AN ANALYTICAL STUDY OF THE
DYNAMIC CHARACTERISTICS OF
TOWED FLEXIBLE CYLINDERS

JEFFREY M. KOMROWER
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OF THE DYNAMIC CHARACTERISTICS
OF TOWED FLEXIBLE CYLINDERS

by

Jeffrey M. Komrower

A Thesis Submitted to the Faculty of the
College of Engineering
in Partial Fulfillment of the Requirements for the Degree
of Master of Science in Engineering

Florida Atlantic University
Boca Raton, Florida
June 1979
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This thesis was prepared under the direction of the candidate's thesis advisor, Dr. Stanley Dunn, Department of Ocean Engineering. It was submitted to the faculty of the College of Engineering and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Engineering.

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ACKNOWLEDGEMENTS

I would sincerely like to thank Dr. Stanley E. Dunn, Associate Professor of Ocean Engineering, Florida Atlantic University, for the considerable time, effort, and guidance he gave me towards the preparation of this thesis.

Additionally, I would like to thank Dr. C. L. Su, Associate Professor of Ocean Engineering, and Dr. Jeffrey S. Tennant, Associate Professor of Ocean Engineering, for the time and effort they have spent on my behalf.
ABSTRACT

Author: Jeffrey M. Komrower
Title: An analytical study of the dynamic characteristics of towed flexible cylinders
Institution: Florida Atlantic University
Degree: Master of Science in Engineering
Year: 1979

A linear differential equation of motion to describe the fluid-induced vibrations of a towed flexible cylinder, with applications toward towed array system, is developed. The equation is explored for its free vibration characteristics with the aid of a computer program. The behavior of the locus of the several lowest eigenfrequencies, as system parameters were varied, revealed the basic dependence of the system on these parameters. It is shown that for various combinations of system parameters, the system may be subject to buckling or oscillatory instabilities in its lower flexural modes. It was found that (1) longer cylinders are more susceptible to oscillatory instabilities than are shorter cylinders but are less susceptible to buckling, (2) tension due to skin friction drag stabilizes the system, and (3) the tail of the cylinder acts as an initiator of unstable motions. Nomographs are presented to predict natural frequencies for various combinations of system parameters.
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NOMENCLATURE

\( x \) = Coordinate along centerline of undeflected cylinder
\( y \) = Lateral displacement of cylinder
\( L \) = Length of cylinder
\( M \) = Added mass of fluid/unit length of cylinder
\( m \) = Mass of cylinder/unit length
\( T \) = Axial Tension along cylinder
\( D \) = Diameter of cylinder
\( U \) = Tow speed
\( S \) = Cross sectional area of cylinder
\( \rho_f \) = Fluid density
\( \rho_b \) = Cylinder density
\( E I \) = Flexural rigidity of cylinder
\( \theta \) = Apparent angle of incidence
\( Q \) = Lateral shear force
\( C_f \) = Frictional drag coefficient
\( C_t = 4C_f \)
\( F \) = Parameter associated with tail end of cylinder
\( t \) = Time
\( C_t' \) = Coefficient of form drag at the tail
\( x_e \) = Effective length of tail section of cylinder
\( \ell \) = Length of tail section

\( I_{zz} \) = Moment of inertia of cylinder in z-direction

\( C_i \) = Added mass coefficient associated with cylinder

\( \xi \) = Nondimensional x coordinate, \( x/L \)

\( \eta \) = Nondimensional lateral displacement, \( y/L \)

\( \varepsilon \) = Slenderness ration, \( L/D \)

\( \beta \) = Virtual mass ratio, \( M/(m+M) \)

\( \tau \) = Nondimensional time, \( [EI/(m+M)]^{1/2}(t/L^2) \)

\( \mu \) = Nondimensional velocity, \( (M/EI)^{1/2}UL \)

\( \omega \) = Circular frequency

\( \Omega \) = Nondimensional Circular Frequency, \( [(m+M)/EI]^{1/2}L^2 \)

\( X = x_e/L \)

\( M \) = Bending moment

\( U_{CO} \) = Dimensional critical velocity for unstable oscillation

\( U_{Cb} \) = Dimensional critical velocity for zeroth mode buckling

\( \Omega_{CO} \) = Frequency of oscillation at critical velocity
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INTRODUCTION

When an acoustic array is towed behind a vessel in the direction of its axis, the array has a tendency to exhibit a meandering motion that can introduce errors into acoustical data. This motion, in one form or another, has been observed experimentally and predicted analytically and can emanate from two sources: (a) fluid-induced vibrations and (b) tow vessel motion. Physically, a towed array is a very long, thin, flexible cylinder.

The purpose of this study is to extend existing theory concerning the dynamic motion of flexible cylinders subjected to axial flow to study the fluid-induced vibrations of a towed array. This would provide a first step in the prediction of total array motion for various tow conditions and system parameters. Once fully developed, a model predicting array motion could then be used to correct acoustical data, thus eliminating errors due to array meandering.

To date, the vast majority of analytical and experimental work dealing with the dynamics of flexible cylinders in axial flow has several common characteristics:

(a) The analytical models have used the equation of motion as developed by M. P. Paidoussis.¹
(b) Only relatively short cylinders were considered (under 100 feet).

(c) Experiments performed did not accurately represent the case of a towed cylinder in a free field due to the effects caused by such apparatus necessities as tow struts, walls, discontinuities, tow cables, etc.

In his development of the equation of motion, Paidoussis used relations for the fluid forces on a cylinder taken from a work done by G. I. Taylor. These relations were derived for a grossly roughened cylinder in a laminar flow. A recent investigation by J. S. Tennant revealed that these relations were not valid for towed arrays because the flow situation associated with an array is that of a smooth cylinder exposed to a very thick (possibly thicker than a cylinder diameter) turbulent boundary layer.

In this paper, a rederivation of the governing equation of motion, incorporating expressions derived by Tennant, is presented. With certain simplifying assumptions, a linear differential equation that is slightly different than the equation of Paidoussis results. The general character of the dynamical behavior of the system and the possible mechanisms involved is then examined. This is accomplished by numerically solving for the complex eigenfrequencies of the system and observing the effect of changes in system parameters such as length, drag coefficients, tow velocity and tail shape on
the natural frequencies of the system. This is done with the aid of a computer program that is given in the appendix. Finally, conclusions are drawn and recommendations for future work are given.
The study of the dynamics of towed cylinders was first examined by Hawthorne in connection with the observed 'snakeing' of towed Dracones, which are flexible, sausage-like towed barges used for the transport of fluid cargo.

M. P. Paidoussis extended work done by Hawthorne into a general theory describing the dynamics of flexible cylinders subjected to an axial flow. Theory developed by Paidoussis is valid for fluid-induced vibrations of a stationary cylinder in axial flow, such as occurs in fissile-fuel elements in liquid-cooled nuclear reactor channels, as well as vibrations of a cylinder which is towed. The equation of motion derived by Paidoussis, which is similar to the equation describing vibrations of a beam except for the presence of additional terms due to fluid-cylinder interaction, is given by:

\[ EI \frac{\partial^4 y}{\partial x^4} + M \left( \frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \right)^2 + \frac{1}{2} C_N \left( \frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \right) + \]
\[ \frac{1}{2} C_t \left( \frac{\partial y}{\partial x} \right)^2 y - \frac{1}{4} MU \left( C_t' + (C_t/D)(L-X) \right) \frac{\partial^2 y}{\partial x^2} + ma^2 \frac{\partial^2 y}{\partial t^2} = 0. \]

By subjecting this equation to appropriate boundary conditions and then examining the locus of eigenfrequencies as system parameters were varied, Paidoussis contemplated...
the problem of buckling and stability of flexible cylinders subjected to a flow field. Numerical means were used to arrive at a characteristic equation for the eigenfrequencies. Experiments that followed this analytical work loosely confirmed the stability theory.

Later papers published on the subject used the theory developed by Paidoussis as a basis but considered various extensions of the problem. H. P. Pao³ closely examined the effect of the length to diameter ratio on buckling and stability by using the same technique as Paidoussis. Experimental work performed by Pao confirmed the general character of his analytical results.

C. R. Ortloff and J. Ives⁹ considered the case of a very long, thin cable with zero bending rigidity. Thus, the equation of motion was reduced to one of second order. By assuming the virtual mass ratio, M/(m+M), to be very small (i.e. cables much more dense than water) a closed-form Bessel solution was obtained. Various modal shapes were predicted, but no experimental work was performed.

The response of a flexible cylinder to a forcing frequency applied at the upstream end was first examined by Pao and Q. Tran¹⁰. They forced the system in the range of the lower natural frequencies and examined the effect on mode shapes. Also, by using a Runge-Kutta technique to solve the Paidoussis equation, amplitudes of vibration for a range of forcing frequencies were calculated. By noting the re-
relative maxima of these amplitudes, Pao and Tran were able to substantiate the values of system natural frequencies for a given set of parameters that were computed by Pao in his earlier paper.

Lindemann and Gardner\textsuperscript{11} studied the effect of forcing the system at frequencies much higher than its lower natural frequencies. In such cases, the resulting cylinder motion was believed to be stable except for certain infrequent combinations of parameters. Hence, the purpose of the paper was to predict the steady-state behavior at selected frequencies.

A slightly modified form of the above equation was used by C. Lee\textsuperscript{12} to study the forced system response. Lee introduced a complex modulus of elasticity, $E^* = E(1 + i\delta)$ where $\delta$ is a factor for the inherent damping of the cylinder, and then employed a central difference technique to obtain a solution for an array with varying buoyancy. The forcing frequencies, like those used by Lindemann and Gardner, were much higher than the lower natural frequencies of the system and also much higher than those expected from tow vessel motion. The main purpose here was to study array self-noise.

Paidoussis continued with his work on flexible cylinders and published a paper with B.K. Yu\textsuperscript{13} which closely examined the effect of nose and tail shapes on towed slender bodies. They conclude that, in the range of parameters ex-
examined, the stability of the system is mainly controlled by the shape of the tail portion of the body.

In conclusion, while much analytical work has been done on the dynamics of flexible cylinders, both towed and in stationary axial flow, little has been done to extend present theory to the case of a long, towed array. The few papers that considered long arrays examined the response of the array to high frequency excitations with applications towards predicting array self-noise. The problem of low-vibratory meandering caused by fluid-induced vibrations and tow vessel movements has not been addressed at length.
GOVERNING EQUATION OF MOTION

Derivations

The system under consideration consists of a towed cylindrical body which is completely immersed in an incompressible fluid. The tow speed is assumed to be constant and the tow direction is parallel to the x-axis, which coincides with the position of rest of the cylinder axis. The cylinder will be supported at the upstream end and free at the downstream end.

Considering a differential element, \( \delta x \), of the cylinder in the x-y coordinate system as shown:

\[
\sum F_x = \max = 0 \quad \text{for constant tow speed from Newton's Second Law.}
\]
From the free body diagram above,

\[ [T + (\partial T/\partial x)\delta x] \cos [\theta + (\partial \theta/\partial x)\delta x] - (Q + \partial Q/\partial x) \sin [\theta + (\partial \theta/\partial x)\delta x] \]

\[ - T \cos \theta + Q \sin \theta + F_L \delta x = 0. \]

For small \( \theta \), \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \), thus the equation reduces to:

\[ T + (\partial T/\partial x)\delta x - [Q + (\partial Q/\partial x)\delta x][\theta + (\partial \theta/\partial x)\delta x] - T \theta + F_L \delta x = 0, \]

or:

\[ (\partial T/\partial x)\delta x - Q\theta - \partial Q/\partial x(\delta x) - Q(\partial \theta/\partial x)\delta x + \partial Q/\partial x(\partial \theta/\partial x)(\delta x)^2 + Q\theta + F_L \delta x = 0. \]

Neglecting terms on the order of \((\delta x)^2\) results in

\[ aT/ax - a(Q\theta)/ax + F_L = 0. \quad (1) \]

\[ \Sigma F_y = m ay \text{ yields:} \]

\[ [T + (\partial T/\partial x)\delta x] \sin [\theta + (\partial \theta/\partial x)\delta x] + [Q + (\partial Q/\partial x)\delta x] \cos [\theta + (\partial \theta/\partial x)\delta x] \]

\[ - F_n \delta x - T \sin \theta - Q \cos \theta = m \delta x ay. \]

This reduces, for small \( \theta \), to

\[ [T + (\partial T/\partial x)\delta x][\theta + (\partial \theta/\partial x)\delta x] + Q + (\partial Q/\partial x)\delta x - F_n \delta x - T\theta - Q \]

\[ = m(\delta x)a^2y/\partial t^2, \]

or:

\[ T\theta + a/ax(T\theta)\delta x + aT/ax(\partial \theta/\partial x)(\delta x)^2 + (\partial Q/\partial x)\delta x - F_n \delta x - T\theta = m(\delta x)a^2y/\partial t^2. \]

