PARAMETER ESTIMATION FOR GEOMETRIC LÉVY PROCESSES WITH STOCHASTIC VOLATILITY

by

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ABSTRACT

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In finance, various stochastic models have been used to describe the price movements of financial instruments. After Merton's [38] seminal work, several jump diffusion models for option pricing and risk management have been proposed. In this dissertation, we add alpha-stable Lévy motion to the process related to dynamics of log-returns in the Black-Scholes model where the volatility is assumed to be constant. We use the sample characteristic function approach in order to study parameter estimation for discretely observed stochastic differential equations driven by Lévy noises. We also discuss the consistency and asymptotic properties of the proposed estimators. Simulation results of the model are also presented to show the validity of the estimators. We then propose a new model where the volatility is not a constant. We consider generalized alpha-stable geometric Lévy processes where the stochastic volatility follows the Cox-Ingersoll-Ross (CIR) model in Cox et al. [9]. A number of methods have been proposed for estimating parameters for stable laws. However, a complication arises in estimation of the parameters in our model because of the presence of the unobservable stochastic volatility. To combat this complication we use the sample characteristic function method proposed by Press [48] and the con-

ditional least squares method as mentioned in Overbeck and Rydeń [47] to estimate all the parameters. We then discuss the consistency and asymptotic properties of the proposed estimators and establish a Central Limit Theorem. We perform simulations to assess the validity of the estimators. We also present several tables to show the comparison of estimators using different choices of arguments u_i's. We conclude that all the estimators converge as expected regardless of the choice of u_i's.

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PARAMETER ESTIMATION FOR GEOMETRIC LÉVY PROCESSES WITH STOCHASTIC VOLATILITY

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CHAPTER 1
INTRODUCTION

1.1 OVERVIEW

"How heavy are the tails of financial asset returns ?" is a commonly raised concern in finance. The answer is the key to evaluating risk in financial markets and accurately modeling price information (see Crovella et al. [10]). We are therefore concerned with the estimation of parameters of heavy-tailed distributions.

Empirical results show stock returns have fat tails which violate the Black-Scholes normality assumption. The Black-Scholes (or Black-Scholes-Merton) model [5] assumes that the stock price follows a geometric Brownian motion. In the 1960s, Mandelbrot [35] and Fama [15] claimed financial time series data do not follow the Gaussian distribution. Their conjecture was that financial returns should be modeled by a non-normal stable distribution. Many stochastic volatility and jump diffusion models have been proposed after 1970s to address Mandelbrot [35] and Fama's [15] claim.

Stochastic differential equations (SDE's) driven by Lévy noises have gathered attention in recent years. Applications to finance are discussed by Schoutens [53] and Kyprianou et al. [32]. The processes determined by SDE's have other non-financial applications as well. For example, Ditlevesen, in [11] and [12], discusses the applications of SDE's driven by Lévy noises to climate dynamics. Moreover, applications to physics are discussed by Schertzer et al. [52], applications to signal processing are discussed by many authors including Nikias and Shao [42] and Ilow and Hatzinakos [23], and applications to networks are discussed by Mikosch et al.

[40] and Zhou et al. [59].

Introduction of the jump diffusion is credited to Merton [38] for a model in 1976 which is composed of log-normal jumps driven by a Poisson process. After Merton's [38] seminal work, many SDE's with various jump diffusion models have been proposed. For a proposed model one of the important practices is to estimate the parameters in the model. A number of different approaches for the parameter estimation of the SDE's have been studied. For diffusion processes, the most popular method for parameter estimation is the maximum likelihood estimator (MLE) (based on Liptser and Shiryaev [33]). Another well-known method for parameter estimation is the least squares estimator (LSE)". The convergence of the LSE in probability is proved in Dorogovcev [13] and Breton [6] , and the strong consistency was discussed in Kasonga [26].

Hu and Long ([21], [22]) proposed the least squares estimator (LSE) for Ornstein-Uhlenbeck processes driven by α-stable motions. Another recent work on parameter estimation for diffusion processes with jumps from discrete observations was discussed by Shimizu and Yoshida [54] and the LSE estimators for SDE's driven by small Lévy noise was discussed by Long et al. [34]. We also refer to the work of Aït-Sahalia [1] for the estimation for the MLE based on discrete observation. In our work, the maximum likelihood method for estimating parameter is very difficult and computationally burdensome due to the lack of closed-form density function of the α-stable Lévy distribution.

This dissertation focuses on the sample characteristic function estimation approach for estimating parameters of stochastic differential equations driven by α-stable Lévy noises.

First, we consider the α-stable Geometric Lévy process with constant volatility. In this work, we use the sample characteristic function approach proposed by Press [48]. We estimate all five parameters, i.e., drift constant μ, volatility σ, scale parameter τ, the index of stability α and the skewness parameter β.

Assuming constant volatility appears to be a simplified version of reality. In part motivated by the model mentioned in [18], we introduce the α-stable Geometric Lévy process with stochastic volatility. We suppose the stock price is observed explicitly at discrete times, but the fluctuating volatility is not observed directly. We allow the stochastic volatility to be the Cox-Ingersoll-Ross (CIR) model (see Cox et al. [9]). Estimating parameters of the model is very complicated because we are dealing with the hidden Markov model. Since the closed form density is not available, we use the sample characteristic function approach and the conditional least squares method to estimate all parameters involved in the model and discuss the consistency and asymptotic properties of the proposed estimators.

1.2 ORGANIZATION

The dissertation is organized as follows:

In Chapter 2, we present preliminary notions and results that are needed for our research. We present important properties of the Lévy processes and α-stable Lévy distributions based on Applebaum [2], Sato [51], and Samorodnitsky and Taqqu [50]. General properties of SDE's are based on Øksendal [46].

In Chapter 3, we formulate α-stable Geometric Lévy processes, and estimate parameters using the sample characteristic function approach. We also establish the consistency and the asymptotic behavior of the estimators. Simulation results are also presented in Chapter 3.

Chapter 4 is an extension of the model proposed in Chapter 3. We consider the α-stable Geometric Lévy processes, with stochastic volatility. The estimation of parameters is very complicated due to the presence of the unobserved stochastic volatility, the CIR model. We derive estimators for the parameters found in the model, discuss their consistency, and outline the asymptotic behavior.

Finally, in Chapter 5, we summarize the current findings and outline the future work in this area.

CHAPTER 2
PRELIMINARIES ON LÉVY PROCESSES AND STOCHASTIC CALCULUS

We shall deal with stochastic differential equations driven by Lévy processes and their parameter estimation. We first give definitions of some related important concepts in this Chapter. These definitions are based on the textbooks Applebaum [2] and Øksendal [46].

2.1 SOME DEFINITIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a right continuous and increasing family of σ-algebras $(\mathcal{F}_t, t \geq 0)$.

- Filtration

  A collection $\{\mathcal{F}_t\}_{t\geq 0}$ of sub σ-algebras is called filtration if, for every $s \leq t$, we have $\mathcal{F}_s \subset \mathcal{F}_t$.

- $\mathcal{F}_t$-measurable

  Let $\mathcal{F}_t$ be a sigma-field of subsets of Ω. A random variable X_t is $\mathcal{F}_t$-measurable if every set in $\sigma(X_t)$ is also in $\mathcal{F}_t$.

- Adapted process

  The stochastic process $\{X_t : 0 \leq t \leq \infty\}$ is called adapted to the filtration $\mathcal{F}_t$ if, for every t, X_t is measurable with respect to $\mathcal{F}_t$.

  In other words, the stochastic process $X = (X_t : t \geq 0)$ is said to be adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ if $\sigma(X_t) \subseteq \mathcal{F}_t$ for all $t \geq 0$.

The stochastic process X_t is always adapted to the natural filtration $\sigma(X_s, s \le t)$ generated by X_t.

- Càdlàg paths

  A function $f : \mathbb{R}^+ \to \mathbb{R}^d$ is a càdlàg function that is everywhere right-continuous and has left limits everywhere, i.e., such a function has only jump discontinuities.

- Brownian motion

  A stochastic process $X = (X_t; t \in [0, \infty))$ is called (standard) Brownian motion or a Wiener process if the following conditions are satisfied:

  1. It starts at zero: $X(0) = 0$ almost surely.
  2. It has stationary increments (i.e., $X_t - X_s \overset{d}{=} X_{t+h} - X_{s+h}$, for all $s, t \in T \subseteq \mathbb{R}$ and h with $t + h, s + h \in T$) and independent increments (i.e., if for every choice of $t_i \in T$ with $t_1 < t_2 < \ldots < t_n$ and $n \ge 1$, X_{t_1}, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2} \ldots, X_{t_n} - X_{t_{n-1}}$ are independent).
  3. The random variable $X_t - X_s$ is normally distributed with mean 0 and variance $t - s$ with $s < t$.
  4. It has continuous sample paths: " no jump ".

- Martingale

  An n-dimensional stochastic process $\{X_t\}_{t \ge 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{F}_t, t \ge 0\}$ if

  1. X_t is a $\mathcal{F}_t$-measurable, for all t.
  2. $\mathbb{E}(|X_t|) < \infty$ for all t.
  3. $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $0 \le s \le t$.

- Semi-martingale

  We say that the process $X = (X(t), t \geq 0)$ is a semi-martingale if it is an adapted process such that, for each $t \geq 0$,

  $$X(t) = X(0) + M(t) + C(t),$$

  where $M = (M(t), t \geq 0)$ is a local martingale and $C = (C(t), t \geq 0)$ is an adapted process of finite variation.

- Local martingale

  An adapted process $M = (M(t), t \geq 0)$ is called a local martingale if there exists a sequence of stopping times $\tau_1 \leq \ldots \leq \tau_n \to \infty$ (a.s.) such that each of the processes $M(t \wedge \tau_n, t \geq 0)$ is a martingale. (A stopping time is a random variable $\tau : \Omega \to [0, \infty]$ for which the event $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$.)

Next, we define α-stable Lévy process and some basic properties.

2.2 LÉVY PROCESSES AND α-STABLE LÉVY PROCESSES

Let $(X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with given filtration $\{\mathcal{F}_t, t \geq 0\}$. We say that X is a Lévy process if:

1. $X(0) = 0$ (almost surely).

2. X has independent increments, that is for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 \ldots < t_n$ the random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

3. X has stationary increments, i.e., the distribution of $X_{t+h} - X_t$ does not depend on t, for each $h \in \mathbb{R}^+$.

4. X is continuous in probability, i.e., $X_s \to X_t$ in probability, as $s \to t$. In other words, for any $\epsilon \geq 0$ and $t \geq 0$, $\lim_{h \to 0} \mathbb{P}\left(|X_{t+h} - X_t| > \epsilon\right) = 0$.

We note that X exhibits càdlàg paths and hence its simple paths are right continuous with left limits. Next we define α-stable distributions.

Let $X_1, X_2, \ldots, X_n$ and X are independent and identically distributed random variables, then we say that X has α-stable distribution if there exists a positive constant C_n and a real number D_n such that

$$X_1 + X_2 + \ldots + X_n \overset{d}{=} C_n X + D_n$$

where $C_n = n^{\frac{1}{\alpha}}$.

Let $S_\alpha(\gamma, \beta, \delta)$ denote a stable distribution with index of stability $\alpha \in (0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale parameter $\gamma \in (0, \infty)$, and location parameter $\delta \in \mathbb{R}$. Since, in general, densities of α-stable distributions do not have a closed-form expressions, we use characteristic function often.

Note that there are three special cases with a closed form probability density function.

- $\alpha = 2$ gives the Gaussian distribution: $S_2(\gamma, \beta, \mu) = N(\mu, 2\gamma^2)$
- $\alpha = 1$ and $\beta = 0$ gives the Cauchy distribution: $S_1(\gamma, 0, \mu)$, and
- $\alpha = \dfrac{1}{2}$ and $\beta = \pm 1$ gives the Lévy distribution: $S_{\frac{1}{2}}(\gamma, 1, \mu)$, $S_{\frac{1}{2}}(\gamma, -1, \mu)$.

It is also important to note that, for $\alpha < 2$, α-stable random variables have infinite variance, therefore the central limit theorem is not valid.

A random variable η is said to have a stable distribution with parameters α, γ, β, and δ if it has characteristic function of the form:

$$\begin{aligned}\phi_\eta(u) &= E(\exp(iu\eta)) \\ &= \int_{-\infty}^{\infty} e^{iux} dF(x) \\ &= \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 - i\beta sgn(u) \tan\left(\frac{\pi\alpha}{2}\right)\right] + i\delta u\right), & \alpha \neq 1 \\ \exp\left(-\gamma |u| \left[1 + i\beta \frac{2}{\pi} sgn(u) \log |u|\right] + i\delta u\right), & \alpha = 1 \end{cases}\end{aligned} \tag{2.1}$$

where $i = \sqrt{-1}$, $F(x)$ is the cumulative distribution function of the random variable η and

$$sgn(u) = \begin{cases} 1, & u > 0 \\ 0, & u = 0 \\ -1, & u < 0 \end{cases} \tag{2.2}$$

We denote $\eta \sim S_\alpha(\gamma, \beta, \delta)$. For $\delta = 0$, we say η is strictly α-stable. Further, if $\beta = 0$, we call η is symmetric α-stable.

When $\alpha = 2$,

$$\phi_\eta(u) = \exp\left(i\mu u - \frac{1}{2}\gamma^2 u^2\right)$$

which is the characteristic function of the normal distribution. We can show that $E(X^2) < \infty$ if and only if $\alpha = 2$ (i.e., X is normal) and that $E(|X|) < \infty$ if and only if $1 < \alpha \leq 2$.

The probability density function can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(t) \exp(-ixt) dt.$$

We note that the probability density function is not very useful because we can not express it in terms of some elementary functions.

For $\alpha < 2$, the tails of Lévy stable distributions are asymptotically equivalent to a Pareto law ([50], [24]), i.e., if $X \sim S_\alpha(1, \beta, 0)$, then

$$P(X > x) = 1 - F(x) \to C_\alpha (1 + \beta) x^{-\alpha},$$

$$P(X < -x) = F(-x) \to C_\alpha (1 - \beta) x^{-\alpha},$$

where

$$C_\alpha = \left(2 \int_0^\infty x^{-\alpha} \sin(x) dx\right)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}\right). \tag{2.3}$$

Now we are ready to define the standard α-stable Lévy motion.

An $\mathcal{F}_t$-adapted stochastic process $\{L_t^\alpha\}_{t \geq 0}$ is called a standard α-stable Lévy motion if

- $L_0^\alpha = 0$, almost surely.

- $L_t^\alpha - L_s^\alpha \sim S_\alpha((t-s)^{\frac{1}{\alpha}}, \beta, 0)$, $0 \le s < t < \infty$.

- For any finite time points $0 \le s_0 < s_1 < \ldots < s_m < \infty$, the random variable $L_{s_0}^\alpha, L_{s_1}^\alpha - L_{s_0}^\alpha, \ldots, L_{s_m}^\alpha - L_{s_{m-1}}^\alpha$ are independent.

Next, we state (without proof) some useful propositions from Samorodnitsky and Taqqu [50]:

Proposition 2.2.1. *Let $X \sim S_\alpha(\gamma, \beta, \delta)$ and let a be a non-zero real constant. Then*

$$aX \sim S_\alpha\big(|a|\gamma, sgn(a)\beta, a\delta\big), \quad \alpha \neq 1$$
$$\text{and} \;\; aX \sim S_1\big(|a|\gamma, sgn(a)\beta, a\delta - \frac{2}{\pi}a(\ln|a|\gamma\beta\big), \quad \alpha = 1.$$

Proposition 2.2.2. *Let $X \sim S_\alpha(\gamma, \beta, \delta)$ with $\alpha \neq 1$. Then X is strictly stable if and only if $\delta = 0$.*

Proposition 2.2.3. *$X \sim S_\alpha(\gamma, \beta, \delta)$ is symmetric if and only if $\beta = 0$ and $\delta = 0$. It is symmetric about δ if and only if $\beta = 0$.*

Proposition 2.2.4. *If $X_1, X_2, \ldots, X_n$ are i.i.d. $S_\alpha(\gamma, \beta, \mu)$, then*

$$X_1 + X_2, \ldots + X_n \stackrel{d}{=} n^{\frac{1}{\alpha}} X_1 + \delta(n - n^{\frac{1}{\alpha}}) \text{ if } \alpha \neq 1, \text{ and}$$

$$X_1 + X_2, \ldots + X_n \stackrel{d}{=} nX_1 + \frac{2}{\pi}\gamma\beta n \ln n \text{ if } \alpha = 1.$$

For details of parametrization and properties of the stable distributions, we refer to Samorodnitsky and Taqqu [50], Sato [51], Janicki and Weron [24] and Nolan [43]. Next, we state the Itô's Formula and the Lévy-type stochastic integral:

Theorem 2.2.1 (The Itô's Formula)**.** *If a process X_t satisfies the SDE*

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

where $\mu(X_t,t)$ is the drift term, $\sigma(X_t,t)$ is the diffusion function and W_t is the Brownian motion, then given any function $f(X_t,t)$ of the stochastic process X_t which is twice differential in its first argument and once in its second, we have

$$df(X_t,t) = \left[\left(\frac{\partial}{\partial t} + \mu(X_t,t)\frac{\partial}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2}{\partial X_t^2}\right) f(X_t,t)\right] dt + \left[\sigma(X_t,t)\frac{\partial}{\partial X_t} f(X_t,t)\right] dW_t. \tag{2.4}$$

