass of the system varies as the manipulator changes its configuration and it equals to:

$$h_x = \sum_{i=b}^{N} M_i \, (\underline{p}^i)_x \tag{6-96}$$

$$h_y = \sum_{i=b}^{N} M_i \, (\underline{p}^i)_y \tag{6-97}$$

$$h_z = \sum_{i=b}^{N} M_i \, (\underline{p}^i)_z \tag{6-98}$$

where $M = \sum_{i=b}^{N} M_i$

where $\underline{p}^i$ is the position vector of the center of mass of link i measured in the base coordinate, and $\underline{h}$ is the position of the center of mass of the system.

6.2 Procedures of Solving the Problem of Movable Robot

The above example has three wheels on the base where one is the free spinning type and the other two are the fixed type. The procedure to be presented shortly refers to this example. However, the procedures can easily be extended to movable robots with any combination of wheels type. The procedure to solve for the reaction movement of the robot whose wheels are not actuated is summarized below:

Step 0 : Measure the mass of the system and I_{zz} of the body base of the movable robot.

Step 1 : Set the coordinate frame of the base of the movable robot at the ground level where the origin is at the middle of the line connecting the two fixed wheels, and the x-axis points towards the free spinning wheel(6.1.1).

Step 2a: Compute the force and torque at the first joint of the manipulator using the dynamical models derived in chapters 3, 4 and 5.

Step 2b: Perform a statical transformation of force and torque from the manipulator base coordinate frame to the body base coordinate frame using equations 2-105 & 2-110. Make sure to reverse signs since reaction force and torque are assumed in the equations.

Step 3 : Compute the horizontal position(h_x, h_y) of the center of mass of the movable robot(6.1.4).

Step 4 : Check if the robot is stable dynamically or not by equations 6-38, 6-44 & 6-45.

Step 5 : If the robot is stable, use equations 6-37, 6-42 & 6-43 to compute the reaction forces on the wheels. Otherwise, the robot is toppled down and the procedure should be aborted.

Step 6a: Get the initial position of the angle of the free spinning wheel, Ø, with respect to the body base coordinate (fig.13). This angle may be arbitrarily assumed if a position sensor is not avaliable.

Step 6b: Obtain the inital angular velocity of the robot, w, with respect to the spatial axis of rotation. This angular velocity, w, can be computed from the angular velocity of any wheel by equations 6-46:6-48 if only rolling motion of the wheels is assumed. If the wheel has a postion sensor or tachometer, then this information may be measured; otherwise, it is estimated.

Step 7 : Compute the induced angular acceleration of the robot using equation 6-88.

Step 8 : Compute the frictional force, f, perpendicular to the fixed type wheels by equations 6-76, 6-79:6-81.

Step 9 : Check if the robot is slipping or not by equation 6-72.

Step 10: If the robot is not slipping but rolling only, then use equation 6-77 or 6-78 to compute $d_t(\emptyset)$. Otherwise, the model breaks down.

Step 11: Update the estimation of $\emptyset$ and Ω, or use these values for the feedforward control.

Step 12: Go back to step 3.

CHAPTER 7

STATICS OF FLEXIBLE MANIPULATOR

We have seen how the generalized manipulator model can be applied to the problem of mobile robot. It can also be used to study the flexible manipulator. The investigation of the flexible manipulator problem is still an open problem and is beyond the scope of this thesis. A simplified version, namely "the static problem of flexible manipulators" is studied. The purpose is to show how a generalized manipulator model can be applied.

There are two major static problems of flexible manipulators. One problem is to find how much the end effector is deflected when it approaches the target, and the other problem is to find whether the material is strong enough to sustain the operation of the manipulator. This thesis demonstrates how basic statics can be incorporated with the generalized manipulator model to obtain useful information for answering these two questions.

The deflection of the end effector of the manipulator is treated first, and the computation of the critical loading of the manipulator is considered later.

7.1 Static Deflection of the End Effector of the Manipulator

When the manipulator moves, time varying forces and torques are exerted on each link. When the force or torque does not exceed the critical loading of the material which is used to construct the manipulator link, the link is deflected. Otherwise, the manipulator link will be broken. Since the forces and torques are time varying, they should be treated using dynamic analysis. This thesis assumes that the bandwidth of the signals that drive the actuators are well below the dominant structural frequency of the manipulator, such that statical analysis may be used as an approximation.

