Proving Gödel's Theorem, a set-theoretical approach

by

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A Thesis Submitted to the Faculty of
The Wilkes Honors College
in Partial Fulfillment of the Requirements for the Degree of
Bachelor of Arts in Liberal Arts and Sciences
with a Concentration in Mathematics

H. L. Wilkes Honors College of
Florida Atlantic University
Jupiter, Florida
May, 2020

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This thesis was prepared under the direction of the candidate's thesis advisor, Dr. Warren McGovern, and has been approved by the members of his supervisory committee. It was submitted to the faculty of The Honors College and was accepted in partial fulfillment of the requirements for the degree of Bachelor of Arts in the Liberal Arts and Sciences.

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Institution	Wilkes Honors College of Florida Atlantic University
Thesis Advisor:	Dr. Warren McGovern
Degree:	Bachelor of Arts in Liberal Arts and Sciences
Concentration:	Mathematics
Year:	May 2020

Abstract: Before Gödel's incompleteness theorems, logicians such as Bertrand Russel and Alfred Whitehead pursued an ideal axiomatic system which would have created a reliable framework to successfully prove or refute every mathematical sentence. Gödel proved that such systems can never be created. In fact, Gödel's incompleteness theorems establish that axiomatic systems that are complex enough to formulate arithmetic can never generate a proof of all the logical statements that are expressible inside of them. According to the first incompleteness theorem, there are constructible mathematical sentences that can never be proven to be true or false using the axioms and the logical rules of the system. Furthermore, the second incompleteness theorem argues that the consistency of all axiomatic systems which contain Peano or Robinson arithmetic can never be determined using the rules and the proof mechanisms available in the system.

This thesis is dedicated to my family, my mother Daphney Charles, and my father Lohier Nizard. They trusted my career choices and never fail to encourage me.

Special thanks to Dr Terje Hoim whose advice guided me throughout my time at the Wilkes Honors College. Her ability to help her student overcome academic obstacles is remarkable. I will remember her anecdotal explanation of convergent series in which she talks about containing the tails of series or chopping off mathematical objects into pieces.

Special thanks to Dr. McGovern for supervising this thesis and for advising me on my career. His modern algebra course opened my eyes on my ability to pursue theoretical mathematics. He is one of the few professors at the Wilkes Honors College who enjoys writing on the chalkboard.

Contents

Introduction

Gödel's theorem could easily be referenced as the greatest achievement in logic since Aristotle. It is an argument that resembles the Liar's Paradox which in simple terms can be stated as "I am lying". If this statement were to be the words of someone who always tells the truth, it would escape the traditional dichotomy of truth which differentiates between true and false statements. Considering the Liar's Paradox, if it is true, it is automatically false (as lying means to not be telling the truth). However, Gödel's theorem does not directly deal with truth. It instead investigates a property of mathematical statements called provability. The notion of provability only makes sense inside of what mathematicians call an axiomatic system. Gödel used the axiomatic system of Peano arithmetic which constitutes a theoretical framework to people's traditional assumption about numbers, counting and measuring. A cornerstone of Gödel's argument is the construction of a self-referential statement of the form "G is not demonstrable". This argument seems to replicate the liar's paradox as it negates in itself the very property that is being investigated.

What does it mean to be provable?

One can argue that mathematics has been around since the early development of languages and the notion of counting within the earliest human groups. However, the Greeks are known as the founder of pure mathematics. This form of mathematics uses logic to assess the truth or falsehood of mathematical statements. A proof is a sequence of logical steps that lead to a statement about a specific property of mathematical objects. The property of being provable is often naively thought of as an immediate consequence of a statement being expressible within a certain axiomatic system. This naive way of viewing the notion of provability works perfectly in the context of complete systems. The point of Gödel's incompleteness theorem is that this form of provability that is tied to expressibility does not extend to all mathematical systems. In fact, there are systems that express mathematical properties that cannot be proven using the axioms of the system.

Chapter 1

Abstract proof of Gödel's theorem

1.1 Introduction

In mathematics, a **formal language** is defined by an alphabet and formation rules. The alphabet refers to all individual symbols that constitute a language while the formation rules designate which combinations of those symbols are well-formed. Those well-formed combinations of symbols are called **expressions**. Sentences are a specific type of expressions that have a truth value.
A language is **consistent** if there are no sentences in this language that are true and false at the same time. It then follows trivially that in a consistent system, every provable sentence (sentences that can be proven to be true) is in fact true. It is important to note that the converse of this statement is not always true. In fact, the main purpose of Godel's theorem is prove that there are true statements of specific consistent systems which can not be proven to be true (not provable).
Gödel's theorem is applicable to a language L with the following properties:

- The language L contains a countable subset $\mathcal{E}$ of expressions.
- The subset $\mathcal{E}$ contains a subset S of sentences and a subset $\mathcal{H}$ of predicates.
- The subset S contains a subset P of provable sentences and a subset R of refutable sentences, and these two sets are disjoint.

