A TWO-SECTOR GROWTH MODEL
OF THE U. S. ECONOMY

by

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ABSTRACT

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This work includes a survey of the literature of growth theory with emphasis on the two-sector Neoclassical model. A two-sector model of the U. S. economy with disembodied Harrod neutral technological change is estimated and simulated. Stability and the role of price mechanism is pointed out. The results of the estimation do not seem to support the significance of the disaggregation of the data into the two sectors. A clear advantage of the model, however, lies in the efficient disequilibrium adjustment mechanism.

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to discuss some of the economic theories of growth with special emphasis on the two-sector Neoclassical model. The implications of this model for growth policy will be pointed out.

Economic growth can simply be defined as a sustained increase in per capita output. Most people will agree that there has been considerable economic growth since 1800 and the economist is interested in explaining this. Presently, it appears that growth has slowed in the so-called "advanced" countries and again some explanation is desirable. It is also true that some nations have not shared in the above mentioned growth and here the economist is interested in why this has occurred and what policies are necessary to alter this.

Certain other developments appear to have occurred simultaneously with the rise in per capita output or income. They are:

a. a vast increase in labor productivity and, therefore, an increase in real wages;

b. a significant capital accumulation (K/L increase) and technological change;

c. the following constants (Burmeister, Dobell, 1972, P. 65):

(1) constant labor and capital shares,

(2) constant real interest rates,

(3) constant savings rate,

(4) constant capital output ratio.

Economic theories of growth have attempted to establish connections between these "stylized facts" and rising per capita incomes.

Chapter II is a discussion of Harrod's expectations model and Solow's Neoclassical model. Chapter III contains a treatment of the fixed coefficients and Cobb-Douglas case of the two-sector model. Estimation of the two-sector model is the topic of the fourth chapter with resulting policy implications.

CHAPTER II

PRECURSORS TO THE TWO-SECTOR MODEL

When one considers the first contributions to growth theory, classical ideas are quite exemplary. Mill, Malthus, and Marx could be discussed pointing out quite pessimistic conclusions about growth. This was due to the restrictive assumptions about one of the factors of production being fixed in the long run (land) along with the lack of consideration of the importance of technological change. The Classical models were not formulated mathematically, and it was not until after the work of Keynes (1936) that the first mathematical formulations of growth were derived.

Harrod (1939) and Domar (1946) are considered to be the first major contributors to the mathematical growth literature. Harrod's model was highly demand oriented and thus became a forerunner to later Neo-Keynesian Kaldor (1957), Robinson (1956) models. Domar's work emphasized the supply side and lead to later Neoclassical models such as Solow's (1956) and Swan's (1956).

This chapter will deal with Harrod's model as an example of Keynesian thought and Solow's model as an example of Neoclassical ideas.

The Harrod Model.

The Harrod model is an expectations model, that is, growth occurs as a result of businessmen's expectations of future sales in the economy. The implementation of investment plans in response to these expected sales creates future demand via the multiplier principle. The "warranted" rate of growth is the growth of demand that ensures fulfillment of the sales expectations.

The structural equations of the model are:

$$S = sY, \qquad (2.1)$$

where S is the level of savings, Y is the output or income generated, and s is the marginal propensity to save, a constant positive fraction.

$$K = vY^*, \qquad (2.2)$$

where K is the desired capital stock, Y* is the forecasted demand or the expected sales value, and v is the fixed fractional proportion of expected output that is maintained as the stock of capital goods, i.e., the constant capital output ratio.

$$I = \frac{dK}{dt} = \dot{K}. \qquad (2.3)$$

This defines net investment (I) as the time derivative of capital and implies that the investment function is:

$$I = v\dot{Y}^*. \tag{2.4}$$

In a static model the saving investment equilibrium condition,

$$S_E = I_E, \tag{2.5}$$

would be sufficient to close the model, and determine equilibrium income. In this case, however, the equilibrium path of income can only be determined if an additional condition is satisfied, namely, the equality of the expected change in aggregate demand $\dot{Y}^*$ and the actual change $\dot{Y}$:

$$\dot{Y}^*_E = \dot{Y}_E. \tag{2.6}$$

These six equations are sufficient to determine the equilibrium path of demand. The savings-investment equality implies that:

$$sY = v\dot{Y}^*. \tag{2.7}$$

Then (2.6) implies that:

$$sY = v\dot{Y}. \tag{2.8}$$

This reveals that:

$$s/v = \dot{Y}/Y. \tag{2.9}$$

Thus, aggregate demand should grow at this (s/v) rate in order to insure expectations are fulfilled.

This percentage change of output is defined to be the warranted rate of growth of the system. Note that it is a constant, so that aggregate demand is said to be in a steady state. Furthermore, it can be shown that all endogenous variables do grow at this same rate so that they move along a "balanced" growth path.[1] Thus,

$$\frac{s}{v} = \frac{\dot{I}}{I} = \frac{\dot{Y}}{Y} = \frac{\dot{S}}{S} = \frac{\dot{Y}^*}{Y^*} \ . \tag{2.10}$$

This balanced steady growth path has been called a "Golden Age" (Robinson 1956) to emphasize that it is a mythical state of affairs, useful for analysis but unlikely to prevail in practice.

Stability, an important characteristic, should be mentioned. When a system experiences a state of disequilibrium, and returns to equilibrium either immediately or as time progresses, it is said to be stable.

Consider a verbal explanation of what might happen in a disequilibrium situation. Assume the aggregate demand was growing at a slower rate than the warranted rate (s/v). This would imply that businessmen had overestimated future sales. A rational response would be to reduce the rate of investment. This would decrease aggregate demand's growth rate even further, still maintaining the inequality between expected and

---

[1]Note that all variables are proportional to each other. Consider the case where $y = ax$, where a is a constant. It follows that $\dot{y}/y = \dot{x}/x$. Repeated application of this principle can demonstrate that all variables in the system will grow at the same rate as $\dot{Y}/Y$.

actual sales. The result would be the economy spiraling into depression. In the opposite situation an underestimation of expected sales would analagously lead to spiraling inflation. This is one of the obvious problems with the Harrod model that later models tried to avoid.

The model can be extended with the addition of a labor market, and with this comes the concept of the natural rate of growth. This rate reflects the change in the labor force over time. Since capital is assumed to be proportional to output, it is reasonable to assume that the demand for labor is also proportional to output:

$$L_D = uY. \tag{2.11}$$

If the economy is moving along the equilibrium growth path, the supply of labor should grow at the rate (s/v). Harrod assumed[2] that the supply of labor grows exogenously at n percent per annum:

$$L_S = L_o e^{nt}. \tag{2.12}$$

Thus, full employment can only be maintained along the warranted path when (s/v) equals n. In Harrod's terminology, this would mean that the warranted rate equals the natural rate of growth.

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[2]This is another distinction between classical models and more recent contributions. Malthus, for example, focused on the endogeneity of the labor force.

It was already pointed out that there is no assurance that (s/v) would be attained. A further uncertainty is established when (s/v) is equated to n. The equality of a natural phenomenon such as the rate of growth of the labor force, i.e., population growth, to an economic or production condition seems almost impossible to achieve except in a random situation. The possible or probable inequality is referred to in the literature as the "Bastard Golden Age", since it could lead to perpetual unemployment in one case, or unbounded inflation in the other.

In the case where (s/v) is greater than n, excess capacity growth or labor shortage, government policy would attempt to decrease the savings rate perhaps by diminishing disposable income via taxation. In the case of excess labor, or n being greater than (s/v), policy would attempt to increase the savings rate. In either case, notice there is still no force present to produce the above mentioned equality of the growth rates. This can be attributed to the rigidity of the model based on the number of assumed constants.

Domar (1946) produced a model structurally very similar stressing the production side of the economy. His contribution leads to the same warranted rate of growth (s/v) with the same property of instability.

Later models attempt to resolve the inflexibility by allowing some of the constants to vary when the economy is not in equilibrium. The Neo-Keynesian models are considered offshoots of Harrod's work, and allow the savings rate to vary. However, the Neoclassical models are offshoots of Domar's work and allow the capital output ratio to change to ensure stability.

The Solow Model.

The Neoclassical model is an attempt to relax some of these built in rigidities of Harrod so as to allow a variable (s/v) ratio in disequilibrium. This permits adjustments to the warranted rate and provides some mechanism for the equality of (s/v) to n. The adjusting variable is v, the capital-output ratio, and the emphasis is on production and supply rather than on demand.

Solow begins his contribution with the following assumptions:

(a) Labor and capital are mutual substitutes;

(b) Diminishing marginal returns are present;

(c) A production function describes the output in terms of combinations of inputs with constant returns to scale;

(d) The economy sustains full employment at all times.