7.1.1 Deflection of a Manipulator Link

Deflection of the manipulator link occurs whenever the relative position of any two points in the same link changes. The complete treatment of deflection is complicated. In order to simplify the analysis, the following three assumptions are made :

1. Deflection occurs in a "beam" but not in a lump mass. The manipulator link is called a "beam" if its length is much larger than the width of its cross section.

Manipulator links which are not beams are categorized as lump masses. In figure 4, link 2 may be taken as a beam while other links are taken as lump masses.

2. Deflection of a beam is mostly due to bending but not due to compression or twisting. Bending occurs perpendicular to the longitudinal direction of the beam.

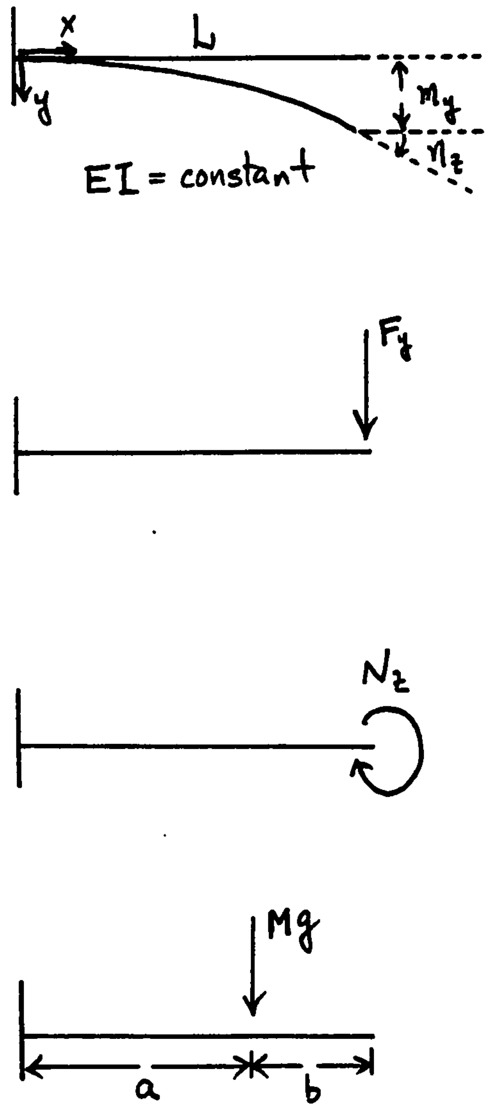
3. The effect of shear deformation is negligible.

4. Deflection is insignificant along the axis of motion, but plays a major role in balancing the force and torque in the other directions. In other words, the beam is dynamically balanced along the axis of motion, but is statically balanced in the other directions.

The first three assumptions are generally valid for materials used to build manipulators(typically steel). The fourth assumption is a rough approximation, and should not be made in general.

Bending of a Cantilever Beam

The cantilever beam has one fixed end while the other end is free(fig.15). If a bending force F_y is applied, the bending displacement(m) and angle(n) can be computed by the following equations[Timo72]:



m_y = bending displacement

n_z = bending angle

$$m_y = \frac{L^3}{3EI} F_y$$

$$n_z = \frac{L^3}{2EI} F_y$$

$$m_y = \frac{L^3}{2EI} N_z$$

$$n_z = \frac{L^3}{EI} N_z$$

$$m_y = \frac{a^2(3L-a)}{6EI} Mg$$

$$n_z = \frac{a^2}{2EI} Mg$$

Figure 15 : Static Deflection of a Cantilever Beam

$$m_y = \frac{L^3}{3EI} F_y \tag{7-1}$$

$$n_z = \frac{L^3}{2EI} F_y \tag{7-2}$$

If a bending torque N_z is applied, then

$$m_y = \frac{L^3}{2EI} F_y \tag{7-3}$$

$$n_z = \frac{L^3}{EI} F_y \tag{7-4}$$

where E is the modulus of elasticity and I is the moment of inertia of the cross section of the beam.