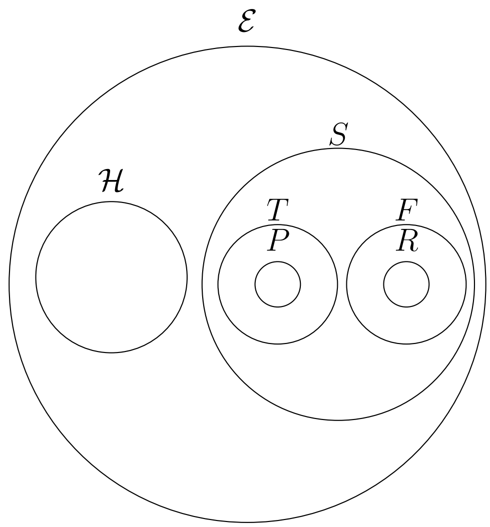

Figure 1.1: Structure of a language L with aforementioned subsets

- For $e \in \mathcal{E}$, $n \in \mathbb{N}$ and $H \in \mathcal{H}$, the language L contains the function $\Phi : \mathcal{E} \times \mathbb{N} \Rightarrow \mathcal{E}$ such that $\Phi(H, n) \in S$ is defined in L.

- The language L contains the **truth value function** $\mathcal{T} : S \Rightarrow \{T, F\}$ which assigns a value of true or false to each statement in S.

- S contains a subset T of true sentences, which in turn contains the collection of provable sentences. Furthermore, the subset of F of false sentences contains the refutable sentences.

The structure of such a language is represented by the **Figure 1.1**.

1.2 Diagonalization

Let G be a one-to-one function between the set $\mathcal{E}$ of expressions and the set $\mathbb{N}$ of natural numbers. Later, it will be established that such a one-to-one function exists for the language of First Order Arithmetic. Let e_n denote the expression corresponding to the natural number n by the function G. In that case, n is called the **Gödel number** of the expression e_n such that $G(e_n)$=n and $G^{-1}(n) = e_n$.

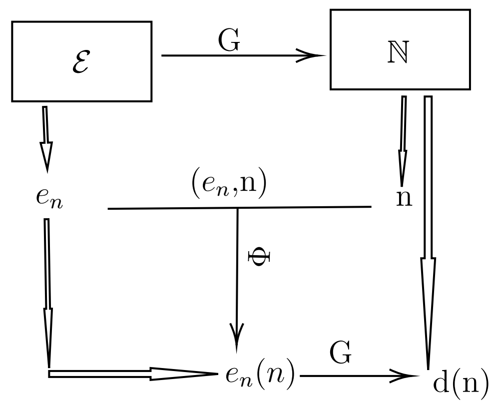

Figure 1.2: diagonal numbers, expressions and Godel numbers

The function Φ introduced earlier in this chapter can be used in a self-referential process called **diagonalization**. This process consists in pairing an expression with its own Gödel number by means of the function Φ. The output of the pairing $\Phi(e_n, n) = e_n(n)$ is an expression and is called the **diagonalization** of e_n. If the expression e_n is a predicate, its diagonalization is obtained by substituting its only free variable by its own Gödel number to obtain a sentence (an expression with no free variables). By definition, all expressions are mapped to a natural number by the function G. Therefore the diagonalization of an expression also corresponds to a natural number. The natural number which corresponds to the diagonalization of an expression is of the form $G(\Phi(E_n, n))$ and is called a **diagonal number**.

The complex relationship between an expression e_n , its Gödel number, and its diagonalization is represented in **Figure 1.2**.

1.3 Expressibility

A subset A of $\mathbb{N}$ is said to be **expressible** in L, if and only if there is a predicate H such that $H(n)$ is true $\Leftrightarrow n \in A$.

Given a set A, the subset A^* is defined as the set of all numbers n such that: $n \in A^* \Leftrightarrow d(n) \in A$.

1.4 Abstract Proof of Gödel's theorem

Theorem 1. *If L is a language and L is consistent* [1]*, then there exists a sentence of L which is neither provable nor refutable in L.*

Proof. Let $\tilde{P}$ designate the set of statements that are not provable in L. Let $\tilde{P}^*$ be expressible in L. Therefore, from **Section 1.2**, there is a predicate $H \in \mathcal{H}$ that expresses the set $\tilde{P}^*$ such that $H(n)$ is true $\Leftrightarrow n \in \tilde{P}^*$.