By neglecting terms on the order of \((\delta x)^2\) one obtains

\[ a/ax(T\theta) + aQ/ax - F_n = ma^2y/\partial t^2. \quad (2) \]

\[ \Sigma M_\theta = I_{zz}(\delta x)a^2\theta/\partial t^2 \text{ yields:} \]

\[ (F_n\delta x)(\delta x/2) - [Q + (\partial Q/\partial x)\delta x](\delta x) + M - [M + (\partial M/\partial x)\delta x] = I_{zz}(\delta x)a^2\theta/\partial t^2. \]
By assuming no angular acceleration and ignoring terms on the order of \((\delta x)^2\), this reduces to:

\[ Q = -\alpha M/\alpha x. \quad (3) \]

From classical elastic beam theory:

\[ M = +EI\alpha^2 y/\alpha x^2, \text{ thus } Q = -EI\alpha^3 y/\alpha x^3. \quad (4) \]

Noting that \(\sin \theta = \alpha y/\alpha x \approx \theta\) for small \(\theta\), equation (1) can be written as

\[ \alpha T/\alpha x - \alpha /\alpha x[EI\alpha^3 y/\alpha x^3(\alpha y/\alpha x)] + F_L = 0, \]

and noting that the product \((\alpha^3 y/\alpha x^3)(\alpha y/\alpha x)\) will be small, one obtains:

\[ \alpha T/\alpha x + F_L = 0. \quad (5) \]

Making a similar substitution for \(\theta\) in equation (2),

\[ \alpha /\alpha x(T\alpha y/\alpha x) - \alpha /\alpha x(EI\alpha^3 y/\alpha x^3) - F_n = m\alpha^2 y/\alpha t^2. \quad (6) \]

Upon integrating equation (5), noting that \(F_L\) may be given by

\[ F_L = \frac{1}{2}\rho_f C_f \pi DU^2 \quad \text{(from J. Tennant)} \]

or \(F_L = \frac{1}{2}\rho_f S(CT/D)U^2\)

where \(C_t = 4C_f\), the tension becomes

\[ T(x) = - \int_x^L F_L = [T(L) + \frac{1}{2}C_t(\rho_f S/D)U^2(L-x)]. \]

\(T(L)\) results from any form drag which may be present at the free end. By assuming the form drag at the end can be neglected, we obtain:

\[ T(x) = \frac{1}{2}\rho_f SU^2(C_t/D)(L-x) \quad (7) \]

The normal force, which is due to the summation of inertia and viscous drag loading is given as\(^{15}\):
\[ F_n = F_i + F_D = C_i \rho_f S (\alpha/\alpha t + U \alpha/\alpha x)^2 y + \frac{1}{2} \rho f C_D |\alpha y/\alpha t + U \alpha y/\alpha x| (\alpha y/\alpha t + U \alpha y/\alpha x). \]

However, since attention is focused on the responses of the system to small perturbations, very small angles of incidence, and low transverse velocities, the normal drag coefficient is very close to zero. Under these conditions, the flow is such that no separation occurs and the normal drag is only due to slow friction. Thus, the expression for the normal force reduces to:

\[ F_n = C_i \rho_f S (\alpha/\alpha t + U \alpha/\alpha x)^2 y. \]  

Furthermore, in this discussion, since the added mass coefficient, \( C_i \), is very hard to determine, it will be assumed to have a value of unity. Also, provided that the wavelength of motion is large in comparison to the diameter of the cylinder, the virtual mass of the fluid/unit length is equal to \( \rho_f S \).

Combining equations (6), (7) and (8), we obtain:

\[ EI \ddot{\alpha} y/\alpha x^4 - \frac{1}{2} MU^2 (C_t/D) (L-x) \ddot{\alpha} y/\alpha x^2 + \frac{1}{2} C_t (MU^2/D) \alpha y/\alpha x + M (\alpha/\alpha t + U \alpha/\alpha x)^2 y + m \ddot{\alpha} y/\alpha t^2 = 0. \]  

Assumptions:

a) Small values of \( \alpha \) such that \( \sin \alpha \approx \alpha \) (or \( \sin (\alpha y/\alpha x) \approx \alpha y/\alpha x \))

b) Motion constrained to x-y plane

c) Neutral buoyancy

d) An inextensible cylinder
e) Perfect linear elastic behavior in bending
f) Uniform viscous drag distribution in direction of tow
g) Constant tow speed
h) No angular acceleration of cable
i) Flow conditions such that no separation of flow occurs around cylinder
j) Added mass of fluid = \( \rho_f S \).

Physical Interpretation of Equation

As we can see, equation (9) resembles the classical equation for the vibration of a beam in a vacuum, but includes additional terms to account for the effect of the surrounding fluid.

The first term represents the elastic bending reaction of the material of the cylinder and is not influenced by the fluid in any way.

The next two terms represent the tension on the cylinder that is caused by drag due to skin friction. The total tension on the cylinder is position dependent as can be seen by the presence of the variable \( x \) in the second term. Notice that the relative magnitudes of the bending and tension terms will determine to what extent the motions of the cylinder are like those of a beam rather than a string. The conditions for which a certain type of motion can be expected will be ex-
amined later.

The next term, $M(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})^2 y$, is actually three terms. The first part, $Ma^2 y/\partial t^2$, represents the inertia of the added or entrained mass of the fluid. This term appears when a vibrating beam or string is placed in a quiescent fluid and is independent of tow velocity. The next terms, $MUa^2 y/\partial x\partial t$ and $MU^2 a^2 y/\partial x^2$, are velocity dependent. The latter term shows up as a tension on the cylinder and is caused by the centrifugal force of the fluid which is forced to flow parallel to the bent cylinder. The former term is inertial in origin and stems from a momentum change in the flowing fluid caused by changes in the slope of the cylinder.

The last term in the equation is due to the inertia of the cylinder itself.

Boundary Conditions

Two sets of boundary conditions will be considered. First, a cylinder which is clamped at the upstream end ($x = 0$) will be examined, and then the cylinder will be given a pinned end condition. In both cases, the tail or downstream end of the cylinder will be free.

Clamped End: \[ y|_{x=0} = 0 \] \[ \frac{\partial y}{\partial x}|_{x=0} = 0 \] \hspace{1cm} Pinned End: \[ y|_{x=0} = 0 \] \[ \frac{\partial^2 y}{\partial x^2}|_{x=0} = 0 \]
At the downstream end, a boundary condition is obtained by assuming that the tail tapers smoothly from an area \( S \) to zero in a sufficiently short distance, \( \ell \), such that \( y \) and the lateral velocity \( v \) may be considered constant. Also, since the tail is very short, the tension due to skin friction can be neglected. By integrating equation (2) over the length of the tail and including the expression for \( F_n \) from equation (8) one obtains:

\[
\int_{L-\ell}^L \frac{\partial Q}{\partial x} \, dx - \int_{L-\ell}^L F_\alpha (a/\partial t + Ua/\partial x)[M(x)v] \, dx - \int_{L-\ell}^L m(x) a^2 y/\partial t^2 \, dx = 0
\]

where \( F \) is a parameter that is introduced in order to be consistent with previous works. It is a factor that takes into account the shape characteristics of the tail. For an ideally streamlined tail, \( F \) will be unity and will decrease in value with an increase in tail bluntness. Note that the form of the expression for \( F_n \) is slightly different than it appears in (8) because the mass/unit length of the cylinder is not constant over the tail section; thus it must be included in the total derivative.

Noting that \( M(x) = \rho_f S(x) \) and \( m(x) = \rho_b S(x) \) one can write:

\[
\int_{L-\ell}^L \frac{\partial Q}{\partial x} \, dx - \int_{L-\ell}^L F_\rho (a/\partial t + Ua/\partial x)[S(x)v] \, dx - \int_{L-\ell}^L \rho_b a^2 y/\partial t^2 S(x) \, dx = 0
\]

which yields:

\[
\int_{L-\ell}^L \frac{\partial Q}{\partial x} \, dx - \int_{L-\ell}^L F_\rho a/\partial t (Sv) \, dx - \int_{L-\ell}^L F_\rho Ua/\partial x (Sv) \, dx - \int_{L-\ell}^L \rho_b a^2 y/\partial t^2 S(x) \, dx = 0.
\]
Following Paidoussis\textsuperscript{17} and noting that $y$ and $v$ are constant over the tapered end yields:

$$\int_{L-\ell}^{L} \partial Q/\partial x \, dx - F_p \partial^2 v/\partial t^2 \bigg|_{L-\ell} \int_{L-\ell}^{L} S(x) \, dx - F_p U(Sv) \bigg|_{L-\ell}$$

$$- \rho_b \partial^2 y/\partial t^2 \bigg|_{L-\ell} \int_{L-\ell}^{L} S(x) \, dx = 0;$$

which can be written:

$$\int_{L-\ell}^{L} \partial Q/\partial x \, dx - F_p U(Sv) \bigg|_{L-\ell} - (m + FM) \partial^2 y/\partial t^2 \bigg|_{L-\ell} \frac{1}{S} \int_{L-\ell}^{L} S(x) \, dx = 0.$$  

Assuming that $\ell$ is sufficiently small so that the boundary condition can be said to apply at $x = L$, the result is:

$$EI \partial^3 y/\partial x^3 + FMU(\partial y/\partial t + U\partial y/\partial x) - (m + FM) \partial^2 y/\partial t^2 x_e \bigg|_{x=L} = 0. \quad (14)$$

Where $x_e = 1/S \int_{L-\ell}^{L} S(x) \, dx$ is the effective length of the tail.

The second boundary condition at the tail comes from an assumption that there is no moment, thus:

$$\partial^2 y/\partial x^2 \bigg|_{x=L} = 0. \quad (15)$$

Non-Dimensionalization

Consider the following set of nondimensional variables:

$$\xi = x/L \quad \beta = M/(m + M) \quad \tau = \left[\frac{EI}{(m + M)}\right]^{1/2} t/L^2$$

$$n = y/L \quad \varepsilon = L/D \quad u = (M/EI)^{1/2}UL$$
\[
\begin{align*}
\frac{\partial y}{\partial x} &= \frac{a(\eta L)}{\partial x} = L \eta / \partial x = L \eta / \partial \xi (\partial \xi / \partial x) = \eta / \partial \xi. \\
\frac{\partial^2 y}{\partial x^2} &= \frac{a(\eta / \partial \xi)}{\partial x} = a(\eta / \partial \xi) \partial \xi (\partial \xi / \partial x) = (1/L) a^2 \eta / \partial \xi^2.
\end{align*}
\]

Similarly:
\[
\begin{align*}
\frac{\partial^2 y}{\partial x^2} &= (1/L^2) a^3 \eta / \partial \xi^3, \text{ and } \frac{\partial^4 y}{\partial x^4} &= (1/L^3) a^4 \eta / \partial \xi^4.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial y}{\partial t} &= \frac{a(\eta L)}{\partial t} = \eta / \partial \xi \cdot (EI/(m+M))^{1/2}(1/L). \\
\frac{\partial^2 y}{\partial t^2} &= \frac{a(\partial y/\partial t)}{\partial t} = a(\partial y/\partial t) / \partial (\partial t / \partial t) = a^2 \eta / \partial t^2 \cdot (EI/(m+M))(1/L^3). \\
\frac{\partial^2 y}{\partial x \partial t} &= \frac{a(\partial y/\partial x)}{\partial t} = a(\partial \eta / \partial \xi \partial \xi / \partial t) = a^3 \eta / \partial \xi \partial \xi / \partial t = a^3 n / \partial \xi \partial \xi / \partial t.
\end{align*}
\]

Substitution into (9) yields:
\[
\begin{align*}
&\frac{EI/L^3}{} a^4 \eta / \partial \xi^4 - \frac{1}{2}(M \mu^2/L) C_t(L/D)(1-x/L) a^2 \eta / \partial \xi^2 + \frac{1}{2} C_t(M \mu^2/D) \eta / \partial \xi \\
&+ M([EI/(m+M)](1/L^3) a^2 \eta / \partial t^2 + 2U/L^2[EI/(m+M)]^{1/2} a^2 \eta / \partial \xi \partial \xi + (U^2/L) a^2 \eta / \partial \xi^2) \\
&+ [M/(M+m)](EI/L^3) a^2 \eta / \partial t^2 = 0.
\end{align*}
\]

