DISTRIBUTED LAG MODELS

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by

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ABSTRACT

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This thesis presents various formulations of finite distributed lag models. The objective is to demonstrate how prior restrictions may be imposed on econometric models in order to estimate the lag distributions. Four formulations are thus reviewed, namely, the arithmetic lag model, the inverted-V lag model, the Almon polynomial, and the cubic spline lag model. For the latter formulations, the interpolation methods are reviewed. In addition, four models of consumption are estimated under the various lag models, for different lag lengths and orders of polynomials, in order to demonstrate the properties of each formulation. In the discussion of the results, certain inferences are made about the consumption functions.

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I. INTRODUCTION

In economic theory and applications it is often hypothesized that the effect of a change in an independent variable does not occur all at once but rather the impact is distributed over a period of time. Such a situation is referred to as a distributed lag. When it is assumed that the impact is fully exhausted after a certain period of time, the distributed lag is said to be finite. Often this knowledge is incorporated into behavioral equations, most commonly those describing the behavior of firms and households.

The structure of the lag effect may take on a variety of shapes. The distribution function may be linear with a negative slope, or a "kinked" linear function. The assumption of linearity is also not necessary, as the function may also be a polynomial of any degree.

The general formulation of a finite distributed lag model is

$$Y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2}$$

$$+ \dots + \beta_n x_{t-n} + u_t$$

or

$$Y_t = \alpha_0 + \sum_{i=0}^{n} \beta_i x_{t-i} + u_t$$

where the β_i are the unknown lag coefficients or weights, x_{t-i} are lagged values of the independent variable, and n is the lagged length. This formulation is used throughout this study. By imposing certain restrictions on the model, one can obtain various lag distributions, which is the topic of this research.

Chapter two describes the early approaches to specification and estimation of finite distributed lags, namely the arithmetic and inverted-V formulations. It begins with a discussion of the assumptions of the arithmetic models and proceeds to illustrate its properties. Two methods of estimation are presented, namely: a composite variable and a restricted least squares approach. The same format is used to present the inverted-V lag model.

Chapter three presents polynomial distributed lag models. It begins with a general formulation and describes what is termed a direct approach to estimation. Almon, who originated the idea, suggested the use of Lagrangian interpolation in estimation. The method of Lagrangian interpolation and estimation of the polynomial model using this technique are presented next. The use of zero constraints to pre-determine the polynomial shape is also discussed. Both methods, the direct approach and the interpolation methods yield equivalent results. The chapter concludes with a restricted least squares approach.

Chapter four focuses on a generalization of the Almon model, specifically, cubic splines. Cubic splines have been used in the most varied applications and their use in estimating a polynomial lag model is discussed. There are once again two methods of estimation. A direct approach, which employs dummy variables and an interpolation approach. The mathematics of the spline interpolation are discussed next, followed by the method of estimation. The chapter concludes with a presentation of a restricted least squares approach.

Chapter five presents various models of consumption, when the aforementioned formulations are imposed, under varying lag lengths and polynomial specifications. The objective of this chapter is to illustrate the properties and characteristics of each of the lag formulations, when imposed on an econometric model. The chapter concludes with a summary of results and a few conclusions on the application of the lag models.

Chapter six concludes the study with a general summary and some conclusions. Further topics for research are also discussed.

II. EARLY APPROACHES TO THE SPECIFICATION AND ESTIMATION OF FINITE DISTRIBUTED LAGS

This chapter is concerned with the earliest approaches that were taken to the specification and estimation of finite distributed lag models. The first paper concerned with these problems was published by Irving Fisher in 1937. He introduced the arithmetic lag model which is the subject of the first section of this chapter. The statistical properties of the model are examined in the second section.

The arithmetic lag specification assumes that the effect of a change in the independent variable on the dependent variable declines in each succeeding period after the change occurs. This property of the arithmetic lag model is suitable for modeling a wide variety of economic phenomena. However, most researchers in the area of investment expenditures expect that the effect of a change in the independent variable will rise for several periods after the change occurs (Eisner & Strotz, 1963). The second half of this chapter reviews the first attempt to model this type of lag structure, namely, the inverted-V model of De Leeuw (1962). Section three discusses the specification of the model and the chapter concludes with an examination of its statistical properties.

The Arithmetic Lag Model

The arithmetic lag model assumes that the effect of a change of the independent variable diminishes linearly over succeeding time periods. In other words, the adjustment is the largest during the period when the explanatory variable changes and the subsequent adjustments are smaller, their effects diminishing linearly and by a constant amount until the change is exhausted.

The general formulation of a finite distributed lag model was given in the last chapter as

$$(2.1) \qquad y_t = \alpha_0 + \sum_{i=0}^{n} \beta_i x_{t-i} + u_t$$

where all β_i must have the same sign and they must have a finite sum. For simplicity, it will be assumed that all β_i are positive. The arithmetic lag model requires that β_i satisfy the following equation

$$(2.2) \qquad \beta_i = (n + 1 - i)\alpha_1 \qquad i = 0, 1, 2, \ldots, n$$

The term $(n+1-i)$ is positive for all values of i. This implies that the β_i have the same sign as α_1. The sum of the β_i is

$$\sum_{i=0}^{n} \beta_i = \sum_{i=0}^{n} (n+1-i)\alpha_1 = \alpha_1[\sum_{i=0}^{n} (n+1) - \sum_{i=0}^{n} i]$$

$$= \alpha_1[(n + 1)^2 - n(n + 1)/2]$$

$$= \alpha_1(n + 1)[n + 1 - n/2]$$

$$= \alpha_1(n + 1)[2n + 2 - n]/2$$

$$(2.3) \qquad \sum_{i=0}^{n} \beta_i = \alpha_1 (n + 1) (n + 2)/2$$

This is finite provided that α_1 is finite.

The linear relationship among the lag coefficients is more clearly evident if (2.2) is rewritten

$$(2.4) \qquad \beta_i = (n + 1)\alpha_1 - \alpha_1 i$$

This shows that β_i is a linear function of i with intercept $(n + 1)\alpha_1$ and slope $- \alpha_1$.

Table 2.1 gives the values of the lag coefficients.

TABLE 2.1

LAG COEFFICIENTS AND WEIGHTS FOR ARITHMETIC MODEL

i	i	w_i
0	$(n + 1)\alpha_1$	$2(n + 1)/(n + 1)\ (n + 2)$
1	$n\ \alpha_1$	$2n/(n + 1)\ (n + 2)$
2	$(n - 1)\alpha_1$	$2(n - 1)/(n + 1)\ (n + 2)$
.	.	.
.	.	.
.	.	.
n	α_1	$2/(n + 1)\ (n + 2)$

Lag weights are also presented in the table. The lag weights are the coefficients divided by their sum:

$$w_i = \beta_i / \sum_{i=0}^{n} \beta_i = 2[(n + 1)\alpha_1 - \alpha_1 i]/\alpha_1(n + 1)\ (n + 2)$$

using (2.4) and (2.3). The weight for lag i is the fraction of the total change of the dependent variable due to the change in the independent variable during the period i.

A graph of an arithmetic lag structure for the case of n = 4 is given as figure 2.1.

For this lag length, one-third of the total effect of x on y occurs simultaneously (i = 0), 8/30 of the effect occurs with a delay of one period (i = 1), 6/30 of the effect of i = 2, and so on.

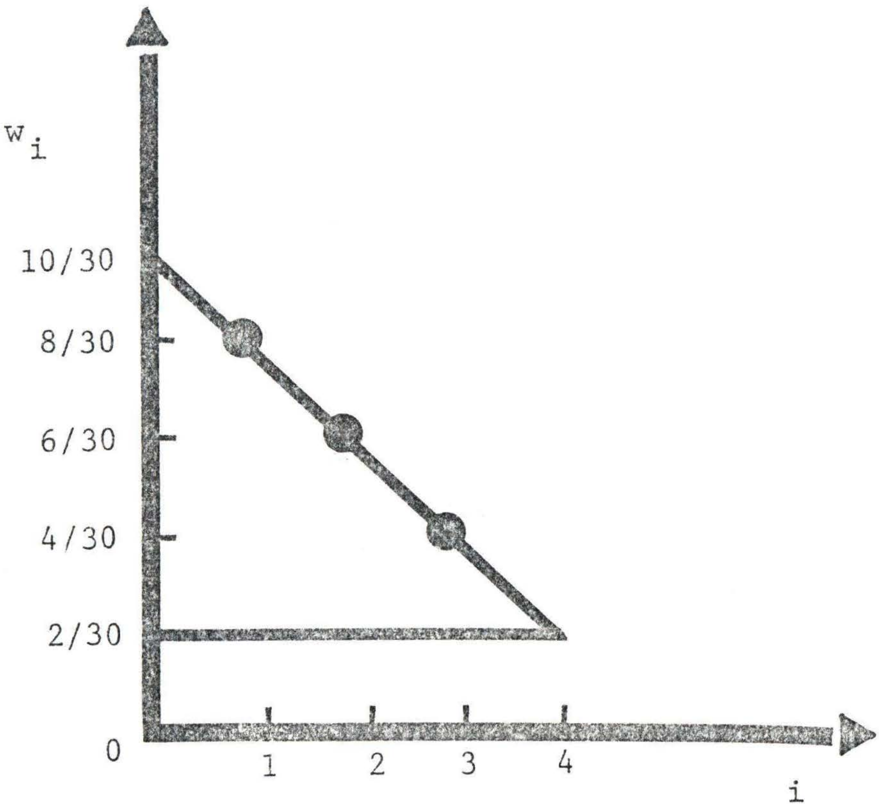

Figure 2.1 Weights for an arithmetic lag of length four.

Estimation of the Arithmetic Lag Model

Two methods have been suggested for estimating the arithmetic lag model: composite variables and restricted least squares. Substituting (2.2) into (2.1) yields

$$y_t = \alpha_0 + \sum_{i=0}^{n} (n + 1 - i)\ \alpha_1\ x_{t-i} + u_t =$$

$$\alpha_0 + \alpha_1 \sum_{i=0}^{n} (n + 1 - i)\ x_{t-i} + u_t$$

define a new variable z_t as

(2.5) $$z_t = \sum_{i=0}^{n} (n + 1 - i) x_{t-i}$$

the model becomes

(2.6) $$y_t = \alpha_0 + \alpha_1 z_t + u_t$$

The variable z_t as defined in (2.5) is composite of the variables x_{t-i}. The model (2.6) is a simple regression model that can be estimated by ordinary least squares. The estimators are

$$\hat{\alpha}_1 = S^2{}_{yz}/S^2{}_z$$

$$\hat{\alpha}_0 = \bar{y} - \hat{\alpha}_1\ \bar{z}$$

If the u_t are normally and independently distributed with zero mean and contstant variance σ^2 (that is, u_t - NID $(0,\sigma_2)$), then $\hat{\alpha}_1$ is distributed as $N(\alpha_1, \sigma^2/T\ S^2{}_z)$. Since the $\hat{\beta}_i$ are linear functions of $\hat{\alpha}_1$ (see equation 2.2), they are distributed as $N(\beta_i, \sigma^2\ (n + 1 - i)^2/T\ S^2{}_z)$.

An alternative approach to estimation is to use restricted least squares. From table 2.1 it is clear that

(2.7) $\beta_i = (n + 1 - i)\ \beta_n \qquad i = 0, 1, 2, \ldots, n - 1$

This provides a set of n linear homogenous restrictions on the regression coefficients. *Equation (2.7) holds for* $i =$ n, but this case is not a meaningful restriction because it only involves a single regression coefficient.

The value of the restricted least squares formulation is that it enables the researcher to test the validity of the arithmetic lag specification. In effect, the unrestricted model (2.1) is estimated, and the restricted model (2.6) is also estimated. An F test of the significance of the restriction is based on the difference of the residual sums of squares of the two models.

The preceding discussion assumes that the lag length is known. If the lag length is unknown, the estimators will generally be biased (Judge et al., 1980, p. 644) and will have unknown sampling properties.

The Inverted-V Lag Model

As the time span that an observation covers decreases (i.e. quarterly vs. annual data), our assumptions about the lag effects and the lag distribution may change. It may be that a few time periods pass before any adjustments take place. The arithmetic lag model cannot take this into account, in this case an inverted-V distribution curve of the weights is suggested (De Leeuw, 1962).

De Leeuw's formulation of the inverted-V model is

$$y_t = \alpha_0 + \sum_{i=0}^{n} \beta_i \, x_{t-i} + u_t$$

$$(2.8) \quad \beta_i = (1 + i)\,\alpha_1 \qquad i = 0, 1, 2, \ldots, s$$

$$= (n + 1 - i)\alpha_1 \qquad i = s + 1, s + 2, \ldots, n$$

and $s = n/2$, n is even. All β_i have the same sign as α_1 and their sum is

$$\sum_{i=0}^{n} \beta_i = \sum_{i=0}^{s} (1+i)\,\alpha_1 + \sum_{i=s+1}^{n} (n+1-i)\,\alpha_1$$

$$= \alpha_1 \Big[\sum_{i=0}^{s} (1+i) + \sum_{i=s+1}^{n} (n+1-i) \Big]$$

$$= \alpha_1 [\sum_{i=0}^{s} 1 + \sum_{i=0}^{s} i + \sum_{i=s+1}^{n} n + \sum_{i=s+1}^{n} 1$$

$$- \sum_{i=s+1}^{n} i]$$

$$= \alpha_1 [(s+1) + s(s+1)/2 + n(n-s) + (n-s)$$

$$- \sum_{i=1}^{n} i - \sum_{i=1}^{s} i)]$$

$$= \alpha_1 [(s+1) + s(s+1)/2 + (n+1)\ (n-s)$$
$$- n(n+1)/2 + s(s+1)/2]$$

$$= \alpha_1 [(s+1) + s(s+1) + (n+1)\ (n-s - n/2)]$$

$$(2.9) \qquad = \alpha_1 [(s+1)^2 + (n+1)\ (n-2s)/2] = \alpha_1 (s+1)^2$$

since $2s = n$. Once again this sum is finite provided α_1 is finite.