Theorem 2.2.2 (Itô's Theorem 2 (Applebaum [2])). *If $Y(t)$ is a Lévy-type stochastic integral of the form*

$$dY(t) = G(t)dt + F(t)dW(t) + \int_{|x|<1} H(t,x)\tilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)N(dt,dx) \tag{2.5}$$

then for all $f \in C^2(\mathbb{R}^d), t \geq 0$ with probability 1 we have

$$\begin{aligned} &f(Y(t)) - (Y(0)) = \\ &\int_0^t \partial_i f(Y(s-))dY_c^i(s) + \frac{1}{2}\int_0^t \partial_i\partial_j f(Y(s-))d[Y_c^i, Y_c^i](s) \\ &+ \int_0^t \int_{|x|\geq 1} [f(Y(s-) + K(s,x)) - f(Y(s-)]N(ds,dx) \\ &+ \int_0^t \int_{|x|<1} [f(Y(s-) + H(s,x)) - f(Y(s-)]\tilde{N}(ds,dx) \\ &+ \int_0^t \int_{|x|<1} [f(Y(s-) + H(s,x)) - f(Y(s-) \\ &- H^i(s,x)\partial_j f(Y(s-))]\nu(dx)ds, \end{aligned} \tag{2.6}$$

where $t \geq 0$, $\nu(dx)$ is the Lévy measure, $|G^i|^{\frac{1}{2}}, F_j^i \in \mathcal{P}_2(T)$, $H^i \in \mathcal{P}_2(T,E)$ and K is predictable. Here, B is standard Brownian motion and N is an independent Poisson random measure on $\mathbb{R}\times(\mathbb{R}\backslash\{0\})$ with compensator $\tilde{N}(ds,dx) = N(ds,dx) - \nu(dx)(ds)$ and intensity measure ν, which we will assume is a Lévy measure. Note that $\mathcal{P}_2(T,E)$ is the set of all equivalence classes of mappings $F : [0,T]\times E\times\Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\nu(dx)dt$ and satisfy the following conditions:

- F is predictable;
- $P\left(\int_0^T \int_E |F(t,x)|^2 \nu(dx)dt < \infty\right) = 1.$

Note that Y_c is the continuous part of Y. Note that $[Y_c^i, Y_c^j]$ is the quadratic variation which is defined below:

For each $t \geq 0$, we define a $d \times d$ matrix-valued adapted process

$$[Y,Y] = ([Y,Y](t), t \geq 0)$$

by the following prescription for its (i,j)th entry $(1 \leq i,j \leq d)$:

$$[Y^i, Y^j](t) = [Y_c^i, Y_c^j](t) + \sum_{0 \leq s \leq t} \Delta Y^i(s) \Delta Y^j(s) \tag{2.7}$$

From Øksendal [46], we have the following definition of quadratic variation for a general process X_t:

$$[X_t, X_t] = \lim_{\Delta t_k \to 0} \sum_{t_k \leq t} |X_{t_{k+1}} - X_{t_k}|^2 \text{ in probability,}$$

where $0 = t_1 < t_2 < \ldots < t_n = t$ and $\Delta_{t_k} = t_{k+1} - t_k$. The limit can be shown exist for continuous integrable martingale.

In the next section, we give an introduction to the Geometric Brownian Motion.

2.3 GEOMETRIC BROWNIAN MOTIONS

A stochastic process X_t is said to be a Geometric Brownian Motion (GBM) if it satisfies the following stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \ X(0) = x_0, \tag{2.8}$$

where W_t is a Brownian motion, μ is the drift term which is used to model deterministic trends and σ is the volatility. In financial mathematics, equation (2.8) is the most popular model used for the dynamics of the price of a stock in the Black-Scholes option pricing model.

Using the Itô's formula (2.4), the exact solution of (2.8) is given by

$$X_t = x_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right), \tag{2.9}$$

where $X(t_0) = x_0 > 0$ is the initial condition.

The solution (2.9) is log-normally distributed random variable with expected value and variance given by

$$E(X_t) = x_0 e^{\mu t} \quad \text{and} \quad Var(X_t) = x_0^2 e^{2\mu t}\left(e^{t\sigma^2} - 1\right),$$

respectively.

We consider the stochastic process $\log(X_t)$ because in the Black-Scholes model, it is related to the log-return of the stock price.

Using the Itô's formula (2.4) with $f(x) = \log(x)$, we have that

$$d(\log X_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \tag{2.10}$$

The explicit solution of (2.10) is given by

$$\log(X_t) = \log(x_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \tag{2.11}$$

where $X(0) = x_0$ is the initial condition. It is easy to see that the mean for the process is $\mathbb{E}(\log(X_t)) = \log(x_0) + \left(\mu - \frac{\sigma^2}{2}\right) t$.

Starting with Merton's [38] groundbreaking work and continuing to the present, many models with jumps have been proposed. For other important contributions about financial modeling with jump process, one can go over the list of references in Cont and Tankov [8].

In the next Chapter, we shall propose geometric α-stable Lévy motion model and estimate the involved parameters.

CHAPTER 3

PARAMETER ESTIMATION FOR GEOMETRIC LÉVY PROCESSES

3.1 MODEL SPECIFICATION

Lévy processes encompass the nature of both Brownian motion and Poisson processes. Like Brownian motion, Lévy processes are used in applications ranging from climate change and Geo-sciences to insurance and finance. Similar to the Poisson processes, Lévy processes allow random moves by jumps, which is an important feature for many applications.

In the last twenty years, Lévy processes have received increased attention in academic and industrial settings, because they provide versatile stochastic models of financial markets. One of our main goals is to develop models that capture important features of financial data. Combining the geometric Brownian motion (2.10) and an α-stable Lévy process, we consider the following SDE (an α-stable Geometric Lévy process):

$$d(\log(X_t)) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \tau dL_t^{\alpha}, \quad X(0) = x_0, \tag{3.1}$$

where X_t is the process at time t, $(\mu - \frac{1}{2}\sigma^2)$ is the constant trend or drift, σ denotes the volatility of the diffusion component, W_t is a standard Wiener process (Brownian motion), L_t^{α} is a standard α-stable Lévy motion with index of stability $0 < \alpha < 2$, skewness parameter $\beta \in [-1, 1]$ and $\tau > 0$ is the dispersion constant of the α-stable Lévy motion.

Lévy type stochastic calculus is used to describe α-stable Lévy motion. From Hu and Long [21], the Lévy measure is given by

$$\nu\left(dx\right)=\frac{c_1}{x^{1+\alpha}}\mathbb{1}_{(0,\infty)}(x)dx+\frac{c_2}{|x|^{1+\alpha}}\mathbb{1}_{(-\infty,0)}(x)dx,$$

where $c_1\geq 0$, $c_2\geq 0$, $c_1+c_2>0$, and

$$L_t^{\alpha}=\frac{1}{\varsigma}\int_0^t\int_{\mathbb{R}\backslash\{0\}}x\tilde{N}(ds,dx),$$

where $\tilde{N}(dt,dx)$ is a compensated Poisson random measure,

$$\varsigma^{\alpha}=-\left(c_1+c_2\right)\Gamma\left(-\alpha\right)\cos\left(\frac{\pi\alpha}{2}\right)\text{ and }\beta=\frac{c_1-c_2}{c_1+c_2}.$$

Then we have

$$dL_t^{\alpha}=\frac{1}{\varsigma}\int_{\mathbb{R}\backslash\{0\}}x\tilde{N}(dt,dx).$$

Using Itô's formula (2.6) for a Lévy-type stochastic integral, we can show that the solution of (3.1) satisfies the following equation

$$\begin{aligned}X_t=x_0&+\int_0^t X_s\left(\mu ds+\sigma dW_s\right)\\&+\int_0^t\int_{|x|\geq 1}\Big[X_s(e^{\tau x}-1)\Big]N(ds,dx)\\&+\int_0^t\int_{|x|<1}\Big[X_s(e^{\tau x}-1)\Big]\tilde{N}(ds,dx)\\&+\int_0^t\int_{|x|<1}\Big[X_s(e^{\tau x}-\tau x-1)\Big]\nu(dx)ds,\end{aligned}\tag{3.2}$$

where $X(0)=x_0$ is the initial condition.

Here the Lévy measure ν is a measure satisfying

$$\nu(\{0\})=0\text{ and }\int_{-\infty}^{\infty}(|x|^2\vee 1)\nu(dx)<\infty.$$

Also, $\tilde{N}$ is the compensator $\tilde{N}(dt,dx)=N(dt,dx)-\nu(dx)dt$, where $N(dt,dx)$ is a Poisson random measure defined by

$$N((0,t],A)=\sum_{s\leq t}\mathbb{1}_A(\Delta L_s^{\alpha})$$

for $A \in \mathcal{B}(\mathbb{R} - \{0\})$ (the Borel-sigma algebra generated by open sets in $\mathbb{R} - \{0\}$) and $\Delta L_s^\alpha = L_s^\alpha - L_{s^-}^\alpha$ denoting the jump of L_s^α at time s.

We note that the (3.2) is not useful in estimating parameters of the model (3.1). The main goals of this Chapter are as follows:

- We use the sample characteristic function method or moment estimation method to estimate all the parameters in model (3.1). We estimate the index of stability α, the scale parameter τ, the volatility parameter σ, the drift parameter μ, the dispersion constant τ and the skewness parameter β from discretely observed stochastic differential equation (3.1) driven by α-stable Lévy motions.

- We obtain the strong consistency and asymptotic normality of the proposed estimators when the sample size goes to infinity while sampling time step remains arbitrarily fixed.

- We will perform the simulation to assess the validity of the proposed estimators.

To estimate the parameters of the continuous-time specifications, we need to discretize the model (3.1). We will use the Euler scheme to discretize the stochastic differential equation (3.1). We use the sample characteristic function method to estimate the parameters based on the sampling data $\left(X_{t_j}\right)_{j=1}^n$ taken from n observations.

Assume that the process is observed at some discrete time instants $\{t_j = jh, j = 1, 2, \ldots, n\}$, with h fixed. The discretization of the SDE (3.1) is given by

$$\begin{aligned} Z_j &= \Delta(\log(X_{t_j})) \\ &= \log(X_{t_j}) - \log(X_{t_{j-1}}) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) h + \sigma \Delta W_{t_j} + \tau \Delta L_{t_j}^\alpha, \quad j = 1, 2, \ldots, n \end{aligned} \tag{3.3}$$

where $h = \Delta t_j = t_j - t_{j-1}$ is the step-size, $\Delta W_{t_j} = W_{t_j} - W_{t_{j-1}}$ is the increment of the Brownian motion, and the increment of the α-stable Lévy motion is

$\Delta L^{\alpha}_{t_j} = L^{\alpha}_{t_j} - \Delta L^{\alpha}_{t_{j-1}} \sim (h)^{\frac{1}{\alpha}} S_{\alpha}(1, \beta, 0)$, where $S_{\alpha}(1, \beta, 0)$ is the standard α-stable distribution.

3.2 ESTIMATION OF ALL PARAMETERS

3.2.1 Sample Characteristic Function Approach

The problem of estimating the parameters of stable distributions is usually severely hampered by the lack of known closed form density functions for the vast majority of stable distributions (see Belovas et al. [3]). Standard estimation methods have been proposed to estimate parameters of stable laws (see Fama and Roll [16] , Koutrouvelis [30], Mittnik et al. [41] and Press [48]).

In this section, we review some of those methods of parameter estimation. There are a number of different parametrization of stable distributions. We refer to Zolotarev [61] for some possible parameterizations. Among those, equation (2.1) is one of the most widely used parameterizations of the characteristic function $\phi(u)$.

The estimation of parameters of stable distributions was first initiated by Fama and Roll [16] in 1971. Fama and Roll [16] proposed the quantile estimation method to estimate parameters of symmetric stable laws ($\beta = 0$). Later on, in 1986, McCulloch [37] generalized Fama and Roll's method. Also, the maximum likelihood method to estimate parameters of stable laws was discussed by Mittnik et al. [41] and Nolan [44]. The regression type method of parameter estimation of stable laws was proposed by Koutrouvelis [29] in 1980 for strictly distributed stable random variables. In 1986, Zolotarev [61] constructed an estimate of α based on special transformations of the underlying random variables. Estimating the index of a stable law was also discussed, in 1999, by Marohn [36], where the pot-method was used. In 1996, Tsihrintzis and Nikias [56] proposed the fast estimation method which is based on moment properties of stable random variables. We also refer to Nolan [45], Uchaikin and Zolotarev [57], Dumouchel [14], and Rachev [49] for further references about methods of estimation of stable laws. For the comparison of estimators in stable models, we refer to Höpfner and Rüschendorf [20].

Hereby we use a sample characteristic function approach, which belongs to the generalized method of moments, to estimate parameters of the SDE (3.1). This method is mainly based on the transformations of the characteristic function, which was first proposed by Press [48] in 1972. In this method, we fit the sample characteristic function $\phi_n(u)$ to the theoretical characteristic function $\phi(u)$.

The sample characteristic function of $\Delta(\log(X_{t_j}))$ (the discretized SDE (3.3)) is defined by

$$\phi_n(u) = \frac{1}{n}\sum_{j=1}^{n} \exp\left(iu\Delta(\log(X_{t_j})\right) \tag{3.4}$$

for any $u \in \mathbb{R}$, $i = \sqrt{-1}$ and independent and identically distributed (i.i.d.) random variables $\left\{\Delta(\log(X_{t_j}))\right\}_{j=1}^{n}$. We note that $\phi_n(u)$ is computable for all values of u. Further $\{\phi_n(u), -\infty < u < \infty\}$ is a stochastic process, and for each u, $|\phi_n(u)|$ is bounded by unity. Hence all moments of $\phi_n(u)$ are finite. So for any fixed u, $\phi_n(u)$ is the average of i.i.d. random variable. Therefore by the law of large numbers, $\phi_n(u)$ is a consistent estimator of $\phi(u)$, i.e., $\phi_n(u) \to \phi(u)$ almost surely.

Note that $\left\{\Delta(\log(X_{t_j}))\right\}_{j=1}^{n}$ is a sequence of independent and identically distributed (i.i.d) random variables. Let us first compute the characteristic function $\phi(u)$ of $\Delta(\log(X_{t_j}))$ for any j.

For any $u \in \mathbb{R}$ and $i = \sqrt{-1}$ (the imaginary unit), by the independence of ΔW_{t_j} and $\Delta L^{\alpha}_{t_j}$, the characteristic function of $\Delta(\log(X_{t_j}))$, for any j is given by

$$\begin{aligned}
\phi(u) &= \mathbb{E}\left(e^{iu\Delta(\log(X_{t_j}))}\right) \\
&= \mathbb{E}\left(e^{\left[iu(\mu-\frac{1}{2}\sigma^2)h + iu\sigma\Delta W_{t_j} + iu\tau\Delta L^{\alpha}_{t_j}\right]}\right) \\
&= \left(e^{\left[iu(\mu-\frac{1}{2}\sigma^2)h\right]}\right) \times \mathbb{E}\left(e^{(iu\sigma\Delta W_{t_j})}\right) \times \mathbb{E}\left(e^{(iu\tau\Delta L^{\alpha}_{t_j})}\right) \\
&= e^{iu(\mu-\frac{1}{2}\sigma^2)h} \times e^{-\frac{1}{2}\sigma^2 u^2 h} \times \mathbb{E}\left(e^{(iu\tau\Delta L^{\alpha}_{t_j})}\right).
\end{aligned} \tag{3.5}$$

Thus,

$$\phi(u) = \begin{cases} \exp\left[J(u) - h\tau^\alpha |u|^\alpha \left(1 - i\beta sgn(u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right], & \alpha \neq 1 \\ \exp\left[J(u) - h\tau |u| \left(1 + i\beta \frac{2}{\pi} sgn(u) \log|u|\right)\right], & \alpha = 1 \end{cases} \tag{3.6}$$

where $J(u) = iu\mu h - \frac{1}{2} i\sigma^2 uh - \frac{1}{2}\sigma^2 u^2 h$.

We shall estimate $\alpha, \sigma, \tau, \beta$ and μ by matching $\phi_n(u)$ with $\phi(u)$ at different values of u in the following subsections.

3.2.2 Estimator for α

For any $u \in \mathbb{R}$, from the characteristic function (3.6), we have

$$\log|\phi(u)| = -\frac{1}{2}\sigma^2 u^2 h - h\tau^\alpha |u|^\alpha. \tag{3.7}$$

Replacing the theoretical characteristic function $\phi(u)$ by the empirical characteristic function $\phi_n(u)$ and choosing three non-zero different real numbers u_1, u_2 and u_3, we get

$$\log|\phi_n(u_1)| = -\frac{1}{2}\sigma^2 u_1^2 h - h\tau^\alpha |u_1|^\alpha, \tag{3.8}$$

$$\log|\phi_n(u_2)| = -\frac{1}{2}\sigma^2 u_2^2 h - h\tau^\alpha |u_2|^\alpha, \tag{3.9}$$

and

$$\log|\phi_n(u_3)| = -\frac{1}{2}\sigma^2 u_3^2 h - h\tau^\alpha |u_3|^\alpha. \tag{3.10}$$

Multiplying equation (3.8) by u_2^2 and equation (3.9) by u_1^2 and then subtracting them, we get

$$u_2^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_2)| = h\tau^\alpha \left(u_1^2 |u_2|^\alpha - u_2^2 |u_1|^\alpha\right). \tag{3.11}$$

Similarly, multiplying equation (3.8) by u_3^2 and equation (3.10) by u_1^2 and then subtracting them, we get

$$u_3^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_3)| = h\tau^\alpha \left(u_1^2 |u_3|^\alpha - u_3^2 |u_1|^\alpha\right). \tag{3.12}$$

Dividing equation (3.11) by equation (3.12), we get

$$\frac{u_2^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_2)|}{u_3^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_3)|} = \frac{u_1^2|u_2|^\alpha - u_2^2|u_1|^\alpha}{u_1^2|u_3|^\alpha - u_3^2|u_1|^\alpha}, \quad \text{which yields}$$
$$\frac{(\frac{u_2}{u_1})^2 \log|\phi_n(u_1)| - \log|\phi_n(u_2)|}{(\frac{u_3}{u_1})^2 \log|\phi_n(u_1)| - \log|\phi_n(u_3)|} = \frac{|\frac{u_2}{u_1}|^\alpha - (\frac{u_2}{u_1})^2}{|\frac{u_3}{u_1}|^\alpha - (\frac{u_3}{u_1})^2}, \quad u_1 \neq u_2 \neq u_3 \neq 0. \tag{3.13}$$

We solve the nonlinear equation (3.13) by using the Newton's method to get $\hat{\alpha}_n$ (the estimator for α). By the almost sure convergence of $\phi_n(u)$ to $\phi(u)$, we have that $\hat{\alpha}_n$ converges to α almost surely.