Equation 7-1:7-4 can be written in a vector form as follows:

$$\underline{m} = - \frac{L}{3EI} ((\underline{L} \times (\underline{L} \times \underline{F})) \tag{7-5}$$

$$\underline{n} = \frac{L}{2EI} (\underline{L} \times \underline{F}) \tag{7-6}$$

$$\underline{m} = - \frac{L}{2EI} (\underline{L} \times \underline{N}) \tag{7-7}$$

$$\underline{n} = \frac{L}{EI} \frac{((\underline{N} \times \underline{L}) \times \underline{L})}{\underline{L} \bullet \underline{L}} \tag{7-8}$$

The vector product of the bending force with the length vector of the beam eliminates the compression component of the force, and the vector product of the bending torque with the length vector of the beam eliminates the twisting component of the torque.

According to the second assumption, the deflection due to compression force and twisting torque are not considered. Therefore, the deflection of a beam is contributed by bending only and total bending displacement and angle are computed as follows:

$$\underline{m} = - \frac{L}{EI} \left((\underline{L}X(\underline{L}X\underline{F}))/3 + (\underline{L}X\underline{N})/2 \right) \quad (7\text{-}9)$$

$$\underline{n} = \frac{L}{EI} \left(\frac{(\underline{L}X\underline{F})}{2} + \frac{((\underline{N}X\underline{L})X\underline{L})}{\underline{L}\bullet\underline{L}} \right) \quad (7\text{-}10)$$

<u>Bending of a Manipulator Link</u>

The generalized manipulator model provides for the forces and torques for each coordinate frame. For each manipulator link, two consecutive coordinate frames are located at both ends. The computed forces and torques on the coordinate frame at the end which is closer to the base are the force and torque that the actuator should supply. The computed force and torque on the coordinate frame at the other end are the reaction force and torque of the actuactor at that joint(refer to the open dynamical chain concept in chapter III, secion 3). The forces and torques at the two ends are balanced by the gravity and the movement of that link. However, the movement of the link is only allowable along the axis of motion. In the other direction, no motion is allowed and the bending of the link provides for the balancing. This corresponds to the fourth assumption.

A manipulator link is not a cantilever beam, so equations 7-9:7-10 can not be used directly. If one end of the link is chosen as reference, that means the deflection is measured with respect to that coordinate frame, then the deflection of the other end should compensate for the difference of forces and torques at both ends of the link. It is equivalent to substituting the net force and torque to equations 7-9 & 7-10 to compute the bending displacement and angle.

Since the bending displacement and angle of a link is a relative measure, it requires a non-moving point as a reference. The base of the manipulator is assumed to be stationary, and is a natural choice for a reference. Thus the deflection of the end effector consists of two parts: the deflection that each link contributes and the propogation of each deflection to the end effector.

Denote the deflection of link i as ${}_i\underline{m}$ and ${}_i\underline{n}$, and the net force and torque is ${}_i(\delta\underline{F})$ and ${}_i(\delta\underline{N})$. The length of link i is ${}_i\underline{d}^{i+1}$. Then equations 7-9 & 7-10 can be written as:

$$ {}_i\underline{m} = - \frac{{}_i\underline{d}^{i+1}}{EI} \left(({}_i\underline{d}^{i+1} X ({}_i\underline{d}^{i+1} X_i(\delta\underline{F}))/3 + ({}_i\underline{d}^{i+1} X_i(\delta\underline{N})/2) \right) \quad (7\text{-}11) $$

$$ {}_i\underline{n} = \frac{{}_i\underline{d}^{i+1}}{EI} \quad \frac{({}_i\underline{d}^{i+1} X_i(\delta\underline{F}))}{2} \quad \frac{(({}_i(\delta\underline{N} X_i\underline{d}^{i+1}) X_i\underline{d}^{i+1})}{{}_i\underline{d}^{i+1} \bullet {}_i\underline{d}^{i+1}} \quad (7\text{-}12) $$

According to assumption 3, the component of the forces and torques that cause the motion do not produce deflection. Thus they need to be eliminated. The elimination is done by the following vector manipulation.