The inverted-V lag structure can be thought of as the sum of two arithmetic lag models joined at lag $i = s$. Table 2.2 gives the values of the lag coefficients and weights for the model.

TABLE 2.2

LAG COEFFICIENTS AND WEIGHTS FOR INVERTED-V MODEL

i	β_i	w_i
0	α_1	$1/(s+1)^2$
1	$2\alpha_1$	$2/(s+1)^2$
2	$3\alpha_1$	$3/(s+1)^2$
.	.	.
.	.	.
.	.	.
$s-1$	$s\alpha_1$	$s/(s+1)^2$
s	$(s+1)\alpha_1$	$(s+1)/(s+1)^2$
$s+1$	$s\alpha_1$	$s/(s+1)^2$
.	.	.
.	.	.
.	.	.
n	α_1	$1/(s+1)^2$

Where $s = n/2$ is used to re-write the last s values. The lag weights are the coefficients divided by their sum

$$w_i = \beta_i / \sum_{i=0}^{n} \beta_i = (1+i)/(s+1)^2 \qquad i \leq s$$

$$+ \ (n+1-i)/(s+1)^2 \qquad n > i \geq s+1$$

using (2.9) and (2.8). Once again, the weight for lag i is a fraction of the change of the dependent variable due to the change in the independent variable during period i.

A graph of the inverted-V lag structure is given for the case n = 8 in figure 2.2.

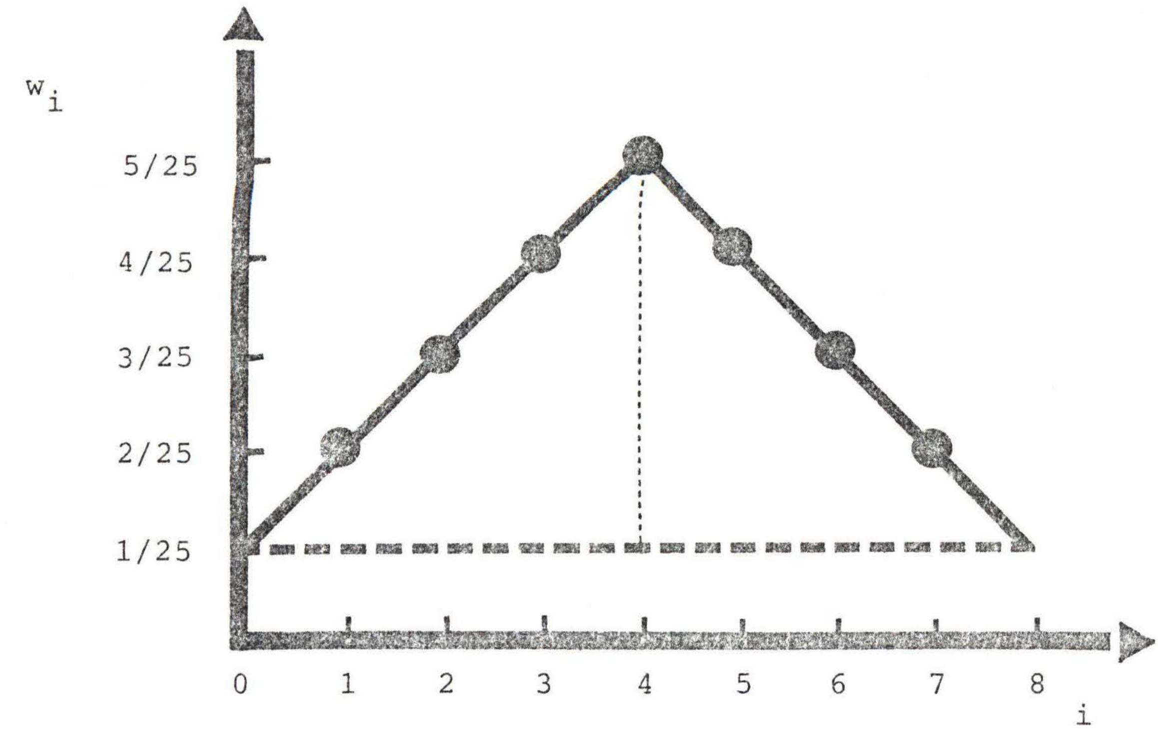

Figure 2.2. Weights for an inverted-V lag of length four.

For this lag length, only 1/25 of the total effect of x on y occurs simultaneously (i = 0), it increases to 2/25 on the next period (i = 1), and continues to rise to a maximum of 5/25 of the total effect occurring with a lag of four periods, then decreasing in the same manner.

Estimation of the Inverted-V Lag Model

Two methods have been suggested for estimating the inverted-V model: composite variables and restricted least squares. Substituting (2.8) into (2.1) yields

$$y_t = \alpha_0 + \sum_{i=0}^{s} (1+i)\ \alpha_1 x_{t-i} + \sum_{i=s+1}^{n} \alpha_1 (n+1-i) x_{t-i} + u_t$$

Define a new variable z_t as

$$(2.10)\quad z_t = \sum_{i=0}^{s} (1+i)\ x_{t-i} + \sum_{i=s+1}^{n} (n+1-i)\ x_{t-i}$$

The model becomes

$$(2.11)\quad y_t = \alpha_0 + \alpha_1 z_t + v_t$$

The variable z_t as defined in (2.10) is composite of the variable x_{t-i}. The model (2.11) is a simple regression model that can be estimated by ordinary least squares. The estimators are

$$\hat{\alpha}_1 = S^2{}_{yz}/S^2{}_z$$

$$\hat{\alpha}_0 = \bar{y} - \hat{\alpha}_1\ \bar{z}$$

If u_t - NID $(0, \sigma^2)$, then $\hat{\alpha}_1$ is distributed as $N(\alpha_1, T\ S^2{}_z)$. Since the $\hat{\beta}_i$ are linear functions of $\hat{\alpha}_1$ (see equation 2.8), they are distributed as $N(\beta_i, \sigma^2 (\sum_{i=0}^{s} (1+i)^2$

$$+ \sum_{i=s+1}^{n} (n+1-i)^2 / T\ S^2_z)$$

Once again, restricted least squares can be used as an alternative approach to estimation. From table 2.2 it is clear that

$$(2.12) \quad \beta_i - (1 + i)\beta_n = 0 \qquad i = 1, 2, \ldots, s$$
$$\beta_i - (25 + 1 - i)\ \beta_n = 0 \qquad i = s + 1, \ldots, n = 2s$$

This provides a set of n linear homogenous restrictions on the regression coefficients. Equation 2.12 holds for i = n, however, as in the case of the arithmetic lag it is not meaningful since it only involves a single regression coefficient. Using the restricted least squares formulation as described previously, it is then possible to test the validity of the inverted-V specification.

The problems associated with estimating an inverted-V model are quite similar to those encountered with the arithmetic lag. Again, the preceeding discussion assumes that the lag length is known. If the lag length is unknown, the estimate will generally be biased (Judge et al., 1980, p. 646) and will have unknown sampling properties.

III. POLYNOMIAL DISTRIBUTED LAGS

The main problem of the arithmetic and inverted-V lag models is the inflexibility of the lag distribution function. The shape of the function is rigorously determined a priori by the formulation of the model, making the regression very restricted. In other words, the data may be too constrained to allow the researcher to gain any insight. It then becomes necessary to find a less restrictive method of estimating the model. Such a method may be found by assuming that the lag distribution function is actually a polynomial.

Section one of this chapter describes the general formulation of a polynomial distributed lag model along the lines first presented by Almon (1965). The model may be estimated in a number of different ways. The "direct approach" (Cooper, 1976) involves the use of composite variables and is described in the second section. Almon's original estimation procedure was an application of Lagrangian interpolation methods and is explained in section three. The equivalence of the direct and Lagrangian methods is shown in the fourth section. Almon originally specified endpoint constraints and the methodology for estimating such models is outlined in section five. The chapter concludes with the presentation of restricted least squares approaches to estimating polynomial distributed lags.

The Polynomial Distributed Lag Model

The general formulation of a polynomial distributed lag model is

$$y_t = \alpha_0 + \sum_{i=0}^{n} \beta_i x_{t-i} + u_t$$

$$(3.1) \qquad \beta_i = \alpha_{10} + \alpha_{11} + \alpha_{12} i^2 + \ldots + \alpha_{1Q} i^Q \qquad i = 0, 1, 2, \ldots n$$

so that the β_i are the values of a polynomial of the Qth degree in the lag index i with coefficients α_{1q}. Defining the vectors β (containing the n + 1 β_i values), α (containing the Q + 1 α_{1q} values), and the (n+1) x (Q+1) matrix H, the relationship between β and α is

$$\beta = H\alpha$$

and

$$(3.2) \quad H = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^Q \\ 1 & 2 & 2^2 & \cdots & 2^Q \\ 1 & 3 & 3^2 & \cdots & 3^Q \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 1 & n & n^2 & \cdots & n^Q \end{bmatrix}$$

Hence, the model is

$$y = x\beta + u$$

$$\beta = H\alpha$$

This formulation of the model assumes that the polynomial is not subject to endpoint constraints (see figure 3.1).

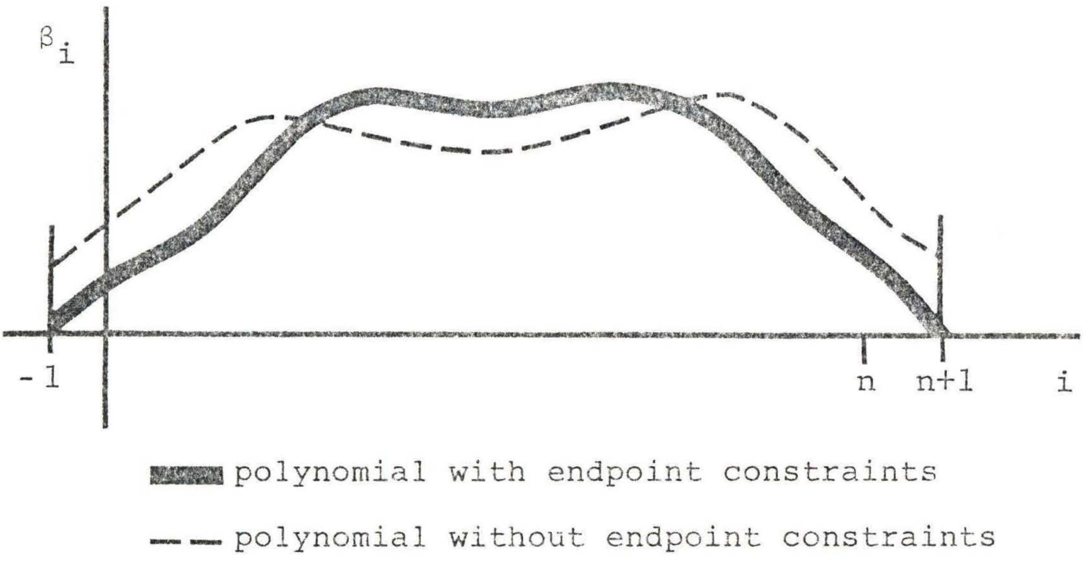

Figure 3.1. Polynomial distributions.

Figure 3.1 is an illustration of the difference between a free polynomial and one with zero left-hand and right-hand constraints. Almon originally imposed each constraints on the model but they are no longer commonly used. A further discussion of the endpoint constraints is given below.

Direct Estimation of the Polynomial Model

Direct estimation of the polynomial lag model is accomplished with the use of composite variables

$$Z = xH$$

and the model becomes

$$Y_t = Z_t \alpha + U_t$$

The ordinarly least squares estimators of the polynomial coefficients are given by

$$\hat{\alpha} = (Z_t' Z_t)^{-1} Z_t' Y_t$$

and the lag coefficients are obtained from

$$\hat{\beta} = H\hat{\alpha}$$

$\hat{\beta}$ is a linear transformation of $\hat{\alpha}$, therefore it is a best linear unbiased estimator of β provided the disturbances U_t satisfy the assumptions of the Gauss-Markov theorem. The variance-covariance matrix of $\hat{\beta}$ is

$$V(\hat{\beta}) = E\ (\hat{\beta} - \beta)\ (\hat{\beta} - B)'$$

$$= E\ (H\hat{\alpha} - H\alpha)\ (H\hat{\alpha} - H\alpha)'$$

$$= E\ [H(\hat{\alpha} - \alpha)\ (\hat{\alpha}'\ H' - \alpha'\ H')]$$

$$= E\ [H(\hat{\alpha} - \alpha)\ (\hat{\alpha}' - \alpha')H']$$

$$= H\ E\ (\hat{\alpha} - \alpha)\ (\hat{\alpha} - \alpha)'H'$$

$$= H\ V\ (\hat{\alpha})H'$$

$$= H\ (\sigma^2\ (Z'Z)^{-1})H'$$

If the U_t are NID $(0, \sigma^2 I)$, then $\hat{\alpha}$ is $N(\alpha, \sigma^2 (Z'Z)^{-1})$, and $\hat{\beta}$ is $N(\beta, H(\sigma^2 (Z'Z)^{-1})H')$.

Estimation Using Lagrangian Interpolation

Almon's original approach to estimation used Lagrangian interpolation polynomials. The effect of this procedure is to employ a different set of composite variables and the coefficients that are estimated are a subset of the lag coefficients β_i rather than the polynomial coefficients α_{1i}.