3.2.3 Estimator for σ

To find the estimator for σ, we multiply (3.8) by $|u_2|^\alpha$ and (3.9) by $|u_1|^\alpha$ and then we subtract them to get the estimator

$$\hat{\sigma}_n^2 = \frac{2|u_2|^{\hat{\alpha}_n} \log|\phi_n(u_1)| - 2|u_1|^{\hat{\alpha}_n} \log|\phi_n(u_2)|}{h\left(u_2^2|u_1|^{\hat{\alpha}_n} - u_1^2|u_2|^{\hat{\alpha}_n}\right)}, \quad u_1 \neq u_2 \neq 0. \tag{3.14}$$

Note that $\hat{\sigma}_n^2$ is positive, so in the simulation we consider only sample paths where the right side of (3.14) is positive.

By the almost sure convergence of $\hat{\alpha}_n$ and $\phi_n(u)$, we get the almost sure convergence of $\hat{\sigma}_n$ to σ.

3.2.4 Estimator for τ

To get the estimator $\hat{\tau}_n$ of τ, we solve the equation (3.11) for τ:

$$\tau^\alpha = \frac{u_2^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_2)|}{h\left(u_1^2|u_2|^\alpha - u_2^2|u_1|^\alpha\right)}, \quad \text{which yields}$$
$$\hat{\tau}_n = \left(\frac{u_2^2 \log|\phi_n(u_1)| - u_1^2 \log|\phi_n(u_2)|}{h\left(u_1^2|u_2|^{\hat{\alpha}_n} - u_2^2|u_1|^{\hat{\alpha}_n}\right)}\right)^{\frac{1}{\hat{\alpha}_n}}, \quad u_1 \neq u_2 \neq 0. \tag{3.15}$$

By the almost sure convergence of $\hat{\alpha}_n$ and $\phi_n(u)$, we have the almost sure convergence of $\hat{\tau}_n$ to τ.

3.2.5 Estimators for β and μ

We now discuss the estimation for the drift parameter $\mu \in \mathbb{R}$ and the skewness parameter $\beta \in [-1, 1]$.

For $\alpha \neq 1$, from the characteristic function $\phi(u)$ in (3.6), we have

$$\arctan\left(\frac{\Im(\phi(u)}{\Re(\phi(u))}\right) = u\mu h - \frac{1}{2}\sigma^2 uh + h\beta\tau^\alpha |u|^\alpha sgn(u)\tan\left(\frac{\pi\alpha}{2}\right), \tag{3.16}$$

where $\Re(\phi(u))$ and $\Im(\phi(u))$ are the real and imaginary parts of the complex valued function $\phi(u)$, respectively.

We choose two non-zero and different values u_4 and u_5 with

$$-\frac{\pi}{2} < u_4\mu h - \frac{1}{2}\sigma^2 u_4 h + h\beta\tau^\alpha |u_4|^\alpha sgn(u_4)\tan\left(\frac{\pi\alpha}{2}\right) < \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} < u_5\mu h - \frac{1}{2}\sigma^2 u_5 h + h\beta\tau^\alpha |u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right) < \frac{\pi}{2}.$$

By replacing $\phi(u)$ by $\phi_n(u)$, we can solve the following system of equations to get the estimators for β and μ :

$$\arctan\left(\frac{\Im(\phi_n(u_4))}{\Re(\phi_n(u_4))}\right) = u_4\mu h - \frac{1}{2}\sigma^2 u_4 h + h\beta\tau^\alpha |u_4|^\alpha sgn(u_4)\tan\left(\frac{\pi\alpha}{2}\right) \tag{3.17}$$

and

$$\arctan\left(\frac{\Im(\phi_n(u_5)}{\Re(\phi_n(u_5))}\right) = u_5\mu h - \frac{1}{2}\sigma^2 u_5 h + h\beta\tau^\alpha |u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right). \tag{3.18}$$

Multiplying equation (3.17) by u_5 and equation (3.18) by u_4 and subtracting them, we get the estimator

$$\hat{\beta}_n = \frac{u_5 \times \arctan\left(\frac{\Im(\phi_n(u_4))}{\Re(\phi_n(u_4))}\right) - u_4 \times \arctan\left(\frac{\Im(\phi_n(u_5))}{\Re(\phi_n(u_5))}\right)}{h\hat{\tau}_n^{\hat{\alpha}_n}\tan(\frac{\pi\hat{\alpha}_n}{2})\left[u_5|u_4|^{\hat{\alpha}_n}sgn(u_4) - u_4|u_5|^{\hat{\alpha}_n}sgn(u_5)\right]}, \tag{3.19}$$

where $u_4 \neq u_5 \neq 0$. Then, by replacing β by $\hat{\beta}_n$, α by $\hat{\alpha}_n$ and τ by $\hat{\tau}_n$ in (3.17), we get the estimator

$$\hat{\mu}_n = \frac{1}{hu_4}\left[\arctan\left(\frac{\Im(\phi_n(u_4))}{\Re(\phi_n(u_4))}\right) + \frac{1}{2}\hat{\sigma}_n^2 u_4 h - h\hat{\beta}_n\hat{\tau}_n^{\hat{\alpha}_n}|u_4|^{\hat{\alpha}_n}\right.$$
$$\left. \times\, sgn(u_4)\tan\left(\frac{\pi\hat{\alpha}_n}{2}\right)\right]. \tag{3.20}$$

By the almost sure convergence of $\hat{\alpha}_n$, $\hat{\sigma}_n$, $\hat{\tau}_n$, $\phi_n(u_4)$ and $\phi_n(u_5)$, we get the almost sure convergence of $\hat{\beta}_n$ to β and $\hat{\mu}_n$ to μ, respectively.

3.2.6 Joint Asymptotic Behavior of the Proposed Estimators

In this section, we develop the asymptotic distributions for the estimators of all the parameters when large samples are available. We study the joint behavior of the estimators of all the parameters α, σ, τ, β and μ. We closely follow the procedure and ideas in Cheng et al. [7], where the parameter estimation for α-stable Ornstein-Uhlenbeck process is studied.

Let $\eta = (\alpha, \sigma, \tau, \beta, \mu)$ and $\hat{\eta}_n = (\hat{\alpha}_n, \hat{\sigma}_n, \hat{\tau}_n, \hat{\beta}_n, \hat{\mu}_n)$. We would like to compute the asymptotic covariance of the estimators of all the parameters, i.e., the covariance matrix of $\sqrt{n}\,(\hat{\eta}_n - \eta)$ as $n \to \infty$.

For any nice function f, we let $S_n(f) = \frac{1}{n}\sum_{j=1}^{n} f(Z_j)$. Let $F_u(x) = \cos(ux)$ and $G_u(x) = \sin(ux)$, then we have $\phi_n(u) = \frac{1}{n}\sum_{j=1}^{n} e^{iuZ_j} = S_n(F_u) + iS_n(G_u)$.
Note that

$$|\phi_n(u)|^2 = S_n^2(F_u) + S_n^2(G_u). \tag{3.21}$$

Now, we construct the asymptotic covariance matrix related to

$$V_n = (V_{n_1}, V_{n_2}, V_{n_3}, V_{n_4}, V_{n_5}, V_{n_6}, V_{n_7}, V_{n_8}, V_{n_9}, V_{n_{10}})^T, \text{ where}$$
$$V_{n_1} = S_n(F_{u_1}), \quad V_{n_2} = S_n(G_{u_1}), \quad V_{n_3} = S_n(F_{u_2}), \quad V_{n_4} = S_n(G_{u_2}),$$

$$V_{n_5} = S_n(F_{u_3}), \quad V_{n_6} = S_n(G_{u_3}), \quad V_{n_7} = S_n(F_{u_4}), \quad V_{n_8} = S_n(G_{u_4}),$$
$$V_{n_9} = S_n(F_{u_5}), \quad \text{and} \quad V_{n_{10}} = S_n(G_{u_5}).$$

For two functions $f(x)$ and $g(x)$, the asymptotic covariance $Cov\left(\sqrt{n}S_n(f), \sqrt{n}S_n(g)\right)$ of $\sqrt{n}S_n(f)$ and $\sqrt{n}S_n(g)$ is defined by

$$\sigma_{fg} = \lim_{n\to\infty} Cov\left(\sqrt{n}S_n(f), \sqrt{n}S_n(g)\right). \tag{3.22}$$

The corresponding asymptotic covariance matrix is written as

$$\Sigma_{10} = \lim_{n\to\infty} \left(Cov(\sqrt{n}V_{nk}, \sqrt{n}V_{nl})\right)_{1\le k,l\le 10} = \left(\sigma_{g_k g_l}\right)_{1\le k,l\le 10}, \tag{3.23}$$

where

$$\begin{aligned}
&g_1(x) = F_{u_1}(x), \quad g_2(x) = G_{u_1}(x), \quad g_3(x) = F_{u_2}(x),\\
&g_4(x) = G_{u_2}(x), \quad g_5(x) = F_{u_3}(x), \quad g_6(x) = G_{u_3}(x),\\
&g_7(x) = F_{u_4}(x), \quad g_8(x) = G_{u_4}(x), \quad g_9(x) = F_{u_5}(x),\\
&g_{10}(x) = G_{u_5}(x).
\end{aligned} \tag{3.24}$$

Let $v = (v_1, v_2, \ldots, v_{10})^T$, where $v_j = \mathbb{E}(g_j(Z_1))$, $j = 1, 2, \ldots, 10$. By the strong law of large numbers, we know that $V_{n_j} \to v_j$ almost surely for $j = 1, 2, \ldots, 10$.

The explicit expressions of the elements in the covariance matrix Σ_{10} are provided at the end of this section.

For $z = (z_1, z_2, \ldots z_{10})^T$, $z_j \neq 0$, $j = 1, 2, \ldots, 10$, we define

$$\begin{aligned}
&\hat{\gamma}_1(z) = \frac{1}{2}\log(z_1^2 + z_2^2), \quad \hat{\gamma}_2(z) = \frac{1}{2}\log(z_3^2 + z_4^2),\\
&\hat{\gamma}_3(z) = \frac{1}{2}\log(z_5^2 + z_6^2), \quad \hat{\gamma}_4(z) = \arctan\left(\frac{z_8}{z_7}\right),\\
&\hat{\gamma}_5(z) = \arctan\left(\frac{z_{10}}{z_9}\right).
\end{aligned}$$

Then we have that

$$\gamma_1(\eta) := \hat{\gamma}_1(v) = -\frac{1}{2}\sigma^2 u_1^2 h - h\tau^\alpha |u_1|^\alpha, \tag{3.25}$$

$$\gamma_2(\eta) := \hat{\gamma}_2(v) = -\frac{1}{2}\sigma^2 u_2^2 h - h\tau^\alpha |u_2|^\alpha, \tag{3.26}$$

$$\gamma_3(\eta) := \hat{\gamma}_3(v) = -\frac{1}{2}\sigma^2 u_3^2 h - h\tau^\alpha |u_3|^\alpha, \tag{3.27}$$

$$\gamma_4(\eta) := \hat{\gamma}_4(v) = u_4\mu h - \frac{1}{2}\sigma^2 u_4 h + h\beta\tau^\alpha |u_4|^\alpha sgn(u_4)\tan\left(\frac{\pi\alpha}{2}\right), \tag{3.28}$$

$$\gamma_5(\eta) := \hat{\gamma}_5(v) = u_5\mu h - \frac{1}{2}\sigma^2 u_5 h + h\beta\tau^\alpha |u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right). \tag{3.29}$$

Let $\hat{\gamma}(z) = (\hat{\gamma}_1(z), \hat{\gamma}_2(z), \hat{\gamma}_3(z), \hat{\gamma}_4(z), \hat{\gamma}_5(z))^T$, for $z \in \mathbb{R}^{10}$.

So, we have

$$\hat{\gamma}^{(1)}(z) = \left(\frac{\partial \hat{\gamma}_j}{\partial z_k}\right)_{1\le j\le 5, 1\le k\le 10} \text{ and}$$
$$\gamma(\eta) = (\gamma_1(\eta), \gamma_2(\eta), \gamma_3(\eta), \gamma_4(\eta), \gamma_5(\eta))^T.$$

The partial derivatives of $\hat{\gamma}_j(z),\ j = 1,2,3,4,5$ with respect to $z_1, z_2, \ldots z_{10}$ are given by

$$\frac{\partial \hat{\gamma}_1}{\partial z_1} = \frac{z_1}{(z_1^2+z_2^2)}, \quad \frac{\partial \hat{\gamma}_1}{\partial z_2} = \frac{z_2}{(z_1^2+z_2^2)}, \quad \frac{\partial \hat{\gamma}_1}{\partial z_3} = \ldots = \frac{\partial \hat{\gamma}_1}{\partial z_{10}} = 0,$$
$$\frac{\partial \hat{\gamma}_2}{\partial z_1} = 0, \ \frac{\partial \hat{\gamma}_2}{\partial z_2} = 0, \quad \frac{\partial \hat{\gamma}_2}{\partial z_3} = \frac{z_3}{(z_3^2+z_4^2)}, \quad \frac{\partial \hat{\gamma}_2}{\partial z_4} = \frac{z_4}{(z_3^2+z_4^2)},$$
$$\frac{\partial \hat{\gamma}_2}{\partial z_5} = \ldots = \frac{\partial \hat{\gamma}_2}{\partial z_{10}} = 0,$$
$$\frac{\partial \hat{\gamma}_3}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_3}{\partial z_4} = 0, \quad \frac{\partial \hat{\gamma}_3}{\partial z_5} = \frac{z_5}{(z_5^2+z_6^2)}, \quad \frac{\partial \hat{\gamma}_3}{\partial z_6} = \frac{z_6}{(z_5^2+z_6^2)},$$
$$\frac{\partial \hat{\gamma}_3}{\partial z_7} = \ldots = \frac{\partial \hat{\gamma}_3}{\partial z_{10}} = 0,$$
$$\frac{\partial \hat{\gamma}_4}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_4}{\partial z_6} = 0, \quad \frac{\partial \hat{\gamma}_4}{\partial z_7} = \frac{-z_8}{(z_7^2+z_8^2)}, \quad \frac{\partial \hat{\gamma}_4}{\partial z_8} = \frac{z_7}{(z_7^2+z_8^2)}, \quad \frac{\partial \hat{\gamma}_4}{\partial z_9} = \frac{\partial \hat{\gamma}_4}{\partial z_{10}} = 0,$$
$$\frac{\partial \hat{\gamma}_5}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_5}{\partial z_8} = 0, \quad \frac{\partial \hat{\gamma}_5}{\partial z_9} = \frac{-z_{10}}{(z_9^2+z_{10}^2)}, \quad \frac{\partial \hat{\gamma}_5}{\partial z_{10}} = \frac{z_9}{(z_9^2+z_{10}^2)}.$$

Set

$$\Phi_n(\eta) = (\Phi_{1,n}(\eta), \Phi_{2,n}(\eta), \Phi_{3,n}(\eta), \Phi_{4,n}(\eta), \Phi_{5,n}(\eta))^T,$$

where $\Phi_{j,n}(\eta) = \hat{\gamma}_j(V_n) - \gamma_j(\eta),\ j = 1,2,3,4,5$. Then, $\hat{\eta}_n$ satisfies $\Phi_n(\hat{\eta}_n) = 0$.