Let h be the diagonal number of the predicate H. Therefore $H(h)$ is the diagonalization of H. It is a statement; an expression with no free variables that has a truth value.

$H(h)$ is true $\Leftrightarrow h \in \tilde{P}^* \Leftrightarrow d(h) \in \tilde{P}$.

It becomes clear that $d(h)$ is the Gödel number of an improvable statement, therefore $H(h)$ is true and not provable.

If instead $H(h)$ is false , it implies that $h \notin \tilde{P}^*$ which implies that $d(h) \notin \tilde{P}$. Therefore by the law of excluded middle $d(h) \in P$. Therefore, the statement $H(h)$ is either true and not provable or false and provable. This contradicts that L is consistent.

We discussed earlier in **Section 1.1** that in consistent systems provable sentences are true sentences. Given that the system L is consistent, it cannot allow for the provability of a false statement. Thus, $H(h)$ is true but not provable. □

1.5 The language of First Order Arithmetic (FOA)

The abstract proof of Gödel's theorem is called abstract because it does not take into consideration any particular system. In the next chapter, a system called First Order Arithmetic will be analyzed.

In Chapter 3, it will be shown that First Order Arithmetic has the structure described in Section 1.1. To achieve that, it will be proven that a function equivalent to the function Φ is expressible in First Order Arithmetic. Later

[1] A language is consistent if there are no sentences that are true and false at the same time

in the third chapter, a Gödel numbering will be created using a process called **concatenation**. This will play the role of the one-to-one function that assigns a Gödel number to every expression. Lastly, the set $\tilde{P}^*$ will be shown to be expressible in **FOA**. It will therefore follow that First order Arithmetic is incomplete which means that it contains at least a true but not provable statement.

Chapter 2

First Order Arithmetic

2.1 Introduction

First order arithmetic is represented by the three basic operations of addition, multiplication, and exponentiation. The language of this axiomatic system can be formulated using the following 13 symbols: $0 \,{}' \,(\,)\, f \,,\, v \sim \Rightarrow \forall = \leq \otimes$. This finite set of characters allows for the representation of all numbers. A number n can be represented by 0 followed by n many prime symbols. For example, 2 can be represented by $0''$. The three operations of addition, multiplication and exponentiation can be represented respectively by f', f'' and f''' that will be abbreviated by $(\cdot)$, $(+)$ and (e). The symbols $\sim \Rightarrow \forall = \leq$ will be respectively interpreted as negation, logical implication, the universal quantifier, equality and the less than or equal symbol. The symbol v represents a variable. It can also be concatenated with the prime subscript symbol to create an infinite but countable set of new variables. The variables $v_{'}$, $v_{''}$, $v_{'''}$, ... can be abbreviated by v_1, v_2, v_3, ...which can be referred by any letter of the English alphabet.

2.2 Terms, Formulas and Variables

- A term is defined as a variable or a numeral. Both $v_{'}$ and $0''$ are terms.
- If t_1 and t_2 are terms, so are $(t_1 + t_2)$, $(t_1 \cdot t_2)$ and $(t_1 \text{ e } t_2)$.
- A formula is an expression of the form $(t_1 = t_2)$ or $(t_1 \leq t_2)$ where t_1 and t_2 are terms.

$(F \vee G)$	$(\sim F \Rightarrow G\ )$
$(F \wedge G)$	$\sim (F \Rightarrow \sim G\ )$
$(F \equiv G)$	$((F \Rightarrow G) \wedge (F \Rightarrow G))$
$\exists v_i F$	$\sim \forall v_i \sim F$
$t_1 \neq t_2$	$\sim (t_1 = t_2)$
$t_1 < t_2$	$((t_1 \leq t_2) \wedge (\sim (t_1 = t_2)))$
$t_1^{t_2}$	$t_1 e t_2$
$(\forall v_i \leq t)F$	$\forall v_i(v_i \leq t \Rightarrow F)$
$(\exists v_i \leq t)F$	$\sim (\forall v_i \leq t) \sim F$

Table 2.1: Abbreviation of expressions

- If F and G are formulas, then $\sim F$ and $F \Rightarrow G$ are also formulas and $\forall\ v_i\ F$ is also a formula.
- If F and G are atomic formulas meaning formulas with no variables, all occurrences of a variable in a formula of the form $\sim F$ or $F \Rightarrow G$ are free.
- All occurrences of v_i in $\forall v_i F$ are said to be bound.
- Sentences refer to formulas with no free variables. An open formula has at least one free variable.

For convenience and clarity purposes, several expressions are going to be abbreviated moving forward.