Given two points on a straight line, simple arithmetic can be employed to solve for the coefficients of the line, namely, the intercept and the slope. A line is a polynomial of degree one. When the degree of the polynomial is greater than one, say Q, Lagrange's interpolation formula can be used to obtain the polynomial coefficients, given the coordinates of Q + 1 points. This is best seen through the use of an example (see figure 3.2).

Consider the case where the lag coefficients lie on a second degree polynomial and the length of the lag is six periods.

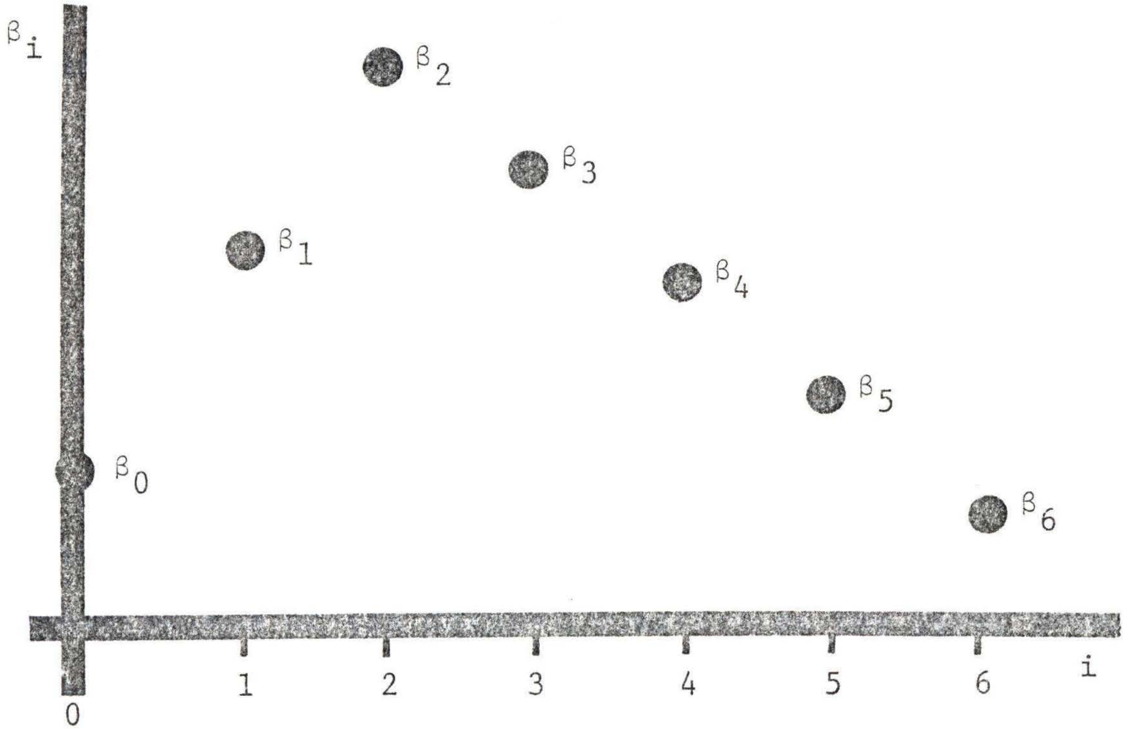

Figure 3.2. Quadratic distributed lag over six periods.

Suppose the initial three coordinates that we have are $(1, \beta_1)$, $(2, \beta_2)$, $(3, \beta_3)$. Thus, we have the required three points from which to interpolate a polynomial of degree 2. Using Lagrange's interpolation formula, which is

$$P(i) = \sum_{j=1}^{Q} {}_{j} \prod_{\substack{k=1 \\ k \neq j}}^{Q} \frac{(i-k)}{(j-k)}$$

we may specify the interpolation polynomial for this example as:

$$P(i) = \beta_1 \frac{(i-2)(i-3)}{(1-2)(1-3)} + \beta_2 \frac{(i-1)(i-2)}{(2-1)(2-3)} + \beta_3 \frac{(i-1)(i-3)}{(3-1)(3-2)}$$

Notice that $P(1) = \beta_1$, because

$$P(i) = \beta_1 \frac{(1-2)(1-3)}{(1-2)(1-3)} + \beta_2 \frac{(1-1)(1-3)}{(2-1)(2-3)} + \beta_3 \frac{(1-1)(1-2)}{(3-1)(3-2)}$$

$$= \beta_1 + \beta_2(0) + \beta_3(0) = \beta_1$$

It can also be demonstrated that $P(2) = \beta_2$ and $P(3) = \beta_3$. Therefore, the interpolation function P(i) takes on the known values of the unknown function at the given points. Evaluating the function at i = 4, 5, 6; yields

$$P(4) = \beta_1 \frac{(4-2)(4-3)}{(1-2)(1-3)} + \beta_2 \frac{(4-1)(4-3)}{(2-1)(2-3)} + \beta_3 \frac{(4-1)(4-2)}{(3-1)(3-2)}$$

$$= \beta_1 - 3\beta_2 + 3\beta_3$$

$$P(5) = \beta_1 \frac{(5-2)(5-3)}{(1-2)(1-3)} + \beta_2 \frac{(5-1)(5-3)}{(2-1)(2-3)} + \beta_3 \frac{(5-1)(5-2)}{(3-1)(3-2)}$$

$$= 3\beta_1 - 8\beta_2 + 6\beta_3$$

$$P(6) = \beta_1 \frac{(6-2)(6-3)}{(1-2)(1-3)} + \beta_2 \frac{(6-1)(6-3)}{(2-1)(2-3)} + \beta_3 \frac{(6-1)(6-2)}{(3-1)(3-2)}$$

$$= 6\beta_1 - 15\beta_1 + 10\beta_3$$

This example could have been expressed in matrix form as

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} P(0) \\ P(1) \\ P(2) \\ P(3) \\ P(4) \\ P(5) \\ P(6) \end{bmatrix}$$

Notice that

$$\begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

or $\beta^{**} = Q_1 \beta^{*}$

Now suppose the initial coordinates are $(4, \beta_4)$, $(5, \beta_5)$, $(6, \beta_6)$. The interpolation formula is:

$$P(i) = \beta_4 \frac{(i-5)(i-6)}{(4-5)(4-6)} + \beta_5 \frac{(i-4)(i-6)}{(5-4)(5-6)} + \beta_5 \frac{(i-4)(i-6)}{(6-4)(6-5)}$$

The result is:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} 15 & -24 & 10 \\ 10 & -15 & 6 \\ 6 & -8 & 3 \\ 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} P(0) \\ P(1) \\ P(2) \\ P(3) \\ P(4) \\ P(5) \\ P(6) \end{bmatrix}$$

In this case

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 10 & -15 & 6 \\ 6 & -8 & 3 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}$$

or $\beta^* = Q_2 \ \beta^{**}$

Notice that Q_1 is the inverse of Q_2 and vice-versa, since

$$\begin{bmatrix} 10 & -15 & 6 \\ 6 & -8 & 3 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $Q_1 \ Q_2 = I$ implies $Q_2 = (Q_1)^{-1}$

In general then,

$$\begin{bmatrix} \beta_0 \\ \\ \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{bmatrix} = \begin{bmatrix} \prod\limits_{\substack{k=1 \\ k\neq 1}}^{Q} \frac{0-k}{1-k} & \prod\limits_{\substack{k=1 \\ k\neq 2}}^{Q} \frac{0-k}{2-k} & \cdots & \prod\limits_{\substack{k=1 \\ k\neq Q}}^{Q} \frac{0-k}{Q-k} \\ \prod\limits_{\substack{k=1 \\ k\neq 1}}^{Q} \frac{1-k}{1-k} & \prod\limits_{\substack{k=1 \\ k\neq 2}}^{Q} \frac{1-k}{2-k} & \cdots & \prod\limits_{\substack{k=1 \\ k\neq Q}}^{Q} \frac{1-k}{Q-k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \prod\limits_{\substack{k=1 \\ k\neq 1}}^{Q} \frac{n-k}{1-k} & \prod\limits_{\substack{k=1 \\ k\neq 2}}^{Q} \frac{n-k}{2-k} & \cdots & \prod\limits_{\substack{k=1 \\ k\neq Q}}^{Q} \frac{n-k}{Q-k} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_Q \end{bmatrix}$$

The implication here is that it does not matter which points are selected. We may interpolate the polynomial using subset of the points of size one plus the degree of the polynomial. This is expressed as

$$\beta = Q\,\beta^*$$

where β^* is the vector of known abcissa values.

Estimation of this model also requires the use of composite variables. Let

$$C = XQ$$

The model becomes

$$Y_t = C_t\,\beta^* + U_t$$

The difference between this approach and the direct approach is as follows. The polynomial coefficients α_{1i} are estimated by ordinary least squares in the direct approach, and the lag coefficients β_i are obtained from

$$\hat{\beta} = H\ \hat{\alpha}$$

For the Lagrange interpolation approach, a subset of the lag coefficients β^* (in the example β_1, β_2, β_3) are estimated by OLS and the entire vector of lag coefficients is estimated from

$$\hat{\beta} = Q_1\ \hat{\beta}^*$$

Ordinary least squares estimates of the polynomial are given by

$$\hat{\beta}^* = (C_t'\ C_t)^{-1}\ C_t'\ Y_t$$

The lag coefficients are obtained from

$$\hat{\beta} = Q\ \hat{\beta}^*$$

$\hat{\beta}$ is a linear transformation of $\hat{\beta}^*$, therefore it is a best linear unbiased estimator of provided the disturbances U_t satisfy the assumptions of the Gauss-Markov theorem. The variance-covariance matrix of $\hat{\beta}$ is

$$V(\hat{\beta}) = Q(\sigma^2(C'C)^{-1})Q'$$

If the U_t are NID $(0, \sigma^2\ I)$, then $\hat{\beta}^*$ is N $(\beta^*, \sigma^2(C'C)^{-1})$, and $\hat{\beta}$ is N $(\beta, Q(\sigma^2(C'C)^{-1})Q')$.

The Relationship Between the Direct and Almon Methods

The relationship between the direct and Almon methods is easily derived if the matrix H is partiioned as follows. Consider the case where the subset of points in the Almon method consists of β_4, β_5, and β_6. The H matrix can be partitioned into those rows involving β_0, β_1, β_2, β_3 and those involving β_4, β_5, β_6:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ \hline 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix} = \begin{bmatrix} H_1 \\ \hline J \end{bmatrix}$$

Let the lag coefficients β_0, β_1, β_2, β_3 be denoted β^*, and β_4, β_5, β_6 be denoted β^{**}. Then because $\beta = H\alpha$ it is true that

(3.3) $\beta^* = H_1\alpha$

(3.4) $\beta^{**} = J\alpha$

where α is the vector of the polynomial coefficients. J is a square nonsingular matrix, called a Vandermonde matrix.

In the example

$$J^{-1} = \begin{bmatrix} 15 & -24 & 20/2 \\ 11/2 & 10 & -9/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

using (3.4)

$$\alpha = J^{-1} \beta^{**}$$

so that the two equataions (3.3 and 3.4) can be written

$$\beta^{*} = H_1 J^{-1} \beta^{**}$$

$$\beta^{**} = JJ^{-1} \beta^{**}$$

or

$$\beta = \begin{bmatrix} J1 \; J^{-1} \\ J \; J^{-1} \end{bmatrix} \beta^{**} = \begin{bmatrix} H_1 \\ J \end{bmatrix} J^{-1} \beta^{**} = HJ^{-1} \beta^{**}$$

but the interpolation method defined Q such that

$$\beta = Q \quad **$$

and thus

$$Q = HJ^{-1}$$

Given this result, the equivalence of the two methods can be shown.

Using the direct approach, $\hat{\beta}$ is computed as

$$\hat{\beta} = H\hat{\alpha} = H(Z'\ Z)^{-1} Z'\ Y$$

$$= H[(xH)'\ (xH)]^{-1} (xH)'\ Y$$

$$= H(H'\ X'X'\ XH)^{-1} H'\ X'\ Y$$

The Almon approach entails computing $\hat{\beta}$ as

$$\hat{\beta} = Q\hat{\beta}_1 = Q[(C'C)^{-1}C' \; Y]$$

$$= Q[(xQ)' \; (xQ)]^{-1} \; (xQ)' \; Y$$

$$= Q[Q' \; x' \; x \; Q]^{-1} \; Q' \; x' \; Y$$

Since $Q = H \; J^{-1}$

$$\hat{\beta} = H \; J^{-1}[(H \; J^{-1})' \; x' \; x \; H \; J^{-1}]^{-1}(H \; J^{-1})' \; x' \; Y$$

$$= H \; J^{-1}[J^{-1})' \; H' \; x' \; x \; H \; J^{-1}]^{-1} \; (J^{-1})' \; H' \; x' \; Y$$

$$= H \; J^{-1}[J' \; (J^{-1})' \; H' \; x' \; x \; H \; J^{-1}]^{-1} \; H' \; x' \; Y$$

$$= H[(J^{-1} \; J)' \; H' \; x' \; x \; H \; J^{-1} \; J]^{-1} \; H' \; x' \; Y$$

$$= H[I \; H' \; x' \; x \; H \; I]^{-1} \; H' \; x' \; Y$$

$$= H \; (H' \; x' \; x \; H)^{-1} \; H' \; x' \; Y$$

Therefore the two models are the same.

Upon comparing the weighting matrix HJ^{-1} from the Almon approach with H, notice that the former will yield more irregular sums of the columns of x when forming the composite variable matrix C. This is due to the use of the Lagrangian interpolation coefficients. The conclusion that may be drawn here is that the composite variable matrix C will have columns that are less collinear than those from the composite variable matrix Z from the direct approach (Cooper, 1972).