Dividing by EI/L^3 and rearranging results in:
\[
\begin{align*}
&\frac{a^4 \eta / \partial \xi^4}{} - \frac{1}{2}(M \mu^2/EI)L^2[C_t(L/D)(1-x/L)] a^2 \eta / \partial \xi^2 + \frac{1}{2} C_t(M \mu^2/L^2/EI)(L/D) \eta / \partial \xi \\
&+ [M/(M+m)] a^2 \eta / \partial t^2 + 2(M/EI)^{1/2} L U [M/(M+m)]^{1/2} a^2 \eta / \partial \xi \partial \xi + (M \mu^2/L^2/EI) a^2 \eta / \partial \xi^2 \\
&+ [M/(M+m)] a^2 \eta / \partial t^2 = 0
\end{align*}
\]

which upon substitution of nondimensional parameters, becomes:
\[
\begin{align*}
&\frac{a^4 \eta / \partial \xi^4}{} - \frac{1}{2} \mu^2 [3C_t(1-\xi)] a^2 \eta / \partial \xi^2 + \frac{1}{2} C_t \xi \mu^2 a^2 \eta / \partial \xi + 2\mu \beta \alpha^2 \eta / \partial \xi \partial \xi + \mu^2 a^2 \eta / \partial \xi^2 \\
&+ a^2 \eta / \partial t^2 = 0
\end{align*}
\]

or:
\[
\begin{align*}
&\frac{a^4 \eta / \partial \xi^4}{} + \mu^2 [1-(\xi C_t/2)] a^2 \eta / \partial \xi^2 + \frac{1}{2} \mu^2 C_t \xi^2 \alpha^2 \eta / \partial \xi^2 + \frac{1}{2} C_t \xi \mu^2 \alpha^2 \eta / \partial \xi \\
&+ 2\mu \beta \alpha^2 \eta / \partial \xi \partial \xi + a^2 \eta / \partial t^2 = 0.
\end{align*}
\]

This is the nondimensional equation of motion.
The Boundary Conditions become:

Clamped end:
\[ \eta|_{\xi=0} = 0 \quad (17) \]
\[ \partial \eta / \partial \xi |_{\xi=0} = 0 \quad (18) \]

Pinned end:
\[ \eta|_{\xi=0} = 0 \quad (19) \]
\[ \partial^2 \eta / \partial \xi^2 |_{\xi=0} = 0 \quad (20) \]

Free end:
\[ (EI/L^2) \partial^3 \eta / \partial \xi^3 + (FMU/L)[EI/(m+M)]^{\frac{1}{3}} \partial \eta / \partial \xi + FMU^2 \partial \eta / \partial \xi \]
\[ - [(m+FM)/L^3][EI/(m+M)] x_e \partial^2 \eta / \partial \xi^2 |_{\eta=1} = 0 \]

or:
\[ \partial^3 \eta / \partial \xi^3 + F(M/EI)^{\frac{1}{3}}[M/(m+M)]^{\frac{1}{3}} L \partial \eta / \partial \xi + (FMU^2 L^2/EI) \partial \eta / \partial \xi \]
\[ - (x_e/L)[(m+FM+M-M)/(M+m)] \partial^2 \eta / \partial \xi^2 |_{\eta=1} = 0. \]

Substitution of dimensionless parameters yields:
\[ \partial^3 \eta / \partial \xi^3 + F_\mu \beta \partial \eta / \partial \xi + F_\mu \partial \eta / \partial \xi - X[1 + (F-1)\beta] \partial^2 \eta / \partial \xi^2 |_{\eta=1} = 0 \]

or:
\[ \partial^3 \eta / \partial \xi^3 + F_\mu^2 \partial \eta / \partial \xi + F_\mu \beta \partial \eta / \partial \xi - [1 + (F-1)\beta]X \partial^2 \eta / \partial \xi^2 |_{\eta=1} = 0. \]

where:
\[ X = x_e/L. \]

Also, for no bending moment at free end:
\[ \partial^2 \eta / \partial \xi^2 |_{\eta=1} = 0. \]

Analysis

Collecting equations:
\[ n^{iv} + u^2 [1 - (\frac{1}{2} \mu C_t)] n'' + \frac{1}{2} \mu^2 \epsilon C_t \xi n''' + \frac{1}{2} \mu \epsilon C_t \mu^2 n' + 2 \mu \beta \xi'' + \ddot{n} = 0 \]

(16)
which is subjected to boundary conditions:
\[ n(0) = 0 \quad (17) \quad \text{and} \quad n'(0) = 0 \quad (18) \quad \text{at clamped end} \]
\[ n(0) = 0 \quad (19) \quad \text{and} \quad n''(0) = 0 \quad (20) \quad \text{at pinned end}. \]
\[ n''(1) + F_{\mu} n'(1) + F_{\mu} \beta \frac{1}{2} n(1) - [1+(F-1)\beta]X\ddot{n}(1) = 0 \quad (21) \]
and
\[ n''(1) = 0 \quad (22) \quad \text{at free end}. \]

Following convention, consider cylinder motions of the form:
\[ n = Y(\xi) e^{\omega \tau} \quad (23) \]
where \( \omega \) is a dimensionless frequency defined by:
\[ \omega = \left[ \frac{(m+M)/EI}{\Omega} \right]^\frac{1}{2} \Omega L^2 \]
where \( \Omega \) is the circular frequency of motion and in general is complex.

Differentiating:
\[ \ddot{n} = Y' e^{\omega \tau} \]
\[ \dddot{n} = Y'' e^{\omega \tau} \]
\[ \dddot{n} = Y''' e^{\omega \tau} \]

Substitution of these equations into (16) and making use of the following identities:
\[ a = \mu^2 [1-\frac{1}{2}c t] \]
\[ b = \frac{1}{2} \mu^2 c t \]
\[ c = \frac{1}{2} c t \mu^2 + 2 \beta \frac{1}{2} \mu \omega \]
\[ e = \omega^2 \]
\[ h = F_\mu^2 \]
\[ j = F \beta^\frac{1}{2} \mu \omega - [1+(F-1)\beta]X \omega^2 \]
results in:

\[ y^{iv} + aY'' + b\xi Y'' + cY' + eY = 0 \]  

(24)

with boundary conditions:

\[ Y(0) = 0 \]  

(25) and \[ Y'(0) = 0 \]  

(26) at clamped end

\[ Y(0) = 0 \]  

(27) and \[ Y''(0) = 0 \]  

(28) at pinned end

\[ Y'''(1) + hY'(1) + jY(1) = 0 \]  

(29) and \[ Y''(1) = 0 \]  

(30) at the free end.

Assume a series solution to equation (24) of the form:

\[ Y(\xi) = \sum_{r=0}^{\infty} A_r \xi^r. \]  

(31)

Thus,

\[ Y' = \sum_{r=0}^{\infty} rA_r \xi^{r-1}, \]

\[ Y'' = \sum_{r=0}^{\infty} r(r-1)A_r \xi^{r-2}, \]

\[ Y''' = \sum_{r=0}^{\infty} r(r-1)(r-2)A_r \xi^{r-3}, \]

\[ Y^{iv} = \sum_{r=0}^{\infty} r(r-1)(r-2)(r-3)A_r \xi^{r-4}. \]

Substitution into (24) yields:

\[ \sum_{r=0}^{\infty} r(r-1)(r-2)(r-3)A_r \xi^{r-4} + a\xi \sum_{r=0}^{\infty} r(r-1)A_r \xi^{r-2} + b\xi \sum_{r=0}^{\infty} r(r-1)A_r \xi^{r-2} \]

\[ + c\xi \sum_{r=0}^{\infty} rA_r \xi^{r-1} + e \sum_{r=0}^{\infty} A_r \xi^r = 0. \]

Upon rearranging, one obtains:

\[ \sum_{r=0}^{\infty} r(r-1)(r-2)(r-3)A_r \xi^{r-4} + a\xi \sum_{r=0}^{\infty} r(r-1)A_r \xi^{r-2} + \sum_{r=0}^{\infty} [r(r-1)b+rc]A_r \xi^{r-1} \]

\[ + e \sum_{r=0}^{\infty} A_r \xi^r = 0. \]  

(32)
Boundary Conditions:

\[ Y(0) = 0 \rightarrow \sum_{r=0}^{\infty} r A_r(0) r^{-1} = 0 \rightarrow A_0 = 0 \quad \text{(33)} \]

\[ Y'(0) = 0 \rightarrow \sum_{r=0}^{\infty} r A_r(0) r^{-1} = 0 \rightarrow A_1 = 0 \quad \text{(34)} \]

\[ Y(0) = 0 \rightarrow \sum_{r=0}^{\infty} r A_r(0) r = 0 \rightarrow A_0 = 0 \quad \text{(35)} \]

\[ Y''(0) = 0 \rightarrow \sum_{r=0}^{\infty} r(r-1) A_r(0) r^{-2} = 0 \rightarrow A_2 = 0 \quad \text{(36)} \]

\[ Y'''(1) + h Y'(1) + j Y(1) = 0 \rightarrow \sum_{r=0}^{\infty} r(r-1)(r-2) A_r + h \sum_{r=0}^{\infty} r A_r + j \sum_{r=0}^{\infty} A_r = 0 \quad \text{(37)} \]

\[ Y'''(1) = 0 \rightarrow \sum_{r=0}^{\infty} r(r-1) A_r = 0 \quad \text{(38)} \]

Consider equation (32). By collecting the coefficients of the terms of \( \xi^p \) and noting that \( A_0 = 0 \) for all cases considered we obtain:

\[ p = 0 \]

\[ 4(3)(2)(1)A_4 + 2(1)A_2 + cA_1 = 0 \quad \text{(39)} \]

\[ p = 1 \]

\[ 5(4)(3)(2)A_5 + 3(2)A_3 + [2(1)b + 2c]A_2 + eA_1 = 0 \quad \text{(40)} \]

\[ p = 2 \]

\[ 6(5)(4)(3)A_6 + 4(3)A_4 + [3(2)b + 3c]A_3 + eA_2 = 0 \quad \text{(41)} \]

\[ p = 3 \]

\[ 7(6)(5)(4)A_7 + 5(4)A_5 + [4(3)b + 4c]A_4 + eA_3 = 0 \quad \text{(42)} \]

...
\[ p = n-4 \quad \rightarrow \quad n = p+4 \]
\[
n(n-1)(n-2)(n-3)A_n + (n-2)(n-3)aA_{n-2} + [(n-3)(n-4)b + (n-3)c]A_{n-3} + eA_{n-4} = 0.\]
So, in general one can write:
\[
A_n = \frac{-1}{n(n-1)(n-2)(n-3)} \{ a(n-2)(n-3)A_{n-2} + [(n-3)(n-4)b + (n-3)c]A_{n-3} + eA_{n-4} \}
\]
\[ n = 6, 7, 8, \ldots \infty. \quad (43) \]

Examination of equation (36) for a pinned end condition reveals that \( A_2 \) as well as \( A_0 = 0 \). Thus, one can write:

From (39):
\[
A_4 = \frac{-1}{4(3)(2)} \{ 2aA_2 + cA_1 \} = \frac{-c}{4(3)(2)}A_1
\]
From (40):
\[
A_5 = \frac{-1}{5(4)(3)} \{ 3(2)aA_3 + eA_1 \}
\]
From (41):
\[
A_6 = \frac{-1}{6(5)(4)} \{ 4(3)aA_4 + [3(2)b + 3c]A_3 \}
\]
\[ = \frac{-1}{6(5)(4)} \{ [-1/(4(3)(2))}A_1(4)(3)ac + [3(2)b + 3c]A_3 \}, \]
and so forth.
So, it follows that \( A_n \) can be written in terms of \( A_1 \) and \( A_3 \) only.