Table 2.1 displays a few expressions and their abbreviations.

2.3 Arithmetic Sets and relations

To this point, we already know that predicates are expressions but we have yet to formally define the concept of predicate. We start by giving an intuitive definition of a predicate as an expression expressing properties of numbers. In this sense, the properties of prime, even, odd, equal to, less than or equal to are all predicates. Formally, we define a **predicate** as a formula with only one free variable. We shall use the notation $F(v_1)$ to denote a predicate

F with free variable v_1. For a numeral n, the notation $F(\overline{n})$ refers to the sentence created by replacing every instance of the variable v_1 in the formula F with the numeral n.

$F(v_1)$, with v_1 as the only free variable is said to *express* a set A of numbers if for all $n \in A$, $F(\overline{n})$ is a true sentence.
$F(\overline{n})$ is true $\Leftrightarrow n \in A$.

A formula can also express a relation R (a set of tuples). In this case a formula $F(v_1, v_2, v_3, \ldots v_n)$ exists such that for all tuples of the form $(k_1, k_2, \ldots k_n)$ in R, $F(\overline{k_1}, \overline{k_2}, \overline{k_3}, \ldots \overline{k_n})$ is true.
$F(\overline{k_1}, \overline{k_2}, \overline{k_3}, ...\overline{k_n})$ is true $\Leftrightarrow R(k_1, k_2, k_3, ...k_n)$.

A function $f(x_1, x_2, x_3, \ldots x_n) = y$ is arithmetic if there is a formula $F(v_1, v_2, v_3, \ldots v_n, v_{n+1})$ such that whenever $F(\overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots \overline{x_n}, \overline{x_{n+1}})$ is true, it implies that $f(x_1, x_2, x_3, \ldots x_n) = y$.

Given those definitions, it is therefore possible to create proofs that verify the arithmetic nature of a set, a relation or a statement.

The set of even numbers can be expressed in first order arithmetic: The set of even numbers is expressed by the formula $\exists\ v_1(v_1 = 0''v_2)$. This formula F is true for all even numbers. It is created using the operation of $+$ and (.) and $=$ alone.

The set of prime numbers can be expressed in first order arithmetic: Let z represent a prime number, let x, y represents two arbitrary natural number: $(\forall x \forall y(xy = z) \Rightarrow ((x = 0' \wedge y = z) \vee (x = z \wedge y = 0')))$

The relation "x divides y" is arithmetic: It is in fact represented by the formula $\exists k : k.x = y$.

2.4 Considerations on truth

To formulate the set of true statements in FOA, one has to systematically define the concept of truth in formal languages. To accomplish that, Tarksi's inductive definition of truth will be used. For a language containing negation ($\sim$), conjunction ($\wedge$), disjunction ($\vee$), the universal quantifier ($\forall$) and the existential quantifier($\exists$),

- $\sim A$ is true if and only if A is not true.

- $A \wedge B$ is true if and only if A is true and B is true.
- $A \vee B$ is true if and only if one of A and B is true or both are true.
- $\forall v_i F$ is true if $F(\overline{n})$ is true for all numbers n ; .
- $\exists v_i F$ is true if and only if there is a number n for which $F(\overline{n})$ is true.

2.5 Axioms and axiom schema

The inductive definition of truth in **section 2.5** can only be used inductively on formulas that already have a determined truth value. Those formulas are called axioms.

The axioms that are going to be used are:

- **Implication introduction:** $F \to (G \to F)$
- **Implication distribution:**$(F \to (G \to H)) \to ((F \to G) \to (F \to H))$
- **Negation introduction:** $(\sim F \to \sim G) \to (G \to F)$
- **Universal quantifier distribution:** $(\forall v_i(F \to G) \to (\forall v_i F \to \forall v_i G))$
- If v_i does not occur in F, $(\to \forall v_i)$
- If v_i does not occur in t, $(\exists v_i(v_i = t))$
- If E_1 and E_2 are expressions, $(v_i = t) \to (E_1 v_i E_2 \to E_1 t E_2))$

2.5.1 Axioms of Arithmetic

To formulate arithmetic, in the language of FOA, a few more axioms are going to be added to the system. Those axioms express key properties of operations on numbers.