Another point of interest is the objective of each approach. The Almon method estimates some of the lag coefficients directly, while the direct approach estimates the polynomial coefficients. From the Almon approach, if widely spaced points are used for interpolation, a grid of the lag structure is obtained. The direct approach provides different intermediate information, since it estimates the polynomial coefficients, it may provide better information about the proper degree of the polynomial, i.e. if α_Q turns out to be insignificant statistically, perhaps we should estimate a polynomial of a lesser degree.

Endpoint Constraints and Zero Restrictions

The advantage of the polynomial lag model is that it is less restrictive than the arithmetic or inverted-V formulations. Sometimes, however, a priori information may have something to say about the shape of the lag distribution. On this situation, knowledge about the shape may be incorporated into the model by imposing a constraint on the polynomial coefficients. For example, if a monotonically increasing polynomial is desired, a left endpoint zero restriction is incorporated into a second order polynomial. In the opposite case, a right endpoint zero restriction is imposed. By constraining both endpoints, a "humped" shape will result.

In general, the zero restrictions can be imposed at any point or combination of points of a polynomial of the proper degree. This can be done with either of the previously described approaches (direct vs. Almon). The objective here is to show how this is accomplished for either approach. However, in this section, the zero restrictions will only be imposed at the endpoints.

If a polynomial is pre-specified to have roots at any point, the shape is being partly pre-determined. As an example, consider figure 3.3, where the solid line is a second order polynomial with roots at i = -1 and i = n + 1, and the broken line is the unconstrained quadratic.

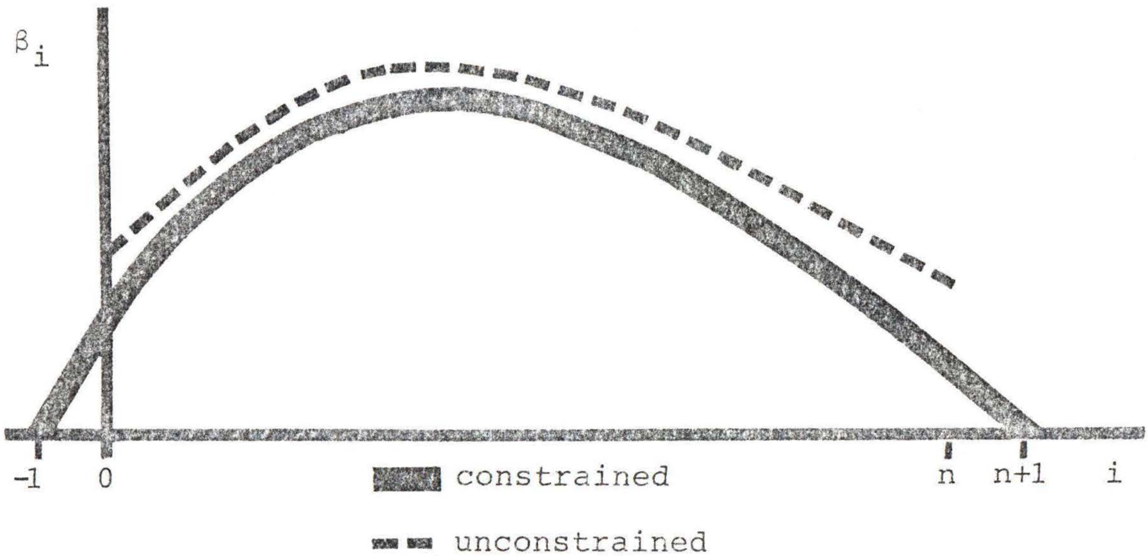

Figure 3.3. Second order polynomials.

Regardless of the approach that is being employed, the H matrix will have to be augmented. This is because the range over which the polynomial is being estimated has increased from $i = 0, 1, 2, \ldots, n$ to $i = -1, 0, 1, 2, \ldots, n, n + 1$. Recall that the typical row of H is $(i^0, i^1, i^2, \ldots, i^Q)$ in (3.2) in the unconstrained case, there will be $n + 1$ rows in H. By constraining the polynomial to have roots at $i = -1$ and $i = n + 1$ it is necessary to estimate over a larger interval. Therefore, H in the constrained case will have two extra rows, one at the top, and one at the bottom.

Notice that in the unconstrained case, the polynomial of degree Q may be expressed in terms of its roots, as:

$$(3.5) \qquad \alpha_{10} + \alpha_{11}\, i + \alpha_{12}\, i^2 + \ldots + \alpha_{1Q}\, i^Q =$$

$$\alpha_{1Q}\, (i^Q + \frac{\alpha_{1Q-1}}{\alpha_{1Q}}\, i^{Q-1} + \ldots + \frac{\alpha_{12}}{\alpha_{1Q}}\, i^2 + \frac{\alpha_{11}}{\alpha_{1Q}}\, i + \frac{\alpha_{10}}{\alpha_{1Q}})$$

$$= \alpha_{iQ}\, (i-i_1)\, (i-i_2)\, \ldots\, (i-i_Q)$$

In the case of a constrained polynomial, such as the one depicted in figure 3.3, two of the roots are known. To be more specific, these are roots at $i = -1$ and $i = n + 1$. Equation (3.5) may be rewritten, for the constrained case as

$$\alpha_{1Q}\ (i+1)\ (i-n-1)\ (i-i_3)\ (i-i_4)\ \ldots\ (i-i_Q)$$

Since (i+1) and (i-n-1) are known, the actual polynomial *that is estimated is*

$$(3.6) \quad \alpha_{1Q}\ (i-i,)\ (i-i_4)\ \ldots\ (i-i_Q)$$

$$= \alpha_{1Q}\ (i^{Q-2} + b_{Q-1}\ i^{Q-1} + \ldots + b_1 i + b_0)$$

and will be a polynomial of degree Q-2. The estimated coefficients of the composite variables formed using (3.6) are α_{1Q}, $\alpha_{1Q}\ b_{Q-1}$, ... , $\alpha_{1Q}\ b_1$, $\alpha_{1Q}\ b_0$. Denote the vector of these coefficients as b. Each element in b is multipled by (i-i), $(i-i_2) = (i+1)\ (i-n-1)$, to obtain the α vector. Since (3.6) is a polynomial of degree Q-2, the typical row of H for the direct approach in the constrained case is $(i-i_1)\ (i-i_2)\ (i^0\ i^1\ i^2\ \ldots\ i^{Q-2})$ and will have two less elements than in the unconstrained case. Therefore the H matrix will have two more rows and two less columns when zero right and left endpoints are imposed than it would in the unconstrained case. Notice that the lag coefficients β still satisfy $B = H\alpha$.

In the example discussed previously, a second degree polynomial

$$\alpha_{10} + \alpha_{11} i + \alpha_{12} i^2$$

was estimated over lags i = 0 to i = 6. Imposing two endpoint constraints requires estimation over lag i = -1 to i = 7. The H matrix called $\bar{H}$ is

$$\begin{matrix} (-1 + 1) & (-1 - 7) \\ (0 + 1) & (0 - 7) \\ (1 + 1) & (1 - 7) \\ (1 + 1) & (2 - 7) \\ (3 + 1) & (3 - 7) \\ (4 + 1) & (4 - 7) \\ (5 + 1) & (5 - 7) \\ (7 + 1) & (7 - 7) \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -12 \\ -15 \\ -16 \\ -15 \\ -12 \\ 0 \end{bmatrix}$$

The composite variable matrix Z consists of a single vector whose value at time t is:

$$z_t = -7x_t - 12x_{t-1} - 15x_{t-2} - 16x_{t-3} - 15x_{t-4} - 12x_{t-5} - 7x_{t-6}$$

Because the elements in the b vector of constrained polynomial is equal to unity in this case (Q = 2), the coefficient of z given by ordinary least squares is $_{1Q}$. In general, the model actually estimated is

$$Y = X\bar{H}b + u$$

For the Almon approach, as discussed previously, the H matrix is augmented to cover the cases i = -1 and i = 7. The choice of the set of rows of H used to form the J matrix is restricted to contain those where i = -1 and i = 7, where the β_i are constrained. The columns of Q corresponding to the constrained i's are then deleted, since the corresponding artificial variables are set equal to zero.

For the example where a second degree polynomial is estimated for lag i = -1 to i = 7, H is

$$H = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \end{bmatrix}$$

The matrix J includes, the first row (i = -1), a middle row (i = 3), and the last row (i = 7) of H. Thus,

$$J = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 9 \\ 1 & 7 & 49 \end{bmatrix}$$

and

$$J^{-1} = (1/32) \begin{bmatrix} 21 & 14 & -3 \\ -10 & 12 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

The weighting matrix $Q = HJ^{-1}$, defined as $\bar{Q}$ in the constrained case is

$$\bar{Q} = (1/32) \begin{bmatrix} 32 & 0 & 0 \\ 21 & 14 & -3 \\ 12 & 24 & -4 \\ 5 & 30 & -3 \\ 0 & 32 & 0 \\ -3 & 30 & 5 \\ -4 & 24 & 12 \\ -3 & 14 & 21 \\ 0 & 0 & 32 \end{bmatrix} = 1/32 \begin{bmatrix} 0 \\ 14 \\ 24 \\ 30 \\ 32 \\ 30 \\ 24 \\ 14 \\ 0 \end{bmatrix}$$

where columns 1 and 3 of (HJ^{-1}) were deleted since β_{-1}, β_7 were set equal to zero. The new model is

$$Y = X\bar{Q}\ \beta^* + u$$

There is a single column in Q and a single coefficient β_3 in β^*. The full set of lag coefficients is given by

$$\beta = \bar{Q}\ \beta^*$$

Restricted Least
Squares Approaches

An alternative approach to estimation is to use restricted least squares. It is possible to formulate a restricted model to test either approach.

The direct approach requires that $\beta = H\alpha$ be satisfied. From this we obtain

$$\alpha = H^{+}\beta = (H'H)^{-1} H'\beta$$

where H^{+} is the generalized inverse of H. It is necessary to use the generalized inverse since H is generally not square. Now

$$\begin{aligned} 0 &= \beta - H\alpha \\ &= \beta - HH^{+} \\ &= (I - H(H'H)^{-1} H')\beta \end{aligned}$$

The restrictions matrix R is then

$$R = (I - H(H'H)^{-1} H')$$

which yields n - Q linear homogenous restrictions. The model to be estimated now is

$$Y = X\beta + u \qquad \text{subject to } R\beta = 0$$

Consider the method suggested by Almon, where each lag coefficient is calculated as a linear combination of the coefficients estimated for interpolation, or

$$\beta = Q\beta^*$$

It follows that such interpolated coefficient minus a linear combination of the coefficients in β^* is equal to zero.

To obtain the restrictions for the Almon approach, partition the matrix Q into two parts as $[Q_R : - I_{Q+1}]$. The first partition is the set of rows that are used to obtain the interpolated coefficients, which was denoted, for the two examples in the third section of this chapter, as Q_1 and Q_2. The second partition is a negative identity matrix with the order equal to the number of estimated coefficients used for interpolation.

To demonstrate the formulation of the restrictions matrix, consider the example where β was interpolated from $(4, \beta_4)$, $(5, \beta_5)$, $(6\ \beta_6)$ which is reproduced for the reader here.

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} 15 & -24 & 10 \\ 10 & -15 & 6 \\ 6 & -8 & 3 \\ 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}$$

It follows that:

$$15\beta_4 - 24\beta_5 + 10\beta_6 - \beta_0 = 0$$

$$10\beta_4 - 15\beta_5 + 6\beta_6 - \beta_1 = 0$$
$$6\beta_4 - 8\beta_5 + 3\beta_6 - \beta_2 = 0$$
$$3\beta_4 - 3\beta_5 + 1\beta_6 - \beta_3 = 0$$

which implies n - Q = 4 linear homogenous restrictions. The above set of equations could have been written in matrix format as $R\beta = 0$, or

$$\begin{bmatrix} 15 & -24 & 10 & -1 & 0 & 0 & 0 \\ 10 & -15 & 6 & 0 & -1 & 0 & 0 \\ 6 & -8 & 3 & 0 & 0 & -1 & 0 \\ 3 & -3 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The latter respresentation of the restrictions may be preferred to the former. The generalized inverse H^+ is usually difficult to compute and is more susceptible to computational error than the latter method. Another advantage of the second representation is that when one searches for the proper degree of the polynomial, i.e. testing the model under alternative restrictions, the different restrictions will be "nested" in the same matrix.

There are other methods of deriving the restrictions matrix, namely by Shiller (1973) and Hill and Johnson (1976). Hill and Johnson noted that the Almon weights could be derived using orthogonal polynomials, from which the restrictions matrix may also be derived. These other representations are not discussed here since they will be algebraically equivalent (Judge et al., 1980).

IV. THE CUBIC SPLINE FINITE DISTRIBUTED LAG MODEL

The Almon polynomial distributed lag is based on Lagrangian interpolation. However, there are other methods used to interpolate an unknown function, one of them being the use of spline functions. Splines have a long history of use in curve fitting applications by draftsmen. The cubic splines in particular present some advantages for economic applications as discussed later (Poirier, 1978).

The first section of this chapter presents the approach and general formulation of the spline polynomial distributed lag. A "direct" approach (Judge et al., 1980) to estimating the cubic spline model is the subject of the second section. The cubic spline interpolation approach is presented in the third section. The chapter concludes with a restricted least squares formulation corresponding to each approach.

General Formulation of the Cubic Spline Distributed Lag

The cubic spline approach to estimating a polynomial distributed lag is similar to a piece-wise polynomial regression. It is assumed that the relationship between the lag coefficients β_i and the lag index i varies over certain intervals. The variation in the relationship shows up as changes in the parameters, which occur at certain locations along the abcissa, called knots. Consider figure 4.1.