Define:
\[
C_{nI} = \frac{-a}{n(n-1)}, C_{nIII} = \frac{-(n-4)b + c}{n(n-1)(n-2)}, \]
\[
C_{nIII} = \frac{-e}{n(n-1)(n-2)(n-3)}\]
Thus, (43) can be rewritten as:
\[
A_n = C_{nI}A_{n-2} + C_{nII}A_{n-3} + C_{nIII}A_{n-4} \quad (44)
\]
Writing out several terms of $A_n$ explicitly:

\[ A_4 = C_{4\, I} A_2 + C_{4\, I\!I} A_1 + C_{4\, I\!I\!I} A_0 = C_{4\, I\!I} A_1 \quad (45) \]
\[ A_5 = C_{5\, I} A_3 + C_{5\, I\!I} A_2 + C_{5\, I\!I\!I} A_1 = C_{5\, I} A_3 + C_{5\, I\!I\!I} A_1 \quad (46) \]
\[ A_6 = C_{6\, I} A_4 + C_{6\, I\!I} A_3 + C_{6\, I\!I\!I} A_2 = C_{6\, I\!I\!I} A_1 + C_{6\, I\!I\!I} A_3 \quad (47) \]
\[ A_7 = C_{7\, I\!I} A_5 + C_{7\, I\!I\!I} A_4 + C_{7\, I\!I\!I\!I} A_3 = C_{7\, I}(C_{5\, I} A_3 + C_{5\, I\!I\!I} A_1) \]
\[ + C_{7\!I\!I}(C_{4\, I\!I} A_1) + C_{7\!I\!I\!I} A_3 = (C_{7\, I} C_{5\, I\!I\!I} + C_{7\, I} C_{4\, I\!I\!I}) A_1 \]
\[ + (C_{7\, I} C_{5\, I} + C_{7\, I\!I\!I}) A_3 \quad (48) \]
\[ A_8 = C_{8\, I\!I\!I} A_6 + C_{8\, I\!I\!I\!I} A_5 + C_{8\, I\!I\!I\!I\!I} A_4 = C_{8\, I}(C_{6\, I\!I\!I} C_{4\, I\!I\!I} A_1 + C_{6\, I\!I\!I} A_3) \]
\[ + C_{8\!I\!I\!I}(C_{5\, I} A_3 + C_{5\, I\!I\!I} A_1) + C_{8\!I\!I\!I\!I}(C_{4\, I\!I\!I} A_1) = \]
\[ (C_{8\, I} C_{6\, I\!I\!I} C_{4\, I\!I\!I} + C_{8\!I\!I\!I\!I} C_{5\, I\!I\!I} + C_{8\!I\!I\!I\!I\!I} C_{4\, I\!I\!I\!I}) A_1 + (C_{8\, I} C_{6\, I\!I\!I} \]
\[ + C_{8\!I\!I\!I\!I} C_{5\, I}) A_3 \quad (49) \]
\[ A_9 = [C_{9\, I}(C_{7\, I} C_{5\, I\!I\!I} + C_{7\!I\!I\!I\!I\!I}) + C_{9\!I\!I\!I\!I} C_{6\, I\!I\!I} C_{4\, I\!I\!I} + C_{9\!I\!I\!I\!I\!I\!I} C_{5\, I\!I\!I\!I\!I}) A_1 \]
\[ + [C_{9\, I}(C_{7\, I} C_{5\, I} + C_{7\!I\!I\!I\!I}) + C_{9\!I\!I\!I\!I\!I} C_{6\, I\!I\!I} + C_{9\!I\!I\!I\!I\!I\!I} C_{5\, I}) A_3 \quad (50) \]
\[ \text{etc.} \]

Define:

- $C_{1n}$ as the coefficient of $A_1$ in $A_n$
- $C_{3n}$ as the coefficient of $A_3$ in $A_n$

From (45): $C_{14} = C_{4\, I\!I}; \quad C_{34} = 0$

From (46): $C_{15} = C_{5\, I\!I\!I}; \quad C_{35} = C_{5\, I}$

From (47): $C_{16} = C_{6\, I\!I\!I} C_{14}; \quad C_{36} = C_{6\, I\!I\!I}$

From (48): $C_{17} = C_{7\, I\!I\!I} C_{15} + C_{7\!I\!I\!I\!I} C_{14}; \quad C_{37} = C_{7\, I} C_{35} + C_{7\, I\!I\!I}$

From (49): $C_{18} = C_{8\, I\!I\!I} C_{16} + C_{8\!I\!I\!I\!I} C_{15} + C_{8\!I\!I\!I\!I\!I} C_{14};$
\[ C_{38} = C_{8\, I} C_{36} + C_{8\!I\!I\!I} C_{35} \]

From (50): $C_{19} = C_{9\, I\!I\!I\!I\!I} C_{17} + C_{9\!I\!I\!I\!I\!I\!I} C_{16} + C_{9\!I\!I\!I\!I\!I\!I\!I} C_{15};$
\[ C_{39} = C_{9\, I} C_{37} + C_{9\!I\!I\!I} C_{36} + C_{9\!I\!I\!I\!I\!I} C_{35} \]
In general,

\[ C_{1n} = C_{nI}C_1(n-2) + C_{nII}C_1(n-3) + C_{nIII}C_1(n-4) \]  \hspace{1cm} (51) \\
\[ C_{3n} = C_{nI}C_3(n-2) + C_{nII}C_3(n-3) + C_{nIII}C_3(n-4) \]  \hspace{1cm} (52)

So,

\[ A_n = C_{1n}A_1 + C_{3n}A_3 \quad n = 6, 7, 8, \ldots \]  \hspace{1cm} (53)

Note that comparing (53) with (47) results in:

\[ A_6 + (C_{11}C_1 + C_{6II}C_1 + C_{6III}C_1)A_1 + (C_{6IV}C_3 + C_{6II}C_3 + C_{6III}C_3)A_3 = C_{6IV}A_1 + C_{6II}A_3. \]

Thus,

\[ C_{13} = 0 \quad C_{34} = 0 \quad C_{32} = 0 \]
\[ C_{12} = 0 \quad C_{33} = 1 \]

These will be helpful in the algorithm development.

In summary, it is possible to write:

\[ A_n = C_{1n}A_1 + C_{3n}A_3 \quad n = 6, 7, 8, \ldots \]  \hspace{1cm} (53)

where \( C_{1n} \) and \( C_{3n} \) are given by (51) and (52). By substituting (53) into boundary conditions (37) and (38) at the free end,

\[ \sum_{r=3}^{8} [r(r-1)(r-2) + hr + j][C_{1r}A_1 + C_{3r}A_3] + (h+j)A_1 = 0 \]

and,

\[ \sum_{r=3}^{8} r(r-1)[C_{1r}A_1 + C_{3r}A_3] = 0. \]
or, in matrix form:

\[
\begin{bmatrix}
\sum_{r=3}^{\infty} r(r-1)C_{1r} & \sum_{r=3}^{\infty} r(r-1)C_{3r} \\
(h+j)\sum_{r=3}^{\infty} [r(r-1)(r-2)+hr+j]C_{1r} & \sum_{r=3}^{\infty} [r(r-1)(r-2)+hr+j]C_{3r}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (54)

Thus we have a set of linear equations in \(A_1\) and \(A_3\). For a nontrivial solution, the determinant of the coefficient matrix must vanish. Hence, one obtains a relationship between the nondimensional natural frequency, \(\omega\), and the other system parameters.

For the case of a clamped-free cylinder, the solution technique is the same, however the sets of linear equations are in terms of \(A_2\) and \(A_3\). The form of the coefficient matrix is the same (see appendix).
DISCUSSION

Throughout the literature on the dynamics of flexible cylinders, a couple of terms that have classical meanings in the field of vibrations have been used to describe the theoretical motions of a cylinder in a fluid. This, however, tends to be misleading because the phenomena associated with the classical interpretation of these terms are not believed to occur in the system presently under investigation. The following discussion is facilitated by incorporating the use of these terms, however they will assume a slightly different interpretation than usual.

The term unstable, which is classically used to denote motion that increases without bound, will be employed to denote the point at which the linear equation of motion developed is no longer valid. It is at this point that nonlinear forces become significant. These forces limit the amplitude of motion which, based solely on linear theory, would grow without bound.

Buckling is a form of instability that connotes the total collapse of a system. A trend that appears in this analysis is the bifurcation of the root locus of the natural frequencies of a system onto the real axis in an Argand dia-
gram with one branch moving into the unstable region ($\omega_R > 0$), such as the zeroeth mode in Figure 2. This is indicative of a buckling instability. Once this buckling point is reached, however, nonlinear forces which are not taken into account by the present linear theory, will limit the amplitude of motion. Thus, in this analysis, buckling will be used to indicate a non-oscillatory ($\omega_I = 0$) deviation of the cylinder from its equilibrium position. In past works, this phenomenon is also referred to as yawing.

The calculation procedure used was as follows: (a) a set of system parameters, $\varepsilon$, $C_t$, $\beta$, $F$, and $X$, which are all independent of tow velocity, were selected and (b) the complex frequencies of the first three modes of the system were calculated as functions of $\mu$ by using the computer program given in the appendix. The computations were done on a Univac 1100/81A which was accessed from Florida Atlantic University. The complex frequencies, when plotted on an Argand diagram, demonstrate the general character of the dynamical behavior of the system for varying $\mu$. To verify convergence of the resulting frequencies to the desired accuracy, the computation is repeated with an increasing number of terms in the series solution.

In the solution technique, motions of the form $e^{\omega \tau}$ were examined. When $\omega$ is complex, this can be written as $e^{(\omega_R + i\omega_I)\tau}$ or $e^{\omega_R \tau}e^{i\omega_I \tau}$. Thus, it can be seen that the real part of the frequency, $\omega_R$, controls the envelope of de-
cay and the imaginary part, $\omega_I$, is associated with oscil-
latory form. For $\omega_R < 0$, motions are of a damped oscillatory
form. For $\omega_R > 0$, it is obvious that the ensuing motion
grows without bound. The sign of $\omega_R$, therefore, defines the
region of stability of the system.

Following the convention used previously, the three low-
est modes are labelled "zeroeth", "first", and "second". The
zeroeth mode has a tendency to branch onto the $\omega_R$ axis and,
as defined before, is associated with a buckling instability.
The other two modes are generally oscillatory.

Comparison of Governing Equation with Paidoussis

The differences between the equation given by Paidoussis
in the literature review and (9) are the absence of (a) the
tail drag term, $\frac{1}{2}M\mathbf{U}^2\mathbf{C}_t \partial^2 y/\partial x^2$ and (b) the term $\frac{1}{2}C_n(MU/D)(\partial y/\partial t + \mathbf{U} \partial y/\partial x)$. Neglecting the tail drag reduces the total tension
on the cylinder. However, tail drag was a very small factor
in the analysis by Paidoussis and hence the absence of this
term will not cause much of a difference between the two solu-
tions.

The second term under consideration is a fluid damping
term and its net effect is to absorb energy from the system.
Hence, the absence of this term should destabilize the system
as a whole. The effect of the differences in the two equa-
tions is immediately apparent in Figures 2 and 3, which com-
pare the equations for a clamped condition and identical system parameters. Notice that for a given velocity in the stable region \((\omega_R < 0)\), the vibrational amplitude is greater in the present study since \(\omega_R\) is less negative than the values calculated by Paidoussis. More importantly, the point of instability in the present model is reached sooner in all modes, and the range of instability in the first and second modes is larger. In other words, this model goes unstable at a lower velocity and requires a higher velocity before the locus of the natural frequencies passes back into the stable region. In this sense, the present theory is more conservative and pessimistic than the previous model.

A valid question pertaining to the validity of the linear model once the system first loses stability can be raised at this point. It would seem that linear theory is only capable of predicting the critical velocity at this initial point of instability, and is generally unable to predict behavior that is postcritical. However, Paidoussis\(^{18,19}\) has shown that his theory was capable of predicting fairly well, in the case of short cylinders in unconfined flow, the onset of higher mode instabilities that occurred beyond the lowest critical point. A similar capability will be presumed to exist here also. A possible explanation of this occurrence lies in the realization that past a critical point, as mentioned before, the system is not actually going unstable in the classical sense, but nonlinear forces are becoming a dom-
Figure 2. The dimensionless complex frequency of the three lowest modes of a clamped-free cylinder as a function of dimensionless tow velocity for the Paidoussis equation with \( \varepsilon = 500 \), \( C_t = 0.002 \), \( \beta = 0.5 \), \( F = 0.8 \), and \( X = 0.01 \).
Figure 3. The dimensionless complex frequency of the three lowest modes of a clamped-free cylinder as a function of dimensionless tow velocity with $\varepsilon = 500$, $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$, $X = 0.01$. 
inant factor, thus the linear model no longer holds. However, as the flow velocity is further increased, the system passes through this region, and can re-enter the stable region where linear forces again dominate.

Boundary Condition at Upstream End

After comparison of the present model with Paidoussis' model for clamped-free cylinders, a pinned condition was imposed at the upstream end \((x = 0)\). As evident from Figure 4, the application of a pinned rather than a clamped condition further destabilizes the system. In the zeroth mode, buckling occurs for any velocity greater than zero. Instabilities in the first and second modes are reached at lower velocities than for a clamped end condition.

The examination of pinned and clamped conditions allows the leading edge to be bounded as far as rotation of the end piece is concerned. The pinned end, which is used in all subsequent computations, defines an upper bound of rotation. Also, since the pinned condition results in the least stable system, predictions of array stability using this condition would represent the most conservative predictions.

This case does not constitute the actual situation in a towed array, where, among other things, the tow end can be expected to be undergoing motion which is induced by motion of
Figure 4. The dimensionless complex frequency of the three lowest modes of a pinned-free cylinder as a function of dimensionless tow velocity with $\varepsilon = 500$, $C_t = .002$, $\beta = 0.5$, $F = 0.8$, $X = .01$. 
the tow vessel. However, examination of the system with the immobile conditions reveals the basic dynamical characteristics. Thus, once the system is examined for these conditions, the situation of the case where the same boundary conditions exist except that the end is undergoing a lateral, time-dependent motion, can be introduced via the classical moving boundary problem studied by Meirovitch.20

Significance of the Bending Term

Intuitively, it is feasible that the bending rigidity of a long, thin cylinder would be small. To examine the conditions for which this is true, thus allowing the governing equation to be reduced to one of second order, an order of magnitude analysis is performed.