- If two numbers have the same successor, they are the same number, that is:

  $(v_1' = v_2' \to v_1 = v_2)$

- 0 is not the successor of a number, that is:

  $\overline{0} \neq v_1^{'}$

- 0 added to any number equals the number itself, that is:

  $\overline{0} + v_1 = v_1$

- 0 multiplied by a number equals 0, that is:

  $\overline{0}.v_1 = \overline{0}$

- 0 is the first number, that is:

  $(v_1 \leq \overline{0} \equiv v_1 = 0)$

- a number to the power of 0 is equal to the successor of 0, that is:

  $v_1^{\overline{0}} = \overline{0}^{'}$

- The successor of a number added to another number is the successor of the sum of the two numbers, i.e.,

  $v_1 + v_2^{'} = (v_1 + v_2)'$

- The product of a number (v_1) and the successor of another number (v_2) is the sum of the product of the two numbers added to (v_1), that is:

  $(v_1.v_2^{'}) = (v_1.v_2) + v_1$

- A number (v_1) to the power of the successor of another number (v_2) is the product of $v_1^{v_2}$ and (v_1), i.e:

  $(v_1^{v_2^{'}}) = (v_1^{v_2}).v_1$

- Given two numbers, either one is less than the other or they are the same number, meaning that:

  $((v_1 \leq v_2) \vee (v_2 \leq v_1))$

Chapter 3

Gödel's Theorem in FOA

3.1 Introduction

In Chapter 1, it was proven that if the set $\tilde{P}^*$ is expressible in a language, the language will contain a true but not provable statement. It was also established that the proof of Gödel's theorem relies on a Gödel numbering process, a substitution function and a diagonalization function. In this chapter, the abstract concepts discussed in Chapter 1 will be discussed within the context of first order arithmetic.

3.2 Gödel's numbering

In Chapter 1, the concept of a one-to-one function was introduced. The one-to-one function is the function $G : \mathcal{E} \Rightarrow \mathbb{N}$ that maps every well formulated expression to a natural number. Notice however that the function need not be onto. To prove Godel's theorem, it suffices that every expression is mapped to a natural number and not the other way around. The simplest way to create such a mapping is to start by mapping the symbols one by one. Once the symbols are mapped to a natural number, concatenation can be used to complete the mapping for complex expressions. The mapping that will be adopted maps the symbols 0 ‘() f , v $\sim$ $\Rightarrow$ $\forall$ $=$ $\leq$ $\otimes$ to the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, δ, η, ε, respectively. The mapping is represented in **Table 3.1** below with the base-13 numbering system to the left and the associated symbols to the right.

0	’
1	0
2	(
3	)
4	f
5	,
6	v
7	$\sim$
8	$\Rightarrow$
9	$\forall$
η	$=$
ε	$\leq$
δ	$\otimes$

Table 3.1: Mapping of 13 symbols

Concatenation is now used on the numbers the same way it is used on the symbols to create expressions. Thus, it is straightforward that every expression is now mapped to a base-13 number. As an example, considering the expression $v_, = 0''$, (v) [1] is mapped to 6, (,) is mapped to 5, (=) is mapped to η, (0) is mapped to 1, the sequence (”) is mapped to 00. Therefore the whole expression is mapped to $65\eta100$, that is $G(v_, = 0'') = 65\eta100$.

It has been established that the numbering system introduced maps every expression to a number. However, it is still important to prove that the process of concatenation itself can be expressed using First order Arithmetic.

3.3 Well formulated expressions

A well formulated expression refers to the 13 symbols and all possible concatenation of the 13 symbols except for the string $\prime X$ where X represents any other string such that $X \in \mathcal{E}$. In other words, a well formulated expression is a finite string of the 13 symbols except the first symbol cannot be $\prime$. This restriction ensures that there is a valid one-to-one relationship between the

[1]It would be more correct to write v is mapped to 6. In this case parenthesis are only used to create emphasis on the symbol 6

set of expressions in the language of FOA and the set of naturals $\mathbb{N}$.

We let $\mathcal{E}$ denote the collection of well formulated expressions.

Theorem 2. *There is a one to one function between the set of well formulated expressions and the set of naturals.*

Proof. Each symbol is mapped to a natural number. From the definition, an expression is a string of symbols. Using the concatenation algorithm, we map each symbol to a natural number (base 13). The resulting natural number is the Gödel number of the expression. □

We let G denote the one-to-one function from $\mathcal{E}$ onto $\mathbb{N}$.

3.4 Arithmetic nature of concatenation

Theorem 3. *The process of concatenation can be expressed arithmetically.*

Proof. Let L_b define the **length function** which returns the number of digits of a number in a base b. Let C_b define the concatenation operation that concatenates x and y to create the number x followed by y in base b. The property $xC_by = x.b^{L_t^{(y)}} + y$ holds for all numbers. For example, in base 10, $40\ C_{10}\ 987 = 40987$ which is $40.\ 10^3 + 987$.