We have

$$\frac{\partial \gamma_1}{\partial \alpha} = -h(\tau|u_1|)^\alpha \times \log(\tau|u_1|), \quad \frac{\partial \gamma_1}{\partial \sigma} = -\sigma u_1^2 h,$$
$$\frac{\partial \gamma_1}{\partial \tau} = -\alpha h|u_1|^\alpha \tau^{\alpha-1}, \quad \frac{\partial \gamma_1}{\partial \beta} = \frac{\partial \gamma_1}{\partial \mu} = 0.$$

$$\frac{\partial \gamma_2}{\partial \alpha} = -h(\tau|u_2|)^\alpha \times \log(\tau|u_2|), \quad \frac{\partial \gamma_2}{\partial \sigma} = -\sigma u_2^2 h,$$
$$\frac{\partial \gamma_2}{\partial \tau} = -\alpha h|u_2|^\alpha \tau^{\alpha-1}, \quad \frac{\partial \gamma_2}{\partial \beta} = \frac{\partial \gamma_2}{\partial \mu} = 0.$$

$$\frac{\partial \gamma_3}{\partial \alpha} = -h(\tau|u_3|)^\alpha \times \log(\tau|u_3|), \quad \frac{\partial \gamma_3}{\partial \sigma} = -\sigma u_3^2 h$$
$$\frac{\partial \gamma_3}{\partial \tau} = -\alpha h|u_3|^\alpha \tau^{\alpha-1}, \quad \frac{\partial \gamma_3}{\partial \beta} = \frac{\partial \gamma_3}{\partial \mu} = 0.$$

$$\frac{\partial \gamma_4}{\partial \alpha} = h\beta sgn(u_4)\left(\frac{\pi}{2}(\tau|u_4|)^\alpha \sec^2(\frac{\pi\alpha}{2}) + (\tau|u_4|)^\alpha \tan(\frac{\pi\alpha}{2})\log(\tau|u_4|)\right),$$
$$\frac{\partial \gamma_4}{\partial \tau} = h\alpha\beta\tau^{\alpha-1} sgn(u_4)|u_4|^\alpha \tan\left(\frac{\pi\alpha}{2}\right), \frac{\partial \gamma_4}{\partial \sigma} = -\sigma u_4 h,$$
$$\frac{\partial \gamma_4}{\partial \beta} = h\tau^\alpha|u_4|^\alpha sgn(u_4)\tan\left(\frac{\pi\alpha}{2}\right), \quad \frac{\partial \gamma_4}{\partial \mu} = u_4 h.$$

$$\frac{\partial \gamma_5}{\partial \alpha} = h\beta sgn(u_5)\left(\frac{\pi}{2}(\tau|u_5|)^\alpha \sec^2(\frac{\pi\alpha}{2}) + (\tau|u_5|)^\alpha \tan(\frac{\pi\alpha}{2})\log(\tau|u_5|)\right),$$
$$\frac{\partial \gamma_5}{\partial \tau} = h\alpha\beta\tau^{\alpha-1} sgn(u_5)|u_5|^\alpha \tan\left(\frac{\pi\alpha}{2}\right), \quad \frac{\partial \gamma_5}{\partial \sigma} = -\sigma u_5 h,$$
$$\frac{\partial \gamma_5}{\partial \beta} = h\tau^\alpha|u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right), \quad \frac{\partial \gamma_5}{\partial \mu} = u_5 h.$$

It's easy to see that $\nabla_\eta \Phi_n(\eta) = -\nabla_\eta \gamma(\eta)$,

$$\text{where } \nabla_\eta \gamma(\eta) = \begin{pmatrix} \frac{\partial \gamma_1(\eta)}{\partial \alpha} & \frac{\partial \gamma_1(\eta)}{\partial \sigma} & \frac{\partial \gamma_1(\eta)}{\partial \tau} & \frac{\partial \gamma_1(\eta)}{\partial \beta} & \frac{\partial \gamma_1(\eta)}{\partial \mu} \\ \frac{\partial \gamma_2(\eta)}{\partial \alpha} & \frac{\partial \gamma_2(\eta)}{\partial \sigma} & \frac{\partial \gamma_2(\eta)}{\partial \tau} & \frac{\partial \gamma_2(\eta)}{\partial \beta} & \frac{\partial \gamma_2(\eta)}{\partial \mu} \\ \frac{\partial \gamma_3(\eta)}{\partial \alpha} & \frac{\partial \gamma_3(\eta)}{\partial \sigma} & \frac{\partial \gamma_3(\eta)}{\partial \tau} & \frac{\partial \gamma_3(\eta)}{\partial \beta} & \frac{\partial \gamma_3(\eta)}{\partial \mu} \\ \frac{\partial \gamma_4(\eta)}{\partial \alpha} & \frac{\partial \gamma_4(\eta)}{\partial \sigma} & \frac{\partial \gamma_4(\eta)}{\partial \tau} & \frac{\partial \gamma_4(\eta)}{\partial \beta} & \frac{\partial \gamma_4(\eta)}{\partial \mu} \\ \frac{\partial \gamma_5(\eta)}{\partial \alpha} & \frac{\partial \gamma_5(\eta)}{\partial \sigma} & \frac{\partial \gamma_5(\eta)}{\partial \tau} & \frac{\partial \gamma_5(\eta)}{\partial \beta} & \frac{\partial \gamma_5(\eta)}{\partial \mu} \end{pmatrix}$$

Let $I(\eta) = \nabla_\eta \gamma(\eta)$. Let $U = (U_1, U_2, \ldots U_{10})^T \sim N(0, \Sigma_{10})$, then we have the following Central Limit Theorem. Our proof idea mainly follow Cheng et al. [7].

Theorem 3.2.1. *Let* $U = (U_1, U_2, \ldots, U_{10})^T \sim N(0, \Sigma_{10})$, $V_n = (V_{n1}, V_{n2}, \ldots, V_{n_{10}})^T$, $v = (v_1, v_2, \ldots, v_{10})^T$, *then we have the Central Limit Theorem:*

$$\sqrt{n}(V_n - v) \xrightarrow{d} U \tag{3.30}$$

Proof. Since $U \sim N(0, \Sigma_{10})$, for any non-zero vector $a = (a_1, a_2, \ldots, a_{10})^T \in \mathbb{R}^{10}$, it follows that

$$a^T U \sim N(0, a^T \Sigma_{10} a).$$

Define $H = a^T (g_1, g_2, \ldots, g_{10})^T$, $\bar{H} = H - \mathbb{E}[H(Z_1)] = a^T (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{10})^T$. By the Central Limit Theorem for i.i.d random variables, it follows that

$$a^T \sqrt{n}(V_n - v) = \sqrt{n} S_n(\bar{H}) \xrightarrow{d} N(0, \sigma_H^2), \tag{3.31}$$

where $\sigma_H^2 = \mathbb{E}[\bar{H}^2(Z_1)] = a^T \Sigma_{10} a$. So for any nonzero $a \in \mathbb{R}^{10}$, we have $a^T \sqrt{n}(V_n - v) \xrightarrow{d} a^T U$. This implies $\sqrt{n}(V_n - v) \xrightarrow{d} U$ by Theorem 29.4 of Billingsley [4]. □

We also have the following Central Limit Theorem:

Theorem 3.2.2. *As* $n \to \infty$, *we have*

$$\sqrt{n}\Phi_n(\eta) \xrightarrow{d} \hat{\gamma}^{(1)}(v) U.$$

Proof. Since $\sqrt{n}\Phi_n(\eta) = \sqrt{n}(\hat{\gamma}(V_n) - \hat{\gamma}(v))$, then by Theorem 3.2.1 and the delta method we obtain the result. □

Finally, we have the following main result:

Theorem 3.2.3. *Fix an arbitrary* $h > 0$. *Denote* $\eta = (\alpha, \sigma, \tau, \beta, \mu)$ *and* $\hat{\eta}_n = (\hat{\alpha}_n, \hat{\sigma}_n, \hat{\tau}_n, \hat{\beta}_n, \hat{\mu}_n)$, *where* $\hat{\alpha}_n$, $\hat{\sigma}_n$, $\hat{\tau}_n$, $\hat{\beta}_n$ *and* $\hat{\mu}_n$ *are given by equations* (3.13), (3.14), (3.15), (3.19) *and* (3.20), *respectively. Then the following results hold:*

i. $\lim_{n\to\infty} \hat{\eta}_n = \eta$ *almost surely.*

ii.

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{d} N(0, \Sigma_5), \tag{3.32}$$

where $\Sigma_5 = (I(\eta))^{-1}\hat{\gamma}^{(1)}(v)\Sigma_{10}\left(\hat{\gamma}^{(1)}(v)\right)^T\left((I(\eta))^{-1}\right)^T.$

Proof. The proof ideas mainly follow Cheng et al. [7].

i. Note that each component of $\hat{\eta}_n$ converges to the corresponding component of η almost surely as $n \to \infty$ as discussed in subsections 3.2.2 - 3.2.5. Therefore $\lim_{n\to\infty} \hat{\eta}_n = \eta$ almost surely.

ii. By mean-value theorem for vector-valued functions, it follows that

$$\Phi_n(\hat{\eta}_n) - \Phi_n(\eta) = \int_0^1 \nabla_\eta \Phi_n(\eta + s(\hat{\eta}_n - \eta))ds.(\hat{\eta}_n - \eta) \tag{3.33}$$

Denote $I_n(\eta) = -\int_0^1 \nabla_\eta \Phi_n(\eta + s(\hat{\eta}_n - \eta))ds$, which is invertible. From the first part, we obtain $\Phi_n(\hat{\eta}_n) = 0$, which implies that

$$\sqrt{n}(\hat{\eta}_n - \eta) = (I_n(\eta))^{-1}.\sqrt{n}\Phi_n(\eta). \tag{3.34}$$

Since $\hat{\eta}_n \to \eta$ a.s., it follows that $(I(\eta_n))^{-1} \to (I(\eta))^{-1}$ a.s. By using Theorem 3.2.2 and Slutsky's Theorem, we find

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{d} N(0, \Sigma_5).$$

□

Next, we compute the elements of the covariance matrix Σ_{10}.

Computation of the Covariance Matrix of Σ_{10} : We have

$$\begin{aligned}
&Cov(\sqrt{n}S_n(f), \sqrt{n}S_n(f)) \\
&= \mathbb{E}[\sqrt{n}(S_n(f) - \mathbb{E}(S_n(f))).\sqrt{n}(S_n(g) - \mathbb{E}(S_n(g)))] \\
&= n\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n}(f(Z_j) - \mathbb{E}(f(Z_j))).\frac{1}{n}\sum_{k=1}^{n}(g(Z_k) - \mathbb{E}(g(Z_k)))\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\mathbb{E}\Bigg[\sum_{j=1}^{n}(f(Z_j) - \mathbb{E}(f(Z_j)))(g(Z_j) - \mathbb{E}(g(Z_j))) \\
&\quad + \sum_{k\neq j}(f(Z_j) - \mathbb{E}(f(Z_j)))(g(Z_k)) - \mathbb{E}(g(Z_k))\Bigg] \\
&= \mathbb{E}\Big[(f(Z_1) - \mathbb{E}(f(Z_1)))(g(Z_1) - \mathbb{E}(g(Z_1)))\Big] \\
&= Cov(f(Z_1), g(Z_1)) = \sigma_{fg}. \qquad (3.35)
\end{aligned}
$$

Note that $\sigma_{fg} = \sigma_{gf}$. If $f = g$, then $\sigma_{fg} = \sigma_{ff} = Var(f(Z_1))$. Thus the covariance matrix is given by

$$
\Sigma_{10} = \begin{pmatrix}
\sigma_{g_1g_1} & \sigma_{g_1g_2} & \cdots\cdots & \sigma_{g_1g_{10}} \\
\sigma_{g_2g_1} & \sigma_{g_2g_2} & \cdots\cdots & \sigma_{g_2g_{10}} \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\sigma_{g_9g_1} & \sigma_{g_9g_2} & \cdots\cdots & \sigma_{g_9g_{10}} \\
\sigma_{g_{10}g_1} & \sigma_{g_{10}g_2} & \cdots\cdots & \sigma_{g_{10}g_{10}}
\end{pmatrix}.
$$

By using the characteristic function $\phi(u)$ for $\alpha \neq 1$ given in (3.6), we define

$$
\begin{aligned}
A(u) &= \mathbb{E}(\cos uZ_1) \\
&= \exp\left(-\frac{1}{2}\sigma^2u^2h - h\tau^\alpha|u|^\alpha\right) \\
&\quad \times \cos\left(u\mu h - \frac{1}{2}\sigma^2uh + h\tau^\alpha|u|^\alpha\beta sgn(u)\tan\left(\frac{\pi\alpha}{2}\right)\right) \qquad (3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
B(u) &= \mathbb{E}(\sin uZ_1) \\
&= \exp\left(-\frac{1}{2}\sigma^2u^2h - h\tau^\alpha|u|^\alpha\right) \\
&\quad \times \sin\left(u\mu h - \frac{1}{2}\sigma^2uh + h\tau^\alpha|u|^\alpha\beta sgn(u)\tan\left(\frac{\pi\alpha}{2}\right)\right). \qquad (3.37)
\end{aligned}
$$

Next, we find the explicit expressions of the elements of the covariance matrix Σ_{10}.

Computation of $\sigma_{g_k g_l}$: $(k, l = 1, 2, \ldots, 10)$

For $F_u(x) = \cos ux$ and $F_v(x) = \cos vx$, we have

$$\begin{aligned}
\sigma_{F_u F_v} &= cov(\cos uZ_1, \cos vZ_1) \\
&= \mathbb{E}\left(\cos uZ_1, \cos vZ_1\right) - \mathbb{E}\left(\cos uZ_1\right)\mathbb{E}\left(\cos vZ_1\right) \\
&= \mathbb{E}\Big[\frac{\cos((u+v)Z_1) + \cos((u-v)Z_1)}{2}\Big] - \mathbb{E}\left(\cos uZ_1\right)\mathbb{E}\left(\cos vZ_1\right) \\
&= \frac{1}{2}\big[\mathbb{E}(\cos(u+v)Z_1) + \mathbb{E}(\cos(u-v)Z_1)\big] - \mathbb{E}(\cos uZ_1)\mathbb{E}(\cos vZ_1) \\
&= \frac{1}{2}\big[A(u+v) + A(u-v)\big] - A(u)A(v). \qquad (3.38)
\end{aligned}$$

So, from equation (3.24), we have

$$\begin{aligned}
\sigma_{g_{2k-1}g_{2l-1}} &= \sigma_{F_{u_k}F_{u_l}} \\
&= \frac{1}{2}\big[A(u_k+u_l) + A(u_k-u_l)\big] - A(u_k)A(u_l), \quad k, l = 1, 2, 3, 4, 5.
\end{aligned} \qquad (3.39)$$

Next for $G_u(x) = \sin ux$ and $G_v(x) = \sin vx$, we find

$$\begin{aligned}
\sigma_{G_u G_v} &= cov(\sin uZ_1, \sin vZ_1) \\
&= \mathbb{E}\left(\sin uZ_1, \sin vZ_1\right) - \mathbb{E}\left(\sin uZ_1\right)\mathbb{E}\left(\sin vZ_1\right) \\
&= \frac{1}{2}\big[\mathbb{E}(\cos(u-v)Z_1) - \mathbb{E}(\cos(u+v)Z_1)\big] - \mathbb{E}(\sin uZ_1)\mathbb{E}(\sin vZ_1) \\
&= \frac{1}{2}\big[A(u-v) - A(u+v)\big] - B(u)B(v). \qquad (3.40)
\end{aligned}$$

So we have

$$\begin{aligned}
\sigma_{g_{2k}g_{2l}} &= \sigma_{G_{u_k}G_{u_l}} \\
&= \frac{1}{2}\big[A(u_k-u_l) - A(u_k+u_l)\big] - B(u_k)B(u_l), \quad k, l = 1, 2, 3, 4, 5. \qquad (3.41)
\end{aligned}$$

For $F_u(x) = \cos ux$ and $G_v(x) = \sin vx$, we have

$$\begin{aligned}\sigma_{F_uG_v} &= cov(\cos uZ_1, \sin vZ_1)\\ &= \mathbb{E}\left(\cos uZ_1, \sin vZ_1\right) - \mathbb{E}\left(\cos uZ_1\right)\mathbb{E}\left(\sin vZ_1\right)\\ &= \frac{1}{2}\mathbb{E}\big[\sin(u+v)Z_1 - \sin(u-v)Z_1\big] - \mathbb{E}(\cos uZ_1)\mathbb{E}(\sin vZ_1)\\ &= \frac{1}{2}\big[B(u+v) - B(u-v)\big] - A(u)B(v). \end{aligned} \tag{3.42}$$

Then from (3.24), we find

$$\begin{aligned}\sigma_{g_{2l}g_{2k-1}} =& \sigma_{g_{2k-1}g_{2l}} = \sigma_{F_{u_k}G_{u_l}}\\ &= \frac{1}{2}\big[B(u_k+u_l) - B(u_k-u_l)\big] - A(u_k)B(u_l), \end{aligned} \tag{3.43}$$

where $k, l = 1, 2, 3, 4, 5$. Thus, we obtain the explicit expressions in (3.39), (3.41) and (3.43) for computing all the elements of the covariance matrix Σ_{10}.

In the next section we present the simulation study for the estimators obtained in subsections 3.2.2 - 3.2.5.

3.3 SIMULATION STUDY

In this section, we present simulations to assess the effectiveness of the estimators $\hat{\alpha}_n$, $\hat{\sigma}_n$, $\hat{\tau}_n$, $\hat{\beta}_n$ and $\hat{\mu}_n$. Since there is no analytic expression for the inverse of the cumulative distribution function of an α-stable distribution, the simulation procedure is complicated. The first credit goes to Kanter [25] for introducing simulation of symmetric stable random variables in 1975. The following year Chambers et al. [55] discussed a method for simulating stable random variables. In this work, we use the simulation of α-stable random variables developed by Janicki and Weron [24] and Weron and Weron [58].

We partition the interval $[0, T]$ as $0 = t_0 < t_1 < t_2 \ldots t_n = T = nh$, where $t_k = kh$ and h is fixed step size. Integrating both sides of (3.1) from r to t, $r < t$ gives us

$$\log(X_t) = \log(X_r) + \left(\mu - \frac{1}{2}\sigma^2\right)(t-r) + \sigma(W_t - W_r) + \tau\int_r^t dL_s^\alpha. \tag{3.44}$$

Thus we have that

$$\log(X_{t_{k+1}}) = \log(X_{t_k}) + \left(\mu - \frac{1}{2}\sigma^2\right) h + \sigma \Delta W_{t_{k+1}} + \tau \int_{kh}^{(k+1)h} dL_s^{\alpha}, \tag{3.45}$$

where $\Delta W_{t_{k+1}} = W_{t_{k+1}} - W_{t_k}$.

We note that

$$\tau \int_{kh}^{(k+1)h} dL_s^{\alpha} \stackrel{d}{=} \tau (h)^{\frac{1}{\alpha}} DL_k,$$

where DL_k are independent and identically distributed $S_\alpha(1, \beta, 0)$ random variables. For the numerical simulation of independent α-stable random variables DL_k, we refer to Janicki and Weron [24] and Weron and Weron [58] as well as Cheng et al. [7].