The equations of the model are:

$$L = L_0 e^{nt}, \tag{2.13}$$

defining the supply of labor with variables are described above.

$$Y_{FC} = F(K,L), \tag{2.14}$$

where F is a linearly homogeneous production function with constant returns to scale describing output. It is equated to full capacity output, income, Y_{FC}.

$$S = sY, \tag{2.15}$$

defined as above.

$$S_E = I_E, \tag{2.16}$$

where the supply of savings just equals the amount of investment goods demanded.

$$I = \dot{K} = \frac{dK}{dt} . \tag{2.17}$$

$$Y_E = Y_{FC}. \tag{2.18}$$

Solving for the equilibrium growth rate utilizing simple substitution techniques reveals:

$$\frac{dK}{dt} = sF(K,L) = sF(K,L_o e^{nt}). \tag{2.19}$$

This equation determines the desired rate of change of the capital stock that would be necessary to maintain full employment of manpower determined by Equation (2.13) above.

To augment further analysis, Solow introduced the capital labor ratio defined as:

$$k = K/L. \tag{2.20}$$

With proper substitution one obtains:

$$K = Lk = L_o e^{nt} k. \tag{2.21}$$

$$\frac{dK}{dt} = \frac{dk}{dt}(L_o e^{nt}) + nkL_o e^{nt} \tag{2.22}$$

$$= (\frac{dk}{dt} + nk)\, L_o e^{nt}.$$

Substituting Equation (2.22) into (2.19) yields:

$$(\frac{dk}{dt} + nk)\, L_o e^{nt} = sF(K, L_o e^{nt}). \tag{2.23}$$

Dividing by $L_o e^{nt}$ produces:

$$(\frac{dk}{dt} + nk) = \frac{sF(K, L_o e^{nt})}{L_o e^{nt}}. \tag{2.24}$$

Remembering that F is assumed to be linearly homogeneous with constant returns to scale implies that:

$$\frac{F(K,L)}{\gamma} = F(\frac{K}{\gamma}, \frac{L}{\gamma}). \tag{2.25}$$

Hence:

$$\frac{dk}{dt} + nk = sF(\frac{K}{L_o e^{nt}}, 1) = sF(k,1), \tag{2.26}$$

from (2.20) above. Clearly,

$$\frac{dk}{dt} = sF(k,1) - nk = sf(k) - nk. \tag{2.27}$$

This is a nonlinear differential equation in the variable k. It can be graphically represented as in Figure 1.

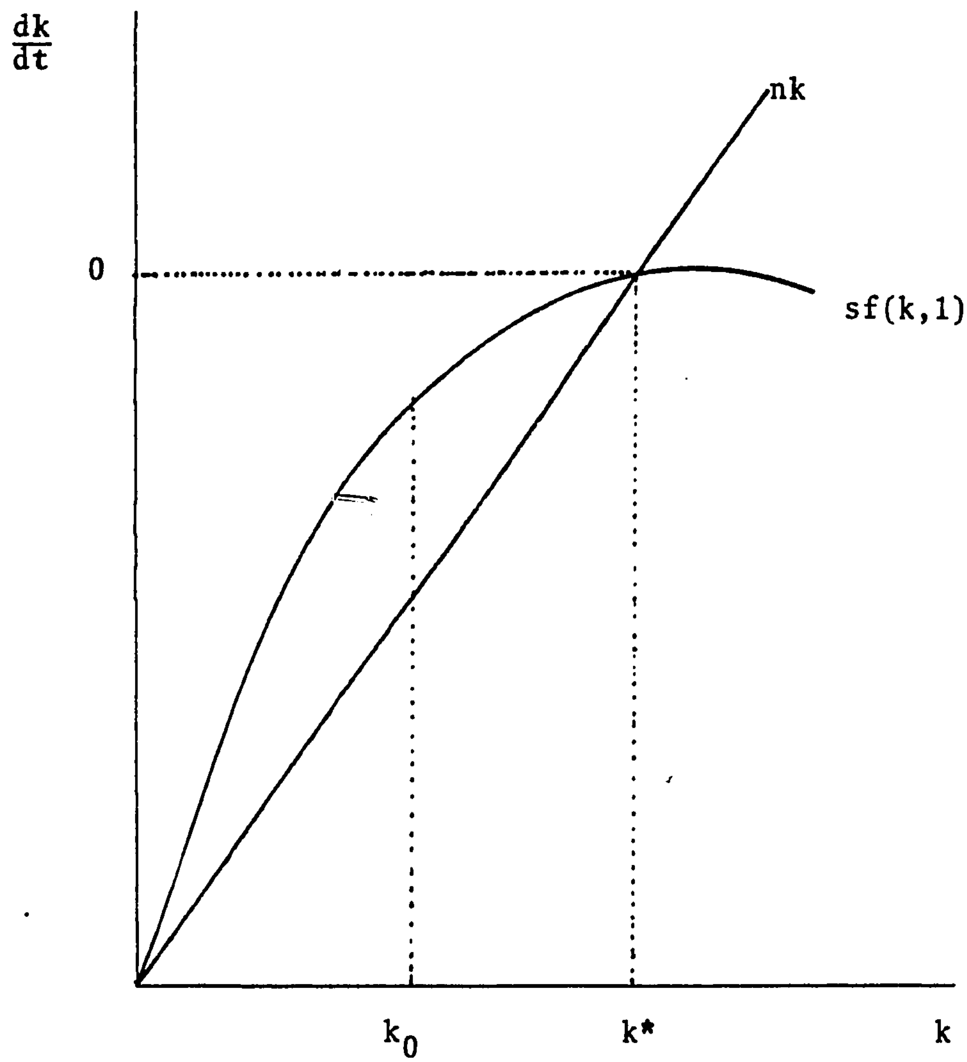

Figure 1. Equilibrium in the Neoclassical Model

The function sf(k,1) represents the amount of capital accumulation that actually occurs as a technical result of production, given a capital labor ratio. Note that sf(k,1) is drawn with upward concavity reflecting the diminishing marginal product of capital. The function nk represents the amount of capital needed to sustain a given capital labor ratio. The characteristics of these two functions insure the existence of a k* such that $\frac{dk}{dt} = 0$. (Solow, 1956, p. 27).

To interpret the differential equation, it is helpful to rewrite it as follows:

$$\frac{dk}{dt} = sf(k) - nk = \frac{S}{L} - n\frac{K}{L} . \tag{2.28}$$

Multiplying both sides by L/K reveals:

$$\frac{L}{K}\frac{dk}{dt} = \frac{S}{K} - n; \tag{2.29}$$

which equals

$$\frac{1}{k}\frac{dk}{dt} = \frac{S}{K} - n = \frac{S/Y}{K/Y} - n = \frac{s}{v} - n. \tag{2.30}$$

Thus, the percentage rate of growth of the capital labor ratio equals the difference between the warranted and natural growth rates.

There exists a mechanism in this model which brings about the equality of these two rates. Let k_o be a capital labor ratio less than the equilibrium value k*. At this point, Figure 1 reveals that the rate of growth of k determined by the production process exceeds the capital labor ratio necessary to maintain full employment of labor, i.e., sf(k) is greater than nk. It follows from Equations (2.28) through (2.30) that the warranted rate is larger than the natural rate of growth. From the assumption of a convex production function, it follows that $\dot{Y}/Y$, and $\dot{K}/K$ are greater than zero. Since diminishing returns are present, $\dot{Y}/Y$ must be less than $\dot{K}/K$. It can be demonstrated that this leads to the fact that $\dot{v}/v$ also is positive.[3] Since it is true that:

$$\dot{(\frac{s}{v})} = \frac{v\dot{s} - s\dot{v}}{v^2} = \frac{\dot{s}}{v} - \frac{s}{v}\frac{\dot{v}}{v} ; \qquad (2.31)$$

it follows that:

$$\dot{(\frac{s}{v})} = -\frac{s}{v}\frac{\dot{v}}{v} . \qquad (2.32)$$

Because the value of Equation (2.32) is less than zero, it is obvious that the percentage change in (s/v) will also be negative. This shows that the warranted rate will decrease and move to equality with the natural rate of growth. This mechanism works similarly when (s/v) is less than n.

In the case where the labor force grows at a fixed rate n_o, the equilibrium condition would dictate:

$$\frac{s}{v} = n_o. \qquad (2.33)$$

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[3] K/Y = v from above implying K = vY. Then, it follows that $\dot{K} = v\dot{Y} + Y\dot{v}$ and $\dot{K}/K = (v/K)\,\dot{Y} + (Y/K)\,\dot{v} = \dot{Y}/Y + \dot{v}/v$. Since $\dot{K}/K > \dot{Y}/Y$, this implies that $\dot{v}/v > 0$.