Let $\underline{s}$ be the unity vector in the direction of motion, for a vector $\underline{V}$, the vector cross product ($\underline{s}$ X $\underline{V}$) X $\underline{s}$ will eliminate the component of $\underline{V}$ along the direction $\underline{s}$; the vector scalar product $(\underline{V} \bullet \underline{s})\underline{s}$ keeps only the component along $\underline{s}$.

If joint i is prismatic, then both the applied and reacted force along the direction of motion have to be eliminated. The net bending force is:

$$ {}_i(\delta\underline{F}) = (\underline{s} \; X \; ({}_iA^{i+1} \, {}_{i+1}\underline{F} - {}_i\underline{F})) \; X \; \underline{s} \quad (7\text{-}13) $$

For most of prismatic joints, the actuators are located along the axis of motion. Therefore there is no deduction of the net torque.

$${}_{i}(\delta \underline{N}) = {}_{i}A^{i+1}\ {}_{i+1}\underline{N} - {}_{i}\underline{N} \quad (7\text{-}14)$$

If joint i is revolute, and the axis of motion is perpendicular to the longitudinal direction of the beam, then only the force component along the axis of motion will cause deflection. Hence the net force is:

$${}_{i}(\delta \underline{F}) = (\ (\ {}_{i}A^{i+1}\ {}_{i+1}\underline{F} - {}_{i}\underline{F}\ ) \bullet \underline{s}\ )\ \underline{s} \quad (7\text{-}15)$$

For the torque, the component along $\underline{s}$ has to be eliminated. Hence,

$${}_{i}(\delta \underline{N}) = (\ \underline{s} \times (\ {}_{i}A^{i+1}\ {}_{i+1}\underline{N} - {}_{i}\underline{N}\ )\ ) \times \underline{s} \quad (7\text{-}16)$$

Equations 7-11:7-16 compute the deflection of a manipulator link under the four previously stated assumptions. The orientation matrix ${}_{i}A^{i+1}$ is introduced to transfer the reference of vector in coordinate frame i+1 to coordinate frame i.

The above equations are not complete for calculating the deflection of a manipulator beam because gravity bending of that link has not been considered(as yet).

The gravity effect

Each manipualtor link will be deflected by gravity. Let gravity acceleration be ${}_{i}\underline{g}$, the mass of link i is denoted as ${}_iM$ and the center of mass is ${}_i\underline{p}'^{i}$, then the bending displacement and angle are computed by the following two equations[Timo72]:

$$-{}_i\underline{m} = \frac{{}_iM(\ 3\,{}_i\underline{d}^{i+1} - {}_i\underline{p}'^{i}\ )}{6EI}((\ {}_i\underline{p}'^{i} \times (\ {}_i\underline{p}'^{i} \times \underline{g}'))\quad (7\text{-}17)$$

$${}_i\underline{n} = \frac{{}_iM\,{}_i\underline{p}'^{i}}{2EI}\ (\ \underline{ip}'^{i} \times \underline{g}'\ ) \qquad (7\text{-}18)$$

wher $\underline{g}'$ is the bending gravity.

The bending gravity is computed as above. If link i is a prismatic link, then the bending gravity is:

$${}_i\underline{g}' = (\ \underline{s} \times {}_i\underline{g}\ ) \times \underline{s} \qquad (7\text{-}19)$$

and if link i is a revolute joint, then the bending gravity is:

$${}_i\underline{g}' = (\ {}_i\underline{g} \bullet \underline{s}\ )\ \underline{s} \qquad (7\text{-}20)$$

Equations 7-17:7-20 compute the bending displacement and angle by gravity under the above four assumptions.