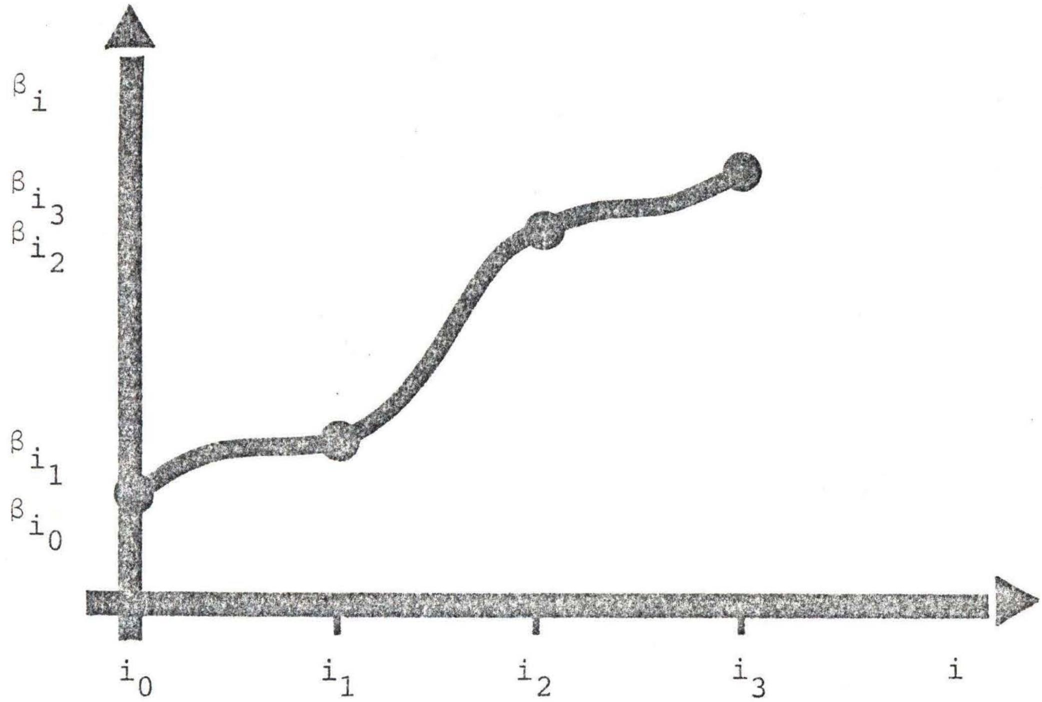

Figure 4.1. A cubic spline with four knots.

The relationship between β_i and i is viewed then, as a series of polynomial functions. The functional form relating β_i to i is different for each of the three intervals (i_0, i_1), (i_1, i_2), and (i_2, i_3) as shown in figure 4.1. Moreover, it is assumed that the function that describes each interval is a polynomial of degree at most three.

The lag coefficients are expressed as

(4.1) $\beta_i = g_1(i) \quad 0 \leq i \leq i_1$

$$= g_2 \ (i) \quad i_1 \leq i \leq i_2$$

$$= g_3 \ (i) \quad i_2 \leq i \leq i_3$$

$$\vdots$$

$$= g_k \ (i) \quad i_{k-1} \leq i \leq i_k$$

where k is the number of knots.
The g_j (i) (for j = 1, 2, ..., k) are cubic polynomials of the form

(4.2) $$g_j \ (i) = a_j + b_j i + c_j i^2 + d_j i^3$$

The lag coefficient generating function as defined in Equation (4.1) will, in general, be discontinuous at the knots. Therefore, one restriction that is imposed when estimating the functions g is that the functions be continuous at the knots where they join. For the example presented as figure 4.1, this amounts to the following:

(4.3) $$g_1 \ (i_1) = g_2 \ (i_1), \ g_2 \ (i_2) = g_3 \ (i_2)$$

When these k - 1 restrictions are imposed, a continuous lag distribution will result.

Estimating the functions g_j subject to the constraints (4.3) will in general yield a continuous, but jagged or kinked function. If the lag distribution is assumed to be *smoother*, as is the usual case, further restrictions are imposed. Specifically, the first and second derivatives at the knots are assumed to be continuous. Those additional restrictions are

(4.3) $g_1'(i_1) = g_2'(i_1)$, $g_2'(i_2) = g_3'(i_2)$ and

(4.4) $g_1''(i_1) = g_2''(i_1)$, $g_2''(i_2) = g_3''(i_2)$

Thus, another 2(k-1) restrictions are added on.

Once again, composite variables are used. The construction of the coefficient matrices used in both approaches will be addressed in the succeeding sections of this chapter.

Direct Estimation of the Cubic Spline Distributed Lag Model

The direct approach to fitting cubic splines to a lag distribution employs composite variables in estimating the model

$$Y_t = X_t\beta + u_t$$

The composite variable matrix will, as usual, be the product of the data matrix and a weighting matrix. The weighting matrix is better understood by considering the following example.

Let the lag length be M = 8, and assume that there are

$k + 1 = 4$ knots occurring at $i_0 = 0$, $i_1 = 2$, $i_2 = 5$, $i_3 = 8$. The polynomial function for the values of $_i$ as given by (4.1) and (4.2) is:

for $i_0 \leq i \leq i_1$, $\beta_0 = a_1$

$$\beta_1 = a_2 + b_1 + c_1 + d_1$$

$$\beta_2 = a_1 + 2b_1 + 4c_1 + 8d_1$$

or in matrix format

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$$

for $i_1 \leq i \leq i_2$

$$\beta_3 = a_2 + 3b_2 + 9c_2 + 27d_2$$

$$\beta_4 = a_2 + 4b_2 + 16c_2 + 64d_2$$

$$\beta_5 = a_2 + 5b_2 + 25c_2 + 125d_2$$

in matrix format

$$\begin{bmatrix} \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$$

for $i_2 \leq i \leq i_3$

$$\beta_6 = a_3 + 6b_3 + 36c_3 + 216d_3$$

$$\beta_7 = a_3 + 7b_3 + 49c_3 + 393d_3$$

$$\beta_8 = a_3 + 8b_3 + 64c_3 + 512d_3$$

in matrix format

$$\begin{bmatrix} \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}$$

The three sets of matrix equations above can be stacked together into an equation matrix system as follows:

$$(4.5) \qquad \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 16 & 64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 25 & 125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 36 & 216 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 49 & 393 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 64 & 512 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}$$

or

(4.6) $\beta = N\gamma$

The weighting matrix N is a block diagonal matrix. Each block is similar to the matrix H, as defined in (3.2), over the defined interval.

The twelve elements in γ are subjected to the restrictions (4.2) (4.3), and (4.4). The restrictions required to satisfy (4.2), namely, equality of the functions at the knots are

$$a_2 = a_1 + (b_1-b_2)i_1 + (c_1-c_2)i_1^2 + (d_1-d_2)i_1^3$$

$$a_3 = a_2 + (b_2-b_3)i_2 + (c_2-c_3)i_2^2 + (d_2-d_1)i_1^3$$

The restrictions required to satisfy (4.3), namely, equality of the first derivations at the knots are

(4.7) $b_2 = b_1 + 2(c_1-c_2)i_1 + 3(d_1-d_2)i_1^2$

$$b_3 = b_2 + 2(c_2-c_3)i_2 + 3(d_2-d_3)i_2^2$$

The restrictions required to make the second derivatives equal to the knots are

(4.8) $c_2 = c_1 + 3(d_1-d_2)i_1$

$$c_3 = c_2 + 3(d_2-d_3)i_2$$

These restrictions take the form

(4.9) $R\gamma = 0$

where

$$R = \begin{bmatrix} -1 & -i_1 & -i_1^2 & -i_1^3 & 1 & i_1 & i_1^2 & i_1^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -i_2 & -i_2^2 & -i_2^3 & 1 & i_2 & i_2^2 & i_2^3 \\ 0 & -1 & -2i_1 & -3i_1^2 & 0 & 1 & i_1 & 3i_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2i_2 & -3i_2^2 & 0 & 1 & 2i_2 & 3i_2^2 \\ 0 & 0 & -1 & -3i_1 & 0 & 0 & 1 & -3i_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3i_2 & 0 & 0 & 1 & 3i_2 \end{bmatrix}$$

Thus, direct estimation of the cubic spline model involves transforming X using (4.5)

$$Y = X\beta + u = XN\gamma + u = G\gamma + u$$

This model is estimated subject to the restrictions (4.9) $R\gamma = 0$. Given least squares estimators of γ, the estimators of β are obtained by solving (4.5). These estimators are linear transformations of the estimators of γ so that their properties are easily determined, as noted above on page 26 in the case of the Almon lag.

Estimation Using Spline Interpolation Functions

The focus of the Lagrangian interpolation was to calculate a polynomial function that took on the same given values as an unknown function and approximate the values of the function at other points on the abcissa. The cubic spline method of interpolation not only calculates a polynomial function that generates the given values of the unknown function, but also allows the coefficient of the polynomial to take on different values of the in different ranges of the independent variable. Spline interpolations usually allow the polynomial to be at most of third degree.

Following Poirier (1976, p. 55) it is useful to write the spline function in terms of its second derivatives. The second derivative of a cubic function is a linear function. Let the spline function over the interval $[i_{j-1}, i_j]$ be denoted S(i). Its first derivative is S'(i) and its second derivative is S''(i). Because the second derivative is linear over the interval it may be written as

$$S''(i) = u + vi \quad i_{j-1} \leq i \leq i_j \tag{4.10}$$

At the left hand endpoint, the value of the second derivative is

$$S''(i_{j-1}) = u + vi_{j-1}$$

and at the right hand endpoint the second derivative is

$$S''(i_j) = u + vi_j$$

Given these two points on the linear function, the slope coefficient v can be calculated as

$$v = \frac{S''(i_j) - S''(i_{j-1})}{i_j - i_{j-1}}$$

The intercept u is

$$u = S''(i_{j-1}) + \left[\frac{S''(i_j) - S''(i_{j-1})}{i_j - i_{j-1}}\right] i_{j-1}$$

$$= \frac{S''(i_{j-1})\ i_j - i_{j-1}\ S''(i_j)}{i_j - i_{j-1}}$$

Hence, (4.10) can be written

$$\text{(4.11)} \quad S''(i) = \left(\frac{i_{j-1}}{h_j}\right) S''(i_{j-1})$$

$$+ \left(\frac{i - i_{j-1}}{h_j}\right) S''(i_j)$$

where $h_j = i_j - i_{j-1}$

The spline function S(i) is obtained from (4.11) by integrating twice and is given by Poirier as

$$\text{(4.12)} \quad S(i) = \frac{i_{j-i}}{6h_j} [(i_j - i)^2 - h_j^2]\ S''(i_{j-1})$$

$$+ \frac{i - i_{j-1}}{6h_j} [(i - i_{j-1})^2 - h_j^2]\ S''(i_j)$$

$$+ \left[\frac{i_{j-1}}{h_j}\right] \beta(i_{j-1}) + \left[\frac{i - i_{j-1}}{h_j}\right] \beta(i_j)$$

where $\beta(i_{j-1})$ is the lag coefficient at i_{j-1} and $\beta(i_j)$ is the lag coefficient at i_j.

Poirier also gives the first derivative of the spline function as

$$(4.13) \qquad S'(i) = \left[\frac{h_j}{6} - \frac{(i_j - i)^2}{2h_j}\right] S''(i_{j-1})$$

$$+ \left[\frac{(i - i_{j-1})^2}{2h_j} - \frac{h_j}{6}\right] S''(i_j)$$

$$+ \frac{\beta(i_j) - \beta(i_{j-1})}{h_j}$$

Because (4.12) contains $\beta(i_{j-1})$ and $\beta(i_j)$ the requirement that the cubic polynomials be equal at the join points will be satisfied. To ensure that the first and second derivatives be equal at the join points, it is necessary to have the left-hand derivative of S(i) equal to the right-hand derivative of S(i).

$$(4.14)\quad \lim_{i \to i_j -} S'(i) = \frac{h_j\ S''(i_{j-1})}{6} + \frac{h_j\ S''(i_j)}{3}$$

$$+ \frac{\beta(i_j) - (i_{j-1})}{h_j}$$

$$(4.15)\quad \lim_{i \to i_j +} S'(i) = \frac{-h_{j+1}\ S(i_j)}{3} - \frac{h_{j+1}\ S(i_j)}{6}$$

$$+ \frac{\beta(i_{j+1}) - \beta(i_j)}{h_j}$$

Where the limit as i approaches i_j- implies that i is in the interval $[i_{j-1}, i_j]$, and the limit as i approaches i_j+ implies that i is in the interval $[i_j, i_{j+1}]$. When i is in the interval $[i_{j-1}, i_j]$, the first derivative of the spline corresponds to the formula (4.13) given above. When i is in the interval $[i_j, i_{j+1}]$ the latter formula is used with $_{j+1}$ replacing $_j$ in all subscripts. Equating (4.14) with (4.15), and combining terms yields

$$(4.16)\quad (i-\lambda_j)\ S''(i_{j-1}) + 2\ S''(i_j) + \lambda_j\ S''(i_{j+1})$$

$$= \frac{6\ \beta(i_{j-1})}{h_j\ (h_j+h_{j+1})} - \frac{6\ \beta(i_j)}{h_j\ h_{j+1}} + \frac{6\ \beta(i_{j+1})}{h_{j+1}\ (h_j+h_{j+1})}$$

for $j = 1, 2, \ldots, k-1$

where $\lambda_j = h_{j+1}/h_j+h_{j+1}$

The model is defined, or set up is under-identified. Notice that each cubic spline has four parameters. Thus, for k intervals, we must solve for a total of 4k parameters. The restrictions, which can be expressed as equations, consist of k-1 continuity conditions at the knots, k-1 first derivatives equal at the knot, and k-1 second derivatives which must be equal at the knots. Additionally, S(i) must pass through k+1 points. Therefore, we have 4k-2 restrictions, or equations to be used to solve for 4k unknowns. This underspecification can be solved by placing endpoint constraints on the function S(i).