Assume, however, the savings rate rises to a higher level, s_1. The result would be an increase in S and I by the same amount from Neoclassical assumptions. Then, K would also rise implying that K/Y = v rises. The change in v would tend to offset the change in s. The final result of this situation would be to establish a new (s/v) ratio again equal to n_o. An obvious policy conclusion would be that adjustment of the savings rate is not an effective tool for controlling the rate of growth when the economy is in equilibrium, according to this model.

The characteristics of this model can be amplified by considering the production function to be of Cobb-Douglas form. Additions of technical progress, taxation, variable population growth, variable savings ratio also can add more validity to this model of economic expansion.

Technological Change.

One of the most striking developments of the last one hundred years has been the dramatic advance in technology. This is generally agreed to have been a significant source of growth, and the purpose of this section is to show technological change is incorporated in the Neoclassical model.

Technology can be considered to be embodied or disembodied in form. The former refers to changes in the quality of labor, machines or both. These quality changes are usually incorporated in the production process as new inputs are hired.

Disembodied technological change refers to advancements in production that are not attributed to changes inherent to labor or capital. Most authors refer to it as falling upon the production process like "manna from heaven" and encompasses such things as managerial innovations enabling existing inputs to be used more effectively. This type of change is reflected by shifts of the production function over time. The time factor enters into the production function simply as a parameter and can be written as:

$$Q = f(K,L,t). \tag{2.34}$$

The use of disembodied technological change rather than embodied tends to be less cumbersome in most cases, yet still leads to similar analytical results.

> In some respects the long run behavior... of models in which technological progress is embodied is, in fact, identical to that in models with disembodied change.[4]

Disembodied change can be considered factor augmenting or non-factor augmenting. When it is factor augmenting, one can transform the production from one like Equation (2.34) to:

$$Q = G \{ a(t)K, b(t)L \}, \tag{2.35}$$

or:

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[4]Burmeister, E. and A. Dobell; Mathematical Theories of Economic Growth, p. 66.

$$Q = G\ (\overline{K},\overline{L}); \tag{2.36}$$

where $\overline{K}$, $\overline{L}$ are variables expressed in efficiency units, and G is a linearly-homogeneous function. This function will shift outward over time due to increased productivity of inputs. One might consider this equivalent to increased employment of inputs.

Technology is said to be non-factor augmenting when the production function cannot be rewritten as above. If the technology parameter shifted the intercept and not the slopes of a linear production function, the technological change would not be factor augmenting.

Three cases of the factor-augmenting process can be distinguished. One case is capital augmenting, where the function can be written as:

$$f(K,L,t) = G\ \{a(t)K,L\}. \tag{2.37}$$

Labor augmenting technology can be written as:

$$f(K,L,t) = G\ \{K,b(t)L\}. \tag{2.38}$$

Finally, capital and labor augmenting progress is simply written as Equation (2.35).

One further classification of technology involves the concept of neutrality. Neutrality exists when progress occurs without bias in favor of labor or capital; i.e., the factor shares of capital and labor remain unaltered after the shift in the production function. Progress can be one of three types: Harrod neutral, Solow neutral, or Hicks neutral. Neutrality in the sense of Harrod is present when labor-augmenting technology leads to constant factor shares. Also, a constant capital-output ratio is maintained. Solow neutrality tends to be capital augmenting in nature with a constant labor output ratio. Lastly, Hicksian neutrality is both labor and capital augmenting. The constant capital labor ratio is maintained, reflecting a constant marginal rate of technical substitution. This can be depicted graphically as in Figure 2. As time goes on, the isoquant line Q_0 shifts outward to the right in a "parallel" fashion, without alteration of shape. This type of technological change can also be shown simply by changing the scale of the axes to reflect increased efficiency of the inputs.

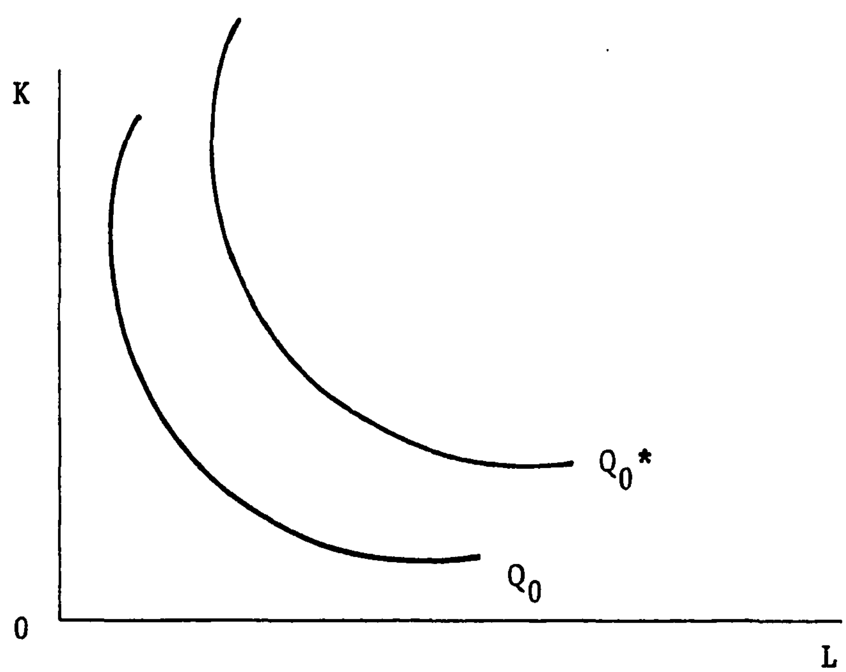

Figure 2. Hicks Neutral Technological Change

The Cobb-Douglas production function of homogeneity degree one can be shown to satisfy the conditions for all three types of neutrality. In general form with technological progress, it reads:

$$Q = \alpha(t)\ K^{\mu}L^{1-\mu}, \tag{2.39}$$

where $\alpha(t)$ is defined as the technological factor. Given a constant rate of technological growth m and $\theta = f(m)$, the above equation can then be rewritten as:

$$Q = e^{\theta t}K^{\mu}L^{1-\mu}. \tag{2.40}$$

Defining θ will designate the type of neutrality present.

In a Harrod neutral case, the following conditions are true:

$$Q = K^{\mu}(e^{mt}L)^{1-\mu}; \tag{2.41}$$

$$Q = K^{\mu}e^{mt-m\mu t}L^{1-\mu}; \tag{2.42}$$

$$Q = e^{m(1-\mu)t}\ \{K^{\mu}L^{1-\mu}\} \tag{2.43}$$

Hence, θ is defined as equalling $m(1-\mu)$.

For Solow neutrality, this can be shown:

$$Q = (e^{mt}K)^{\mu}L^{1-\mu}, \tag{2.44}$$

$$Q = e^{m\mu t}\{K^{\mu}L^{1-\mu}\}. \tag{2.45}$$

In this instance, θ equals $m\mu$.

Lastly, for Hicksian neutrality, it can be shown that θ simply equals m.

Clearly through proper algebraic manipulation, it can be shown that the Cobb-Douglas function reflects all three types of neutrality. The incorporation of a Cobb-Douglas function produces an easily solvable and explainable model. Consider the Neoclassical model incorporating disembodied Harrod neutral technological change. The technology would enter the system in the following equations:

$$Y_{FC} = F(K,\bar{L}) = AK^{\mu}\bar{L}^{1-\mu}, \tag{2.46}$$

$$\bar{L} = e^{mt}L, \tag{2.47}$$

$$L = L_o e^{nt}. \tag{2.48}$$

Here the production function is of Cobb-Douglas form with labor augmenting technological change. Labor is measured in efficiency units, $\bar{L}$. Equation (2.47) shows how a given labor force measured in natural units is augmented at m percent per annum. Considering both Equations (2.47) and (2.48), reflects that the labor force grows at a rate of (m+n) per year.

It can be shown that this refinement of the model will eventually lead to the following solution[5] based on the solution techniques used above:

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[5]Stronge, William B.; Jackson, Gary L.; A Neo-Classical Growth Model of the U. S. 1929-66.

$$\frac{d\bar{k}}{dt} = \frac{1}{L}\frac{dk}{dt} - (m+n)\ \bar{k}, \tag{2.49}$$

$$\frac{d\bar{k}}{dt} = sh(\bar{k}) - (m+n)\ \bar{k}. \tag{2.50}$$

The variable $\bar{k}$ is the capital labor ratio in efficiency units. Equation (2.50) is analagous to the Equation (2.27) above and can be shown to have the identical properties of existence, uniqueness, and stability.