7.1.2 Propagation of Static Deflection

In the last section, the equations to compute the bending displacement and angle of a manipulator link have been derived. How does this deflection propagate to the end effector? The propagation of deflection is caused by translation and rotation. These can be expressed by a single equation that is derived as follows:

In figure 16, link i & link i+1 have a fixed angle between them. Suppose link i has a small bending displacement ${}_i\underline{m}$ and bending angle ${}_i\underline{n}$. Then, link i+1 will have the following bending displacement and angle.

$$ {}_{i+1}\underline{m} = {}_iA^{i+1} \left({}_i\underline{m} + {}_i\underline{n} \times {}_i\underline{d}^{i+1} \right) \qquad (7\text{-}21) $$

and
$$ {}_{i+1}\underline{n} = {}_iA^{i+1} \; {}_i\underline{n} \qquad (7\text{-}22) $$

where the orientation matrix ${}_iA^{i+1}$ transfers the reference from coordinate frame i to coordinate frame i+1. Note that the sum in eqüation 7-21 is a vector sum, and it shows the rotational effect of the coordinate frame.

7.1.3 Computation on the Deflection of the End Effector

It is more efficient to compute the end effector deflection from the base towards the end effector since deflection propagates in that direction(refer to the open kinematic chain concept in chpater III, section 3). For closed form dynamical model, it is computed after the forces and torques are computed. For the recursive dynamical model, the kinematic model is computed from the base to the end effector, and the dynamical model is computed from the end effector to the base that complete a computational cycle

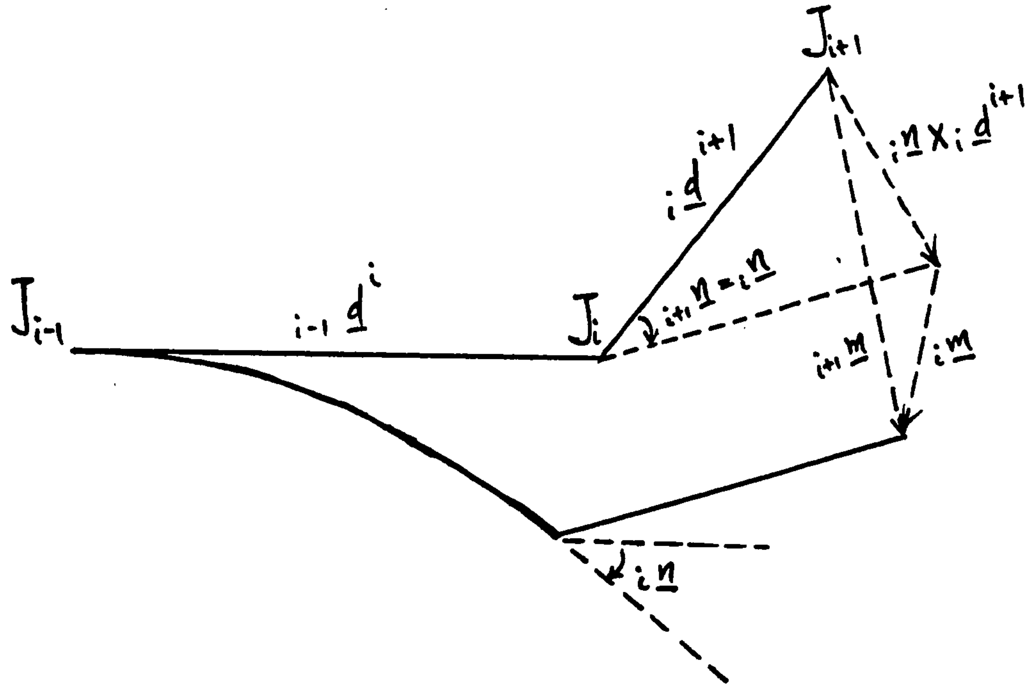


Figure 16 : Propagation of the Static Deflection

for a task point. The computation of deflection can be thought of the correction factor for the next cycle of kinematic model calculation. In this model, deflection is treated as a static effect(not a time varying signal). This means that it does not depend on previous state. The static deflection should be of concern only when approaching the target or when moving close to an obstacle. Note, the deflection from one computation to the next one are not related by a recursion formula, so there is no advantage in computing deflection elsewhere. These are of no practical use.

7.2 Critical Loading of the Manipulator Link

In selecting the material of the right strength to construct the manipulator, one assumes that the mounting is sufficiently strong. Attention is focused on the strength of the link itself. Refer to the dissertation by Book[Book74].