To determine if the relation $b^{L_t^{(y)}} = z$ is arithmetic, notice that $b^{L_t^{(y)}}$ is the smallest power of b greater than y for $b \geq 2$. This fact can be expressed by the formula $F(v_1) = \forall v_2$: $b^{v_2} \geq y\ (v_1 \leq v_2 \wedge b^{v_1} \geq y)$.

All components of the expression $x\ C_b\ y = x.b^{L_t^{(y)}} + y$ are arithmetic, therefore concatenation is arithmetic. □

3.5 Arithmetic nature of diagonalization

3.5.1 Quasi-substitution

Recall that for a predicate $F(v_1)$, $F(\overline{n})$ is the result of the substitution of the numeral representation of n for v_1 in F. The sentence $F(\overline{n})$ is equivalent to the formula $\forall\ v_1\ (\ v_1 = \overline{n} \Rightarrow F(v_1))$. (We will denote this logically equivalent sentence by $F[\overline{n}]$.) The sentence $F[\overline{n}]$ is true whenever $F(\overline{n})$ is true and false whenever $F(\overline{n})$ is false. This method of substitution which consists in creating a sentence that is equivalent to the sentence $F(\overline{n})$ will be called **quasi-substitution**.

We will also use quasi-substitution on $\mathcal{E}$ that are not necessarily predicates. If E is a well-formulated expression, then we pick out the first free variable, say v_1, and let $E(\overline{n})$ denote the well formulated expression that results of the substitution of the numeral representation of n for v_1 in F. However $E[\overline{n}]$ is meaningless in the context of our use of quasi-substitution.
Let $\Phi : \mathcal{E} \times \mathbb{N} \to \mathbb{N}$ be the function defined by $\Phi((E, n) = E(n)$. Observe that if F is a predicate, then for any $n \in \mathbb{N}$, $\Phi((F, n)$ is a sentence. Then it follows that the diagonlization of a well formulated expression is the the number $\Phi((E, G(E)))$. Letting $n = G(E)$ and setting $e_n = E$ we then know that $d(n) = \Phi((e_n, n))$.

3.5.2 Quasi-substitution is Arithmetic

Theorem 4. *Quasi-substitution is arithmetic. Moreover, the relation of diagonalization is arithmetic.*

Proof. For an arbitrary expression E, $E(\overline{n})$ is a well formulated expression, nonetheless meaningless if E is not a formula. Well-definedness is enough to allow the construction of the function relation $r(e, n)$ which represents the Gödel number of $E[\overline{n}]$, that is $E[\overline{n}] \equiv \forall\ v_1\ (\ v_1 = \overline{n} \Rightarrow E(v_1))$. Let k_1 be the Gödel number of "$\forall v_1(v_1 =$", the Gödel number of $\overline{n}$ is 13^n, the Gödel number of $\Rightarrow$ is 8, the Gödel number of E_n is n, and the Gödel number of) is 3, then $r(n, n) = d(n) = k * 13^n * 8 * n * 3$. Therefore quasi-substitution is arithmetic. □

3.6 The set A^* is arithmetic

Theorem 5. *If A is arithmetic, then so is A^*.*

Proof. Let A be an arithmetic set. By definition, this means there exists a predicate $F(v_1)$, such that: $F(n)$ is true $\Leftrightarrow n \in A$. The set A^* is the set of natural numbers that have their diagonal numbers in the set. From 3.5.2, there is a formula expressing the relation $d(x) = y$. Let $F(v_1)$ be the predicate that expresses the set A, therefore $\forall x \exists y(d(x) = y \wedge F(\mathrm{y}))$ expresses the set A^*. □

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Discussion: The proof of Gödel's theorem that will be formulated in this chapter relies on the set $\tilde{P}^*$. According to 3.5.2, the existence of this set can be deducted from the existence of the set of non-provable sentences $\tilde{P}$. It implies that to complete Gödel's theorem, one should focus on proving that the provable sentences as a set can be expressed in FOA. To achieve that, one should be able to construct a predicate of FOA that expresses the set $\tilde{P}$. This is equivalent to proving that the condition of being not provable is an arithmetic condition.

3.7 Arithmetization of the notion of proof

To this point, the concept of Gödel's numbering has only been used on symbols and expressions. In this section, this concept is going to be extended to map series of expressions to natural numbers.

Let the symbol $\otimes$ indicate that two expressions are part of the same series. Thus, the series $(E_{a_1}, E_{a_2}, ...E_{a_n})$ can be represented by $E_{a_1} \otimes E_{a_2} \otimes E_{a_3} \otimes ..E_{a_n}$. Using the coding introduced in **Section 3.2**, the Gödel number of such series of expressions is $a_1\delta a_2\delta a_3...\delta a_n$. Series of expressions can be recognized because they have the symbol $\otimes$ in their representation and the number δ in their Godel number.