One final remark should be made concerning the prerequisites for stability in terms of the type of technology present in this model. In order to insure stability in the model with disembodied technological change, it is necessary and sufficient that the progress be of Harrod neutrality. (Burmeister and Dobell, 1972, p.95). This condition assures that the capital output ratio remains unchanged; capital stock increases at the same rate as output. The fourth stylized fact mentioned above would be readily violated by either Solow or Hicksian neutrality. Such a problem is completely avoided by the use of a Cobb-Douglas function, since it can always be shown to be Harrod neutral.

CHAPTER III

TWO-SECTOR MODELS

The excessive rigidity of the Harrod-Domar models also can be surmounted by dividing the economy into two sectors, because in disequilibrium, changes in relative prices may induce a reallocation of resources between the sectors that will move the economy to the steady state. There have been many versions of the two-sector model discussed in the literature, which differ in assumptions and functional forms, yet most have certain common characteristics.

The output of the economy is assumed to be divided into two categories, investment or capital goods, and consumption goods. These are the two sectors of the economy. This classification is used to distribute the labor force and capital stock between the two sectors. There are respective sector production functions and demands for inputs.

In this chapter three two-sector growth models are discussed. The first, developed by Shinkai (1960), is a fixed coefficients version. It is included because it is the most direct generalization of Harrod's model and the instability of the latter can be avoided. Uzawa's 1963 model is covered next. It consists of a straight forward extension of the

Neoclassical model without technological change. The final section generalizes Uzawa's model to the case of Harrod-neutral technological change in both sectors. The technological progress must be assumed to occur at the same rate in both sectors. This assumption is necessary, given the state of knowledge about two-sector growth and, although counter intuitive, can be supported empirically.

The Shinkai Model.

This model was developed in 1960 and is characterized by the use of fixed coefficient production functions in the two sectors. Thus, the elasticity of substitution is zero.

The structural equation can be stated as:

$$Y_c = \frac{K_c}{v_c} = \frac{L_c}{u_c}, \tag{3.1}$$

$$Y_i = \frac{K_i}{v_i} = \frac{L_i}{u_i}, \tag{3.2}$$

$$Y = Y_c + pY_i, \tag{3.3}$$

$$K = K_c + K_i, \tag{3.4}$$

$$L = L_c + L_i, \tag{3.5}$$

$$w = \frac{1}{u_c}, \tag{3.6}$$

$$w = \frac{1}{u_i} p, \tag{3.7}$$

$$r = \frac{1}{v_c}, \tag{3.8}$$

$$r = \frac{1}{v_i} p, \tag{3.9}$$

$$Y_c = w(L_c + L_i). \tag{3.10}$$

The first two equations are the fixed proportional production functions for the two sectors, where K_i, L_i, K_c and L_c are the input levels for the respective investment (i), consumption (c) sectors. The letter v represents the fixed capital output ratio, and u represents the fixed labor output ratio. Equation (3.3) divides Y, national output measured in terms of consumer goods, into (Y_c) output produced by the consumption sector and (pY_i) output of the investment sector converted to units of consumption goods by the price ratio p--the ratio of the price of consumption goods to the price of capital goods. Equations (3.4) and (3.5) distribute the supply of labor and capital goods into the respective sectors. This reflects the assumption of full employment. Equations (3.6) through (3.9) are the marginal conditions for profit maximization in a perfectly competitive economy. The symbol w is the real wage rate and r represents the rental value of capital goods. The final equation reveals the source of consumption spending and shows that consumption equals the total value of wage income.

The result of the above equations and assumptions is that all prices are constant and determined by technology and that the capital labor ratios are also constant. The first result follows from u_c, v_c being constant in (3.6) and (3.8) revealing a fixed w, r and p considering equations (3.7) and (3.9). The fixed capital labor ratios can be demonstrated by manipulation of (3.1) and (3.2) to show:

$$\frac{K_c}{L_c} = \frac{v_c}{u_c} \tag{3.11}$$

and

$$\frac{K_i}{L_i} = \frac{v_i}{u_i}. \tag{3.12}$$

It should be pointed out that knowledge of the values of L_c and L_i is sufficient to solve the system in this static form. Given L_c and L_i, Equations (3.11) and (3.12) will determine K_c and K_i. The variables Y_c and Y_i will follow from (3.1) and (3.2), and the remaining variables are fixed.

The true path of the system is generated by the dynamic conditions:

$$\dot{L} = nL, \tag{3.13}$$

$$\dot{K} = Y_i, \tag{3.14}$$

where labor is defined to grow at a fixed rate, n percent per annum, and the addition to the capital stock is defined as the investment level.

Since the static solution of the model depends only on L_c and L_i, the strategy for generating the time path is to transform these dynamic conditions into a pair of differential equations involving only the variables L_c and L_i. Equation (3.4) can be rewritten as:

$$K = \frac{K_c L_c}{L_c} + \frac{K_i L_i}{L_i} = \frac{v_c L_c}{u_c} + \frac{v_i L_i}{u_i}. \tag{3.15}$$

Differentiation with respect to time yields:

$$\dot{K} = \frac{v_c \dot{L}_c}{u_c} + \frac{v_i \dot{L}_i}{u_i} = Y_i, \tag{3.16}$$

where (3.14) is used to write the last equality. From the production function (3.2), is obtained the first differential equation involving L_c and L_i:

$$\frac{L_i}{u_i} = \frac{v_c \dot{L}_c}{u_c} + \frac{v_i \dot{L}_i}{u_i} . \tag{3.17}$$

To obtain the second differential equation, differentiate (3.13) to obtain:

$$\dot{L} = \dot{L}_c + \dot{L}_i. \tag{3.18}$$

Substituting (3.5) into (3.13) shows:

$$\dot{L} = n(L_c + L_i). \tag{3.19}$$

Then, equating the two equations yields:

$$n(L_c + L_i) = \dot{L}_c + \dot{L}_i, \qquad (3.20)$$

which is the second differential equation. The two equations (3.17) and (3.20) together form the system:

$$\begin{bmatrix} k_c & k_i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{L}_c \\ \dot{L}_i \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_i}{v_i} \\ n & n \end{bmatrix} \begin{bmatrix} L_c \\ L_i \end{bmatrix}, \qquad (3.21)$$

where we have defined:

$$k_c = \frac{K_c}{L_c} = \frac{v_c}{u_c}, \qquad (3.22)$$

$$k_i = \frac{K_i}{L_i} = \frac{v_i}{u_i}. \qquad (3.23)$$

The solution for this system is shown in appendix A to be:

$$L_c(t) = m_{11} \exp(nt) + m_{12} \left(\frac{k_i/v_i}{k_c - k_i}\right), \qquad (3.24)$$

$$L_i(t) = m_{21} \exp(nt) + m_{22} \left(\frac{-k_i/v_i}{k_c - k_i}\right), \qquad (3.25)$$

where the m_{ij} are determined by the initial conditions $L_c(0)$ + $L_i(0)$.

Stability can be shown to exist if the second root is always negative. Since k_i, v_i, and k_c are known to be positive, the above ratio will be less than zero as long as k_c exceeds k_i. The necessary and sufficient condition then will be a greater capital labor ratio in the consumption sector then in the investment sector.[6] Thus, as time approaches infinity, the effect of the second root becomes negligible. Notice:

$$\text{Limit } \{\exp\left(\frac{-k_i/v_i}{k_c - k_i}\right)\} = 0, \qquad (3.26)$$

as time approaches infinity. This leaves the natural rate of growth as a limit value.

The first root n being positive reveals that labor grows at this natural rate n. It can be demonstrated with proper substitution that the rest of the variables in the system will also grow at this natural rate; hence, a balanced rate of growth is assured.

Clearly, the model contains the flexibility to ensure stability even with the use of a fixed coefficient production function as was used in Domar's model. Nevertheless, the stability condition has been regarded as unsatisfactory because it is counter intuitive. Although the empirical evidence suggests that k_c is greater than k_i, its necessity for

[6]This is referred to as the capital intensity theorem and is valid in more general two-sector models.

stability no longer holds in general models, such as Uzawa's which will now be discussed.

Uzawa's Two-Sector Model.

The Uzawa model defines the two sectors as above but is a Neoclassical version of the Shinkai model. The equations are as follows:

$$Y_c = F_c(K_c, L_c), \tag{3.27}$$

$$Y_i = F_i(K_i, L_i), \tag{3.28}$$

$$Y = Y_c + pY_i, \tag{3.29}$$

$$K = K_i + K_c, \tag{3.30}$$

$$L = L_i + L_c, \tag{3.31}$$

$$w = \frac{\partial F_c}{\partial L_c}, \tag{3.32}$$

$$w = \frac{\partial F_i}{\partial L_i} p, \tag{3.33}$$

$$r = \frac{\partial F_c}{\partial K_c}, \tag{3.34}$$

$$r = \frac{\partial F_i}{\partial K_i} p, \tag{3.35}$$

$$pY_i = sY. \tag{3.36}$$

The first two equations are the sector production functions assumed to reflect short run diminishing returns with constant returns to scale, homogeneity, and the Inada conditions. Equation (3.29) reflects full employment of factors in the economy. Equations (3.30) and (3.31) divide up the available inputs. Then (3.32) through (3.35) reflect the demand for inputs meeting the marginal product condition. Lastly, Equation (3.36) is the savings investment equilibrium condition.