Thus, we have the iteration

$$\log(X_{t_{k+1}}) = \log(X_{t_k}) + \left(\mu - \frac{1}{2}\sigma^2\right) h + \sigma \Delta W_{t_{k+1}} + \tau (h)^{\frac{1}{\alpha}} DL_k. \tag{3.46}$$

To obtain the process in the interval $[0, T]$ with $T = nh$, we use the expression given in equation (3.46) with fixed parameters to generate 10000 values. Each of Fig. 3.1, Fig. 3.2, Fig. 3.3 and Fig. 3.4 show the plots of a sample paths of the process $\log(X_t)$ for chosen parameters as shown in the figures. Fig. 3.5 and Fig. 3.6 show multiple sample paths of the process $\log(X_t)$ for specified parameters.

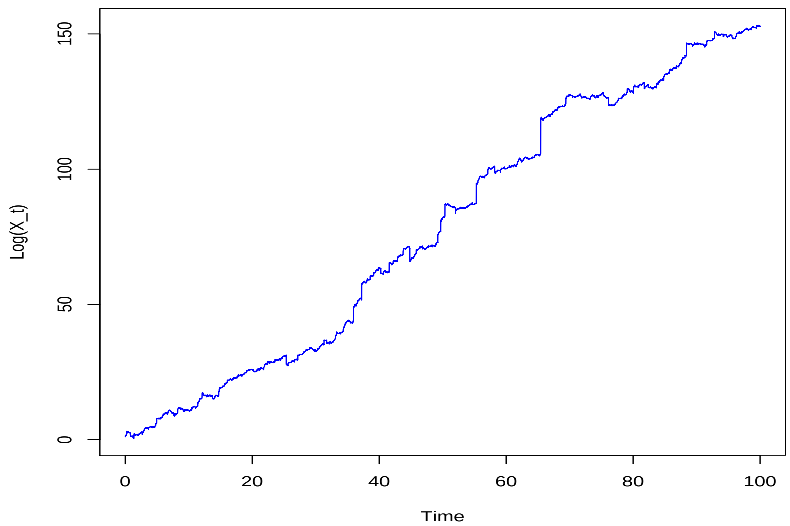


Figure 3.1: Simulation of the process $\log(X_t)$ taking $h = 0.01$, $n = 10000$, $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 1$, $\beta = 0.5$ and $\mu = 2$.

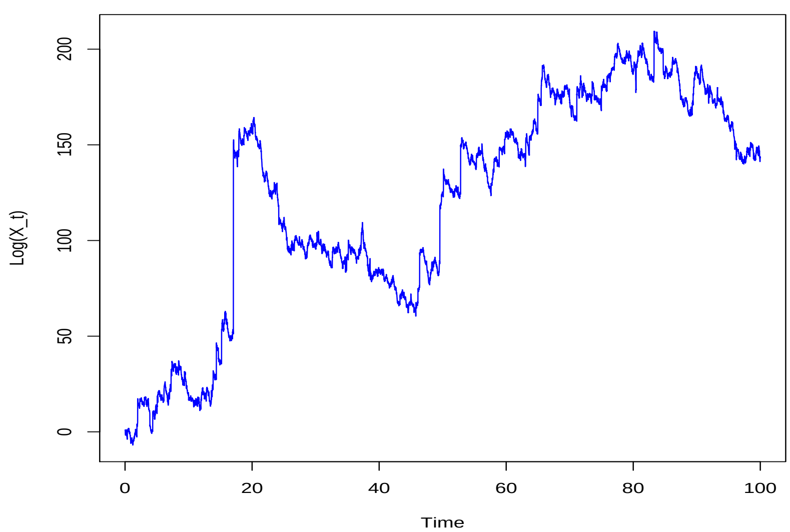


Figure 3.2: Simulation of the process $\log(X_t)$ taking $h = 0.01$, $n = 10000$, $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 7$, $\beta = 0.5$ and $\mu = 2$.

In order to assess the effectiveness of the estimators, we consider the fixed step sizes $h = 0.1$ and $h = 0.4$. For each of these we generate 1000 sample paths. Each

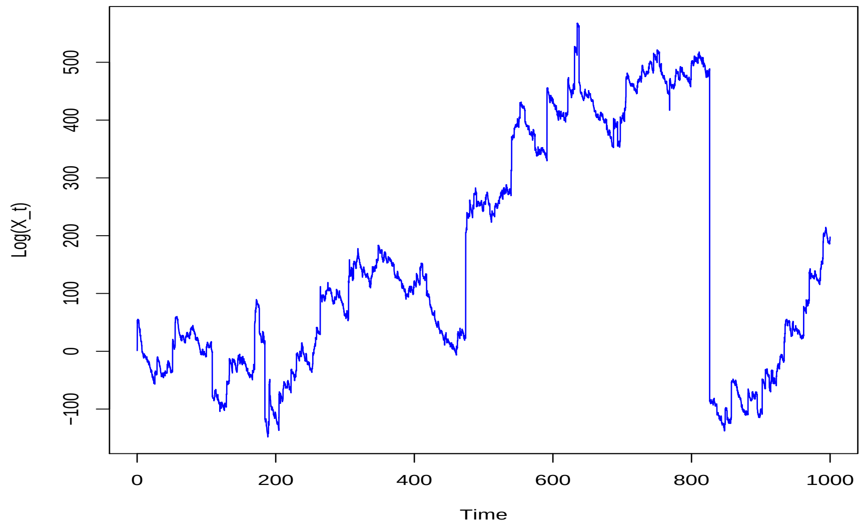


Figure 3.3: Simulation of the process $\log(X_t)$ taking $h = 0.1$, $n = 10000$, $\alpha = 1.3$, $\sigma = 0.7$, $\tau = 4$, $\beta = 0.5$ and $\mu = 2$.

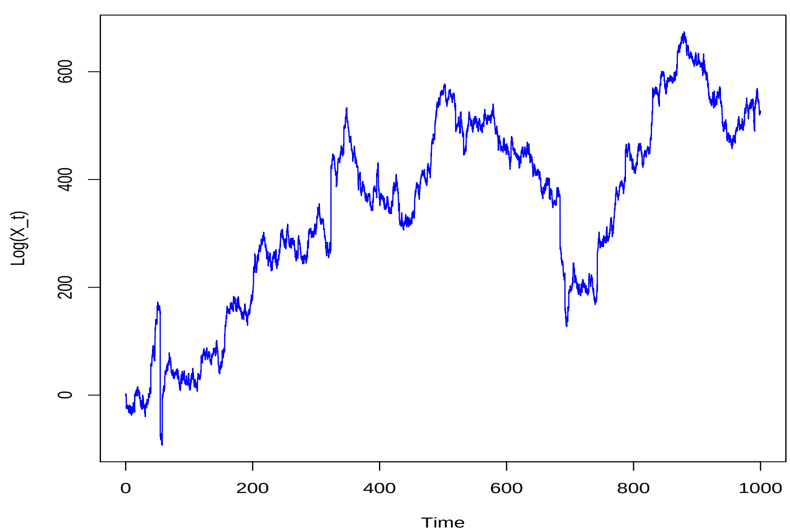


Figure 3.4: Simulation of the process $\log(X_t)$ taking $h = 0.1$, $n = 10000$, $\alpha = 1.7$, $\sigma = 0.5$, $\tau = 7$, $\beta = 0.5$ and $\mu = 2$.

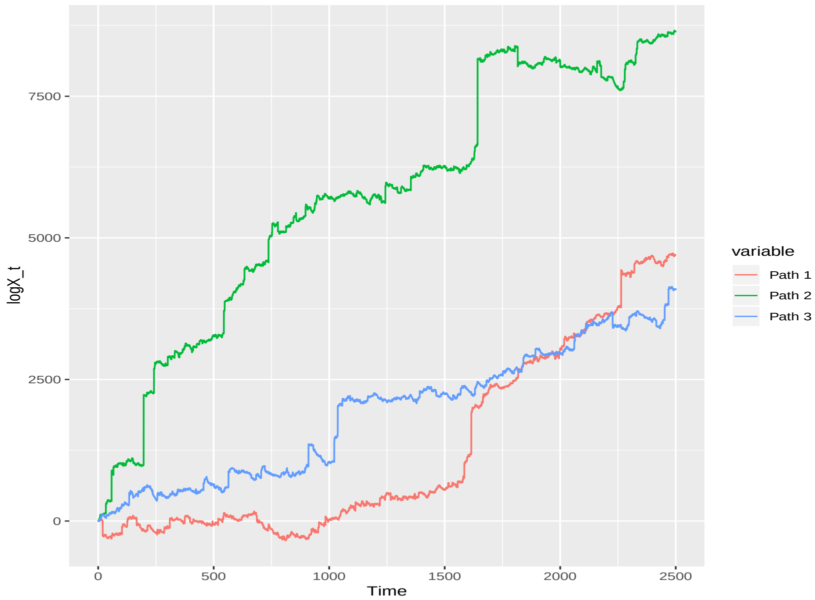


Figure 3.5: Simulated paths of the process $\log(X_t)$ taking $h = 0.25$, $n = 10000$, $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 7$, $\beta = 0.5$ and $\mu = 2$.

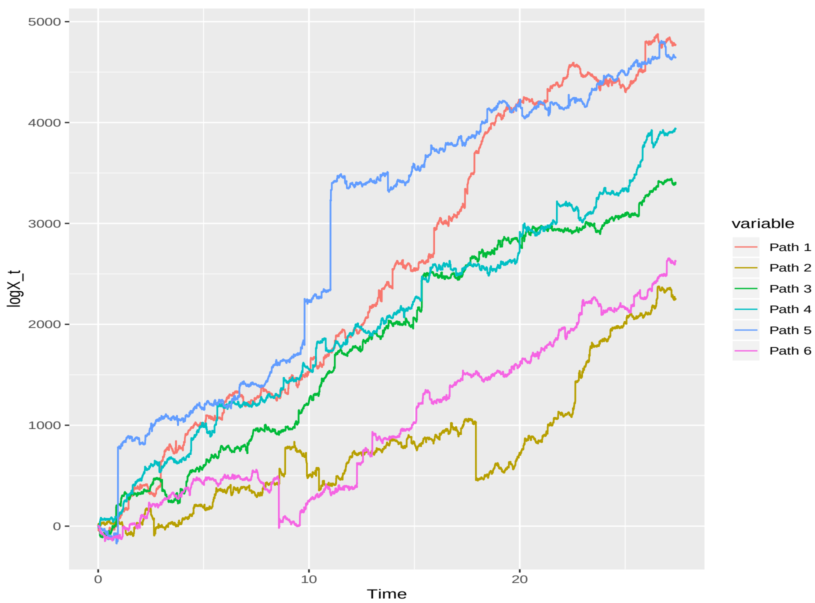


Figure 3.6: Simulated paths of the process $\log(X_t)$ taking $h = 0.25$, $n = 10000$, $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 4$, $\beta = 0.5$ and $\mu = 2$.

path is used to estimate each of the parameters. The mean of over all 1000 estimates and their variances are shown in Tables 3.1, 3.2, 3.3, 3.4, and 3.5. For this, we choose the true values $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 1$, $\beta = 0.5$ and $\mu = 2$ and we pick the arguments u_i: $u_1 = 0.1, u_2 = 0.3, u_3 = 0.2, u_4 = 0.6$, and $u_5 = 0.7$.

Table 3.1: Numerical values of the estimator $\hat{\alpha}_n$.

Step Size	$T = Nh$				
	N=5000	N=10000	N=15000	N=20000	N=25000
h=0.1	1.449	1.487	1.489	1.496	1.5051
	(0.420)	(0.296)	(0.205)	(0.177)	(0.072)
h=0.4	1.479	1.490	1.494	1.507	1.498
	(0.185)	(0.083)	(0.057)	(0.045)	(0.034)

The Table 3.1 describes the estimation $\hat{\alpha}_n$ for different choices of the step size h. Thus, we see that the estimators are closer to the true values and the variances of the estimator are decreased. This conforms with the consistency of $\hat{\alpha}_n$.

Table 3.2: Numerical values of the estimator $\hat{\sigma}_n$.

Step Size	$T = Nh$				
	N=5000	N=10000	N=15000	N=20000	N=25000
h=0.1	0.927	0.912	0.740	0.688	0.511
	(0.228)	(0.153)	(0.104)	(0.087)	(0.073)
h=0.4	0.719	0.607	0.549	0.513	0.493
	(0.154)	(0.064)	(0.061)	(0.048)	(0.053)

Table 3.2 describes the estimation for different choices of the step size h. Thus, we see that the estimators are closer to the true values and the variances of the estimator are decreased. This conforms with the consistency of $\hat{\sigma}_n$.

Table 3.3: Numerical values of the estimator $\hat{\tau}_n$.

Step Size	$T = Nh$				
	N=5000	N=10000	N=15000	N=20000	N=25000
h=0.1	0.887	0.942	0.998	0.988	1.009
	(0.064)	(0.031)	(0.025)	(0.019)	(0.017)
h=0.4	0.915	0.967	0.978	0.995	1.016
	(0.027)	(0.010)	(0.008)	(0.005)	(0.004)

Table 3.3 describes the estimation for different choices of the step size h. Thus, we see that the estimators are closer to the true values and the variances of the estimator are decreased.This conforms with the consistency of $\hat{\tau}_n$.

Table 3.4: Numerical values of the estimator $\hat{\beta}_n$.

Step Size	$T = Nh$				
	N=5000	N=10000	N=15000	N=20000	N=25000
h=0.1	0.690	0.587	0.519	0.513	0.503
	(0.255)	(0.071)	(0.041)	(0.023)	(0.010)
h=0.4	0.457	0.521	0.523	0.506	0.505
	(0.022)	(0.015)	(0.009)	(0.007)	(0.004)

Table 3.4 describes the estimation for different choices of the step size h. Thus, we

see that the estimators are closer to the true values and the variances of the estimator are decreased. This conforms with the consistency of $\hat{\beta}_n$.

Table 3.5: Numerical values of the estimator $\hat{\mu}_n$.

Step Size	$T = Nh$				
	N=5000	N=10000	N=15000	N=20000	N=25000
h=0.1	2.610	2.28	2.171	2.145	2.10
	(0.183)	(0.014)	(0.004)	(0.003)	(0.002)
h=0.4	2.317	2.094	2.065	2.030	2.001
	(1.14)	(0.002)	(0.008)	(0.0008)	(0.0003)

Table 3.5 describes the estimation for different choices of the step size h. Thus, we see that the estimators are closer to the true values and the variances of the estimator are decreased. This conforms with the consistency of $\hat{\mu}_n$.

In Table 3.6, we present the parameter estimated for the case $\alpha < 1$ where we choose the true values: $\alpha = 0.5,\ \sigma = 0.5,\ \tau = 1,\ \ \beta = 0.5$ and $\mu = 2$. We also choose the positive arguments $u_1 = 0.1,\ \ u_2 = 0.3,\ u_3 = 0.2,\ u_4 = 0.6$, and $u_5 = 0.7$.

In Table 3.7, we present the parameter estimation for the symmetric α-stable case ($\beta = 0$) for the following choice of true values: $\alpha = 1.5,\ \ \sigma = 0.5,\ \ \tau = 1$, and $\mu = 2$. We choose positive arguments $u_1 = 0.1,\ \ u_2 = 0.3,\ u_3 = 0.2,\ u_4 = 0.6$, and $u_5 = 0.7$.

Table 3.6: Numerical Results for:$\alpha = 0.5$, $\sigma = 0.5$, $\tau = 1$, $\beta = 0.5$, $\mu = 2$ and $h = 0.2$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	0.514	0.497	0.499	0.505	0.501
	(0.035)	(0.014)	(0.009)	(0.007)	(0.006)
$\hat{\sigma}_n$	0.777	0.641	0.587	0.535	0.507
	(0.124)	(0.077)	(0.066)	(0.051)	(0.044)
$\hat{\tau}_n$	1.074	0.981	0.989	1.025	1.007
	(0.042)	(0.013)	(0.011)	(0.006)	(0.005)
$\hat{\beta}_n$	0.489	0.516	0.508	0.490	0.492
	(0.105)	(0.060)	(0.040)	(0.027)	(0.023)
$\hat{\mu}_n$	2.194	2.091	2.05	2.026	2.001
	(0.115)	(0.003)	(0.001)	(0.001)	(0.0006)

Table 3.7: Numerical Results for: $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 1$, $\beta = 0$, $\mu = 2$ and $h = 0.25$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.485	1.494	1.501	1.502	1.496
	(0.165)	(0.082)	(0.059)	(0.043)	(0.034)
$\hat{\sigma}_n$	0.708	0.595	0.541	0.502	0.494
	(0.087)	(0.063)	(0.059)	(0.046)	(0.039)
$\hat{\tau}_n$	0.964	0.976	1.007	1.008	1.013
	(0.021)	(0.011)	(0.010)	(0.005)	(0.002)
$\hat{\beta}_n$	0.0037	-0.002	0.000	0.001	-0.001
	(0.015)	(0.006)	(0.004)	(0.003)	(0.004)
$\hat{\mu}_n$	2.15	2.065	2.046	2.011	2.002
	(0.009)	(0.011)	(0.010)	(0.001)	(0.001)

3.3.1 Comparison of Estimators for Different Choices of Arguments u_i's

For the simulation results presented in Tables 3.8, 3.9, 3.10, 3.11, 3.12 and 3.13, we choose the true values: $\alpha = 1.5$, $\sigma = 0.5$, $\tau = 1$, $\beta = 0.5$ and $\mu = 2$. We fix the step-size as $h = 0.3$. For these true values of α, σ, τ, β and μ, we generate 1000 realizations and simulate samples of sizes $n = 5000, 10000, 15000$, 20000 and 25000. We also present the average of estimators: Mean($\hat{\alpha}_n$), Mean($\hat{\sigma}_n$), Mean($\hat{\tau}_n$), Mean($\hat{\beta}_n$) and Mean($\hat{\mu}_n$). Also, the variance of the estimators: Var($\hat{\alpha}_n$), Var($\hat{\sigma}_n$), Var($\hat{\tau}_n$), Var($\hat{\beta}_n$) and Var($\hat{\mu}_n$)are presented.