Definition: Two sentences E_{a_1} and E_{a_2} are said to be **logically interdependent** if there is a rule of inference or an axiom schema that allows one to be deducted from the other.

Formally, a proof is a series of true sentences, each one being logically derivable from the one preceding it. A sentence X will be said to have a proof, if there exists a series Y of true logically interdependent sentences ending in X[2]. As an example, the sentence *"The prime numbers are infinite"* is provable within FOA because there is a series of true and logically interdependent sentences such that the sentence *"The prime numbers are infinite"* is the end of that series.

Formally, the proof of this sentence can be expressed as followed:
v_1 is the greatest prime number. $[E_{a_1}]$

[2]In Chapter 2, a true statement was defined as either an axiom or a statement that can be created using the deduction methods

Let the number $v_2 = 3.5.7.11...v_1{+}1$ be a number. [E_{a_2}]
If the number v_2 is prime, v_1 is not the greatest prime number as $v_2 > v_1$. [E_{a_3}]
If v_2 is not prime, there is a prime greater than v_1 that divides it. [E_{a_4}]
Therefore, no constant v_1 is the greatest prime number. [E_{a_5}]

In the case presented above, the series $(E_{a_1}, E_{a_2}, E_{a_3}, E_{a_4}, E_{a_5})$ represented by $E_{a_1} \otimes E_{a_2} \otimes E_{a_3} \otimes E_{a_4} \otimes E_{a_5}$ is a proof for E_{a_5}.

In order to take advantage of the formal definition of proofs, one has to be able to express the properties of *ending*, *beginning*, *preceding* and *being part of a series* arithmetically.

Definition: *A* number x is said to **begin a number** y if x is an initial segment of the representation of y. This relation is expressed by $x\beta y$. For example, in 56778, the relation $56\beta 56778$ holds. For consistency, all numbers begin themselves and 0 does not begin any number other then itself. Arithmetically, the predicate that expresses this property is the formula :
$x = y \vee (x \neq 0 \wedge (\exists \leq y)(\exists w \leq y)(Pow_b(w) \wedge (x.w)C_b z = y))$. Therefore the relation $x\beta y$ is arithmetic.

Definition: *A* number x is said to **end a number** y if x is a final segment of the representation of y. This relationship is expressed by xEy. For example, in 56778, the relation 778 E 56778 holds. For consistency, all numbers end themselves. Arithmetically, the predicate that expresses this property is the formula :
$x = y \vee (\exists z \leq y)(zC_b x = y)$. Therefore the relation xEy is arithmetic.

Definition: A number x is **part of a number** y if x ends some number that begins the representation of y. This relationship is expressed by xPy. For example, in 56778, the relation 67 P 56778 holds. Arithmetically, the predicate that expresses this property is the formula :
$(\exists z \leq y)(zEy \wedge x\beta z)$. Therefore the relation xPy is arithmetic.

Definition: *A* number x is said to **precede** a number y in a series z if x and y are members of a series z in which the first occurrence of x happens before the first occurrence of y. This relation is denoted by $x \prec_z y$ and expressed by $x \in z \wedge y \in z \wedge (\exists w \leq z)(w\beta z \wedge x \in w \wedge \sim y \in w)$.

From the formal definition of a proof, the two principal characteristics are:

- It is a series of expressions.

- The expressions that make up the series are true statements.

3.7.1 $Seq(x)$ is arithmetic

Theorem 6. *Define $Seq(x)$ as the property for a number x to be a sequence number. Formally, we express $Seq(x)$ as a number (base 13) that has the form $\delta a_1 \delta a_2 \ldots \delta a_n \delta$ for some number a_n that do not have a δ in them. Then $Seq(x)$ is arithmetic.*

Proof. The predicate $Seq(x)$ is expressed by the formula :
$\delta\beta x \wedge \delta E x \wedge \delta \neq x \wedge \delta\delta\tilde{P}x \wedge (\forall y \leq x)(\delta 0 y P x \rightarrow \delta\beta y)$. Therefore, $Seq(x)$ is arithmetic. □

3.7.2 Categorization of true statements

A statement will be said to be true if it is an axiom or if it is derivable from a true statement using the rules of inferences. Let $A(x)$ designate the property of being an axiom . Let Dev designate the property of being derivable from two statements using logical rules. In particular, $Dev(x, y, z)$ means that z is derivable from x and y.