The use of intensive variables similar to Solow's enhances solution and analysis. Consider the following identities: $k = K/L$; $y = Y/L$; $k_c = K_c/L_c$; $k_i = K_i/L_i$; $l_c = L_c/L$; $l_i = L_i/L$; $\omega = w/r$; $y_c = Y_c/L$; and $y_i = Y_i/L$. Then Equations (3.27) through (3.36) can be rewritten as:

$$y_c = f_c(k_c)l_c, \tag{3.37}$$

$$y_i = f_i(k_i)l_i, \tag{3.38}$$

$$y = y_c + py_i, \tag{3.39}$$

$$k = k_i l_i + k_c l_c, \tag{3.40}$$

$$1 = l_i + l_c, \tag{3.41}$$

$$w = pf_c(k_c) - f_c'(k_c)k_c, \tag{3.42}$$

$$r = f_c'(k_c), \tag{3.43}$$

$$w = pf_i(k_i) - pf_i'(k_i)k_i, \tag{3.44}$$

$$r = pf_i'(k_i), \tag{3.45}$$

$$py_i = sy. \tag{3.46}$$

Equation (3.37) is derived from the following:

$$\frac{Y_c}{L} = F_c(K_c,L_c)\frac{1}{L} = F_c(K_c/L_c,1)L_c/L \tag{3.47}$$

$$= f(k_c)l_c.$$

Multiplying both sides of this equation by L reveals:

$$Y_c = f(k_c)L_c. \tag{3.48}$$

The marginal product of labor in the consumption sector is:

$$\frac{\partial Y_c}{\partial L_c} = f(k_c) + L_c f'(k_c)\frac{\partial k_c}{\partial L_c}. \tag{3.49}$$

From the third identity above, it can be shown that:

$$\frac{\partial k_c}{\partial L_c} = \frac{-K_c}{L^2_c} = \frac{-k_c}{L_c} . \tag{3.50}$$

Substituting this result into Equation (3.49) provides Equation (3.42). Equation (3.44) follows analagously. The remaining equations represent simple substitution. It is also useful to introduce the wage rental ratio $\omega = w/r$. The four Equations (3.42) to (3.45) can be replaced by:

$$\omega = \frac{f_c(k_c)}{f'_c(k_c)} - k_c; \qquad (3.51)$$

dividing (3.42) by (3.43):

$$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i; \qquad (3.52)$$

follows analagously. Equations (3.43) and (3.45) reveal:

$$p = \frac{f'_c(k_c)}{f'_i(k_i)}. \qquad (3.53)$$

This relation shows that the value of the marginal product of capital is the same in both sectors.

The system has so far been reduced to the following nine equations:

$$y_c = f_c(k_c)l_c, \qquad (3.54)$$

$$y_i = f_i(k_i)l_i, \qquad (3.55)$$

$$y = y_c + py_i, \qquad (3.56)$$

$$k = k_c l_c + k_i l_i, \tag{3.57}$$

$$l = l_i + l_c, \tag{3.58}$$

$$\omega = \frac{f_i(k_i)}{k_i'(k_i)} - k_i, \tag{3.59}$$

$$\omega = \frac{f_c(k_c)}{f'_c(k_c)} - k_c, \tag{3.60}$$

$$p = \frac{f'_c(k_c)}{f'_i(k_i)}, \tag{3.61}$$

$$py_i = sy. \tag{3.62}$$

There are nine endogenous variables y_c, k_c, l_c, y_i, k_i, l_i, p, ω, y and one exogenous variable k. Thus, the reduced form equations, if they exist, express each endogenous variable as a function of k alone. The key reduced form equation, however, is the equation for ω, because once ω is known all other endogenous variables can be obtained. The reduction of the nine equations to:

$$k + \omega = \frac{(k_i + \omega)(k_c + \omega)}{s(k_c + \omega) + (1 - s)(K_i + \omega)}, \tag{3.63}$$

is shown in appendix B to this chapter. The Inada conditions guarantee that k_i and k_c are unique functions of ω, so that this equation expresses k as a function of ω:

$$k = k(\omega). \tag{3.64}$$

The existence of a solution to the model, therefore, hinges on whether this function can be inverted so that the reduced form equation for ω can be obtained. The proof that the Inada conditions guarantee a solution is to be found in Gandolfo (1971).

The dynamic conditions of the model are established with the addition of the following equations:

$$\dot{K} = Y_i, \tag{3.65}$$

$$\dot{L} = L_o e^{nt}. \tag{3.66}$$

The first equation is the change in the capital stock attributed to production in the investment sector. The second shows that labor grows at a natural rate n per annum. Changing to intensive variables provides a more usable form.

From the intensive equations above:

$$K = kL, \; Y_i = y_i L. \tag{3.67}$$

Differentiating reveals:

$$\dot{K} = k\dot{L} + L\dot{k}. \tag{3.68}$$

Then Equation (3.65) becomes:

$$k\dot{L} + L\dot{k} = Y_i, \tag{3.69}$$

or

$$k\dot{L} + L\dot{k} = y_i L, \tag{3.70}$$

$$k = y_i - \frac{k\dot{L}}{L}, \tag{3.71}$$

$$k = y_i - nk, \tag{3.72}$$

since $\dot{L}/L = n$ from definition. From above it was shown that y_i and k are known functions of ω, so this yields:

$$\dot{k} = h(\omega) - nk(\omega). \tag{3.73}$$

Clearly (3.73) resembles the dynamic differential equation discussed in Solow's model. Uzawa points out the stability and uniqueness of the solution in his work.

A more general set of five sufficient conditions for stability are stated in Burmeister and Dobel's chapter on the model. The first is that the marginal propensity to save be the same in both sectors. The second is that the savings rate out of profit income be larger than that out of wage income and the consumption sector be more capital intensive. The third is that profit income should have a smaller savings propensity and the investment sector have greater capital intensity. The fourth is that the capital labor ratio be the same in both sectors and lastly that the sum of the elasticities of substitution be greater than or equal to one. A two-sector model need only satisfy one of these five conditions to establish stability, and obviously the two models mentioned so far do this. The savings rate out of wage and profit income are the same, and the use of Cobb-Douglas functions ensures that the sum of the elasticities equals two (greater than one).

Two-Sector Model with Technological Change.

In this discussion, the production functions will be assumed to be of Cobb-Douglas form with disembodied Harrod-Neutral technological change. It will also be assumed that this change occurs at the same rate in both sectors.

The altered equations of the model will then be:

$$Q_c = A_c K_c^{\alpha_c} (\bar{L})_c^{1-\alpha_c} = A_c K_c^{\alpha_c} (e^{mt} L_c)^{1-\alpha_c}, \tag{3.74}$$

$$Q_i = A_i K_i^{\alpha_i} (\bar{L})_i^{1-\alpha_i} = A_i K_i^{\alpha_i} (e^{mt} L_i)^{1-\alpha_i}, \tag{3.75}$$

$$\bar{L} = \bar{L}_c + \bar{L}_i = e^{mt} L_c + e^{mt} L_i = e^{mt} L. \tag{3.76}$$

The variable Q represents output or income Y as above. The variable $\bar{L}$ measures labor in efficiency units.