The endpoint conditions express the second derivatives at the endpoints as proportional to their values at the adjacent interior knots. That is:

$$S''(i_0) = \Pi_0 \; S''(i_1)$$

$$S''(i_k) = \Pi_k \; S''(i_{k-1})$$

The researcher may choose values for the Π's in the range (-2, 2). If the Π's are chosen equal to unity, the spline in the endpoint intervals is a quadratic (since its second derivative is constant). Setting the Π's equal to zero implies that the first derivative is a constant at the endpoints.

The endpoint constraints are combined with the continuity equations (4.16) to express the unknown second derivatives in terms of the known $\beta(i_j)$ values. For the case k = 3, which was depicted in figure 4.1, the full set of equations is:

$$2M_0 - 2\Pi_0 M_1 = 0$$

$$1-\lambda_1 M_0 + 2M_1 + \lambda_1 M_2 = \frac{6}{h_1(h_1+h_2)}\beta(i_0) - \frac{6}{h_1 h_2}\beta(i_1)$$

$$+ \frac{6}{h_2(h_1+h_2)}\beta(i_2)$$

$$1-\lambda_2 M_1 + 2M_2 + \lambda_2 M_3 = \frac{6}{h_2(h_2+h_3)}\beta(i_1) - \frac{6}{h_2 h_3}\beta(i_2)$$

$$+ \frac{6}{h_3(h_2+h_3)}\beta(i_3)$$

$$-2\Pi_3 M_2 + 2M_3 = 0$$

where M_j is an alternative notation for $S''(i_j)$.

In matrix format

$$\begin{bmatrix} 2 & -2\Pi_0 & 0 & 0 \\ 1-\lambda_1 & 2 & \lambda_1 & 0 \\ 0 & 1-\lambda_2 & 2 & 2 \\ 0 & 0 & -2\Pi_3 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6/h_1(h_1+h_2) & -6/h_1h_2 & 6/h_2(h_1+h_2) & 0 \\ 0 & 6/h_2(h_2+h_3) & -6/h_2h_3 & 6/h_3(h_2+h_3) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta(i_0) \\ \beta(i_1) \\ \beta(i_2) \\ \beta(i_3) \end{bmatrix}$$

or as

(4.17) $\Delta M = \theta\beta^*$

The solution for the second derivatives is given by

(4.18) $M = \Delta^{-1}\,\theta\beta^*$

The equation for the spline function (4.12), can be written in matrix notation as follows

(4.19) $$\beta_i = \begin{bmatrix} \dfrac{i_j-i}{6h_j}\,[(i_j-i)^2 - h_j^2] & \dfrac{i-i_{j-1}}{6h_j}\,[(i-i_{j-1})^2 \end{bmatrix}$$

$$\begin{bmatrix} M_{j-1} \\ M_j \end{bmatrix}$$

$$+ \begin{bmatrix} \dfrac{i_j - 1}{n_j} & \dfrac{i - i_{j-1}}{h_j} \end{bmatrix} \begin{bmatrix} \beta(i_{0-1}) \\ \beta(i_j) \end{bmatrix}$$

for $i_{j-1} \leq i \leq i_j$

For the example presented in the previous section, where the lag length was 8 and the k + 1 = 4 knots occurring at $i_0 = 0$, $i_1 = 2$, $i_2 = 5$, $i_3 = 8$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2-0}{6(2-0)} [(2-0)^2 - (2-0)^2] & \dfrac{0-0}{6(2-0)} [(0-0)^2 - (2-0)^2] \\ \dfrac{2-0}{6(2-0)} [(2-1)^2 - (2-0)^2] & \dfrac{1-0}{6(2-0)} [(1-0)^2 - (2-0)^2] \\ \dfrac{2-2}{6(2-2)} [(2-2)^2 - (2-0)^2] & \dfrac{2-0}{6(2-0)} [(2-0)^2 - (2-0)^2] \end{bmatrix} \begin{bmatrix} M_0 \\ M_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \dfrac{2-0}{2-0} & \dfrac{0-0}{2-0} \\ \dfrac{2-1}{2-0} & \dfrac{1-0}{2-0} \\ \dfrac{2-2}{2-0} & \dfrac{2-0}{2-0} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3/12 & -3/12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_0 \\ M_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \end{bmatrix}$$

For $2 \leq i \leq 5$

$$\begin{bmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -10/18 & -8/18 \\ -8/18 & -10/18 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_2 \\ M_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_2 \\ \beta_8 \end{bmatrix}$$

For $5 \leq i \leq 8$

$$\begin{bmatrix} \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -10/18 & -8/18 \\ -8/18 & -10/18 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_5 \\ M_8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_5 \\ \beta_8 \end{bmatrix}$$

The three sets of matrix equations above can be stacked together into a 9 equation system as follows:

$$(4.20)\quad \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3/12 & -3/12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -10/18 & -8/18 & 0 \\ 0 & -8/18 & -10/18 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10/18 & -8/18 \\ 0 & 0 & -8/18 & -10/18 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \\ \beta_5 \\ \beta_8 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \\ \beta_5 \\ \beta_8 \end{bmatrix}$$

or

$$\beta = PM + Q\beta^*$$

Using equation (4.18), this becomes

$$(4.21)\quad \beta = P\,\Delta^{-1}\,\theta\beta^* + Q\,\beta^* = (P\Delta^{-1}\theta + Q)\,\beta^* = W\beta^*$$

for

(4.27) $W = (P\Delta^{-1}\theta + Q)$

Thus, for any point on the function $_i$ can be obtained as a linear function of β^*. Using (4.19) and repeating the derivation above between (4.19) and (4.22). For the case of $k = 3$, Poinier (1976, p. 48) provides $\Delta^{-1}\theta$.

In order to estimate the lag coefficients of a polynomial distributed lag, the data matrix x is transformed using W as follows:

$$Y_t = x_t\beta + u_t = x_tW\beta^* + u_t = F_t\beta^* + u_t$$

and F_t is a matrix of composite variables. The full vector of lag coefficients is obtained by using Equation (4.21).

Restricted Least Squares Approaches

An alternative approach to estimation is to use restricted least squares. Once again, it is possible to formulate a restricted model to test each approach.

The direct approach requires that $\beta = N\gamma$ be satisfied. From this we obtain

$$\gamma = N^+\beta$$

where N^+ is the generalized inverse of N. The concept of the generalized inverse is employed once again, since N is generally not square. Now

$$\begin{aligned} 0 &= \beta - N\gamma \\ &= \beta - NN^{+}\beta \\ &= (I - N(N'N)^{-1}N')\beta \end{aligned}$$

The restriction matrix R is

$$R = (I - N(N'N)^{-1}N')\beta$$

The model to be estimated is

$$Y = x\beta + u \quad \text{subject to } R\beta = 0$$

For the interpolation approach, where each coefficient is calculated as a linear combination of the coefficients estimated for interpolation, or

$$\beta = W\beta^*$$

it follows that each interpolated coefficient minus a linear combination of the coefficients in β^* is equal to zero.

To obtain the restrictions for the interpolation approach, partition the matrix W into two parts as $[W_r : I_{k+1}]$. The first partition is the set of rows used to obtain the interpolated coefficients. The second partition is a negative identity matrix with the order equal to the number of estimated coefficients used for interpolation. The model is estimated as described above.

These restricted least squares approaches are similar to those discussed in the previous chapter. Once again, the latter representation of the restrictions may be preferable to the former. The generalized inverse N^+ is difficult to compute and is more susceptible to computational error than the latter method.

V. EMPIRICAL APPLICATIONS TO CONSUMPTION MODELS

The objective of this chapter is to illustrate the properties and characteristics of the previously discussed lag formulations by incorporating them into models of consumption. This is accomplished by fitting various lag lengths of each of the lag models to various measures of consumption.

Although empirical results are presented here, no claim is made about their significance for economic theory. Once again, the objective is demonstration, not determination.

The first section discusses the model and data used in determining each of the lag structures. The subsequent sections present results when the models are run under each of the lag formulations. A general summary of results and comparison comprise the concluding section.

Models and Data

The consumption models presented in this chapter are of a simple nature. Four time series on consumption were regressed on income and varying lagged values of income. The four time series are total consumption expenditures, durable consumption expenditures, non-durable consumption expenditures, and consumption expenditures on services. The income variable used here is personal disposable income. All variables are quartered data at annual rates.

The time span covered by the series is from the first quarter of 1948 to the fourth quarter of 1982. The data is not presented here since it was obtained from the standard source Business Conditions Digest (October 1982 and June 1983). All variables are measured in 1972 dollars.

In general, all the models had a high degree of positive autocorrelation. To correct for this problem, the Cochrane-Orcutt iterative procedure was used. This corrective measure improved the results greatly, and only these final results are presented. Other problems figured in as well, specifically, some level of multicollinearity. This was generally ignored since it does not bias the estimators. Most of the values of the F ratios were omitted from the tables when their values were extremely large.

Most of the problems encountered are perhaps due to the theoretical construct of the model itself. No attempt was made to check for identification or excluded variables. For the demonstration purposes at hand, including other independent variables would only occlude the objective.

Arithmetic Lags

This section presents the four models of consumption under two lag lengths of four and eight respectively. To this end four tables are presented on the next pages. Table 5.1 gives the results from each consumption model when they are regressed on the lagged values of income with a lag length of four. Table 5.2 gives the results from each consumption model when an arithmetic distribution, with lag length of four, is imposed. Tables 5.3 and 5.4 do likewise for when a lag length of eight is used.

With either a lag of four or of eight, the unconstrained lag coefficients decrease in value as the lag increases. One would expect that if an arithmetic lag was imposed, not much "violence" would be done to the data. this is supported by the restricted F tests presented in Tables 5.2 and 5.4. From the adjusted coefficient of determination, we notice that very little explanatory power was lost.

The thrust of the arithmetic lag is to obtain a linearly decreasing lag distribution. The formulation required to obtain such results was discussed in chapter two. Along with the formulation, the slope of the distribution function was also derived. It turned out to be the negative of the slope coefficient of the variable with the largest lag, i.e. $-\beta n$.

In Tables 5.2 and 5.4, which give the values of the lag coefficients when the arithmetic lag is imposed, it is seen that the formulation was successful. The difference

TABLE 5.1

ESTIMATED COEFFICIENTS FOR UNCONSTRAINED LAG MODELS WITH LAG LENGTH OF FOUR[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	$\bar{R}^2$	DW
Total	103.54 (64.117)	.50094 (.0563)	.19731 (.0569)	.10702 (.0551)	.00932 (.0575)	.08482 (.0561)	.9995	2.0552
Durable	-245.22 (44.049)	.18986 (.0378)	.03889 (.0385)	.02038 (.0370)	-.06616 (.0386)	-.02303 (.0377)	.9930	2.3131
Non-Durable	653.69 (11.733)	.17274 (.0226)	.08343 (.0226)	-.0057 (.0218)	-.00077 (.0228)	.03102 (.0225)	.9992	2.0539
Services	208.27 (270.01)	.12119 (.01720)	.06398 (.0720)	.07620 (.01662)	.06506 (.01730)	.06101 (.01710)	.9998	1.2808

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 136

TABLE 5.2

ESTIMATED COEFFICIENTS FOR ARITHMETIC LAG MODELS WITH LAG LENGTH OF FOUR[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Rest. F	R^2	DW
Total	107.86 (69.984)	.30066 (.0033)	.24503 (.0027)	.18040 (.0020)	.12026 (.0013)	.06013 (.0007)	3.468	.9994	1.8558
Durable	-239.87 (48.657)	.05414 (.0023)	.04331 (.0019)	.03248 (.0014)	.02165 (.0009)	.01083 (.0005)	3.641	.9925	2.1606
Non-Durable	651.95 (44.449)	.09580 (.0021)	.07664 (.0017)	.05748 (.0013)	.03832 (.0008)	.01916 (.0004)	4.121	.9991	1.8164
Services	274.86 (303.65)	.12578 (.0065)	.0062 (.0052)	.07547 (.0039)	.05031 (.0026)	.02516 (.0013)	1.924	.9998	1.2386

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 136

Restricted F d.f. numerator = 4; d.f. denominator = 130; tabulated = 2.45

TABLE 5.3

ESTIMATED COEFFICIENTS FOR UNCONSTRAINED LAG MODELS WITH LAG LENGTH OF EIGHT[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	$\bar{R}^2$	DW
Total	949.09	.46195	.19781	.0420	-.03225	.02858	-.03860	.08695	.07644	-.01604	.9995	2.0713
	(68.938)	(.0603)	(.0594)	(.06046)	(.0627)	(.0626)	(.0622)	(.0609)	(.0599)	(.0589)		
Durables	-255.78	.20935	.05777	.05342	-.05499	-.01262	-.06782	-.03971	.00291	-.02568	.9929	2.3490
	(423.11)	(.0406)	(.0401)	(.0409)	(.0424)	(.0423)	(.0421)	(.0412)	(.0405)	(.0397)		
Non-Durables	652.50	.17327	.08167	-.00690	-.01165	.01689	.01140	.01612	.01740	-.01158	.9991	2.0355
	(93.713)	(.0247)	(.0241)	(.0249)	(.0253)	(.0252)	(.0251)	(.0246)	(.0243)	(.0240)		
Services	-217.69	.10549	.05530	.05808	.03562	.02710	.01990	.07487	.05352	.01833	.9998	1.5270
	(96.756)	(.0160)	(.0155)	(.0156)	(.0161)	(.0161)	(.0160)	(.0157)	(.0156)	(.0155)		

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 132

TABLE 5.4

ESTIMATED COEFFICIENTS FOR ARITHMETIC LAG MODELS WITH LAG LENGTH OF EIGHT[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Rest. F	$\bar{R}^2$	DW
Total	142.66	.18127	.16113	.14099	.12085	.10071	.08057	.06043	.04029	.02014	4.775	.9993	1.7486
	(90.736)	(.0026)	(.0023)	(.0020)	(.0017)	(.0014)	(.0012)	(.0009)	(.0006)	(.0003)			
Durables	-224.86	.03240	.02880	.02520	.02160	.01780	.01440	.01080	.00720	.00360	3.745	.9917	2.0971
	(53.558)	(.0015)	(.0014)	(.0012)	(.0010)	(.0009)	(.0007)	(.0005)	(.0003)	(.0002)			
Non-Durables	668.54	.05760	.05120	.04480	.03840	.03200	.02560	.01920	.01280	.00640	3.711	.9989	1.7197
	(52.487)	(.0015)	(.0013)	(.0012)	(.0010)	(.0008)	(.0007)	(.0005)	(.0003)	(.0002)			
Services	-102.06	.08626	.07668	.06710	.05751	.04810	.03834	.02876	.01920	.00959	2.837	.9998	1.4681
	(140.46)	(.0032)	(.0028)	(.0025)	(.0021)	(.0018)	(.0014)	(.0011)	(.0007)	(.0004)			

[a]Standard errors are in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 132

Restricted F numerator d.f. = 8; denominator d.f. = 132; tabulated = 2.02

between each lag coefficient is the value of the last lag coefficient. Another way to view this is to multiply the last coefficient by (n + 1 - i) to obtain the lag coefficient for the i^{th} period. This demonstrates the equivalence of the restricted least squares and direct or composite variable approaches.