Table 3.8: Choices of u_i's

Choice	u_1	u_2	u_3	u_4	u_5
1	0.5	1.2	1.5	1.6	1.1
2	0.5	0.7	1.5	1.6	1.1
3	0.5	1.2	1.5	0.6	0.7
4	0.8	0.9	0.1	1.6	1.1
5	0.7	1	1.5	1.6	0.5

We see from the estimates presented in tables below that all the estimators converge as expected regardless of the choice of u_i's.

Table 3.9: $u_1 = 0.5, \quad u_2 = 1.2, \ u_3 = 1.5, \ u_4 = 1.6$, and $u_5 = 1.1$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.486	1.493	1.497	1.498	1.501
	(0.033)	(0.017)	(0.113)	(0.007)	(0.007)
$\hat{\sigma}_n$	0.651	0.582	0.534	0.487	0.487
	(0.046)	(0.041)	(0.035)	(0.033)	(0.031)
$\hat{\tau}_n$	0.994	0.991	1.012	0.998	1.004
	(0.002)	(0.002)	(0.001)	(0.001)	(0.0006)
$\hat{\beta}_n$	0.488	0.496	0.485	0.495	0.496
	(0.007)	(0.004)	(0.002)	(0.001)	(0.001)
$\hat{\mu}_n$	2.118	2.075	2.028	2.007	2.006
	(0.026)	(0.016)	(0.010)	(0.008)	(0.008)

Table 3.10: $u_1 = 0.5, \quad u_2 = 0.7, \ u_3 = 1.5, \ u_4 = 1.6$, and $u_5 = 1.1$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.512	1.495	1.503	1.500	1.494
	(0.047)	(0.023)	(0.017)	(0.012)	(0.009)
$\hat{\sigma}_n$	0.684	0.598	0.577	0.533	0.509
	(0.051)	(0.046)	(0.038)	(0.037)	(0.036)
$\hat{\tau}_n$	1.062	1.029	1.028	1.023	1.017
	(0.017)	(0.003)	(0.002)	(0.001)	(0.001)
$\hat{\beta}_n$	0.463	0.481	0.484	0.484	0.488
	(0.008)	(0.004)	(0.003)	(0.002)	(0.001)
$\hat{\mu}_n$	2.123	2.078	2.063	2.036	2.025
	(0.029)	(0.020)	(0.015)	(0.012)	(0.010)

Table 3.11: $u_1 = 0.5, \quad u_2 = 1.2, \; u_3 = 1.5, \; u_4 = 0.6$, and $u_5 = 0.7$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.505	1.497	1.501	1.500	1.506
	(0.039)	(0.018)	(0.012)	(0.007)	(0.007)
$\hat{\sigma}_n$	0.659	0.567	0.534	0.489	0.482
	(0.047)	(0.043)	(0.036)	(0.033)	(0.031)
$\hat{\tau}_n$	1.023	1.021	1.007	1.009	1.018
	(0.005)	(0.002)	(0.001)	(0.001)	(0.0006)
$\hat{\beta}_n$	0.489	0.487	0.490	0.492	0.492
	(0.015)	(0.007)	(0.004)	(0.003)	(0.003)
$\hat{\mu}_n$	2.118	2.052	2.034	2.008	2.001
	(0.036)	(0.019)	(0.014)	(0.010)	(0.009)

Table 3.12: $u_1 = 0.8, \quad u_2 = 0.9, \; u_3 = 0.1, \; u_4 = 1.6$, and $u_5 = 1.1$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.471	1.485	1.493	1.495	1.497
	(0.025)	(0.012)	(0.008)	(0.006)	(0.005)
$\hat{\sigma}_n$	0.722	0.618	0.564	0.533	0.502
	(0.059)	(0.046)	(0.041)	(0.039)	(0.038)
$\hat{\tau}_n$	0.932	0.960	0.977	0.985	0.985
	(0.004)	(0.002)	(0.001)	(0.001)	(0.001)
$\hat{\beta}_n$	0.550	0.531	0.519	0.512	0.511
	(0.010)	(0.005)	(0.003)	(0.002)	(0.001)
$\hat{\mu}_n$	2.207	2.116	2.065	2.043	2.031
	(0.034)	(0.023)	(0.014)	(0.014)	(0.012)

Table 3.13: $u_1 = 0.7, \quad u_2 = 1, \; u_3 = 1.5, \; u_4 = 1.6$, and $u_5 = 0.5$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.495	1.499	1.499	1.509	1.502
	(0.070)	(0.036)	(0.024)	(0.019)	(0.015)
$\hat{\sigma}_n$	0.712	0.639	0.584	0.551	0.520
	(0.055)	(0.047)	(0.046)	(0.041)	(0.040)
$\hat{\tau}_n$	1.022	1.002	1.016	1.031	1.008
	(0.004)	(0.002)	(0.001)	(0.001)	(0.001)
$\hat{\beta}_n$	0.482	0.490	0.487	0.484	0.491
	(0.003)	(0.002)	(0.001)	(0.008)	(0.001)
$\hat{\mu}_n$	2.16	2.114	2.065	2.036	2.030
	(0.027)	(0.020)	(0.014)	(0.013)	(0.012)

In the next chapter, we shall propose an α-stable Geometric Lévy process with stochastic volatility model and estimate the involved parameters.

CHAPTER 4
PARAMETER ESTIMATION FOR GEOMETRIC LÉVY PROCESSES WITH STOCHASTIC VOLATILITY

The constant volatility assumption of the Black-Scholes model is rejected by many empirical studies. Generally, stochastic volatility modeling treats the volatility of the stocks as a random variable. We assume that the stochastic volatility is a positive function of Y_t, namely $\sigma_t = f(Y_t) > 0$, where Y_t follows the Cox-Ingersoll-Ross (CIR) model [9]. Heston [19] also studied the CIR model [9] in a closed-form solution for options with stochastic volatility.

Fouque et al. [18] analyze models in which stock prices are conditionally lognormal, and the volatility process is a positive increasing function of a mean-reverting Ornstein-Uhlenbeck process. The model has only a drift and a diffusion component. In order to account for jumps we modify the model to include the alpha-stable Lévy motion. In this chapter we give details of the proposed model, estimate its associated parameters, discuss the consistency and asymptotic properties of those parameters and perform a simulation study to assess the validity of the estimators.

4.1 THE MODEL

We consider the processes X_t and Y_t that satisfy the following stochastic differential equations

$$d(\log(X_t)) = \left(\mu - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t + \tau dL_t^{\alpha}, \tag{4.1}$$

$$\sigma_t = f(Y_t) = \sqrt{Y_t}, \tag{4.2}$$

$$dY_t = \kappa(m - Y_t)dt + \beta_1 \sqrt{Y_t} dB_t, \tag{4.3}$$

where

- the process $\{X_t, t \geq 0\}$ is is called geometric Lévy process with stochastic volatility,
- $\mu(t)$ is the (deterministic) instantaneous drift of the process X_t,
- Y_t is an unmeasured independent variable,
- $\sigma_t = f(Y_t) = \sqrt{Y_t}$ is the stochastic volatility which is a positive function of Y_t,
- κ is the rate of mean reversion (i.e., how strongly the system reacts to the decay-rate or growth rate),
- m is the long-run mean of Y_t (the asymptotic mean),
- β_1 is the volatility of volatility,
- W_t and B_t are independent standard (unit variance) Brownian motions with respective differentials dW_t and dB_t,
- L_t^α is an α-stable Lévy motion with the stability index $0 < \alpha < 2$,
- τ is the dispersion constant of the α-stable Lévy process.

Note that (4.1) is an α-stable geometric Lévy process and equation (4.3) is the Cox-Ingersoll-Ross model. Our aim is to estimate the parameters α, τ, μ, β, κ, m and β_1 from the stochastic volatility models (4.1) and (4.3). We assume the process X_t is observed at discrete times $\{t_k\}_{k=1}^n$ with $t_k = kh$ for some fixed $h > 0$ and the stochastic volatility $\sigma_t = f(Y_t) = \sqrt{Y_t}$ is not directly observable. This complicates the parameter estimation because we are dealing with a hidden Markov model.

Under the assumption $2\kappa m \geq \beta_1^2$, Feller [17] proved that the process Y_t is non-negative, i.e., for all positive values of κ and m, the standard deviation factor $\beta_1\sqrt{Y_t}$ is non-negative.

The probability density function (pdf) $f_Y(Y_t|Y_0)$ for Y_t given $Y_t = Y_0$ is a non-central χ^2-distribution with degree of freedom $\frac{4\kappa m}{\beta_1^2}$ and the non-centrality parameter $\frac{4\kappa Y_0}{\beta_1^2(1-e^{-\kappa t})}e^{-\kappa t}$ (see [60].)
Note that

$$\begin{aligned}\mathbb{E}(Y_t) &= Y_0 e^{-\kappa t} + m(1-e^{-\kappa t}) \quad \text{and} \\ Var(Y_t) &= Y_0\frac{\beta_1^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\beta_1^2 m}{2\kappa}(1-e^{-\kappa t})^2.\end{aligned}$$

The stationary distribution of the CIR model is a gamma distribution [60], i.e., as $t \to \infty$, $Y_t \to Y_\infty$ follows a Gamma distribution with shape parameter $\frac{2\kappa m}{\beta_1^2}$ and the scale parameter $\frac{2\kappa}{\beta_1^2}$.

It follows immediately that $\mathbb{E}(Y_\infty) = m$ and $Var(Y_\infty) = \frac{\beta_1^2 m}{2\kappa}$.
The corresponding probability density function of Y_∞ is given by

$$f(y;\kappa,m,\beta_1) = \frac{\omega^\nu}{\Gamma(\nu)}y^{\nu-1}e^{-\omega y}, \tag{4.4}$$

with $\omega = \frac{2\kappa}{\beta_1^2}$, $\nu = \frac{2\kappa m}{\beta_1^2}$, where the Gamma function is given by

$$\Gamma(x) = \int_0^{+\infty} e^{-z}z^{x-1}dz.$$

Note that $\nu = m\omega$. The Euler scheme to approximate the solutions of equations (4.1) and (4.3) are given by

$$\begin{aligned}Z_k &= \Delta(\log(X_{t_k})) = \log(X_{t_k}) - \log(X_{t_{k-1}}) \\ &= \mu h + \int_{t_{k-1}}^{t_k}\left(\sigma_s dW_s - \frac{1}{2}\sigma_s^2 ds\right) + \tau\Delta L_{t_k}^\alpha \qquad (4.5) \\ &\approx \mu h + \sqrt{Y_{t_{k-1}}}\Delta W_{t_k} - \frac{1}{2}Y_{t_{k-1}}h + \tau\Delta L_{t_k}^\alpha, \qquad (4.6)\end{aligned}$$

and

$$\begin{aligned}\Delta Y_{t_k} &= Y_{t_k} - Y_{t_{k-1}} = \kappa(m - Y_{t_k})\Delta_{t_k} + \beta_1\int_{t_{k-1}}^{t_k}\sqrt{Y_s}dB_{t_k} \qquad (4.7) \\ &\approx \kappa(m - Y_{t_{k-1}})h + \beta_1\sqrt{Y_{t_{k-1}}}\Delta B_{t_k}, \qquad (4.8)\end{aligned}$$

where $h = \Delta_{t_k} = t_k - t_{k-1}$ is the step size, $\Delta W_{t_k} = W_{t_k} - W_{t_{k-1}}$ is the increment of the diffusion component, $\Delta L^\alpha_{t_k} = L^\alpha_{t_{k+1}} - L^\alpha_{t_k} \sim (h)^{\frac{1}{\alpha}} S_\alpha(1,\beta,0)$ is the increment of the jump component and $\Delta B_{t_k} = B_{t_{k+1}} - B_{t_k} \sim N(0,h)$ follows normal distribution with mean 0 and variance h, and $S_\alpha(1,\beta,0)$ is the standard α-stable distribution.

By the ergodicity of Y_t, it follows that $\{Z_k\}_{k\geq 1}$ is ergodic. Let Z_∞ be the limiting random variable of Z_k as $k \to \infty$.

By ergodicity of the process $\{Z_k\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \exp\left(iuZ_k\right) = \mathbb{E}\left(e^{iuZ_\infty}\right) = \psi(u), \tag{4.9}$$

where $\psi(u)$ is the characteristic function of Z_∞ and $i = \sqrt{-1}$. We will find an expression for the characteristic function of Z_∞ and use the sample characteristic function method and the conditional least squares method to estimate the seven parameters: α, τ, μ, β, κ, m and β_1.

For any $u \in \mathbb{R}$ and any positive integer k, using equation (4.5), the characteristic function of Z_k can be obtained as below:

$$\begin{aligned}
\mathbb{E}\left(e^{iuZ_k}\right) &= \mathbb{E}\left(\exp\left[iu\mu h + iu\int_{t_{k-1}}^{t_k}\left(\sigma_s dW_s - \frac{1}{2}\sigma_s^2 ds\right) + iu\tau\Delta L^\alpha_{t_k}\right]\right) \\
&= \mathbb{E}\left(e^{iu\mu h}\right) \times \mathbb{E}\left(e^{iu\int_{t_{k-1}}^{t_k}\left(\sigma_s dW_s - \frac{1}{2}\sigma_s^2 ds\right)}\right) \times \mathbb{E}\left(e^{iu\tau\Delta L^\alpha_{t_k}}\right).
\end{aligned} \tag{4.10}$$

Since $\{W_s\}$ and $\{\sigma_s\}$ are independent, as $k \to \infty$, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{iu\int_{t_{k-1}}^{t_k}\left(\sigma_s dW_s - \frac{1}{2}\sigma_s^2 ds\right)}\right) \\
&= \mathbb{E}_Y\left[\mathbb{E}\left(e^{iu\int_{t_{k-1}}^{t_k}\sigma_s dW_s - \frac{1}{2}iu\int_{t_{k-1}}^{t_k}\sigma_s^2 ds} | \sigma_s, t_{k-1} \leq s \leq t_k\right)\right] \\
&= \mathbb{E}_Y\left(e^{-\frac{1}{2}iu\int_{t_{k-1}}^{t_k}\sigma_s^2 ds} \times \mathbb{E}\left[e^{iu\int_{t_{k-1}}^{t_k}\sigma_s dW_s} | \sigma_s, t_{k-1} \leq s \leq t_k\right]\right) \\
&= \mathbb{E}_Y\left[e^{-\frac{1}{2}iu\int_{t_{k-1}}^{t_k}\sigma_s^2 ds} e^{-\frac{1}{2}u^2\int_{t_{k-1}}^{t_k}\sigma_s^2 ds}\right] \\
&= \mathbb{E}_Y\left[\exp\left(-\frac{1}{2}(u^2 + iu)\int_{t_{k-1}}^{t_k}\sigma_s^2 ds\right)\right] \\
&= \mathbb{E}_Y\left[\exp\left(-\frac{1}{2}(u^2 + iu)\int_{t_{k-1}}^{t_k}Y_s ds\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= E_Y\left[\exp\left(-\frac{1}{2}(u^2+iu)\int_{t_{k-1}}^{t_k}(Y_s - Y_\infty + Y_\infty)ds\right)\right]\\
&\to E_Y\left[\exp\left(-\frac{1}{2}(u^2+iu)hY_\infty\right)\right]\\
&= \int_0^\infty \exp\left(-\frac{y}{2}(u^2+iu)h\right) f(y;\kappa,m,\beta)dy\\
&= \int_0^\infty \exp\left(-\frac{yh}{2}(u^2+iu)-\omega y\right)\frac{\omega^\nu}{\Gamma(\nu)}y^{\nu-1}dy\\
&= \frac{\omega^\nu}{\Gamma(\nu)}\int_0^\infty \exp\left(y[-\frac{h}{2}(u^2+iu)-\omega]\right)y^{\nu-1}dy\\
&= \frac{(2\omega)^\nu}{[(u^2+iu)h+2\omega]^\nu}\\
&= (2\omega)^\nu \times [(u^2h+2\omega)+iuh]^{-\nu},
\end{aligned}
\tag{4.11}
$$

where $f(y;\kappa,m,\beta_1)$ is the probability density function of the gamma distribution. We also used the fact that the integral $\int_{t_{k-1}}^{t_k}(Y_s - Y_\infty)ds$ converges to 0 almost surely for large values of t_k.

By using equation (2.1), we have that

$$
\mathbb{E}\left(e^{iu(\tau\Delta L_k^\alpha)}\right) = \begin{cases} -h\tau^\alpha|u|^\alpha\left(1-i\beta sgn(u)\tan\frac{\pi\alpha}{2}\right), & \alpha\neq 1\\ -h\tau|u|\left(1+i\beta\frac{2}{\pi}sgn(u)\log|u|\right), & \alpha = 1\end{cases}
\tag{4.12}
$$

Thus using equations (4.11) and (4.12), the characteristic function of Z_k converges to

$$
\begin{aligned}
\psi(u) &= \mathbb{E}\left(e^{iuZ_\infty}\right)\\
&= \begin{cases} C(u)\times\exp\left[iu\mu h - h\tau^\alpha|u|^\alpha\left(1-i\beta sgn(u)\tan\frac{\pi\alpha}{2}\right)\right], & \alpha\neq 1\\ C(u)\times\exp\left[iu\mu h - h\tau|u|\left(1+i\beta\frac{2}{\pi}sgn(u)\log|u|\right)\right], & \alpha=1\end{cases}
\end{aligned}
\tag{4.13}
$$

where $C(u) = \dfrac{(2\omega)^\nu}{[(u^2+2\omega)+iuh]^\nu}$, $\omega = \dfrac{2\kappa}{\beta_1^2}$ and $\nu = \dfrac{2\kappa m}{\beta_1^2}$.