Again, intensive variables can be noted:

$$\bar{q}_c = Q_c/\bar{L},\ \bar{q}_i = Q_i/\bar{L},\ \bar{k}_c = K_c/\bar{L}_c, \tag{3.77}$$

$$\bar{k}_i = K_i/\bar{L}_i,\ \bar{k} = K/\bar{L},\ \bar{y} = Y/\bar{L},$$

$$\bar{l}_c = \bar{L}_c/\bar{L} = l_c,\ \bar{l}_i = \bar{L}_i/\bar{L} = l_i.$$

With appropriate substitution and reduction, the model then becomes:

$$\bar{q}_c = A_c(\bar{k}_c^{\alpha_c})\bar{l}_c, \tag{3.78}$$

$$\bar{q}_i = A_i(\bar{k}_i^{\alpha_i})\bar{l}_i, \tag{3.79}$$

$$\bar{y} = \bar{q}_c + p\bar{q}_i, \tag{3.80}$$

$$\bar{k} = \bar{k}_c\bar{l}_c + \bar{k}_i\bar{l}_i, \tag{3.81}$$

$$1 = \bar{l}_c + \bar{l}_i, \tag{3.82}$$

$$r = \frac{\alpha_c\bar{q}_c}{\bar{k}_c\bar{l}_c} = \frac{\alpha_i\bar{q}_i}{\bar{k}_i\bar{l}_i} \cdot p, \tag{3.83}$$

$$w = \frac{(1-\alpha_c)Q_c}{\bar{L}_c} = \frac{(1-\alpha_i)Q_i}{\bar{L}_i} \cdot p, \tag{3.84}$$

$$p\bar{q}_i = s\bar{y}. \tag{3.85}$$

It so happens this system can be solved in a manner similar to the solution of the previous model to produce the following result (Benos, 1971):

$$\bar{\omega} = \frac{(1-\alpha_c) + s(\alpha_c-\alpha_i)\ \bar{k}}{\alpha_c + s(\alpha_i-\alpha_c)}\ . \tag{3.86}$$

This is clearly the same result as was obtained in the solution of Uzawa's model above, except now it is expressed in terms of efficiency variables. Note that:

$$\bar{\omega} = \omega e^{-m}, \ \bar{k} = ke^{-m}, \tag{3.87}$$

so that

$$\bar{\omega}/\bar{k} = \omega/k. \tag{3.88}$$

Consider now the dynamic condition:

$$\frac{dK}{dt} = Q_i. \tag{3.89}$$

In intensive efficiency units this becomes:

$$\frac{1}{\bar{L}}\frac{dK}{dt} = \bar{q}_i, \tag{3.90}$$

thus,

$$\frac{1}{L}\frac{dK}{dt} = A_i\bar{l}_i\bar{k}^{\alpha_i}. \tag{3.91}$$

The variable $\bar{k}_i$ is obtained from the solution of the model above. It equals:

$$\bar{k}_i = \frac{\alpha_i\omega}{1-\alpha_i}. \tag{3.92}$$

Substitution reveals:

$$\bar{k}_i = \frac{\alpha_i}{1-\alpha_i} \{\frac{s(\alpha_c-\alpha_i) + (1-\alpha_c)}{s(\alpha_i-\alpha_c) + \alpha_c}\} \bar{k}. \tag{3.93}$$

Then,

$$\frac{1}{L}\frac{dK}{dt} = A_i T_i \frac{\alpha_i}{1-\alpha_i} \{\frac{s(\alpha_c-\alpha_i) + 1-\alpha_c}{s(\alpha_i-\alpha_c) + \alpha_c}\}^{\alpha_i}.(\bar{k})^{\alpha_i} \tag{3.94}$$

$$= B(\bar{k})^{\alpha_i},$$

where,

$$B = \frac{A_i\alpha_i}{1-\alpha_i} \{\frac{s(\alpha_c-\alpha_i) + 1-\alpha_c}{s(\alpha_i-\alpha_c) + \alpha_c}\}^{\alpha_i}. \tag{3.95}$$

A remark should be made that this constant B has the same value as the case without technological change (Benos, 1971). Note that:

$$\frac{d\bar{k}}{dt} = \frac{1}{L}\frac{dK}{dt} - k\frac{1}{L}\frac{d\bar{L}}{dt} = \frac{1}{L}\frac{dK}{dt} - \bar{k}(m+n), \tag{3.96}$$

thus,

$$\frac{d\bar{k}}{dt} = B\bar{k}^{\alpha_i} - (m+n)\ \bar{k}. \tag{3.97}$$

The equilibrium capital labor ratio can be shown to be:

$$\bar{k}_E = (\frac{B}{m+n})^{1/1-\alpha_i}, \tag{3.98}$$

with solution being:

$$\bar{k}(t) = \{e^{-(1-\alpha_i)(m+n)t}(\bar{k}^{(1-\alpha_i)}(0)-\bar{k}_E^{(1-\alpha_i)})+\bar{k}_E^{(1-\alpha_i)}\}^{1/1-\alpha_i}. \tag{3.99}$$

Assuming that α_i is between zero and one, and that m+n is greater than zero, it follows that:

$$\lim \bar{k}(t) = \bar{k}_E, \tag{3.100}$$

as time approaches infinity; hence, the model is stable.

It can be readily demonstrated that all variables grow at the same m+n percent per annum. The capital output ratio is also s/v; therefore, the warranted rate equals the natural rate of growth. Finally, it should be pointed out that the price ratio provides the added flexibility to assure the above equality. Remarks about policy implications of this model will be made in the next chapter.

CHAPTER IV

ESTIMATION AND SIMULATION OF A TWO-SECTOR MODEL

Two-Sector Data.

In the estimation of the two production functions of the model, it becomes necessary to first aggregate national income--product values then disaggregate them into the two sectors. It is quite obvious that this process will create a certain number of problems both conceptual and practical in nature. These will be overcome by hopefully feasible assumptions made along the way.

The disaggregation involves splitting up of the capital stock, labor force, and total output or income into the consumption and capital sectors. Benos (1972) suggests using the input-output tables of the National Accounts to determine the distribution of output into consumption good industries and capital good industries. The output of each industry is sold to four categories of buyers: households, business, government and foreigners:

$$Y_j = C_j + I_j + G_j + X_j, \tag{4.1}$$

for industry j. Benos initially considered government expenditures to be consumption in nature.[7] Exports were distributed between the two sectors using the following two percentages: (C+G)/(C+I+G), (I)/(C+I+G). Thus, the output was allocated as follows: $(C+G+X_c)/(C+I+G+X)$, $(I+X_i)/(C+I+G+X)$. These percentages were aggregated over industry categories to obtain percentages for overall industry divisions. See Table 1. The latter percentages formed the basis for the distribution of factor income, output and the capital stock.

Industry income is decomposed into labor and capital factor payments. Using the appropriate percentages derived above:

$$Y_{jl} = Y_{jlc} + Y_{jlk}, \tag{4.2}$$

where Y_{jlc} is the labor income derived from the production of consumption goods and Y_{jlk} is the labor income derived from the production of capital goods in industry j. Similarly,

$$Y_{jk} = Y_{jkc} + Y_{jkk}, \tag{4.3}$$

where Y_{jkc} is the capital income derived from the production of capital goods in industry j, and so on.

Aggregating (4.2) and (4.3) by industry yields:

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[7] In his appendix, Benos considered G to be investment. Empirical results were similar.

$$Y_l = \Sigma Y_{jl} = \Sigma Y_{jlc} + \Sigma Y_{jlk}; \tag{4.4}$$

$$Y_k = \Sigma Y_{jk} = \Sigma Y_{jkc} + \Sigma Y_{jkk}. \tag{4.5}$$

Here Y_l is labor income in the economy and Y_k is the income of capital. Aggregating (4.2) and (4.3) this time by type of income reveals:

$$Y_c^* = \Sigma Y_{jc} = \Sigma(Y_{jlc} + Y_{jlk}); \tag{4.6}$$

$$Y_k^* = \Sigma Y_{jk} = \Sigma(Y_{jlk} + Y_{jkk}). \tag{4.7}$$

Here Y_c^* is the income of the national consumption sector and Y_k^* is the income of the national capital sector. The basic data on national income by industry is available in the Survey of Current Business.

The labor force data is available in the Bureau of Labor Statistics listings and is listed in the categories as mentioned above. Once more the total is segmented by using the appropriate percentages. The final figures reflect full-time equivalent employees having adjusted part-time labor force.

Capital stock is also sectored using the percentage figures above; however, the sources of the stock are numerous (Benos, 1972).

The final product of this data manipulation is found in Tables 2, 3 and 4.

Estimation of the Model.

The production functions were estimated using ordinary least squares with the Cobb-Douglas assumptions along with disembodied technological progress.[8] The production functions for the consumption and investment sectors are:

$$(Y_c)_t = 1.73\, e^{0.0153t}(K_t)^{0.3132}(L_t)^{0.6868}; \qquad (4.8)$$

$$(Y_i)_t = 1.75\, e^{0.0142t}(K_t)^{0.3106}(L_t)^{0.6894}; \qquad (4.9)$$

for Equation (4.8):

Standard error of technological coefficient equals .0016;

Standard error of capital coefficient equals .1023;

R^2 equals .9854; and

DW equals 2.14.

Upon first glance at the capital elasticities, it appears that the capital intensity theorem is fulfilled. However, when the standard error of the consumption sector

[8]1954 data was eliminated to avoid inconclusive results.

capital coefficient is considered, it follows that the capital elasticity of the investment sector falls within the standard deviation interval (.3106 $\pm$ .0123). This would imply that these results do not significantly validate the two-sector disaggregation of data.