Theorem 7. *$A(x)$ and $Dev(x, y, z)$ are Arithmetic*

Proof. $A(x)$ is clearly arithmetic. $Dev(x, y, z)$ can be expressed as $y = x \rightarrow z \vee (\exists z \leq y)(Var(z) \wedge y = 9zx)$. Therefore $Dev(x, y, z)$ is arithmetic. □

3.7.3 The set P of provable sentences is arithmetic

The set P is arithmetic; there is an arithmetic predicate that expresses it. Combining the two results in 3.6.1 and 3.6.2, the predicate expressing $P(x)$ is represented by the formula:
$Seq(x) \wedge (\forall \in x)(A(y) \vee (\exists z, w \prec_x y) Dev(z, w, y)) \wedge (x \in y)$.

Discussion of 3.6.3 In section 3.6.3, it has been proven that the set P is expressible in FOA. Given that the formula P(v_1) expresses P, it is straightforward that the set $\tilde{P}$ is expressed by $\sim P(v_1)$. Using the theorem in 3.5, it can be concluded that $\tilde{P^*}$ is expressible in FOA. Applying the theorem from 1.4 from the abstract proof of Godel's theorem [3] to FOA, it implies that there is at least a true statement of FOA that is not provable inside of FOA.

3.8 Important results

The reasoning behind the first Incompleteness Theorem helped twentieth-century mathematicians to solve a whole class of difficult problems. Those results can be found in numerous branches of mathematics such as set theory, measure theory, topology, etc... One such result is the Continuum Hypothesis which had puzzled logicians and mathematicians for decades before it was finally proven by Cohen in 1967 to be independent from ZFC. Another major result is the second incompleteness axiom which is a direct corollary of the first incompleteness theorem.

3.8.1 Independence

The concept of independence refers to sentences of which the truth and falsehood cannot be established using the axioms and theorems of a given axiomatic system. For example, in Euclidean geometry, the Parallel Postulate states that given a point not on a line, there is a unique line through the point parallel to the given line. In Hyperbolic Geometry, there is more than one such line. Therefore, the Parallel Postulate cannot be established from the first four axioms, i.e. is independent of the first four axioms. What is usually known as "Euclidean Geometry" only accounts for straight planes and surfaces; elliptic and hyperbolic geometries, use the first four euclidean axioms along with an alternate version of the fifth postulate. Independence proofs rely on a technique which mathematicians call forcing which consists in simultaneously creating models in which a mathematical hypothesis is true and models in which it is false.

3.8.2 The continuum hypothesis

The Continuum Hypothesis (CH) was conjectured by Greg Cantor in 1878 and states that there is no set whose cardinality is strictly between the cardinalities of the natural numbers and that of the reals. This problem is equiv-

[3] See Chapter 1, subsection 1.4

alent to saying that given the real number line and an infinite set of points marked out on it, only two things can be true: either the set is countable, or it has as many elements as the whole line. There is no third cardinalities between the two. Formally, the continuum hypothesis is represented by the equality $\mathfrak{c} = 2^{\aleph_0} = \aleph_1$. Over a span of several decades, mathematicians have tried to prove the theorem using the axioms of Zermelo-Fraenkel set theory extended with the Axiom of Choice (ZFC). Gödel constructed a model $V = L$ where the Continuum Hypothesis is true. Then, in 1967, Cohen created a model, using forcing, where the Continuum hypothesis is false. Therefore, the continuum hypothesis or its negation can be added to ZFC without generating a contradiction.

3.8.3 Second Incompleteness Theorem

Mathematicians have often been concerned about the correctness of the languages that they use. In 1900, Hilbert considered finding a proof of the consistency of arithmetic to be one of the most important problems for mathematicians of his time. The second incompleteness theorem is an important result which tackles the problem of inconsistency of axiomatic systems.

Theorem 8. *Any formal system that is complex enough to formulate Peano Arithmetic can prove its own consistency if and only if it is inconsistent.*

Proof sketch : The second incompleteness theorem follows directly from the first incompleteness theorem.
Recall the self referential property of the statement H(h)introduced in 1.4. Let $\mathcal{G}$ designate the statement $H(h)$ constructed in chapter 1. The statement $\mathcal{G}$ expresses that $\mathcal{G}$ itself is not provable. This can be expressed formally as $\mathcal{G} \equiv$ $\mathcal{G}$ *is not provable.*

Let $Cons(L)$ define the property of being consistent for a language L, Formally, Gödel's theorem is of the form $Cons(L) \Rightarrow \mathcal{G}$ is not provable. This expression is equivalent to $Cons(L) \Rightarrow \mathcal{G}$. At this point a proof of $Cons(L)$ successfully bypasses Gödel's first theorem and creates a proof for $\mathcal{G}$. It was already said that $\mathcal{G}$ is not provable, therefore such proof of $Cons(L)$ cannot exist.

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