As we have discussed in Chapter 3, we shall use the sample characteristic function method developed by Press [48]. Our procedure and ideas are similar to Cheng et al. [7], where parameter estimation for α- stable Ornstein-Uhlenbeck processes is dealt with. We use the characteristic function obtained in equation (4.13) to get the estimators $\hat{\alpha}_n$, $\hat{\tau}_n$, $\hat{m}_n$, $\hat{\beta}_n$ and $\hat{\mu}_n$ and discuss their consistency and asymptotic behavior. We use the conditional least squares method to obtain the estimators $\hat{\kappa}_n$ and $\hat{\beta}_{1n}$.

4.2 ESTIMATION OF PARAMETERS

In this section, we estimate all parameters α, τ, ω, ν, m, β, μ, κ and β_1.

4.2.1 Estimators for Parameters α, τ, ω and ν

For $\alpha \neq 1$, equation (4.13) can be written as

$$\begin{aligned} |\psi(u)| &= |(2\omega)^{\nu}(u^2h + 2\omega + iuh)^{-\nu}| \times \exp(-h\tau^{\alpha}|u|^{\alpha}) \\ &= (2\omega)^{\nu}\left((u^2h + 2\omega)^2 + u^2h^2\right)^{\frac{-\nu}{2}} \times \exp(-h\tau^{\alpha}|u|^{\alpha}). \end{aligned} \tag{4.14}$$

Taking logarithm on both sides of (4.14), we obtain

$$\log|\psi(u)| = \nu\log(2\omega) - \frac{\nu}{2}\log[(u^2h + 2\omega)^2 + u^2h^2] - h\tau^{\alpha}|u|^{\alpha}. \tag{4.15}$$

Let

$$g(\omega, u) = \log(2\omega) - \frac{1}{2}\log[(u^2h + 2\omega)^2 + u^2h^2]. \tag{4.16}$$

Then equation (4.15) becomes

$$\log|\psi(u)| = \nu g(\omega, u) - h\tau^{\alpha}|u|^{\alpha}. \tag{4.17}$$

We have four unknown parameters ω, ν, τ and α in (4.17).

Let

$$\psi_n(u) = \frac{1}{n}\sum_{k=1}^{n}\exp(iuZ_k) = \frac{1}{n}\sum_{k=1}^{n}\exp\left(iu\Delta\log(X_{t_k})\right). \tag{4.18}$$

Note that $\psi_n(u) \to \psi(u)$ almost surely as $n \to \infty$ by ergodic theorem.

We replace $\psi(u)$ by $\psi_n(u)$ and set up the following system of equations

$$\log|\psi_n(u_1)| = \nu g(\omega, u_1) - h\tau^\alpha |u_1|^\alpha, \tag{4.19}$$

$$\log|\psi_n(u_2)| = \nu g(\omega, u_2) - h\tau^\alpha |u_2|^\alpha, \tag{4.20}$$

$$\log|\psi_n(u_3)| = \nu g(\omega, u_3) - h\tau^\alpha |u_3|^\alpha, \tag{4.21}$$

$$\log|\psi_n(u_4)| = \nu g(\omega, u_4) - h\tau^\alpha |u_4|^\alpha, \tag{4.22}$$

where $u_1 \neq u_2 \neq u_3 \neq u_4 \neq 0$ are real constants. We need to solve equations (4.19) to (4.22) to obtain the estimators $\hat{\nu}_n$, $\hat{\omega}_n$, $\hat{\alpha}_n$ and $\hat{\tau}_n$. From equations (4.19) and (4.20), we have

$$h\tau^\alpha |u_1|^\alpha = \nu g(\omega, u_1) - \log|\psi_n(u_1)|, \tag{4.23}$$

$$h\tau^\alpha |u_2|^\alpha = \nu g(\omega, u_2) - \log|\psi_n(u_2)|. \tag{4.24}$$

Dividing (4.23) by (4.24) and then taking logarithm on both sides, we get,

$$\alpha \log|\frac{u_1}{u_2}| = \log\left(\frac{\nu g(\omega, u_1) - \log|\psi_n(u_1)|}{\nu g(\omega, u_2) - \log|\psi_n(u_2)|}\right), \quad u_1 \neq u_2. \tag{4.25}$$

Similarly, we can get

$$\alpha \log|\frac{u_3}{u_4}| = \log\left(\frac{\nu g(\omega, u_3) - \log|\psi_n(u_3)|}{\nu g(\omega, u_4) - \log|\psi_n(u_4)|}\right), \quad u_3 \neq u_4. \tag{4.26}$$

Combining (4.25) and (4.26), we have

$$\frac{\log\left(\frac{\nu g(\omega,u_1) - \log|\psi_n(u_1)|}{\nu g(\omega,u_2) - \log|\psi_n(u_2)|}\right)}{\log|\frac{u_1}{u_2}|} = \frac{\log\left(\frac{\nu g(\omega,u_3) - \log|\psi_n(u_3)|}{\nu g(\omega,u_4) - \log|\psi_n(u_4)|}\right)}{\log|\frac{u_3}{u_4}|}. \tag{4.27}$$

Similarly, we can get

$$\frac{\log\left(\frac{\nu g(\omega,u_1) - \log|\psi_n(u_1)|}{\nu g(\omega,u_3) - \log|\psi_n(u_3)|}\right)}{\log|\frac{u_1}{u_3}|} = \frac{\log\left(\frac{\nu g(\omega,u_2) - \log|\psi_n(u_2)|}{\nu g(\omega,u_4) - \log|\psi_n(u_4)|}\right)}{\log|\frac{u_2}{u_4}|}. \tag{4.28}$$

We use some numerical approximation techniques to get the estimators $\hat{\nu}_n$ and $\hat{\omega}_n$ from equations (4.27) and (4.28).

Then we can estimate α by using the expression

$$\hat{\alpha}_n = \frac{\log\left(\frac{\hat{\nu}_n g(\hat{\omega}_n,u_1) - \log|\psi_n(u_1)|}{\hat{\nu}_n g(\hat{\omega}_n,u_2) - \log|\psi_n(u_2)|}\right)}{\log|\frac{u_1}{u_2}|} \tag{4.29}$$

and we estimate τ by using the following expression

$$\hat{\tau}_n = \frac{1}{|u_1|}\left(\frac{\hat{\nu}_n g(\hat{\omega}_n, u_1) - \log|\psi_n(u_1)|}{h}\right)^{\frac{1}{\hat{\alpha}_n}}. \tag{4.30}$$

Next, we present the steps to estimate all parameters involved in the equations (4.1) and (4.3).

4.2.2 Steps for Estimating all Parameters

In this subsection, we explain the procedure for estimating parameters involved from equations (4.1) and (4.3).

1. We use Newton's method to obtain the estimators $\hat{\omega}_n$ and $\hat{\nu}_n$ of ω and ν by solving the set of simultaneous nonlinear equations (4.27) and (4.28).

2. Then we substitute the estimators $\hat{\omega}_n$ and $\hat{\nu}_n$ in equation (4.29) and (4.30) to obtain the estimators $\hat{\alpha}_n$ and $\hat{\tau}_n$.

3. Using the relation $\nu = m\omega$, we obtain the estimator $\hat{m}_n$ for m.

4. We find the real and imaginary part of the characteristic function of $\psi(u)$, then the estimators for β and μ can be obtained based on $\psi_n(u)$, $\Im(\psi_n(u))$ and $\Re(\psi_n(u))$.

5. Using the discretized relation (4.7) and conditional least squares method we obtain the estimator $\hat{\kappa}_n$ for the parameter κ.

6. Next we use the relation $\beta_1^2 = \dfrac{2\kappa}{\omega}$ and find the estimator $\hat{\beta}_{1n}$ for β_1.

4.2.3 Estimator for m

Using the relation $\nu = m\omega$, the estimator $\hat{m}_n$ for m can be obtained by using the relation $\hat{m}_n = \dfrac{\hat{\nu}_n}{\hat{\omega}_n}$. Note that $\hat{m}_n$ converges to m almost surely since $\hat{\nu}_n \to \nu$ and $\hat{\omega}_n \to \omega$ almost surely.

4.2.4 Estimators for μ and β

We now discuss the estimation for the skewness parameter $\beta \in [-1, 1]$ and the drift parameter $\mu \in \mathbb{R}$. We shall first provide an equation and get a simplified expression for $\psi(u)$.

For $\alpha \neq 1$, the logarithm of the characteristic function (4.13) is given by

$$\log(\psi(u)) = \nu \log(2\omega) - \nu \log\big((u^2h + 2\omega) + iuh\big) + iu\mu h - h\tau^\alpha |u|^\alpha + i\beta h\tau^\alpha |u|^\alpha sgn(u) \tan\left(\frac{\pi\alpha}{2}\right). \tag{4.31}$$

Note that for a complex number $z = x + iy$,

$$\log(z) = \frac{1}{2}\log(x^2 + y^2) + i \arctan\left(\frac{y}{x}\right).$$

So we can write

$$\log\big((u^2h + 2\omega) + iuh\big) = \frac{1}{2}\log\left((u^2h + 2\omega)^2 + u^2h^2\right) + i \arctan\left(\frac{uh}{u^2h + 2\omega}\right). \tag{4.32}$$

Thus, equation (4.31) can be written as

$$\psi(u) = \exp\left\{\left[\nu \log(2\omega) - \frac{\nu}{2}\log\left((u^2h + 2\omega)^2 + u^2h^2\right) - h\tau^\alpha |u|^\alpha\right] + i\Big[u\mu h + \beta h\tau^\alpha |u|^\alpha sgn(u) \tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u)\Big]\right\} \tag{4.33}$$

where

$$D(u) = \arctan\left(\frac{uh}{u^2h + 2\omega}\right). \tag{4.34}$$

To estimate μ and β, we have

$$\Im(\psi(u)) = \exp\left[\nu \log(2\omega) - \frac{\nu}{2}\log\left((u^2h + 2\omega)^2 + u^2h^2\right) - h\tau^\alpha |u|^\alpha\right] \times \cos\left\{u\mu h + \beta h\tau^\alpha |u|^\alpha sgn(u) \tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u)\right\} \tag{4.35}$$

and

$$\Re(\psi(u)) = \exp\left[\nu \log(2\omega) - \frac{\nu}{2}\log\left((u^2h+2\omega)^2 + u^2h^2\right) - h\tau^{\alpha}|u|^{\alpha}\right] \times \sin\left\{u\mu h + \beta h\tau^{\alpha}|u|^{\alpha} sgn(u)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u)\right\}. \tag{4.36}$$

Dividing equation (4.35) by (4.36), we get

$$\frac{\Im(\psi(u))}{\Re(\psi(u))} = \tan\left(u\mu h + \beta h\tau^{\alpha}|u|^{\alpha} sgn(u)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u)\right). \tag{4.37}$$

Choose two non-zero values of u, $u_5 \neq u_6$ such that

$$\frac{-\pi}{2} < u_5\mu h + \beta h\tau^{\alpha}|u_5|^{\alpha} sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u_5) < \frac{\pi}{2}$$

and

$$\frac{-\pi}{2} < u_6\mu h + \beta h\tau^{\alpha}|u_6|^{\alpha} sgn(u_6)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u_6)) < \frac{\pi}{2}.$$

We solve the following two equations to obtain the estimators $\hat{\beta}_n$ and $\hat{\mu}_n$:

$$\arctan\left(\frac{\Im(\psi(u_5))}{\Re(\psi(u_5))}\right) = u_5\mu h + \beta h\tau^{\alpha}|u_5|^{\alpha} sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u_5) \tag{4.38}$$

$$\arctan\left(\frac{\Im(\psi(u_6))}{\Re(\psi(u_6))}\right) = u_6\mu h + \beta h\tau^{\alpha}|u_6|^{\alpha} sgn(u_6)\tan\left(\frac{\pi\alpha}{2}\right) - \nu D(u_6). \tag{4.39}$$

Multiplying equation (4.38) by u_6 and equation (4.39) by u_5 and then subtracting them and replacing $\psi(u)$ by $\psi_n(u)$, ν by $\hat{\nu}_n$, α by $\hat{\alpha}_n$, τ by $\hat{\tau}_n$, we get the estimator

$$\hat{\beta}_n = \frac{u_6 \times \arctan\left(\frac{\Im(\psi_n(u_5))}{\Re(\psi_n(u_5))}\right) - u_5 \times \arctan\left(\frac{\Im(\psi_n(u_6))}{\Re(\psi_n(u_6))}\right) + \hat{\nu}_n u_6 \hat{D}(u_5) - \hat{\nu}_n u_5 \hat{D}(u_6)}{h\hat{\tau}_n^{\hat{\alpha}_n}\tan\left(\frac{\pi\hat{\alpha}_n}{2}\right)\left[u_6|u_5|^{\hat{\alpha}_n} sgn(u_5) - u_5|u_6|^{\hat{\alpha}_n} sgn(u_6)\right]}, \tag{4.40}$$

where $\hat{D}(u) = \arctan\left(\frac{uh}{u^2h + 2\hat{\omega}_n}\right)$.

Replacing β by $\hat{\beta}_n$, τ by $\hat{\tau}_n$, α by $\hat{\alpha}_n$, ν by $\hat{\nu}_n$, $\psi(u)$ by $\psi_n(u)$ and $D(u)$ by $\hat{D}(u)$ in equation (4.38), we obtain the estimator $\hat{\mu}_n$ of μ:

$$\hat{\mu}_n = \frac{\arctan\left(\frac{\Im(\psi_n(u_5))}{\Re(\psi_n(u_5))}\right) + \hat{\nu}_n\hat{D}(u_5) - \hat{\beta}_n h\hat{\tau}_n^{\hat{\alpha}_n}|u_5|^{\hat{\alpha}_n} sgn(u_5)\tan\left(\frac{\pi\hat{\alpha}_n}{2}\right)}{u_5 h}. \tag{4.41}$$

4.2.5 Estimator for κ

Note that the process Y_t is not observed directly, so the estimation of κ is not straightforward. We have that as $t \to \infty$, $Y(t)$ converges to a Gamma distribution with shape parameter $\nu = \dfrac{2\kappa m}{\beta_1^2}$ and the scale parameter $\omega = \dfrac{2\kappa}{\beta_1^2}$. We have estimated the parameters m, ν, and ω. But the characteristic function (4.13) is not very useful in estimating the parameters κ and β_1 since we can not isolate κ from β_1. We follow the conditional least squares estimation method based on the transition density function of the CIR model to estimate the parameter κ (see Overbeck and Rydén [47]).

Let $\theta = (\kappa, m, \beta_1)$. The function $f(y, \theta)$ in equation (4.4) is the probability density function of the stationary distribution of $Y(t)$.

Let the sampled process $\{Y_{t_k}\}_{k=1}^n$ have the same stationary law as $Y(t)$ and we follow [9] and [47] and write the transition density. We note that the CIR process has the closed form transition density function.

Given Y_s at time s, the transition density of Y_t at time t, $(s < t)$ is

$$p(Y_t|Y_s; \theta, h = t - s) = ce^{-u-v} \left(\frac{u}{v}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right), \tag{4.42}$$

where

$$c = \frac{2\kappa}{\beta_1^2(1 - e^{-\kappa h})}, \quad u = cY_t e^{-\kappa h}, \quad v = cY_s, \quad q = \frac{2\kappa m}{\beta_1^2} - 1$$

and $I_q(2\sqrt{uv})$ is a modified Bessel function of the first kind and of order q (see Overbeck and Rydén [47]). The transition density function (4.42) was first investigated by Feller [17]. Also, we use the ideas similar to Overbeck and Rydén [47] and Cox et al. [9] and use the conditional least squares estimation method to estimate κ and discuss its strong consistency.

The conditional mean function for the CIR model by averaging in equation (4.3) is given by

$$C(y; \theta) = \mathbb{E}_\theta\left(Y_{t_k}|Y_{t_{k-1}} = y\right) = \xi_0 + \xi_1 y, \tag{4.43}$$

with $\xi_0 = m(1 - e^{-\kappa h})$ and $\xi_1 = e^{-\kappa h}$.

The discretized model (4.7) can be written as

$$Y_{t_k} = \xi_0 + \xi_1 Y_{t_{k-1}} + \mathcal{E}_k, \tag{4.44}$$

where $\mathcal{E}_k$ is a martingale increment sequence with respect to $\mathcal{F}_k$, i. e. $\mathcal{E}_k$ is $\mathcal{F}_k$ measurable and that $\mathbb{E}_\theta[\mathcal{E}_k|\mathcal{F}_{k-1}] = 0$.

To get the conditional least squares estimator, we define the contrast function

$$\begin{aligned}\rho_n(\kappa) &= \rho_n(\kappa; (Y_{t_k})_{k=1}^n)\\ &= \sum_{k=1}^n \left|Y_{t_k} - \xi_0 - \xi_1 Y_{t_{k-1}}\right|^2,\end{aligned} \tag{4.45}$$

where $Y_{t_k} - \xi_0 - \xi_1 Y_{t_{k-1}} = (Y_{t_k} - m) + e^{-\kappa h}(m - Y_{t_{k-1}})$.