With the assumption of a rate of growth of labor of .015, a calculated savings rate of .10, and a depreciation rate of .05, the model then becomes:

$$(Y_c)_t = 1.73\ e^{0.0153}(K_c)_t^{0.3132}(L_c)_t^{0.6868}, \quad (4.10)$$

$$(Y_i)_t = 1.75\ e^{0.0142}(K_i)_t^{0.3106}(L_i)_t^{0.6894}, \quad (4.11)$$

$$K_t = (K_i)_t + (K_c)_t, \quad (4.12)$$

$$W_t = 1.871\ e^{0.0153}(K_c/L_c)_t^{\ 0.3132}, \quad (4.13)$$

$$W_t = 1.3865\ e^{0.0142}(K_i/L_i)_t^{\ 0.3106}, \quad (4.14)$$

$$r_t = .5418\ e^{0.0153}(K_c/L_c)_t^{\ 0.3132}, \quad (4.15)$$

$$r_t = .5436\ e^{0.0142}(K_i/L_i)_t^{\ 0.3106}, \quad (4.16)$$

$$D_t = .05\ K_t, \quad (4.17)$$

$$(Y_i)_t = .10\ Y_t, \quad (4.18)$$

$$\dot{K} = (Y_i)_t, \tag{4.19}$$

$$L_t = L_0 e^{.015t}. \tag{4.20}$$

A Basic program was used to simulate the model through time to point out the growth path with 1947 data initial values. The results of this process are summarized in Table 5. A second similar simulation was performed with a savings rate of .15 instead, and summarized in Table 6.

A few remarks should be made about this simulation. It appears that from Table 5 that it took approximately sixty years to reach a capital labor ratio within 1.2 percent of the equilibrium level, 5.271 with a growth rate of GNP about 3.714. This adjustment time period was appreciably less than that in the one-sector case which took over two hundred years in simulation (Stronge, Jackson, 1972). Again, this can be attributed to the price mechanism of the model. Note that the initial capital labor ratio lies above the equilibrium level and the adjustment follows a decreasing price ratio and a decreasing wage rental ratio. Hence, there is a slight re-allocation of resources from the production of capital to that of consumption goods as the system pursues the equilibrium capital labor ratio. It also appears that the model initially was only six percent above the equilibrium level of the capital labor ratio. This may be considered to be an advantage of the model or perhaps a consequence of the data manipulation.

Adjustment of the savings rate by an increase of fifty percent did not greatly alter the length of the period of adjustment to equilibrium, nor the rate of growth of output 3.714 in equilibrium. The equilibrium capital labor ratio in efficiency units did increase to 5.491. In this case, the initial capital labor ratio determined in 1948 was below the equilibrium level. The result was a rising price ratio and wage rental ratio re-allocating resources to production of more capital goods to increase $K/\overline{L}$. The most impressive reflection of the second simulation is that during periods of disequilibrium the rate of growth of GNP was much higher at this higher savings rate. In addition, there was a greater change in the growth rate of GNP in the same time period.

The effect of the increase in the savings rate is clearly to increase the output of the investment sector. This will in turn increase the capital labor ratio in both sectors. See Table 7. This increases the wage and rental rates w, r and the ratio ω. The variable B as defined in Appendix B increases and thus the equilibrium capital labor ratio. Clearly, the equilibrium rate of growth of output should not change with an alteration of s, from the fact that it always equals the value of $(m + n)$.

The value of the savings rate as a policy tool for government seems to be relevant only in periods of disequilibrium. However, if it is assumed that the economy is constantly in a state of disequilibrium, alteration of s is quite effective. Perhaps it could take the form of tax incentives for corporate investment.

TABLE 1

COMPOSITION OF EACH MAJOR INDUSTRY DIVISION IN CONSUMPTION AND CAPITAL GOODS PRODUCING SECTORS

	Industry Division	Consumption Industries %	Investment Industries %
(1)	Agriculture, forestry and fisheries	93.5	6.5
(2)	Mining	83.5	16.5
(3)	Construction	46.9	53.1
(4)	Manufacturing	80.7	19.3
(5)	Transportation, communication, electric, gas, sanitary services	85.3	14.7
(6)	Wholesale and retail trade	88.6	11.4
(7)	Finance insurance and real estate	94.4	5.6
(8)	Services	93.2	6.8
(9)	Government enterprise	91.1	8.9

Source: An Empirical Two-Sector Model for the United States Economy and the Effects of Alternative Taxation Policies, Theophanis E. Benos, PH.D., p. 43.

TABLE 2

INCOME IN THE CONSUMPTION AND CAPITAL SECTORS 1947-1965
(Data in billions of 1954 dollars)

Year	Consumption Sector	Capital Sector
	ΣY_c	ΣY_k
1947	187.21	34.02
1948	198.06	36.32
1949	189.88	34.75
1950	209.17	38.78
1951	219.62	41.85
1952	223.85	42.25
1953	230.45	43.60
1954	227.02	42.27
1955	246.83	46.13
1956	252.52	47.28
1957	253.00	47.05
2958	248.13	45.00
1959	268.55	49.24
1960	273.69	49.49
1961	275.86	49.68
1962	292.82	52.88
1963	305.38	55.06
1964	321.47	58.18
1965	343.08	62.81

Source: See Table 1, p. 65.

TABLE 3

NUMBER OF FULL-TIME EQUIVALENT EMPLOYEES
BY SECTOR, 1947-1965
(Data in Millions)

Year	Consumption Industry	Capital Industry	The Whole Economy
1947	45.459	8.027	53.486
1948	46.167	8.280	54.447
1949	44.839	7.934	52.773
1950	46.146	8.288	54.434
1951	47.878	8.763	56.641
1952	48.379	8.844	57.223
1953	49.223	9.008	58.231
1954	47.937	8.649	56.586
1955	49.186	8.888	58.074
1956	50.291	9.120	59.411
1957	50.388	9.083	59.471
1958	49.068	8.703	57.771
1959	50.151	8.961	59.112
1960	50.805	9.020	59.825
1961	50.631	8.909	59.540
1962	51.580	9.099	60.679
1963	52.132	9.209	61.341
1964	53.135	9.411	62.546
1965	54.682	9.736	64.418

Source: See Table 1, p. 75.

TABLE 4

FIXED CAPITAL UTILIZATION BY SECTORS

Year	Consumption Sector	Capital Sector
1947	264.28	35.30
1948	271.99	37.37
1949	255.39	35.28
1950	292.53	39.64
1951	304.70	41.41
1952	317.77	43.69
1953	328.59	45.38
1954	336.12	46.32
1955	336.84	46.35
1956	324.29	44.84
1957	323.43	45.06
1958	329.37	45.63
1959	357.79	49.50
1960	359.79	49.58
1961	361.68	49.96
1962	371.30	50.93

[a]Data in billions of 1954 dollars.

[b]Source: See Table 1, p. 112.

TABLE 5

SIMULATION OF TWO-SECTOR MODEL
(Savings rate = .10)

Year	$K/\bar{L}$	% Growth of GNP	p	ω
1947	5.601	-	.86266	12.297
1948	5.582	3.602	.86265	12.255
1949	5.564	3.608	.86264	12.215
1950	5.547	3.614	.86263	12.176
1951	5.531	3.620	.86262	12.142
1952	5.515	3.625	.86262	12.109
1953	5.501	3.630	.86262	12.078
1954	5.488	3.635	.86261	12.048
1955	5.475	3.639	.86261	12.021
1956	5.463	3.643	.86260	11.995
1957	5.452	3.648	.86260	11.970
1967	5.370	3.677	.86256	11.791
1977	5.326	3.694	.86254	11.693
1987	5.301	3.703	.86253	11.639
1997	5.288	3.708	.86253	11.609
2007	5.280	3.710	.86252	11.593
2017	5.276	3.712	.86252	11.584
2027	5.274	3.713	.86252	11.579
2037	5.273	3.714	.86252	11.577
2047	5.272	3.714	.86252	11.575
2057	5.272	3.714	.86252	11.574
2067	5.272	3.714	.86252	11.574
2077	5.271	3.714	.86252	11.574
2087	5.271	3.714	.86252	11.573
2097	5.271	3.714	.86252	11.573

TABLE 6

SIMULATION OF TWO-SECTOR MODEL
(Savings rate = .15)