Another issue that was discussed, is that the coefficients must all have the same sign. in particular they must have the same sign as the lag coefficient at i = n. By comparing the four tables, it is seen that some of the signs of the coefficients were thus altered. Again, by checking the restricted F ratios, apparently this does very little violence to the data.

No attempts were made to obtain results for a longer lag length with an arithmetic lag. With a lag length of eight, the majority of the t-ratios became insignificant for all models. One conclusion that may be drawn then, is that the arithmetic lag model in general will not handle large lag lengths very well.

Inverted-V Models

This section discusses the four models of consumption when the inverted-V lag structure is imposed. The lag lengths used here are once again four and eight. The results for the inverted-V lag models when a lag length of four is assumed are presented in Table 5.5, and for a lag

TABLE 5.5

ESTIMATED COEFFICIENTS FOR INVERTED-V LAG MODELS WITH LENGTH OF FOUR[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Rest. F	R^2	DW
Total	134.89	.1028	.20056	.30083	.20056	.1028	14.074	.9993	1.6256
	(80.399)	(.0013)	(.0026)	(.0039)	(.0026)	(.0013)			
Durables	-230.14	.0180	.0360	.0540	.0360	.0180	5.698	.9921	2.0948
	(50.366)	(.0008)	(.0016)	(.0024)	(.0016)	(.0008)			
Non-Durables	668.64	.03181	.06362	.09544	.06362	.03181	12.712	.9989	1.6353
	(47.339)	(.0008)	(.0015)	(.0023)	(.0015)	(.0008)			
Services	312.91	.0410	.0830	.1240	.0830	.0410	6.859	.9998	1.1494
	(290.13)	(.0023)	(.0046)	(.0069)	(.0046)	(.0023)			

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 136

Restricted F numerator d.f. = 4; denominator d.f. = 130; tabulated = 2.45

TABLE 5.6

ESTIMATED COEFFICIENTS FOR INVERTED-V LAG MODELS WITH LAG LENGTH OF EIGHT[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Rest. F	$\bar{R}^2$	DW
Total	217.38	.0362	.0724	.1086	.1447	.1810	.1447	.1086	.0724	.0362	11.034	.9991	1.4982
	(122.08)	(.0007)	(.0014)	(.0021)	(.0028)	(.0035)	(.0028)	(.0021)	(.0014)	(.0007)			
Durables	-205.24	.0064	.0129	.0193	.0257	.0321	.0257	.0193	.0129	.0064	5.104	.9912	2.0168
	(58.907)	(.0003)	(.0007)	(.0010)	(.0014)	(.0017)	(.0014)	(.0010)	(.0007)	(.0003)			
Non-Durables	699.08	.0115	.0230	.0343	.0456	.0573	.0458	.0344	.0230	.0115	7.477	.9987	1.5318
	(61.228)	(.0003)	(.0007)	(.0010)	(.0014)	(.0017)	(.0014)	(.0010)	(.0007)	(.0003)			
Services	15.217	.0167	.0334	.0501	.0668	.0835	.0668	.0501	.0334	.0167	7.565	.9998	1.3236
	(170.89)	(.0007)	(.0015)	(.0022)	(.0029)	(.0037)	(.0029)	(.0022)	(.0015)	(.0007)			

[a]Standard errors are in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 132.

Restricted F numerator d.f. = 8; denominator d.f. = 122; tabulated = 2.02

length of eight in Table 5.6. These results should be compared with those in Tables 5.1 and 5.3 which presented the results for the unconstrained lag models.

For the models where the lag length is four, the peak of the distribution occurs during the lag period of $n \div 2$, as discussed in chapter three. For the models in Table 5.5, this implies that the coefficient of lag two should be the highest. For the models in Table 5.6 the peak should occur at lag four. By examining these tables, one can confirm this.

Similar to the arithmetic model, the inverted-V also assumes a constant slope. As mentioned before, the shape of the distribution can be likened to a "splice" of two linear functions. In addition it is also assumed that the slope is equal for both sides. This can readily be observed from either Table 5.5 or Table 5.6. The absolute value of the slope should be equal to both the coefficient at lag 0 and at lag n. Notice that this also implies that the coefficient at lag 0 must equal the coefficient at lag n. Once again, as discussed in chapter three, the value of the coefficient of the variable lagged i periods can be obtained by multiplying β_0 or β_n by $1 + i$ if i is less or equal to $n \div 2$ or by $n + 1 - i$ if i is greater than $n \div 2$.

By comparing the results of Tables 5.5 and 5.6 to those for the unconstrained cases presented in Tables 5.1 and 5.3, one might assume a priori that an inverted-V lag structure may not yield satisfactory results. This assumption is supported upon examination of the restricted F ratios. All of the ratios indicate that imposing an inverted-V lag structure on the data decreases the explanatory power of the independent variables.

Almon Models

This section presents the four models of consumption under two lag lengths of eight and twelve respectively. To this end, five tables are presented on the next pages. Tables 5.7 and 5.8 give the results from each consumption model when an Almon model is imposed for the cases where the polynomial degrees employed are two and three, and the lag length is eight. Table 5.9 gives the results from the four models when a lag length of twelve is used, without imposing any lag formulation. Tables 5.10 and 5.11 give the results from each of the models when a lag length of twelve is assumed and the polynomial degree employed is two and three.

The approach of the Almon model is to estimate a subset of the coefficients, and from these interpolate the values of the complement. In chapter four it was observed that the coefficients could be obtained equivalently from using either the "direct", Almon, or restricted least squares approaches. All the results presented in the following tables are obtained from the restricted least

TABLE 5.7

ESTIMATED COEFFICIENTS FOR ALMON LAG MODELS WITH DEGREE TWO AND LAG LENGTH OF EIGHT[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Rest. F	R^2	DW
Total	103.20	.40281	.25482	.13785	.05189	-.00305	-.02698	-.01989	-.01822	-.08734	1.865	.9994	2.0061
	(66.858)	(.0433)	(.0219)	(.0163)	(.0213)	(.0240)	(.0216)	(.0169)	(.0221)	(.0430)			
Durables	-253.72	.17312	.09646	.03599	-.00829	-.03637	-.04827	-.04397	-.02349	.01319	0.831	.9929	2.3335
	(40.784)	(.0283)	(.0142)	(.0106)	(.0140)	(.0157)	(.0141)	(.0110)	(.01436)	(.0281)			
Non-Durables	657.12	.13406	.08457	.04533	.01636	-.00236	-.01082	-.00902	.00304	.02535	1.853	.9991	1.9075
	(43.305)	(.0181)	(.0094)	(.0069)	(.0088)	(.0098)	(.0089)	(.0072)	(.0095)	(.0180)			
Services	-224.94	.08820	.07072	.05662	.04592	.03860	.034677	.03414	.03600	.04324	2.369	.9998	1.6322
	(90.551)	(.0121)	(.0644)	(.0048)	(.0058)	(.0065)	(.0059)	(.0049)	(.0064)	(.0199)			

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 132

Restricted F numerator d.f. = 6; denominator d.f. = 122; tabulated = 2.17

TABLE 5.8

ESTIMATED COEFFICIENTS FOR ALMON LAG MODELS WITH DEGREE THREE AND LAG LENGTH OF EIGHT[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Rest. F	$\bar{R}^2$	DW
Total	94.263	.50408	.21694	.05770	-.00571	-.00535	.02672	.05844	.05775	-.00740	.427	.9995	2.0954
	(66.704)	(.0538)	(.0246)	(.0308)	(.0281)	(.0232)	(.0273)	(.0304)	(.0251)	(.0521)			
Durables	-256.59	.20763	.08379	.00893	-.02779	-.03723	-.03022	-.01760	-.01022	-.01892	.545	.9930	2.3625
	(41.086)	(.0362)	(.0164)	(.0207)	(.0189)	(.0156)	(.0184)	(.0205)	(.0168)	(.0351)			
Non-Durables	651.77	.17560	.06838	.01217	-.00754	-.00348	.01119	.02329	.01964	-.01292	.346	.9991	2.0475
	(42.657)	(.0221)	(.0104)	(.0126)	(.0115)	(.0095)	(.0112)	(.0125)	(.0106)	(.0213)			
Services	-229.98	.11240	.06198	.03722	.03193	.03793	.04753	.05305	.04679	.02109	1.276	.9998	1.6017
	(89.628)	(.0147)	(.0071)	(.0084)	(.0076)	(.0063)	(.0074)	(.0083)	(.0072)	(.0014)			

[a]Standard errors are in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128

Restricted F numerator d.f. = 5; denominator d.f. = 122; tabulated = 2.17

TABLE 5.9

ESTIMATED COEFFICIENTS FOR UNCONSTRAINED LAG MODELS WITH LAG LENGTH OF TWELVE[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7
Total	112.18 (73.392)	.50214 (.0614)	.20754 (.0619)	.03363 (.0631)	-.00822 (.0642)	.02542 (.0641)	-.02716 (.0621)	.0786 (.0624)	.05922 (.0621)
Durables	-265.62 (41.502)	.22401 (.0408)	.08818 (.0413)	.01256 (.0423)	-.04035 (.04311)	-.00828 (.0430)	-.06654 (.0417)	-.00901 (.0419)	.01075 (.0417)
Non-Durables	665.48 (472.67)	.17030 (.0255)	.07856 (.0255)	-.02182 (.0259)	.00584 (.0261)	.00192 (.0261)	.01968 (.0252)	.01815 (.0253)	.00536 (.0252)
Services	-219.95 (90.426)	.09531 (.0163)	.03502 (.0163)	.04333 (.0165)	.02844 (.0165)	.03496 (.0165)	.02161 (.0159)	.06256 (.0160)	.04239 (.0160)

Dependent Variable	Lag 8	Lag 9	Lag 10	Lag 11	Lag 12	R^2	DW
Total	-.01121 (.0622)	.02447 (.0620)	-.02482 (.0617)	-.04920 (.0629)	-.10193 (.0616)	.9995	1.9902
Durables	-.00163 (.0418)	.00121 (.0417)	-.06714 (.0413)	-.06423 (.0420)	.07856 (.0409)	.9934	2.3251
Non-Durables	-.00806 (.0253)	-.00807 (.0252)	.03708 (.0253)	-.00735 (.0257)	-.00564 (.0256)	.9991	1.8324
Services	.00251 (.0159)	.03596 (.0159)	.01034 (.0161)	.01898 (.0166)	.00217 (.0164)	.9998	1.5650

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128

TABLE 5.10

ESTIMATED COEFFICIENTS FOR ALMON LAG MODELS WITH DEGREE TWO AND LAG LENGTH OF TWELVE[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
Total	115.85 (75.032)	.31345 (.0360)	.23022 (.0236)	.15861 (.0155)	.09862 (.0130)	.05024 (.0147)	.01349 (.0168)	-.01164 (.0176)	-.02515 (.0169)	-.02704 (.0149)
Durables	-263.89 (40.467)	.13938 (.0224)	.09285 (.0145)	.05323 (.0093)	.02053 (.0079)	-.00527 (.0092)	-.02415 (.0106)	-.03613 (.0112)	-.04119 (.0107)	-.03934 (.0093)
Non-Durables	667.83 (45.708)	.10260 (.0157)	.07641 (.0105)	.05369 (.0071)	.03446 (.0058)	.01870 (.0063)	.00643 (.0071)	-.00237 (.0074)	-.00769 (.0071)	-.00953 (.0064)
Services	-291.94 (78.281)	.06400 (.0102)	.05672 (.0010)	.05011 (.0048)	.04416 (.0040)	.03889 (.0041)	.03429 (.0045)	.03037 (.0047)	.02711 (.0045)	.02453 (.0042)