Recall the assumed cylinder motion given by equation (23). Expand \( Y(\xi) \) in a Fourier series as follows:21

\[
Y(\xi) = \sum_{n=1}^{\infty} a_n \cos(n\pi \xi) + b_n \sin(n\pi \xi) \quad (55)
\]

Define \( \Sigma = \sum_{n=1}^{\infty} a_n + b_n \). Realizing that the upper bound for \( \sin(n\pi \xi) \) and \( \cos(n\pi \xi) \) is one, the following upper limits can be established for the derivatives of \( n \):

\[
\begin{align*}
\eta_{\text{max}} &= \Sigma e^{\omega \tau} \\
\eta'_{\text{max}} &= n\pi \Sigma e^{\omega \tau} \\
\eta''_{\text{max}} &= (n\pi)^2 \Sigma e^{\omega \tau} \\
\eta'''_{\text{max}} &= (n\pi)^3 \Sigma e^{\omega \tau} \\
\eta^{(iv)}_{\text{max}} &= (n\pi)^4 \Sigma e^{\omega \tau} \\
\eta^{(v)}_{\text{max}} &= \omega \Sigma e^{\omega \tau} \\
\eta^{(vi)}_{\text{max}} &= \omega n\pi \Sigma e^{\omega \tau} \\
\eta^{(vii)}_{\text{max}} &= \omega^2 \Sigma e^{\omega \tau}
\end{align*}
\]
Substitution into (24) yields for each component

$$(n\pi)^4 + (a + b\varepsilon)(n\pi)^2 + n\pi c + e = O(0)$$

where $a$, $b$, $c$, and $e$, are as given before, and $O$ signifies "order of magnitude of". Redefining $c$ as $c = c_1 + c_2\omega$, where $c_1 = \frac{1}{2}\varepsilon C_t \mu^2$ and $c_2 = 2\beta\frac{1}{2}\mu$, we can write:

$$(n\pi)^4 + (a + b\varepsilon)(n\pi)^2 + n\pi c_1 + n\pi c_2\omega + \omega^2 = O(0). \quad (56)$$

Consider the following typical parameters for a towed array:

$L = 200$ meters $= 20,000$ cm.

$D_{\text{outside}} = 7.62$ cm.

$D_{\text{inside}} = 7.3$ cm.

$I = \pi(D_o^4 - D_i^4)/4 = 418$ cm.$^4$

$M = \rho_s = 46.8$ dyne-sec.$^2$/cm.$^2$

$C_t = .002$

$E = 3.7 \times 10^8$ dynes/cm.$^2$

It can be seen that $\varepsilon C_t = O(10^0)$ and:

- $a = O(\mu^2)$
- $b = O(\mu^2)$
- $c_1 = O(\mu^2)$
- $c_2 = O(\mu)$

Thus, one can write (56) as:

$$\underbrace{(n\pi)^4 + O(\mu^2)(n\pi)^2 + O(\mu^2)n\pi + O(\mu)n\pi\omega + \omega^2}_{\text{Bending}} = O(0). \quad (57)$$

$$\underbrace{\text{Tension}} \quad \underbrace{\text{Inertia}}$$
It can be seen that the bending term is related to the term of the next order of magnitude by a factor of \((n\pi)^3/O(\mu^2)\). So, in order for bending to be significant, \((n\pi)^3/O(\mu^2)\) must be on the order of \(10^0\), or:

\[
(n\pi)^3 = O(\mu^2)
\]  \hspace{1cm} (58)

By using this relationship, for a given velocity, the number of zero crossings, \(n\), that the mode shape would have to contain in order for bending to be significant can be calculated, and is given in the following table:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(U) (knots) for (L = 100) m.</th>
<th>(U) (knots) for (L = 250) m.</th>
<th>(n) (to nearest whole #)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.233</td>
<td>0.089</td>
<td>&lt;1</td>
</tr>
<tr>
<td>5</td>
<td>0.582</td>
<td>0.223</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1.165</td>
<td>0.447</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>2.33</td>
<td>0.894</td>
<td>2</td>
</tr>
<tr>
<td>30</td>
<td>3.49</td>
<td>1.341</td>
<td>3</td>
</tr>
<tr>
<td>40</td>
<td>4.66</td>
<td>1.788</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>5.82</td>
<td>2.235</td>
<td>4</td>
</tr>
<tr>
<td>100</td>
<td>11.64</td>
<td>4.471</td>
<td>7</td>
</tr>
<tr>
<td>200</td>
<td>23.28</td>
<td>8.941</td>
<td>11</td>
</tr>
</tbody>
</table>

Since the range of this study will be well under 10 knots, one can see that bending will be important because it is certainly conceivable for the cable to assume seven or more zero crossings.
Frequency as a Function of Flow Velocity

The complex frequencies of the three lowest modes of a system with $\varepsilon = 656.17$ (L = 50 meters), $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$ and $X = 0.01$ are displayed as a function of nondimensional velocity in Figure 5. It is noted that small velocities act to damp free oscillations of the system in the first and second modes, but as the velocity is increased, the system may become unstable in those modes. In the zeroeth mode, buckling occurs for $\mu > 0$. At just over $\mu = 3.6$, this mode reenters the stable region, however, the locus of the first mode crosses the $\omega_I$ axis resulting in an oscillatory instability. At about $\mu = 6.5$, the second mode also enters the unstable region. A further increase in velocity results in an eventual return of all modes to the stable regime. Thus, at just under $\mu = 12.0$, the system seems to be stable in its lower three modes, but previously, at $\mu = 11.4$, the imaginary oscillatory root of the first mode has become zero and the mode has branched onto the $\omega_R$ axis. One branch eventually couples with a branch from the zeroeth mode at $\mu = 11.61$ and leaves the axis. At a velocity just greater than 12.0, this coupled mode crosses into the unstable region and results in a coupled-mode flutter which has been observed by Paidoussis in a related problem. It is interesting to note that at about $\mu = 3.9$, the two branches of the zeroeth mode coalesce and leave the $\omega_R$ axis for a short range, but again
bifurcate and at $\mu \approx 8.0$, one branch reenters the unstable region indicating a second buckling of this mode.

Figure 6 shows the complex frequencies of a system with the same parameters as before except that $\varepsilon$ has been increased to 1312.54, which corresponds to a length of 100 meters. The behavior as a function of velocity is similar. The system immediately buckles in its zeroeth mode for $\mu > 0$. This buckling instability eventually disappears at $\mu = 5.0$, but the system has developed a first mode instability. At $\mu \approx 7.5$, the second mode also crosses the $\omega_1$ axis. Eventually, a further increase in $\mu$ to about 15.5 results in a return of all modes to the stable regime. The first mode has branched at a value of $\mu$ just greater than 15.0, but has remained in the stable region.

Note that the maximum values of $\mu$ on the graphs for cylinder lengths of 100 meters or more represent the limit of the present solution technique. Higher values of $\varepsilon$ or $\mu$ require an increase in the number of terms needed in the series solution. The maximum number of terms that can be handled by the algorithm is about 100 due to the higher order of the characteristic polynomial that must be solved for an increase in terms taken.

For $\varepsilon = 1968.504$ (L = 150 meters), the Argand diagram is shown in Figure 7. Note that the zeroeth mode does not buckle immediately, but a state of damped oscillation exists before the locus bifurcates onto the $\omega_R$ axis at $\mu = 3.1$ and
buckles at \( \mu \approx 3.5 \). A possible explanation will be given later. The first mode enters the unstable region at \( \mu \approx 5.5 \) and the second mode at about \( \mu = 8.0 \). The system regains stability in its zeroeth and first modes at \( \mu = 8.0 \) and 13.5, respectively. Computational limitations prevented a prediction of the return of the second mode locus to the stable region.

An interesting phenomenon is revealed here. Notice that the shape of the curve for this length is reversed. As the velocity is increased, the oscillatory frequency in the first and second modes increases whereas in the previous graphs the opposite was true. The explanation lies in the effect of an increase in velocity on the tension and inertial terms in the governing equation. It seems that for shorter cables, as the velocity is increased, the inertial terms are affected more than the terms associated with tension. Essentially, mass is added to the system faster than tension and this results in a "softer" system with lower oscillatory frequencies. As the length of the cable is increased, a critical point is reached, beyond which a further increase in velocity results in a greater effect of the tension terms on the system as compared to the inertial terms. Thus, the system becomes stiffer and the oscillatory frequency increases. This is a phenomenon that has not previously been observed.

The Argand diagram for \( L = 200 \) meters is given in Figure 8.
Figure 5. The dimensionless complex frequency of the three lowest modes of a pinned-free cylinder as a function of dimensionless tow velocity with $\varepsilon = 656.17$, $C_t = .002$, $\beta = 0.5$, $F = 0.8$, $X = .01$. 
Figure 6. The dimensionless complex frequency of the three lowest modes of a pinned-free cylinder as a function of dimensionless tow velocity with \( \varepsilon = 1312.34, C_t = .002, \beta = 0.5, F = 0.8, X = .01 \).
Figure 7. The dimensionless complex frequency of the three lowest modes of a pinned-free cylinder as a function of dimensionless tow velocity with $\epsilon = 1968.504$, $C_t = .002$, $\beta = 0.5$, $F = 0.8$, $X = .01$. 
Figure 8. The dimensionless complex frequency of the three lowest modes of a pinned-free cylinder as a function of dimensionless tow velocity with $\epsilon = 2624.67$, $C_T = .002$, $\beta = 0.5$, $F = 0.8$, $X = .01$. 
To facilitate conversion of the parameters from nondimensional to dimensional, several nomographs were developed and are shown in Figures 9 through 16. These nomographs allow one to find the natural frequencies of the cylinder for a given length in its first and second modes of vibration for a given tow speed in the range examined. The nomographs were developed for a three-inch (7.62 cm.) diameter cylinder with \( M = 46.79 \text{ dynes-sec.}^2/\text{cm.}^2 \), \( E = 3.7 \times 10^8 \text{ dynes/cm.}^2 \) and \( I \) being calculated as 454 cm.\(^4\). Caution must be exercised in using these nomographs, however, because for a given mode, they are only valid for regions where that mode is in the stable regime. When the mode goes unstable (\( \omega_R > 0 \)) nonlinear forces are present and thus a nonlinear model must be developed and employed in these regions to accurately predict system behavior.

The nomographs are used as such: (a) For a chosen length (50, 100, 150 or 200 meters), the dimensional speed is converted to a nondimensional velocity by the lower left graph. (b) For this value of velocity, go straight up to the top graph to find the corresponding natural frequency for first or second mode vibration. A separate graph is needed for the real and imaginary parts. Care must be taken here, because confidence in the oscillatory frequency, \( \omega_I \), is only assured when the corresponding real part, \( \omega_R \), is negative. Finally, (c) the real and imaginary parts of the frequency can be converted to dimensional frequencies by the correspon-
Figure 9. Nomograph for conversion to dimensional tow velocity and the corresponding real part of the natural frequency for $L = 50$ meters ($\varepsilon = 656.17$), with $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$, $X = 0.01$. 
Figure 10. Nomograph for conversion to dimensional tow velocity and the corresponding imaginary part of the natural frequency for $L = 50$ meters ($\varepsilon = 656.17$), with $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$, and $X = 0.01$. 
Figure 11. Nomograph for conversion to dimensional tow velocity and the corresponding real part of the natural frequency for \( L = 100 \) meters \((\varepsilon = 1312.34)\), with \( C_t = .002, \beta = 0.5, F = 0.8, \) and \( X = .01 \).
Figure 12. Nomograph for conversion to dimensional tow velocity and the corresponding imaginary part of the natural frequency for \( L = 100 \) meters (\( \epsilon = 1312.34 \)), with \( C_t = .002, \beta = 0.5, F = 0.8, \) and \( X = .01. \)
Figure 13. Nomograph for conversion to dimensional tow velocity and the corresponding real part of the natural frequency for $L = 150$ meters ($c = 1968.504$), with $C_t = .002$, $\beta = 0.5$, $F = 0.8$, and $X = .01$. 
Figure 14. Nomograph for conversion to dimensional tow velocity and the corresponding imaginary part of the natural frequency for $L = 150$ meters ($\varepsilon = 1968.504$), with $C_t = .002$, $\beta = 0.5$, $F = 0.8$, and $X = .01$. 
Figure 15. Nomograph for conversion to dimensional tow velocity and the corresponding real part of the natural frequency for $L = 200$ meters ($\varepsilon = 2624.67$), with $Ct = .002$, $\beta = 0.5$, $F = 0.8$, and $X = .01$. 
Figure 16. Nomograph for conversion to dimensional tow velocity and the corresponding imaginary part of the natural frequency for $L = 200$ meters ($\varepsilon = 2624.67$), with $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$, and $X = 0.01$. 
ding graph on the top right (I for imaginary, R for real).