Year	K/L̄	% Growth GNP	p	ω
1947	5.601	-	.86266	12.305
1948	5.807	4.896	.86274	12.758
1949	6.004	4.802	.86281	13.190
1950	6.191	4.717	.86288	13.602
1951	6.370	4.640	.86295	13.993
1952	6.539	4.570	.86300	14.365
1953	6.700	4.507	.86306	14.719
1954	6.853	4.450	.86311	15.054
1955	6.998	4.396	.86316	15.323
1956	7.135	4.348	.86320	15.675
1957	7.266	4.304	.86324	15.961
1967	8.241	4.011	.86352	18.105
1977	8.796	3.870	.86367	19.323
1987	9.106	3.798	.86375	20.005
1997	9.279	3.760	.86379	20.384
2007	9.374	3.739	.86381	20.594
2017	9.427	3.728	.86383	20.709
2027	9.456	3.722	.86383	20.773
2037	9.472	3.718	.86384	20.808
2047	9.481	3.716	.86384	20.828
2057	9.486	3.715	.86384	20.838
2067	9.488	3.715	.86384	20.844
2077	9.490	3.714	.86384	20.847
2087	9.490	3.714	.86384	20.849
2097	9.491	3.714	.86384	20.850

TABLE 7

COMPARISON OF VALUES OF SELECTED VARIABLES IN THE TWO SIMULATIONS

Year	S = .10			S = .15		
	$(K/\overline{L})_c$	$(K/\overline{L})_i$	ω	$(K/\overline{L})_c$	$(K/\overline{L})_i$	ω
1948	5.589	5.521	12.255	5.818	5.748	12.758
1949	5.570	5.503	12.215	6.015	5.943	13.190
1950	5.553	5.486	12.178	6.203	6.128	13.602
1951	5.537	5.471	12.142	6.381	6.304	13.993
1952	5.522	5.456	12.109	6.551	6.472	14.365
1953	5.508	5.441	12.078	6.712	6.631	14.719
1954	5.494	5.428	12.048	6.865	6.782	15.054
1955	5.482	5.416	12.020	7.010	6.926	15.373
1956	5.470	5.404	11.995	7.148	7.062	15.675
1957	5.459	5.393	11.970	7.279	7.191	15.961

CHAPTER V

CONCLUSION AND SUGGESTIONS

A two-sector Neoclassical model of the U. S. economy was estimated and simulated pointing out stability, and the role of prices. The results of the estimation of the production functions do not seem to support the significance of the disaggregation of the data. This might be attributed to the actual process used to allocate the data into the two sectors. The clear advantage of the two-sector assumption, however, lies in the efficient disequilibrium adjustment mechanism.

It might be beneficial to extend this study using more recent data and perhaps to recalculate the percentage table used in Chapter III. This might produce better results for the production functions. Furthermore, the wage rental ratio might be investigated as a more viable policy tool.

APPENDIX A

SOLUTION OF THE SHINKAI TWO-SECTOR MODEL

Given the following system:

$$\begin{bmatrix} k_c & k_i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{L}_c \\ \dot{L}_i \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_i}{v_i} \\ n & n \end{bmatrix} \begin{bmatrix} L_c \\ L_i \end{bmatrix} . \tag{1}$$

The coefficient matrix on the left can be inverted to yield:

$$\begin{bmatrix} \frac{1}{k_c - k_i} & \frac{-k_i}{k_c - k_i} \\ \frac{-1}{k_c - k_i} & \frac{k_i}{k_c - k_i} \end{bmatrix}, \tag{2}$$

which when multiplied by both sides of (1) reveals:

$$\begin{bmatrix} \dot{L}_c \\ \dot{L}_i \end{bmatrix} \quad \begin{bmatrix} \dfrac{-nk_i}{k_c-k_i} & \dfrac{k_i/v_i-nk_i}{k_c-k_i} \\ \dfrac{nk_c}{k_c-k_i} & \dfrac{nk_c-k_i/v_i}{k_c-k_i} \end{bmatrix} \begin{bmatrix} L_c \\ L_i \end{bmatrix} . \tag{3}$$

Working towards finding the characteristic roots provides:

$$\begin{bmatrix} \lambda + \dfrac{nk_i}{k_c-k_i} & \dfrac{nk_i-k_i/v_i}{k_c-k_i} \\ \dfrac{-nk_c}{k_c-k_i} & \lambda + \dfrac{k_i/v_i-nk_c}{k_c-k_i} \end{bmatrix} . \tag{4}$$

Appropriate manipulation will determine the characteristic polynomial:

$$\lambda^2 + \left(\frac{k_i/v_i}{k_c-k_i} - n\right) \lambda - \frac{nk_i/v_i}{k_c-k_i} = 0. \tag{5}$$

Using the quadratic equation for solution reveals that the roots are:

$$\frac{\dfrac{-k_i/v_i}{k_c-k_i} + n + \dfrac{k_i/v_i}{k_c-k_i} + n}{2} = n, \tag{6}$$

and,

$$\frac{\frac{-k_i/v_i}{k_c-k_i} + n - \frac{k_i/v_i}{k_c-k_i} - n}{2} = -\frac{k_i/v_i}{k_c-k_i} . \tag{7}$$

The solution takes the form:

$$L_c(t) = m_{11}\exp(nt) + m_{12}\ (\frac{-k_i/v_i}{k_c-k_i}), \tag{8}$$

$$L_i(t) = m_{21}\exp(nt) + m_{22}\ (\frac{-k_i/v_i}{k_c-k_i}). \tag{9}$$

APPENDIX B

SOLUTION OF THE UZAWA TWO-SECTOR MODEL

The variable l_i can be eliminated using Equation (3.41) and (3.55) to produce:

$$y_i = f_i(k_i)(1-l_c). \tag{1}$$

Then solving for l_c in (3.57) provides:

$$l_c = \frac{k-k_i}{k_c-k_i}, \tag{2}$$

implying:

$$1-l_c = \frac{k_c-k}{k_c-k_i}. \tag{3}$$

Thus, the system (3.54) to (3.62) becomes:

$$y_c = f_c(k_c)\,\frac{k-k_i}{k_c-k_i}, \tag{4}$$

$$y_i = f_i(k_i)\,\frac{k_c-k}{k_c-k_i}, \tag{5}$$

$$y = y_c + py_i, \tag{6}$$

$$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i, \tag{7}$$

$$\omega = \frac{f_c(k_c)}{f_c'(k_c)} - k_c, \tag{8}$$

$$p = \frac{f_c'(k_c)}{f_i'(k_i)}, \tag{9}$$

$$py_i = sy. \tag{10}$$

These are the seven equations from which Uzawa begins his reduction procedure (Uzawa, 1963, p. 108). Substituting Equation (6) into (10) eliminates y from the system. This produces:

$$(1-s)\ py_i = sy_c, \tag{11}$$

implying:

$$(1-s)\ p\frac{y_i}{y_c} = s. \tag{12}$$

Uzawa then uses Equations (4), (5), and (7) through (9) to produce an expression for $(1-s)\ p\frac{y_i}{y_c}$ in terms of ω and the exogenous capital labor ratio k.

Dividing (5) by (4) yields:

$$\frac{y_i}{y_c} = \frac{f_i(k_i)}{f_c(k_c)}\ (\frac{k_c-k}{k-k_i}). \tag{13}$$

Hence:

$$(1\text{-}s)\ p\frac{y_i}{y_c} = (1\text{-}s)\ \frac{f_c'(k_c)}{f_i'(k_i)} \cdot \frac{f_i(k_i)}{f_c(k_c)} \cdot \frac{k_c - k}{k - k_i} \qquad (14)$$

$$= (1\text{-}s)\ \{\frac{f_c'(k_c)}{f_c(k_c)}\}\ \{\frac{f_i(k_i)}{f_i'(k_i)}\}\ \frac{k_c - k}{k - k_i},$$

where (9) was substituted. This can be rewritten as:

$$(1\text{-}s)\ p\frac{y_i}{y_c} = (1\text{-}s)\ \frac{(k_i+\omega)}{(k_c+\omega)} \cdot \frac{(k_c - k)}{k - k_i}. \qquad (15)$$

Adding and subtracting ω from the numerator and denominator of the final fraction yields:

$$(1\text{-}s)\ p\frac{y_i}{y_c} = (1\text{-}s)\frac{(k_i+\omega)}{(k_c+\omega)} \cdot \frac{(k_c+\omega)\ -\ (k+\omega)}{(k+\omega)\ -\ (k_i+\omega)}. \qquad (16)$$

Equation (16) is substituted into (12) and then multiplied on both sides by $(k_c+\omega)\ \{(k+\omega) - (k_i+\omega)\}$:

$$(1\text{-}s)(k_i+\omega)\ \{(k_c+\omega)\ -\ (k+\omega)\} = \qquad (17)$$
$$s(k_c+\omega)\ \{(k+\omega)\ -\ (k_i+\omega)\}.$$

Uzawa then solves this for $(k+\omega)$ to reveal:

$$(k+\omega) = \frac{(k_i+\omega)(k_c+\omega)}{s(k_c+\omega)+(1\text{-}s)(k_i+\omega)}. \qquad (18)$$

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