Dependent Variable	Lag 9	Lag 10	Lag 11	Lag 12	Rest. F	R^2	DW
Total	-.01731 (.0134)	.00404 (.0160)	.03701 (.0240)	.08160 (.0363)	2.395	.9994	1.7663
Durables	-.03058 (.0082)	-.01491 (.0096)	.00767 (.0148)	.03716 (.0226)	1.536	.9931	2.1817
Non-Durables	-.00788 (.0060)	-.00276 (.0073)	.00584 (.0107)	.01792 (.0158)	2.427	.9990	1.6664
Services	.02261 (.0040)	.02137 (.0049)	.02080 (.0070)	.02090 (.0102)	1.748	.9998	1.6002

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128

Restricted F numerator d.f. = 10: denominator d.f. = 114: tabulated = 1.91

TABLE 5.11

ESTIMATED COEFFICIENTS FOR ALMON LAG MODELS WITH DEGREE THREE AND LAG LENGTH OF TWELVE[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
Total	104.59 (72.820)	.41166 (.0461)	.23969 (.0230)	.11882 (.0193)	.04114 (.0217)	-.00123 (.0212)	-.01620 (.0186)	-.01166 (.0170)	.00450 (.0186)	.02437 (.0214)
Durables	-268.11 (39.924)	.17867 (.0299)	.09662 (.0144)	.03727 (.0123)	-.00252 (.0141)	-.02592 (.0139)	-.03609 (.0121)	-.03618 (.0111)	-.02936 (.0122)	-.01879 (.0140)
Non-Durables	660.75 (43.651)	.15061 (.0194)	.08125 (.0101)	.03456 (.0083)	.00668 (.0090)	-.00621 (.0088)	-.00795 (.0077)	-.00237 (.0070)	.00668 (.0077)	.01537 (.0088)
Services	215.97 (85.421)	.09531 (.0163)	.03502 (.0163)	.04333 (.0165)	.02844 (.0165)	.03496 (.0165)	.02161 (.0160)	.06256 (.0160)	.04239 (.0159)	.00251 (.0160)

Dependent Variable	Lag 9	Lag 10	Lag 11	Lag 12	Rest. F	$\bar{R}^2$	DW
Total	.04006 (.0219)	.04368 (.0196)	.02732 (.0234)	-.01690 (.0463)	1.450	.9995	1.8750
Durables	-.00762 (.0142)	.00099 (.0125)	.00388 (.0147)	-.00211 (.0300)	1.271	.9932	2.2548
Non-Durables	.01987 (.0091)	.01634 (.0084)	.00094 (.0102)	-.03016 (.0194)	1.013	.9991	1.7767
Services	.03596 (.0160)	.01034 (.0161)	.01898 (.0170)	.02170 (.0164)	1.686	.9998	1.5650

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128

Restricted F numerator d.f. = 9; denominator d.f. = 114; tabulated = 1.91

squares formulation, which is more useful for the purposes here. It should also be mentioned that the subset of coefficients to be estimated corresponded to the lag periods 0, 1, and 2 for the case of degree two and 0, 1, 2, 3 for the case of degree three.

Taking the results for the total, non-durable, and durable consumption series first, it can be seen that the lag coefficients tend to decline in value over the first four periods, and they increase in value during periods five through n. However, the coefficients for the latter lags are usually statistically insignificant. Hence, the results are consistent with the arithmetic models reported in the second section of this chapter.

The exception to this trend was the lag distribution for the consumption expenditures on services. As can be seen from all tables, this model seemed to agree with the Almon formulation. However, the results did not appear to vary much from the unconstrained models.

Although not reported here, the models were also tested for a lag length of sixteen and degrees two and three. The results were strikingly similar to those obtained for a lag length of eight or twelve.

Spline Lags

This section presents the four models of consumption when a cubic spline distribution is imposed. Tables 5.12

and 5.13 present the results when endpoint constraints equal to unity and zero are assumed. Recall from chapter four, that if the endpoint constraints are unity, the functions over the outer intervals become quadratics. When the constraints are set equal to zero, the functions over the outer intervals become linear. The latter case is referred to as a "natural cubic spline".

In both cases, knots were located at the lag index values 0, 4, 8, 12. Therefore, there were three points per interval. Poirier (1978) suggests using as few knots as possible, and have four or five points per interval to avoid over-fitting. As can be seen from the results, there was a tendency for this to occur. By imposing a spline distribution, lag coefficients that were insignificant as shown in Table 5.9, became significant. This also illustrates the degrees of freedom problem of the spline formulation. By using the composite variables, a lot of dummy variables are not being used, although they are implicit in the model. The final result is that the degrees of freedom, as conventionally defined, does not take this into account. Therefore, as expected and observed in Tables 5.12 and 5.13, the spline formulation gives longer lags.

When the models are analyzed under the restricted F ratio criteria, it is observed that the natural cubic spline provides a better fit. Specifically, total and services expenditures models have a distinct preference

TABLE 5.12

ESTIMATED COEFFICIENTS FOR CUBIC SPLINE MODELS WITH LAG LENGTH OF TWELVE, THREE KNOTS, AND ENDPOINT CONSTRAINTS EQUAL TO UNITY[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
Total	116.85 (77.450)	.32964 (.0442)	.23574 (.0268)	.13487 (.0145)	.00057 (.0293)	.08596 (.0250)	.04254 (.0203)	-.10701 (.0381)	.05308 (.0247)	.08867 .0309)
Durables	-264.20 (40.209)	.14612 (.0273)	.09661 (.0162)	.04884 (.0087)	-.02699 (.0186)	-.0060 (.0157)	-.00860 (.0127)	-.07844 (.0239)	.0025 (.0156)	.00594 (.0195)
Non-Durables	665.59 (44.231)	.12119 (.0182)	.08378 (.0113)	.04646 (.0062)	-.00596 (.0117)	.02504 (.0101)	.01129 (.0082)	-.03189 (.0154)	.02186 (.0099)	.03426 (.0124)
Services	-227.83 (81.342)	.05574 (.0125)	.05063 (.0080)	.03771 (.0044)	.03456 (.0078)	.06192 (.0070)	.04059 (.0055)	.01080 (.0103)	.03299 (.0066)	.05061 (.0083)

Dependent Variable	Lag 9	Lag 10	Lag 11	Lag 12	Rest. F	R^2	DW
Total	.04705 (.0244)	.02322 (.0122)	.00271 (.0226)	-.03181 (.0412)	3.711	.9994	1.9329
Durables	-.00272 (.0146)	.00111 (.0073)	-.00569 (.0136)	-.01675 (.0255)	2.179	.9928	2.2752
Non-Durables	.00937 (.0104)	.00456 (.0052)	-.0776 (.0096)	-.02718 (.0169)	2.512	.9990	1.7898
Services	.03960 (.0074)	.01689 (.0037)	.01430 (.0067)	.00856 (.0015)	3.973	.9998	1.6451

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128

Restricted F numerator d.f. = 9; denominator d.f. = 114; tabulated = 1.96

TABLE 5.13

ESTIMATED COEFFICIENTS FOR CUBIC SPLINE MODELS WITH LAG LENGTH OF TWELVE, THREE KNOTS, AND ENDPOINT CONSTRAINTS EQUAL TO ZERO[a]

Dependent Variable	Constant	Lag 0	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
Total	103.56 (73.209)	.38413 (.0429)	.26005 (.0254)	.14507 (.0162)	.04845 (.0215)	-.02124 (.0297)	-.02796 (.0214)	-.01273 (.0160)	.01545 (.0211)	.04461 (.0295)
Durables	-268.86 (40.006)	.16867 (.0274)	.10678 (.0159)	.04860 (.0101)	-.00217 (.0140)	-.04172 (.0194)	-.04212 (.0139)	-.03322 (.0104)	-.02129 (.0137)	-.00941 (.0192)
Non-Durables	660.73 (43.258)	.13675 (.0183)	-.09044 (.0111)	.04767 (.0072)	.01201 (.0092)	-.01299 (.0123)	-.01160 (.0089)	-.00323 (.0067)	.00969 (.0087)	.02173 (.0122)
Services	241.61 (80.096)	.07307 (.0123)	.05930 (.0076)	.04730 (.0050)	.03887 (.0060)	.03578 (.0080)	.02877 (.0058)	.02679 (.0044)	.03020 (.0057)	.03517 (.0080)

Dependent Variable	Lag 9	Lag 10	Lag 11	Lag 12	Rest. F	R^2	DW
Total	.04118 (.0216)	.02874 (.0166)	-.10365 (.0256)	-.01105 (.0428	1.702	.9995	1.8518
Durables	-.00268 (.0140)	-.00138 (.0104)	-.00366 (.0161)	-.00777 (.0274)	1.326	.9932	2.2501
Non-Durables	.01666 (.00908)	-.00662 (.0073)	-.00670 (.0011)	-.02170 (.0183)	1.495	.9991	1.7377
Services	.02816 (.0060)	.02231 (.0051)	.01726 (.0076)	.01258 (.0122)	1.933	.9998	1.6098

[a]Standard errors are given in parenthesis. All equations corrected using Cochrane-Orcutt.

Number of observations = 128.

Restricted F d.f. numerator = 9; denominator d.f. = 114; tabulated = 1.96

for a natural cubic spline.

Summary

The objective of this chapter was to illustrate the characteristics and properties of the various finite lag models discussed in this study. The economic model employed was the simple consumption function. Four of these models were used, where the dependent variables represented different categories of consumption expenditures.

The experiment described in this chapter did, however, yield rather consistent results. In the first section, it was found that an arithmetic lag with lag length of four quarters fit the data very well. In line with this result, the inverted-V formulation performed very poorly. The next section presented and discussed the results when Almon lag models of varying lag lengths were employed. It was found that they were consistent with the arithmetic lag. Following this, the results from the spline distribution were presented. Although the spline technique suggests the lag is longer, one must be aware of the degrees of freedom problem encountered.

The results from the spline lag were not very inconsistent with the previous findings. Since the arithmetic lag may be thought of as a special case of the Almon polynomial (i.e. polynomial degree one), and the spline as a generalization of the Almon, one would expect the spline to yield good results.

As the literature suggests, and from the results presented here, a few conclusions may be drawn, namely: when the lag length is not very large, the arithmetic or inverted-V will probably yield acceptable results. As the lag length increases, one might prefer to employ an Almon or spline model.

VI. SUMMARY AND CONCLUSIONS

The objective of this study was to demonstrate how prior restrictions, in the form of a finite lag distribution, may be incorporated into econometric models. To this end, various approaches and distribution shapes were reviewed.

The first chapter reviewed and discussed the earliest approaches to the problem. These were the arithmetic and inverted-V formulations. The arithmetic lag model assumes that the effect of a change of the independent variable diminishes linearly over succeeding time periods. It is assumed that the adjustment is the largest during the period when the explanatory variable changes and subsequent adjustments are smaller, their effects diminishing linearly and by a constant amount until the change is exhausted. It was demonstrated that the lag distribution may be thought of as a linear function of the lag index i. In addition, by dividing each coefficient by the sum of the coefficients, lag weights are obtained. Estimation of the arithmetic lag model involves the construction of a composite variable. The advantage of this method is that the problem of multicollinearity is avoided. An alternative method of estimating the lag coefficients is the restricted least squares formulation, which was presented next.

The next model, or formulation, that was examined was the inverted-V lag model. The formulation imposed an inverted-V shape to the lag distribution. An alternative way to view this is to regard each side of the distribution as a separate linear function. While reviewing De Leeuw's formulation it was demonstrated that both sides had the same constant slope, similar to the arithmetic lag model.

The disadvantages of these approaches is that they require the lag coefficients to lie on a linear function of some sort. One way to avoid this restriction is to assume a polynomial distribution. This idea was presented by Almon in 1965. In her original article, she suggested a method of polynomial interpolation, Lagrangian interpolation, as an approach to estimating the polynomial distribution. The advantage of this is that fewer parameters must be estimated. An alternative method that yields equivalent results was suggested by Cooper, and was also reviewed. Although a polynomial distribution may be assumed, it is still possible to pre-determine its shape. This is accomplished by using endpoint constraints. By constraining the value of the lag coefficient at the left end of a second order polynomial to be zero, a monotonically increasing polynomial is obtained, and vice versa for a left endpoint constraint. Constraining both sides yields a "humped" distribution. The discussion on the Almon model closed with an exposition of restricted least squares approaches.

An alternative method of estimating a polynomial distribution is to employ spline functions. The cubic spline approach to estimating a polynomial distributed lag is similar to a piece-wise polynomial regression. The difference is that the model is formulated so as to ensure that the spline will be continuous and have continuous first and second derivatives at the knots. The problem encountered with that was that there were fewer variables than restrictions, or equations, rendering the model insolvable. The solution was to impose constraints on the second derivatives of the spline at the first and last knots. The chapter closed with an exposition of a restricted least squares formulation of the cubic spline.

The spline and Almon approaches to estimating a polynomial distribution are, to some extent, very similar to each other. Each involves estimating a subset of the coefficients and from the estimates, calculate, or interpolate, the missing values.

In order to demonstrate the properties of these lag models, four models of consumption were estimated, using each of the discussed formulations. The results were presented in chapter five. No attempt was made to determine the exact lag structure of each of the consumption models, however, the results presented in the tables did indicate than an arithmetic model with a lag length of four quarters might be the best fit. The spline technique suggests the lag is longer, but there are degrees of freedom problems with the spline formulation. This finding is more in line with Keynesian hypotheses of a relatively quick and stable impact of income on consumption. It conflicts with permanent income theories that hypothesize a long lag between the two time series.

There are many aspects of this subject that were not pursued in this study, but clearly merit some investigative effort. Further research should be undertaken to determine the length and polynomial order of the lag structure. In addition, there are other methods of estimating a finite distributed lag, such as the use of Boyesian inference. Nevertheless, it is likely that such methods will also confirm the essential arithmetic structure to the consumption-income relationship, and the lag length of four quarters which was estimated in chapter five using a variety of different models.

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