Effect of $C_t$

The fricitional drag coefficient, $C_t$, is not a parameter that can easily be varied in experimental work or is easily calculated. However, studies of the effect of changes in this parameter on the system could give a better insight into the causes behind some of the observed phenomena. An increase in $C_t$ has the net effect of increasing drag on the cylinder and thus increasing the total tension. Plots were developed showing the dimensional critical velocities for unstable oscillation in the first mode, $U_{co}$ (Figure 17), and zeroeth mode buckling, $U_{cb}$ (Figure 18). As can clearly be seen in Figure 17, an increase in $C_t$ for a given length cylinder stabilizes the system and results in a higher corresponding frequency of oscillation at the point of neutral stability. This result is consistent with previous works and is not surprising when viewed in the light of classical string theory.

In the zeroeth mode (Figure 18), notice that up to a certain value of $C_t$, increasing tension has no effect on the critical buckling velocity, for the system buckles immediately. However, past $C_t = .0049$ (for $\varepsilon = 656.17$), a further increase in $C_t$ results in a stabilization of the system to buckling.
Figure 17. Critical velocity and corresponding critical frequency for first mode unstable oscillation of a pinned-free cylinder as a function of the drag coefficient for $\varepsilon = 656.17$, $\beta = 0.5$, $F = 0.8$, and $X = 0.01$. 
Figure 18. Critical velocity for zeroeth mode buckling for a pinned-free cylinder as a function of drag coefficient for $\varepsilon = 656.17$, $\beta = 0.5$, $F = 0.8$, and $X = .01$. 
Effect of Length

To better isolate the effect of cable length on the system, plots were developed showing $U_{CO}$ (Figure 19), and $U_{cb}$ (Figure 20) as a function of length. As we can see from Figure 19, an increase in length while holding the other parameters constant, actually destabilizes the system. This is contrary to statements made by Paidoussis\textsuperscript{23}, but is supported by Pao.\textsuperscript{24} From examination of the previous set of Argand diagrams, it would seem that an increase in length would stabilize the system in the oscillatory modes due to the higher nondimensional velocities needed for the loci of these modes to pass into the unstable region. However, upon conversion back to the dimensional variables, it is found that the opposite is true. A possible explanation of this trend is that for a given velocity, an increase in cable length results in a greater effect on the inertial terms compared with tension term effects. Essentially, this would result in the addition of mass to the system faster than tension. Thus, the net effect is to soften, or destabilize the system in the oscillatory modes. This statement is also supported by examining the frequencies of oscillation associated with a state of neutral stability. These frequencies decrease with an increase in length.

In the zeroeth mode, the trend seems to be reversed. Up to a point, an increase in length has no effect on the imme-
Figure 19. Critical velocity and corresponding critical frequency for first mode unstable oscillation of a pinned-free cylinder as a function of cylinder length for $C_t = .002$, $\beta = 0.5$, $F = 0.8$, $X = .01$. 
Figure 20. Critical velocity for zeroeth mode buckling for a pinned-free cylinder as a function of cylinder length for $C_t = 0.002$, $\beta = 0.5$, $F = 0.8$, $X = 0.01$. 
diate buckling of the cylinder. However, past a length of 120 meters (with $C_t = 0.002$), the system doesn't buckle immediately, but first assumes a state of damped oscillatory motion. An increase in cylinder length for a short range increases the velocity needed for buckling. A further increase in length past 175 meters seems to again have no effect on $U_{cb}$. The physical explanation behind this behavior is not clear at this time but seems to lie in the mechanism of the buckling instability which will be addressed later.

Effect of Tail Shape

The parameter $F$ virtually controls the free end boundary condition by determining the value of the hydrodynamic lift force at the tapered end. A lower value of $F$ corresponds to a blunter tail and hence a smaller value of lift due to the hydrodynamic forces. From Figures 21 and 22 it can be seen that increasing $F$, which has the net effect of increasing the amount of shear transmitted from tail to cylinder, destabilizes the system in both the buckling and oscillatory modes.

In the oscillatory mode, the effect of $F$ is the same regardless of cable length. Notice, though, that for a given value of $F$, as cable length is increased, the system becomes less stable in the first mode and oscillates at a lower frequency. This supports previous findings regarding the effect of cable length (Figure 19). Another interesting obser-
Figure 21. Critical velocity and corresponding critical frequency for first mode unstable oscillation of a pinned-free cylinder as a function of tail shape parameter and length for $C_t = 0.002$, $\beta = 0.5$, $X = 0.01$. 
Figure 22. Critical velocity for zeroth mode buckling for a pinned-free cylinder as a function of tail shape parameter for $\varepsilon = 2624.67$, $C_t = .002$, $\beta = 0.5$, $X = .01$. 
vation is that the corresponding frequencies of oscillation at the point of neutral stability don't seem to be affected by the tail shape. This seems to indicate that forces at the tail act to initiate cable motion and the natural frequencies are controlled by other factors.

For the buckling mode it was found that the system buckled immediately for short cables, as before, and the value of F did not effect this phenomenon. However, once that critical length was reached, where the cable first assumed damped oscillations before buckling, it is evident from Figure 22 that an increase in F destabilizes the system and causes it to buckle sooner.

Zeroeth Mode Buckling

It has been suggested in the literature that the mechanism behind the buckling phenomenon lies in the balance between the flexural restoring forces of the cylinder and the forces that tend to oppose these restoring forces. From Figures 4, 5 and 6, it can be seen that for any finite velocity, buckling in the zeroeth mode is immediate. However, as the length of the cylinder is increased while holding the other system parameters constant, it seems that past a certain critical length, the immediate buckling is suppressed and the cylinder first assumes a damped oscillatory motion in its zeroeth mode before buckling. It has been found that
suppression of immediate buckling corresponds extremely close to the point where the length of the cable causes the tension term to become important. This would seem to indicate that tension, once large enough, enhances the restoring forces and stabilizes the system to buckling. This theory is supported by Figure 20. An increase in the drag coefficient past the critical point \( (C_t = .0049 \text{ for } \epsilon = 656.17) \) stabilizes the system to buckling.

The parameter \( F \) also has an effect on buckling. However, the same critical length must be reached before this effect is seen. Below this length, regardless of \( F \), the system buckles immediately, but past this point, an increase in \( F \) destabilizes the system to buckling. This would indicate that tail shape has no effect on the suppression of immediate buckling, but is a factor past the critical point.

An interesting trend surfaces in Figure 20. Once again we notice that a critical point must be reached for the suppression of immediate buckling. Past this point, however, an increase in length stabilizes the system in the zeroeth mode while the opposite was true in the oscillatory mode. It is believed that increasing cylinder length effectively adds mass to the system more than tension which results in the trend found in Figure 19. It would seem, though, that while the effect of tension with a length increase is overshadowed by the mass effect in the oscillatory mode, it is indeed important in the buckling mode, and the effect of the
mass increase becomes uncertain.

So, while the exact mechanism of buckling is not clear, it seems that tension is a dominating factor. The tail shape, which determines the shear transmitted to the cylinder, also is a factor but its effect can only be seen past the suppression of immediate buckling.
A linear differential equation of motion for a towed array system has been developed and explored for its free vibration characteristics. The behavior of the locus of the several lowest eigenfrequencies as various system parameters were varied revealed the basic dependence of the system on these parameters.

Based on linear theory it was found that: (1) The bending rigidity of the cylinder for lengths of up to 250 meters is important in determining the natural frequencies of the system for speeds of up to about ten knots. (2) Longer cylinders are more susceptible to large oscillatory motions than are shorter cylinders, but are less susceptible to a buckling motion. (3) An increase in the skin friction drag coefficient, $C_f$, which effectively increases the total tension on the cylinder, stabilizes the system in all modes. (4) Tail shape acts as an initiator for the onset of buckling and oscillatory instabilities, but does not affect the frequency of oscillation. Also, blunter tails are less effective as initiators and thus a cylinder with more tapered tails are more susceptible to instabilities. (5) The cylinder in its various modes exhibits a tendency to reenter the stable region after going unstable. This indicates that system in-
stabilities are bounded and are analogous to a system that passes through a resonant condition.

This study provides a first step in predicting array behavior. For system parameters where the cylinder is stable, the natural frequencies calculated can be used to predict mode shapes. However, in the unstable regions, although small displacement theory can still hold, the displacements are large enough so that the cross flow drag term which was neglected in arriving at equation (8) is no longer small. Thus nonlinear forces are present and a nonlinear equation must be employed to predict array behavior. In the case of very large displacements, the equation developed is no longer valid and other means of describing system behavior must be developed.
FUTURE WORK

Recommendations for future work include:

1. Examination of modal shapes of cylinder using natural frequencies calculated in stable regions.

2. Inclusion of the cross flow drag term in the differential equation of motion and then examining this nonlinear equation for system stability and array configuration in the regions not accurately described by the linear equation.

3. Application of time-dependent boundary condition at upstream end of the cylinder in order to incorporate ship motion into system.

4. Development of different solution techniques to allow examination of present equation for higher tow speeds.

5. Closer examination of nature of forces present at the tail of the cylinder.
APPENDIX

Algorithm Development of Pinned-Free Cylinder

After specifying the values of the system parameters and then taking a finite number of terms, NT, in the summations of (54), a characteristic equation in \( \omega \) can be obtained by taking the determinant of the coefficient matrix. The roots of this equation will yield the natural frequencies of the system.

In order to group equations by powers of \( \omega \), the parameters \( c \), \( e \), and \( j \) that are used in equations (24) and (29) are redefined as

\[
\begin{align*}
c &= c_A + c_B \omega \\
e &= e_A \omega^2 \\
j &= j_A \omega + j_B \omega^2
\end{align*}
\]

where:

\[
\begin{align*}
c_A &= \frac{1}{2} e C_t u^2 \\
c_B &= 2 B^{\frac{1}{2}} u \\
e_A &= 1 \\
j_A &= F B^{\frac{1}{2}} u \\
j_B &= [1 + (F-1)B] X
\end{align*}
\]

Equation (44) can now be written

\[
A_n = C_{nI} A_{n-2} + C_{nII} A_{n-3} + C_{nIC} A_{n-3} + C_{nIII} A_{n-4}
\]

where:

\[
C_{nIII} = \left[ (n-4) b + C_A \right] / [n(n-1)(n-2)]
\]

and

\[
C_{nIIC} = C_B \omega / [n(n-1)(n-2)].
\]
Thus, in terms of powers of $\omega$, the order of $C_{nI}$ and $C_{nIIB}$ is zero, $C_{nIIC}$ is on the order of one, and $C_{nIII}$ is of second order.

By introducing the arrays of $C_{1n}(z)$ and $C_{3n}(z)$ where positions in the array are associated with particular powers of $\omega$, the coefficients $C_{1n}$ and $C_{3n}$ in equation (53) can be written as

$$C_{1n} = C_{1n}(0) + C_{1n}(1)\omega + C_{1n}(2)\omega^2 + \ldots + C_{1n}(z)\omega^z$$

$$C_{3n} = C_{3n}(0) + C_{3n}(1)\omega + C_{3n}(2)\omega^2 + \ldots + C_{3n}(z)\omega^z.$$ 

Similarly,

$$C_{1}(n-2) = C_{1}(n-2)(0) + C_{1}(n-2)(1)\omega + C_{1}(n-2)(2)\omega^2 + \ldots \quad + C_{1}(n-2)(z)\omega^z$$

$$C_{1}(n-3) = C_{1}(n-3)(0) + C_{1}(n-3)(1)\omega + C_{1}(n-3)(2)\omega^2 + \ldots \quad + C_{1}(n-3)(z)\omega^z,$$

etc.

Incorporation of the above equations and notation into (51) and (52) yields

$$C_{1n}(0) = C_{nI}C_{1}(n-2)(0) + C_{nIIB}C_{1}(n-3)(0)$$

$$C_{1n}(1) = C_{nI}C_{1}(n-2)(1) + C_{nIIB}C_{1}(n-3)(1) + C_{nIIC}C_{1}(n-3)(0)$$

$$C_{1n}(2) = C_{nI}C_{1}(n-2)(2) + C_{nIIB}C_{1}(n-3)(2) + C_{nIIC}C_{1}(n-3)(1)$$

$$\vdots$$

$$C_{1n}(z) = C_{nI}C_{1}(n-2)(z) + C_{nIIB}C_{1}(n-3)(z) + C_{nIIC}C_{1}(n-3)(z-1)$$

$$\vdots$$

$$+ C_{nIII}C_{1}(n-3)(z-2)$$
In a similar fashion, values for \( C_{3n}(z) \) are calculated.