The generalized manipulator model can be used at the design phase for quick approximated evaluation of the strength of a manipulator. Normally, the slender the manipulator link, the more likely it may break. The derivation of the formula for computing the critical loading of a slender manipulator link is shown.

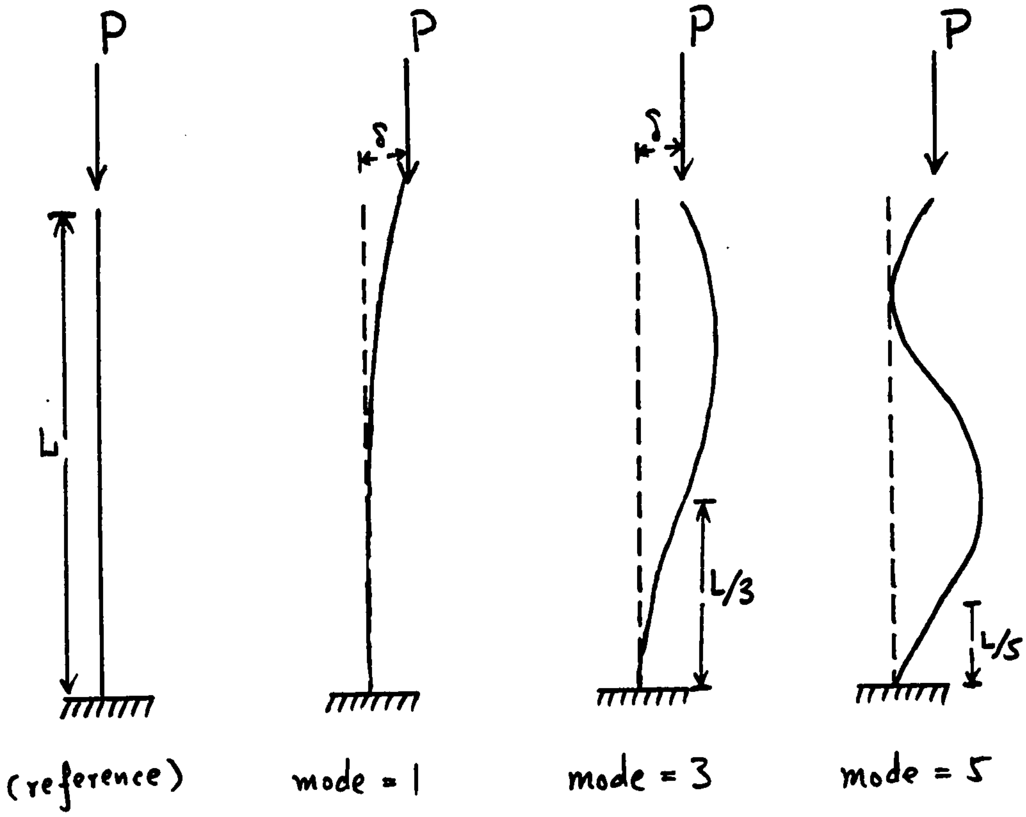
The manipulator link can break due to torsion, strain or stress. Most materials(metallic alloy) can sustain more torsion than strain or stress. Therefore, bending is more likely to be the major cause in a manipulator link breaking rather than twisting. In mechanical engineering literature, such as [Timo72], this subject is discussed under "elastic buckling of columns".

The critical elastic buckling load of a column is inversely proportional to the sqaure of the modal shape number of the column(fig.17). Considering the worst case, then the critical elastic loading(P_{cr}) of a column of mode 1 is:

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad (7\text{-}23)$$

This equation does not consider the shear deformation energy which is stored when the column is bent. The deflection curve remains the same but the critical load reduces a little bit[Spie68].

The critical elastic loading does not depend on the property of the material and the deflection of the column. It does not depend on the material because it assumes



$$P_{cr} = \frac{\pi^2 EI}{4 L^2} (\text{mode})^2$$

Figure 17 : Critical Elastic Loading of a Column

buckling can occur at any point of the column. Definitely more sophisticate treatment should be done. One example is to consider buckling at a localized area, for instance, to find out where does the maximum stress occur, this local buckling problem can be found in Book's disserataion[Book74].