The summations in equations (54) can now be expressed as

\[
\text{Sum}_{11}(0) = \sum_{n=3}^{NT} n(n-1)C_{1n}(0)
\]
\[
\text{Sum}_{11}(1) = \sum_{n=3}^{NT} n(n-1)C_{1n}(1)
\]
\[\vdots\]
\[
\text{Sum}_{11}(z) = \sum_{n=3}^{NT} n(n-1)C_{1n}(z).
\]

Similarly,

\[
\text{Sum}_{12}(z) = \sum_{n=3}^{NT} n(n-1)C_{3n}(z).
\]

Also,

\[
\text{Sum}_{21}(0) = \sum_{n=3}^{NT} [n(n-1)(n-2)+hn]C_{1n}(0)
\]
\[
\text{Sum}_{21}(1) = \sum_{n=3}^{NT} \{[n(n-1)(n-2)+hn]C_{1n}(1) + j_A C_{1n}(0)\}
\]
\[
\text{Sum}_{21}(2) = \sum_{n=3}^{NT} \{[n(n-1)(n-2)+hn]C_{1n}(2) + j_A C_{1n}(1) + j_B C_{1n}(0)\}
\]
\[\vdots\]
\[
\text{Sum}_{21}(z) = \sum_{n=3}^{NT} \{[n(n-1)(n-2)+hn]C_{1n}(z) + j_A C_{1n}(z-1) + j_B C_{1n}(z-2)\}
\]

and similarly for \( \text{Sum}_{22} \).

The coefficient matrix can now be written as

\[
\begin{bmatrix}
\text{Sum}_{11} & \text{Sum}_{12} \\
\text{Sum}_{21} & \text{Sum}_{22}
\end{bmatrix}
\]
where:
\[ \text{Sum}_{11} = \text{Sum}_{11}(0) + \text{Sum}_{11}(1)z + \ldots + \text{Sum}_{11}(z)z^z \]
\[ \text{Sum}_{12} = \text{Sum}_{12}(0) + \text{Sum}_{12}(1)z + \ldots + \text{Sum}_{12}(z)z^z \]
\[ \text{Sum}_{21} = \text{Sum}_{21}(0) + \text{Sum}_{21}(1)z + \ldots + \text{Sum}_{21}(z)z^z \]
\[ \text{Sum}_{22} = \text{Sum}_{22}(0) + \text{Sum}_{22}(1)z + \ldots + \text{Sum}_{22}(z)z^z \]

The determinant can now be taken by simple polynomial multiplication and subtraction and the resulting coefficients of the characteristic equation are stored in array SUM. The final equation used to find the natural frequencies can now be written in the form
\[ \text{Sum}(z)z^z + \text{Sum}(z-1)z^{z-1} + \ldots + \text{Sum}(1)z + \text{Sum}(0) = 0. \]

Subroutine POLRT then computes the real and imaginary roots of this equation by a Newton-Raphson iterative technique. 25

Given a polynomial:
\[ F(z) = \sum_{n=0}^{N} a_n z^n. \]  
(A-1)

Let \( z = X + iY \) be a starting value for a root of \( F(z) \). Then
\[ z^n = (X + iY)^n. \]  
(A-2)

Define \( X_n \) as the real terms of expanded equation (A-2) and \( Y_n \) as the imaginary terms of expanded equation (A-2). Then, for \( n = 0 \) and \( n > 0 \)
\[ X_0 = 1.0 \]
\[ X_n = X \cdot X_{n-1} - Y \cdot Y_{n-1} \]  
(A-3)
\[ Y_0 = 0.0 \]
\[ Y_n = X \cdot Y_{n-1} + Y \cdot X_{n-1} \]  
(A-4)
Let $V$ be the real terms of (A-1), and $W$ be the imaginary terms of (A-1). Then

$$V = \sum_{n=0}^{N} a_n x_n \quad \text{(A-5)}$$
$$W = \sum_{n=0}^{N} a_n y_n \quad \text{(A-6)}$$

or:

$$V = a_0 + \sum_{n=1}^{N} a_n x_n \quad \text{(A-7)}$$
$$W = \sum_{n=1}^{N} a_n y_n \quad \text{(A-8)}$$

Thus,

$$\partial V / \partial x = \sum_{n=1}^{N} n x_{n-1} a_n \quad \text{(A-9)}$$

and,

$$\partial W / \partial x = \sum_{n=1}^{N} n y_{n-1} a_n \quad \text{(A-10)}$$

Note that equations (A-3,4,7,8,9 and 10) can be performed iteratively for $n = 1$ to $N$ by saving $x_{n-1}$ and $y_{n-1}$.

Using the Newton-Raphson method for computing $\Delta X$ and $\Delta Y$, the result, after applying the Cauchy-Riemann equations is:

$$\Delta X = (W_0 W / \partial y - V_0 V / \partial x) / [(V_0 V / \partial x)^2 + (W_0 W / \partial y)^2] \quad \text{(A-11)}$$
$$\Delta Y = -(V_0 V / \partial y + W_0 W / \partial x) / [(V_0 V / \partial x)^2 + (W_0 W / \partial y)^2] \quad \text{(A-12)}$$

Thus, for the next iteration:

$$X' = X + \Delta X$$
$$Y' = X + \Delta Y$$

As an additional check on the accuracy of the roots found by POLRT, subroutine DCHECK was written. As it turns out, for the very high order equations that are dealt with, only the several lowest frequencies are accurate. So, DCHECK substitutes the roots back into the characteristic equation
and prints out the complex root and the corresponding real and imaginary values of the polynomial evaluated at that root. The deviation of the polynomial from zero gives an indication of the error involved at a given root.

Sample Output

First, the input parameters are listed in the output on page 89. Then the roots of the characteristic equation, which are the natural frequencies of the system are listed. Notice that the roots are not necessarily in ascending order due to an arbitrary starting search point. It is also possible for roots to be repeated. In this output, the zeroeth mode frequencies are given in the first two lines, the first mode frequencies are in the next two lines, and the third mode frequencies are in lines 8 and 9. The values of the real and imaginary parts of the function when the root is substituted back into the characteristic equation is given under columns FWR and FWI. The root is known to be accurate if the value of the function upon substitution is sufficiently small compared to the root. Thus, it can be seen in this output that not only are the three lowest modes accurate but also the fourth mode frequency (given in lines 10 and 11) and the next couple of frequencies are also accurate. Conversion must also be checked. For an increase in the number of terms taken, the change in the value of the lowest
frequencies should be extremely small. If not, more terms are needed for convergence.

Clamped-Free Cylinder

The program for a clamped-free cylinder is very similar to that of the pinned-free cylinder. The coefficient arrays, however, are labelled $C_2$ and $C_3$ because the application of the clamped condition leads to linear equations in $A_2$ and $A_3$ rather than $A_1$ and $A_3$. The recurrence relations are given by:

$$A_n = C_{2n}A_2 + C_{3n}A_3$$

where:

$$C_{2n} = C_{12}C_2(n-2) + C_{22}C_2(n-3) + C_{32}C_2(n-4)$$

$$C_{3n} = C_{13}C_3(n-2) + C_{23}C_3(n-3) + C_{33}C_3(n-4)$$

and $C_{12}$, $C_{22}$, $C_{33}$ are as given before. Note that the recurrence relations are identical except for notation. The difference in the two cases originates from the first several coefficients. Thus, for the clamped case:

$C_{21} = 0 \quad C_{22} = 1 \quad C_{23} = 0 \quad C_{24} = C_{41} = -a/4.3$

$C_{31} = 0 \quad C_{32} = 0 \quad C_{33} = 1 \quad C_{34} = 0$

and

$$C_{25} = C_{51}C_{23} + C_{52}C_{22} + C_{53}C_{21} = C_{52}$$

$$C_{35} = C_{51}C_{33} + C_{52}C_{32} + C_{53}C_{31} = C_{51}$$

etc.
The resulting matrix equation as given in (54) is identical except that $C_1r$ and $A_1$ are replaced by $C_2r$ and $A_2$. 


```c
C--------------------------------------------
C PROGRAM = PFF
C--------------------------------------------
C PURPOSE
C COMPUTES THE COMPLEX NATURAL FREQUENCIES OF A PINNED-FREE
C CYLINDER IN A FLUID FOR GIVEN SYSTEM PARAMETERS.
C THE GOVERNING DIFFERENTIAL EQUATION IS IN THE FORM
C Y'''' + a Y'' + b Y' + c Y + e Y = u
C AND IS SUBJECTED TO THE BOUNDARY CONDITIONS
C PINNED END:
C Y(0) = Y''(0) = 0
C FREE END:
C Y''(1) = a Y''(1) + b Y'(1) + c Y(1) = 0
C WHERE Y = Y(X), AND a, b, c, e, h, k, J ARE CONSTANTS THAT
C ARE DEPENDENT ON THE SYSTEM PARAMETERS AND ARE
C DEFINED BELOW.
C
C PARAMETER DESCRIPTION
C UF = FLUID VELOCITY PARALLEL TO CYLINDER
C L = LENGTH OF CYLINDER
C U = DIAMETER OF CYLINDER
C MC = MASS OF CYLINDER PER UNIT LENGTH
C MV = VIRTUAL MASS OF FLUID PER UNIT LENGTH OF CYLINDER
C ET = FLEXURAL RIGIDITY OF CYLINDER
C CT = FRICITONAL RAG COEFFICIENT FOR CYLINDER IN CROSS FLOW
C P = PARAMETER ASSOCIATED WITH TAIL SHAPE OF CYLINDER
C XE = EFFECTIVE LENGTH OF CYLINDER TAIL
C EP = SLENDERNESS RATIO, L/U
C ETA = MASS RATIO, MV/(MC+MV)
C U = NONDIMENSIONAL VELOCITY, (MC+MV)/EIJ*.5*UF*U
C NT = NUMBER OF TERMS TAKEN IN SERIES (INTEGER)
C NI = NUMBER OF PROGRAM RUNS DESIRED (INTEGER)
C N = NONDIMENSIONAL FREQUENCY, (MC+MV)/EIJ*.5*UF*U
C A = U*U*EP*CT/2
C B = (U*U*EP*CT)/2
C C = (EP*CT*U*U)/2 + 2*(BETA**.5)*U*M
C E = M
C I = F*U
C J = F*(BETA**.5)*U*M + D*(F=1)*BETA*E*X*M
C
C PARAMETER INPUT
C INPUT IS IN FREE FORMAT FORM AND REQUIRE TWO CARDS
C ALL VALUES ARE REAL UNLESS STATED OTHERWISE ABOVE
C
C CARD 1: U, EP, ETA, F, X, NT, NI
C CARD 2: U, EP, O, O, O, O, D, D, D, D, N, N
C
C THE VALUES ON CARD 2 ARE INCREMENTS OF THE INPUT PARAMETERS.
C THIS ALLOWS A DESIRED RANGE OF PARAMETERS TO BE EXAMINED WITH
C A SINGLE EXECUTION OF THE PROGRAM. ONCE THE INCREMENT IS
C SPECIFIED, THE TOTAL RANGE IS CONTROLLED BY SPECIFYING THE
C NUMBER OF RUNS, NI, TO BE TAKEN THROUGH THE PROGRAM.
C
C SUBROUTINES NEEDED
```
**Remarks**

The output is such that only a few runs out of the many that are listed are accurate. The roots are calculated from a characteristic function whose coefficients are calculated in the program. Substituting the roots found by Pulat and substituting them back into the characteristic equation, thus the accuracy of the roots is checked by noting the variance of the value of the real and imaginary parts of the function, for any pair, from zero when they are evaluated at a run.

For a given set of parameters, convergence must be checked.

As a rule, the number of terms needed for convergence of the non-dimensional frequency increases with an increase of e.