The critical elastic loading does not depend on the deflection of the column based on the same reason. This thesis suggests to assign a safety factor in the design.

The generalized manipulator model is applied to this problem in the following way. The transverse force and torque will deflect the link according to the equations derived in the last section. Then the net longitudinal force is the buckling loading that should not be larger than the critical buckling load at any operation.

CHAPTER 8

CONCLUSIONS AND FUTURE DIRECTIONS

Four major results have been achieved in this thesis.

1. The closed form and the recursive form dynamical models of the generalized manipulator have been derived using Newton-Euler's method(chapters 3 and 5). These models extend existing models for robot arms.

2. The Newton-Euler formulation has been shown to be equivalent to the Lagrange formulation(chapter 4). The transformation between these two formulations is given in chapter 4.

3. The dynamical model of the generalized manipulator can be reduced to describe any one-degree-of-freedom-per-link rigid manipulators. The model applies not just along principal axes of motion.

4. Two applications have been given: the computation of the trajectory of the mobile robot driven by its manipulator's arm(chapter 6), and the static deflection of the flexible manipulators(chapter 7).

This thesis proves that the concept of the generalized manipulator is very useful, and is easy to handle mathematically. Because the generalized manipulator has no restriction in its motion at the joints, the difficulties of handling the boundary conditions(constraints) are avoided in the derivations. The boundary conditions(constraints) are considered when the dynamical model of the generalized manipulator is actually applied to the physical manipulators. This property is very useful when the dynamical model combines with the other models, because it can match to any boundary conditions.

The generalized manipulator concept offers a great deal of potential for developing sophisticated models. Here are some future directions.

1. We addressed the problem of flexible manipulators through using some results that are taken from a dynamical model for rigid bodies. This approximation must be tested via simulation and compared against Book's model. Future treatment of flexible manipulators(using generalized manipulators theory) should include the consideration of shear and torsion.

The dynamical model of the flexible manipulators should include the structural resonance frequency as a parameter. Refer to the dissertaion of [Book74] to see how structural frequency is expressed, and structural reasonance frequency is determined. In Book's dissertaion, the solution is for the two link planar manipulators. More general solution, such as for generalized manipulator, is of interest.

2. In deriving the dynamical model in this thesis, the manipulator arm is assumed fixed(clamped) in the base and the open kinematic chain concept can be applied. Further extensions of the model may arise when considering other boundary conditions of the base. In chapter 6, the generalized manipulator is equipped with a movable base, and the application of generalized manipulator dynamics to this kind of problem is shown. In chapter 6, only rolling of the movable robot is considered. Future studies should address sliding. Other combinations of wheels that are discussed in chapter 6 should be considered including actuated wheels.

One may address the issue of surface of motion(for instance, a situation where some wheels roll and the other slide, due to different frictional conditions). Including in this topic are subjects as : motion on a rail, motion on top of a moving platform, etc.

3. The open dynamic chain concept is valid only for free ended end effector(no external forces or torques acting on the end effector). This boundary condition for the end effector is no longer valid, if the end effector interacts with an object. The dynamical model of the generalized manipulator should be extended to accept any boundary condition for the end effector. This is the compliance problems in robotics.

4. In this thesis, we treated "generalized manipulator" as a mathematical abstraction that is useful in gaining some insight about difficult dynamic problems, such as flexible manipulators and movable robots. Do generalized manipulators physically exist ? Examples are not easy to find. Robot arms may be cascaded one to another(in an open kinematic chain). Taking, for instance, a robot manipulator that uses a single multi-degree-of-freedom finger, can such a robot be considered as a "two-link generalized manipulator" ? Not until the "Rigid Body assumption" is relaxed to allow for links that consist of finitely many rigid "sub-links". Such "generalized links" may form a tree if, for example, one studies robots that have two or more multi-degree-of-freedom fingers. However, such analysis requires a major extension of the generalized manipulator

model since both the "open kinematic" and "open dynamic" assumptions break down. The problem becomes even more difficult when two fingers interact in grasping an object. Such situation requires "Closed Kinematic Chain" analysis.

Extension of the generalized manipulator model to such problems is a major task, that may or may not be possible.

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