The program is in double precision.
114 13*i+5
115 14*i+4
116 15*i+5
117 if D0%ER1(1) = M
118 D0%ER1(11) = M
119 D0%ER1(12) = M
120 D0%ER1(13) = M
121 D0%ER1(17) = M
122 D0%ER1(19) = M
123 D0%ER1(15) = M
124 M = M+2
125 CONTINUE
126 A = (1.0 + S0U*EP*CT)
127 M = S0U*U + U*EP*CT
128 CA = S0U*EP*CT + U
129 C0 = 2.0*ATTA*S0U*U
130 H = F = U
131 J = = (1.00 + (F-1.00)*ATTA)*A
132 WRITE(h,420) A
133 WRITE(b,430) A
134 WRITE(b,440) CA
135 WRITE(h,450) C0
136 WRITE(h,460) JA
137 WRITE(b,470) JA
138 WRITE(l,480) JA
139 C INITIALIZE COEFFICIENTS 1 THROUGH 8
140 C
141 C
142 C
143 C
144 C1(1,1) = 0.0
145 C1(4,1) = CA/(4.0*5.0*8.0) + 2.0
146 C1(4,2) = CR/(4.0*5.0*2.0)
147 C1(4,3) = 0.0
148 C1(5,1) = 0.0
149 C1(5,2) = 0.0
150 C1(5,3) = 1.0/(5.0*4.0*3.0*2.0)
151 C1(6,1) = CA/(b.0*5.0*2.0)
152 C1(6,2) = CA/(b.0*5.0*2.0)
153 C1(6,3) = 0.0
154 C3(3,1) = 1.0
155 C3(4,1) = 0.0
156 C3(4,2) = 0.0
157 C3(4,3) = 0.0
158 C3(5,1) = M/(5.0*4.0)
159 C3(5,2) = 0.0
160 C3(5,3) = 0.0
161 C3(6,1) = (2.0*5.0*4.0*3.0*2.0)
162 C3(6,2) = CR/(b.0*5.0*2.0)
163 C3(6,3) = 0.0
164 C LOUP TO CALCULATE COEFFICIENT VALUES
165 C
166 C
167 DO 50 I = 7,41
168 I = I + 1
169 I = I + 2
170 I = I + 3
171 C
172 C
173 C
174 C
175 C
176 C
177 C
178 C
179 C
180 C
181 C
182 C
183 C
184 C
185 C
186 C
187 C
188 C
189 C
190 C
191 C
192 C
193 C
194 C
195 C
196 C
197 C
198 C
199 C
200 C
I = M1 - a

ON1 = ON1ER1(I) + 1

ON3 = ON3ER1(I) + 1

DG J = 1, ON1

J1 = J1

J2 = J2 - 2

J3 = J3 - 5

J4 = J4 - 4

IF (J, EQ, 1) GO TO 50

IF (J, EQ, 2) GO TO 70

C1 (I, J) = -A*C1 (I, J) / (I!1) - 14*A*C1 (I, J) /

C1 (I, J) = -B*C1 (I, J) / (I!1) - 14*A*C1 (I, J) /

J1 = J1 - 1

J2 = J2 - 2

J3 = J3 - 5

J4 = J4 - 4

IF (J, EQ, 1) GO TO 51

IF (J, EQ, 2) GO TO 71

C1 (I, J) = -A*C1 (I, J) / (I!1) - 14*A*C1 (I, J) /

C1 (I, J) = -B*C1 (I, J) / (I!1) - 14*A*C1 (I, J) /

DU 41 J = 1, ON1

DU 45 J = 1, ON1

SUM1(J) = SUM1(J) + I1*C1 (I, J)

WHITE(6, 100)

WHITE(6, 80)

CALCULATE SUM OF POSITION 11 IN COEFFICIENT MATRIX

CALCULATE SUM OF POSITION 12 IN COEFFICIENT MATRIX

SUM12(I) = SUM12(I) + 0

DU 36 I = 4, NT
C CALCULATE SUM OF POSITION 21 IN COEFFICIENT MATRIX

C
SU21(1)=SUM12(J)*I
SU21(2)=J*A
SU21(3)=J*B
SU21(4)=SUM12(J)*PSU21(J)
SU21(5)=SUM21(J) + SU21(J)
SU TO 131

I=I-1
J2=J-2
DO 130 J=1,UM
SU21(J)=SUM12(J) + SU21(J) + PSU21(J)
SU TO 131

130 PSU21(J)=SUM12(J)*PSU21(J)
SU TO 131

C C CALCULATE SUM OF POSITION 22 OF COEFFICIENT MATRIX

C
SU22(1)=SUM22(J) + SUM22(J) + PSU22(J)
SU22(2)=J*A
SU22(3)=J*B
SU22(4)=SUM22(J) + SU22(J)
SU TO 131

I=I-1
J2=J-2
DO 130 J=1,UM
SU22(J)=SUM22(J) + SU22(J) + PSU22(J)
SU TO 131

130 PSU22(J)=SUM22(J) + PSU22(J)
PSUM22(J) = (1*11*12*13*14) * C3(1, J)
SUM22(J) = SUM22(J) + PSUM22(J)
GU 10 231
PSUM22(J) = (1*11*12*13*14) * C3(1, J) * JA * C3(1, J)
SUM22(J) = SUM22(J) + PSUM22(J)
C231 WRITE(b,140) J1, PSUM22(J), SUM22(J)
250 CONTINUE
C260 CONTINUE
011 = ORDER1(NT) + 1
015 = ORDER23(NT) + 1
IF(UT11-011) .EQ. 222, 223, 223
016 = ORDER11
017 = ORDER11(NT) + 1
224 DU 240 J1=1, UT1
IA=J
C
TAKE THE DETERMINANT OF THE COEFFICIENT MATRIX TO
GET THE CHARACTERISTIC EQUATION
C
DU 250 I1=1, J1
SUMA(I1) = SUMA(I1) + SUM11(J1) * SUM22(I1)
SUMA(I1) = SUMA(I1) + SUMB2(I1) + SUM12(I1) * SUM21(I1)
I1 = I1 + 1
250 CONTINUE
240 CONTINUE
I1 = I1 + 1
C
WHITE(6, 250)
C
WHITE(6, 270)
DU 280 I1=1, UT
SUM(I) = SUMA(I) + SUMB(I)
I1 = I1 + 1
C
WHITE(6, 290) I1, SUM(I)
280 CONTINUE
WHITE(1, I1)
N1 = 120
C
CALL POINT TO FIND ROOTS OF CHARACTERISTIC EQUATION
C
CALL POLY(SUM, COF, M, ROUTR, ROOUT, IER, NI)
WRITE(6, 490)
WHITE(6, 510)
DU 500 I1=1, M
WHITE(6, 520) ROUTR(I), ROOUT(I), M(I), IER
W(I) = CMPLX(ROUTR(I), ROOUT(I))
500 CONTINUE
CALL CHECK(SUM, M, M, M)
UR = DU
E = EP + DEP
CT = CT + OCT
BETA = BETA + OCT
CIP = CIP + OCTP
GAM = GAM + OCTGAM
STOP
SUBROUTINE POLRI
FILE = UCS#AAMOUNB

PURPOSE
COMPUTES THE REAL AND COMPLEX ROOTS OF A REAL POLYNOMIAL

USAGE
CALL POLRI(XCF, CUF, M, RUDIR, RUDII, IER, N1)

PARAMETER DESCRIPTION
XCF = VECTOR OF M+1 COEFFICIENTS OF THE POLYNOMIAL
ORDERED FROM SMALLEST TO LARGEST POWER
DF = ORDER OF POLYNOMIAL
M = ORDER OF POLYNOMIAL
ROOT = RESULTANT VECTOR OF LENGTH M CONTAINING REAL ROOTS
OF THE POLYNOMIAL
OUTI = RESULTANT VECTOR OF LENGTH M CONTAINING THE
CORRESPONDING IMAGINARY ROOTS OF THE POLYNOMIAL
IER = ERROR CODE WHERE
IER = 0 NO ERROR
IER = 1 M LESS THAN ONE
IER = 2 M GREATER THAN 100
IER = 3 UNABLE TO DETERMINE ROOT WITH 500 ITERATIONS ON 5 STARTING VALUES
N1 = LENGTH OF VECTORS, SHOULD EQUAL M+1.

REMARKS
FOR POLYNOMIALS HIGHER THAN ORDER 30, ONLY THE
LOWEST FEW ROOTS WILL BE ACCURATE. MAXIMUM ORDER
OF POLYNOMIAL IS 100.

METHOD
NEVILLE'S INTERPOLATIVE TECHNIQUE. THE FINAL ITERATIONS
ON EACH ROOT ARE PERFORMED USING THE ORIGINAL POLYNOMIAL
RATHER THAN THE REDUCED POLYNOMIAL TO AVOID ACCUMULATED
ERRORS IN THE REDUCED POLYNOMIAL.

SUBROUTINE POLRI(XCF, CUF, M, RUDIR, RUDII, IER, N1)
DIMENSION XCF(N1), CUF(N1), RUDIR(N1), RUDII(N1)
REAL*8 XCF, YCF, X, Y, XP, YM, U, V, TT, T, T2, SUM, S, T, T2
I = 0, DT, TEMP, ALPHA
REAL*8 XCF, CUF, RUDIR, RUDII

200 IF(I = 0)
45 IF(ABS(XCF(N+1)) > 0.25, 10)
10 IF(N = 15, 15, 32)

C SEI ERROR CODE TO 1
15 IER = 1
20 RETURN
25 M = M - 1
40 GO TO 200
57  C  SET ERROR CODE TO 2
58  C  30 L=9
59     GU TO 2U
60  C  32 IF(W=150) 35,35,36
61  C  35 N=N
62     N1=X+1
63      N2=1
64      K1= N1
65      DU 40 L=1,K1
66      M1=K1-L+1
67  C  40 CUF(M1)=XCOF(L)
68  C  SET INITIAL VALUES
69  C  45 XU=0.05001010DU
70     YU=0.010001010U
71  C  ZERO INITIAL VALUE COUNTER
72  C  30 X=XU
73  C  IN=U
74  C  INCREMENTS INITIAL VALUES AND COUNTER
75  C  35 XU=10.0000+YD
76     YU=10.0000+X
77  C  SET X AND Y TO CURRENT VALUES
78  C  X=XU
79  C  Y=YU
80  C  IN=N+1
81  C  GU TO 54
82  C  35 IF=I
83  C  XP=YX
84  C  YPR=AY
85  C  EVALUATE POLYNOMIAL AND DERIVATIVES
86  C  59 ICT=0
87  C  60 UX=0.0DU
88  C  UT=U-0.0DU
89  C  V=U-0.0DU
90  C  Y1=U-0.0DU
91  C  XT=1.0DU
92  C  U=CUF(N+1)
93  C  IF (U) 65,13U,55
94  C  55 DU 70 I=1,N
95  C  L:Z=L+1
96  C  TEMP=COF(L)
97  C  104 Y12=XX+Y1+Y*XT
98  C  105 Y12=XX+Y1+Y*XT
99  C  111 UX+TEMP*Y2
100  C  112 V+TEMP*Y2
101  C  F=1
UX=UX+F1*XT*TEMP
UY=UY-F1*YT*TEMP
XT=XT2
70 YT=YT2
SUMSQ=UX*UX+UY*UY
IF(SUMSQ) 75,110,75
75 DX=(V*UY-U*UX)/SUMSQ
X=X+DX
DY=(U*UX+V*UX)/SUMSQ
78 IF(DABS(DY)+DABS(DX)<1.0D-05) 100,80,80
C STEP ITERATION COUNTER
80 ICT=ICT+1
IF(ICT=500) 60,85,85
85 IF(IFIT)100,90,100
90 IF(IN=5) 50,95,95
C SET ERROR CODE TO 3
95 TERR=3
GO TO 20
100 DO 105 L=1,NXX
MT=K1-L+1
TEMP=X*COF(M1)
XCOF(MT)=COF(L)
105 COF(L)=TEMP
110 IF(IFIT) 120,55,120
115 X=XPR
120 Y=YPR
122 IF(DABS(Y)-1.0D-4*DABS(X)) 135,125,125
125 ALPHA=X*X
128 SUMSQ=X*X+Y*Y
130 N=N+1
133 GU TO 140
135 Y=0.000
138 SUMSQ=0.000
140 COF(2)=COF(2)+ALPHA*COF(1)
145 DO 150 L2=N
150 COF(L1)=COF(L1)+ALPHA*COF(L1)-SUMSQ*COF(L1)
155 ROOT(L2)=Y
160 N2=N2+1
165 IF(SUMSQ) 160,155,160
168 Y=Y
170 SUMSQ=0.000
171       GO TO 155
172       155 IF(N) 20,20,45
173       RETURN
174       END

PLRT*171
PLRT*172
PLRT*173
PLRT*174
SUBROUTINE CHECK(COEFF,n,M,N)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION COEFF(100),NM(100)
COMPLEX*16 N(100),FM
M=M+1
FM=(0.0,0.0)
WHITE(b,bu)
WHITE(b,40)
 DU 50 I=1,N
 IF(MW(I),EQ.0.)GO TO 21
 J=J+1
 P=FLOAT(J)
 FM=COEFF(I)*MW(I)*P+FM
 CONTINUE
 DU 20 J=1,N
 J=J-1
 P=FLOAT(J)
 FM=COEFF(I)*MW(I)*P+FM
 CONTINUE
 DU 20 J=1,N
 IF(MW(I),EQ.0.)GO TO 21
 J=J+1
 P=FLOAT(J)
 FM=COEFF(I)*MW(I)*P+FM
 CONTINUE
 FORMAT('0',10X,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10)
 FORMAT('0',1X,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10)
 FORMAT('0',1X,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10)
 FORMAT('0',1X,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10)
 FORMAT('0',1X,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10,1B10)
 CONTINUE
 RETURN
END


5. See Reference 1.


15. See Reference 3.

16. See Reference 3.

17. See Reference 1, p. 721.


21. See Reference 3.

22. See Reference 19, p. 373.

23. See Reference 1, p. 736.