The conditional LSE $\hat{\kappa}_n$ of κ is defined as

$$\hat{\kappa}_n = \underset{\kappa>0}{\operatorname{argmin}}\, \rho_n(\kappa).$$

Compute the derivative of equation (4.45) with respect to κ and set it equals to zero. We have

$$\begin{aligned}\frac{d(\rho_n(\kappa))}{d\kappa} &= 2\sum_{k=1}^n \left(Y_{t_k} - \xi_0 - \xi_1 Y_{t_{k-1}}\right)\left[-he^{-\kappa h}\left(m - Y_{t_{k-1}}\right)\right]\\ &= 2\sum_{k=1}^n \left[(Y_{t_k} - m) + e^{-\kappa h}(m - Y_{t_{k-1}})\right]\left[-he^{-\kappa h}\left(m - Y_{t_{k-1}}\right)\right]\\ &= 2he^{-\kappa h}\left[\sum_{k=1}^n (m - Y_{t_{k-1}})(m - Y_{t_k}) - e^{-\kappa h}\sum_{k=1}^n (m - Y_{t_{k-1}})^2\right].\end{aligned} \tag{4.46}$$

Then solving $\dfrac{d(\rho_n(\kappa))}{d\kappa} = 0$ and replacing m by the estimator $\hat{m}_n$, we get the estimator $\hat{\kappa}_n$ for κ:

$$\hat{\kappa}_n = -\frac{1}{h}\ln\left(\frac{\sum_{k=1}^n (Y_{t_k} - \hat{m}_n)(Y_{t_{k-1}} - \hat{m}_n)}{\sum_{k=1}^n (Y_{t_{k-1}} - \hat{m}_n)^2}\right). \tag{4.47}$$

Next we prove the consistency of the estimator $\hat{\kappa}_n$:

Theorem 4.2.1. *Assume that h is fixed and $t_n = nh \to \infty$ as $n \to \infty$. Then the following consistency holds:*

$$\hat{\kappa}_n \to \kappa \quad \textit{almost surely as } n \to \infty.$$

Proof. From the denominator of (4.47), we have

$$\frac{1}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n)^2 \to Var(Y_\infty) = \frac{\beta_1^2 m}{2\kappa}, \quad \text{as } n \to \infty. \tag{4.48}$$

Also, from the numerator of equation (4.47), we have that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}(Y_{t_k} - \hat{m}_n)(Y_{t_{k-1}} - \hat{m}_n)\\
&= \frac{1}{n}\sum_{k=1}^{n}(\xi_0 + \xi_1 Y_{t_{k-1}} + \mathcal{E}_k - \hat{m}_n)(Y_{t_{k-1}} - \hat{m}_n)\\
&= \frac{\xi_0}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n) + \frac{\xi_1}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n + \hat{m}_n)(Y_{t_{k-1}} - \hat{m}_n)\\
&+ \frac{1}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - m + m - \hat{m}_n)\mathcal{E}_k - \frac{\hat{m}_n}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n)\\
&= \frac{\xi_0}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n) + \frac{\xi_1}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n)^2 + \frac{\xi_1 \hat{m}_n}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n)\\
&+ \frac{1}{n}\sum_{k=1}^{n}\mathcal{E}_k(Y_{t_{k-1}} - m) + (m - \hat{m}_n)\frac{1}{n}\sum_{k=1}^{n}\mathcal{E}_k - \frac{\hat{m}_n}{n}\sum_{k=1}^{n}(Y_{t_{k-1}} - \hat{m}_n).
\end{aligned} \tag{4.49}$$

Note that the first, third and the sixth term of the equation (4.49) converges to 0 since $\hat{m}_n \to m$ almost surely and by the law of large numbers for the ergodic process Y_{t_k}.

The second term on the right hand side of (4.49) converges to $\xi_1 Var(Y_\infty)$.

The fourth term on the right side of (4.49) converges to 0 by ergodicity of Y_{t_k} and the fact that

$$\begin{aligned}
\mathbb{E}[(Y_{t_{k-1}} - m)\mathcal{E}_k] &= \mathbb{E}[\mathbb{E}[(Y_{t_{k-1}} - m)\mathcal{E}_k|\mathcal{F}_{k-1}]]\\
&= \mathbb{E}[(Y_{t_{k-1}} - m)\mathbb{E}[\mathcal{E}_k|\mathcal{F}_{k-1}]] = 0.
\end{aligned} \tag{4.50}$$

Since $\hat{m}_n \to m$ almost surely, we have that

$$(m - \hat{m}_n)\frac{1}{n}\sum_{k=1}^{n}\mathcal{E}_k \to 0 \quad \text{a.s.}$$

Thus from (4.47), we have that

$$\lim_{n\to\infty}\hat{\kappa}_n = -\frac{1}{h}\ln\left(\frac{\xi_1 Var(Y_\infty)}{Var(Y_\infty)}\right) = -\frac{1}{h}\ln(e^{-\kappa h}) = \kappa.$$

Hence $\hat{\kappa}_n \to \kappa$ almost surely as $n \to \infty$. □

4.2.6 Estimator for β_1

Since we have estimated the parameters $\hat{\kappa}_n$ and $\hat{\omega}_n$, we use the relation $\omega = \dfrac{2\kappa}{\beta_1^2}$ to get the estimator for β_1, i.e., we use the equation $\hat{\beta}_{1n}^{\,2} = \dfrac{2\hat{\kappa}_n}{\hat{\omega}_n}$ to obtain the estimator $\hat{\beta}_{1n}$.

From the almost sure convergence of $\hat{\kappa}_n$ and $\hat{\omega}_n$, we have the almost sure convergence of $\hat{\beta}_{1n}$.

In the next subsection, we will find the estimators for the case $\alpha = 1$.

4.2.7 Estimation of Parameters for the case $\alpha = 1$

We solve equations (4.19), (4.20), (4.21) and (4.22) with $\alpha = 1$ to obtain the estimators $\hat{\nu}_n$, $\hat{\omega}_n$ and $\hat{\tau}_n$.

For $\alpha = 1$, using equation (4.32), we can write the logarithm of the characteristic function (4.13) as below:

$$\begin{aligned}\log(\psi(u)) = \nu\log(2\omega) - \nu\log\big((u^2h + 2\omega) + iuh\big) + iu\mu h \\ - h\tau|u| - i\frac{2}{\pi}h\beta\tau u\log|u|. \end{aligned} \tag{4.51}$$

Thus, by replacing $\psi(u)$ by $\psi_n(u)$, equation (4.51) reduces to

$$\begin{aligned}\psi_n(u) = \exp\Big\{\Big[\nu\log(2\omega) - \frac{\nu}{2}\log\big((u^2h + 2\omega)^2 + u^2h^2\big) - h\tau|u|\Big] \\ + i\Big[u\mu h - \frac{2}{\pi}h\beta\tau u\log|u| - \nu D(u)\Big]\Big\}\end{aligned} \tag{4.52}$$

where $D(u)$ is given by (4.34).

To estimate μ and β, we have

$$\Im(\psi_n(u)) = \exp\left[\nu\log(2\omega) - \frac{\nu}{2}\log\left((u^2h+2\omega)^2 + u^2h^2\right) - h\tau|u|\right] \times \cos\left\{u\mu h - \frac{2}{\pi}h\beta\tau u\log|u| - \nu D(u)\right\} \tag{4.53}$$

and

$$\Re(\psi_n(u)) = \exp\left[\nu\log(2\omega) - \frac{\nu}{2}\log\left((u^2h+2\omega)^2 + u^2h^2\right) - h\tau|u|\right] \times \sin\left\{u\mu h - \frac{2}{\pi}h\beta\tau u\log|u| - \nu D(u)\right\} \tag{4.54}$$

Dividing equation (4.53) by (4.54), we get

$$\frac{\Im\left(\psi_n(u)\right)}{\Re\left(\psi_n(u)\right)} = \tan\left(u\mu h - \frac{2}{\pi}h\beta\tau u\log|u| - \nu D(u)\right). \tag{4.55}$$

Choose two non-zero values of u, $u_5 \neq u_6$ such that

$$\frac{-\pi}{2} < u_5\mu h - \frac{2}{\pi}h\beta\tau u_5\log|u_5| - \nu D(u_5) < \frac{\pi}{2} \text{ and}$$

$$\frac{-\pi}{2} < u_6\mu h - \frac{2}{\pi}h\beta\tau u_6\log|u_6| - \nu D(u_6) < \frac{\pi}{2}.$$

We solve the following two equations to obtain the estimators $\hat{\beta}_n$ and $\hat{\mu}_n$:

$$\arctan\left(\frac{\Im\left(\psi(u_5)\right)}{\Re\left(\psi(u_5)\right)}\right) = u_5\mu h - \frac{2}{\pi}h\beta\tau u_5\log|u_5| - \nu D(u_5) \tag{4.56}$$

$$\arctan\left(\frac{\Im\left(\psi(u_6)\right)}{\Re\left(\psi(u_6)\right)}\right) = u_6\mu h - \frac{2}{\pi}h\beta\tau u_6\log|u_6| - \nu D(u_6). \tag{4.57}$$

Multiplying equation (4.56) by u_6 and equation (4.57) by u_5 and then subtracting them, and replacing ψ by $\psi_n(u)$, τ by $\hat{\tau}_n$, ν by $\hat{\nu}_n$ and $D(u)$ by $\hat{D}(u)$, we get the estimator of β:

$$\hat{\beta}_n = \frac{u_6\arctan\left(\frac{\Im(\psi_n(u_5))}{\Re(\psi_n(u_5))}\right) - u_5\arctan\left(\frac{\Im(\psi_n(u_6))}{\Re(\psi_n(u_6))}\right) + \hat{\nu}_n u_6\hat{D}(u_5) - \hat{\nu}_n u_5\hat{D}(u_6)}{\frac{2}{\pi}h\hat{\tau}_n u_5 u_6\log|\frac{u_6}{u_5}|}. \tag{4.58}$$

Replacing β_n by $\hat{\beta}_n$, α_n by $\hat{\alpha}_n$, $D(u)$ by $\hat{D}(u)$ and $D(u)$ by $\hat{D}(u)$, we obtain the estimator for μ:

$$\hat{\mu}_n = \frac{1}{\hat{\beta}_n u_5 h}\left[\arctan\left(\frac{\Im\left(\psi_n(u_5)\right)}{\Re\left(\psi_n(u_5)\right)}\right) + \hat{\nu}_n \hat{D}(u_5) + \frac{2}{\pi} h \hat{\tau}_n u_5 \log|u_5|\right]. \tag{4.59}$$

The estimators $\hat{\mu}_n$ and $\hat{\beta}_n$ are consistent estimators of μ and β, respectively because they are based on the estimators $\psi_n(u)$, $\Im(\psi_n(u))$ and $\Re(\psi_n(u))$, i.e., by the almost sure convergence of $\hat{\alpha}_n$, $\hat{\tau}_n$, $\psi_n(u_5)$, $\psi_n(u_6)$, $\hat{D}(u_5)$ and $\hat{D}(u_6)$, we get the almost sure convergence of $\hat{\beta}_n$ to β and $\hat{\mu}_n$ to μ.

4.3 ASYMPTOTIC PROPERTIES OF THE ESTIMATORS

In this section, we develop the asymptotic distribution for the estimators when large samples are available. We will obtain the joint asymptotic properties of the estimators $\hat{\nu}_n$, $\hat{\omega}_n$, $\hat{\alpha}_n$, $\hat{\tau}_n$, $\hat{\beta}_n$ and $\hat{\mu}_n$.

4.3.1 Joint Asymptotic Properties of the Proposed Estimators

In this section, we will study the joint asymptotic behavior of the estimators of the parameters ν, ω, α, τ, β and μ. Hereby we mainly follow the scheme and ideas of Cheng et al. [7].

Let $\eta = (\nu, \omega, \alpha, \tau, \beta, \mu)^T$ and $\hat{\eta}_n = (\hat{\nu}_n, \hat{\omega}_n, \hat{\alpha}_n, \hat{\tau}_n, \hat{\beta}_n, \hat{\mu}_n)^T$. We would like to compute the asymptotic covariance of the estimators of the parameters, i.e., we want to compute the covariance matrix of $\sqrt{n}(\hat{\eta}_n - \eta)$ as $n \to \infty$.

Let $Z_j = \Delta(\log(X_{t_j}))$, $j = 1, 2, \ldots, n$, which follows the law given in (4.13) for $\alpha \neq 1$. Define stationary sequence $\{\tilde{Z}_j\}_{j=0}^{\infty}$ with $\tilde{Z}_0 \overset{d}{\sim} Z_\infty$.

For any nice function f, we let $S_n(f) = \frac{1}{n}\sum_{j=1}^{n} f(Z_j)$. Let $F_u(x) = \cos(ux)$ and $G_u(x) = \sin(ux)$, then we have $\psi_n(u) = \frac{1}{n}\sum_{j=1}^{n} e^{iuZ_j} = S_n(F_u) + iS_n(G_u)$. It's clear that

$$|\psi_n(u)|^2 = S_n^2(F_u) + S_n^2(G_u). \tag{4.60}$$

Now, we construct the asymptotic covariance matrix associated with

$$V_n = (V_{n_1}, V_{n_2}, V_{n_3}, V_{n_4}, V_{n_5}, V_{n_6}, V_{n_7}, V_{n_8}, V_{n_9}, V_{n_{10}}, V_{n_{11}}, V_{n_{12}})^T$$

where

$$\begin{aligned}
&V_{n_1} = S_n(F_{u_1}), \quad V_{n_2} = S_n(G_{u_1}), \quad V_{n_3} = S_n(F_{u_2}), \quad V_{n_4} = S_n(G_{u_2}), \\
&V_{n_5} = S_n(F_{u_3}), \quad V_{n_6} = S_n(G_{u_3}), \quad V_{n_7} = S_n(F_{u_4}), \quad V_{n_8} = S_n(G_{u_4}), \\
&V_{n_9} = S_n(F_{u_5}), \quad V_{n_{10}} = S_n(G_{u_5}), V_{n_{11}} = S_n(F_{u_6}), \text{ and } V_{n_{12}} = S_n(G_{u_6}).
\end{aligned}$$

For two functions $f(x)$ and $g(x)$, the asymptotic covariance $Cov\left(\sqrt{n}S_n(f), \sqrt{n}S_n(g)\right)$ of $\sqrt{n}S_n(f)$ and $\sqrt{n}S_n(g)$ is defined by

$$\begin{aligned}
\sigma_{fg} &= \lim_{n\to\infty} Cov\left(\sqrt{n}S_n(f), \sqrt{n}S_n(g)\right) \\
&= Cov(f(\tilde{Z}_0), g(\tilde{Z}_0)) + 2\sum_{k=1}^{\infty}\left[Cov(f(\tilde{Z}_0), g(\tilde{Z}_k))\right].
\end{aligned} \tag{4.61}$$

The corresponding asymptotic covariance matrix of $\sqrt{n}V_n$ is obtained by

$$\begin{aligned}
\Sigma_{12} &= \lim_{n\to\infty}\left(Cov(\sqrt{n}V_{nk}, \sqrt{n}V_{nl})\right)_{1\le k,\ l\le 12} \\
&= (\sigma_{g_k g_l})_{1\le k,l\le 12}\,,
\end{aligned} \tag{4.62}$$

where

$$\begin{aligned}
g_1(x) &= F_{u_1}(x), \quad g_2(x) = G_{u_1}(x), \quad g_3(x) = F_{u_2}(x), \\
g_4(x) &= G_{u_2}(x), \quad g_5(x) = F_{u_3}(x), \quad g_6(x) = G_{u_3}(x), \\
g_7(x) &= F_{u_4}(x), \quad g_8(x) = G_{u_4}(x), \quad g_9(x) = F_{u_5}(x), \\
g_{10}(x) &= G_{u_5}(x), \quad g_{11}(x) = G_{u_6}(x), \quad g_{12}(x) = G_{u_6}(x).
\end{aligned} \tag{4.63}$$

Let $v = (v_1, v_2, \ldots, v_{12})^T$, where $v_j = \mathbb{E}[g_j(\tilde{Z}_0)],\ j = 1, 2, \ldots, 12$. Thus, by ergodic theorem, we have that $V_{n_j} \to v_j$ almost surely for $j = 1, 2, \ldots, 12$.

For $z = (z_1, z_2, \ldots z_{12})^T$, $z_j \neq 0,\ j = 1, 2, \ldots, 12$, we define the following functions

$$\hat{\gamma}_1(z) = \frac{1}{2}\log(z_1^2 + z_2^2), \quad \hat{\gamma}_2(z) = \frac{1}{2}\log(z_3^2 + z_4^2), \tag{4.64}$$

$$\hat{\gamma}_3(z) = \frac{1}{2}\log(z_5^2 + z_6^2), \quad \hat{\gamma}_4(z) = \frac{1}{2}\log(z_7^2 + z_8^2), \tag{4.65}$$

$$\text{and} \ \ \hat{\gamma}_5(z) = \arctan\left(\frac{z_{10}}{z_9}\right), \quad \hat{\gamma}_6(z) = \arctan\left(\frac{z_{12}}{z_{11}}\right). \tag{4.66}$$

Then we have that

$$\gamma_1(\eta) := \hat{\gamma}_1(v) = \nu\log(2\omega) - \frac{\nu}{2}\log[(u_1^2h + 2\omega)^2 + u_1^2h^2] - h\tau^\alpha|u_1|^\alpha \tag{4.67}$$

$$\gamma_2(\eta) := \hat{\gamma}_2(v) = \nu\log(2\omega) - \frac{\nu}{2}\log[(u_2^2h + 2\omega)^2 + u_2^2h^2] - h\tau^\alpha|u_2|^\alpha \tag{4.68}$$

$$\gamma_3(\eta) := \hat{\gamma}_3(v) = \nu\log(2\omega) - \frac{\nu}{2}\log[(u_3^2h + 2\omega)^2 + u_3^2h^2] - h\tau^\alpha|u_3|^\alpha \tag{4.69}$$

$$\gamma_4(\eta) := \hat{\gamma}_4(v) = \nu\log(2\omega) - \frac{\nu}{2}\log[(u_4^2h + 2\omega)^2 + u_4^2h^2] - h\tau^\alpha|u_4|^\alpha \tag{4.70}$$

$$\begin{aligned}\gamma_5(\eta) := \hat{\gamma}_5(v) &= u_5\mu h + \beta h\tau^\alpha|u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right)\\ &\quad - \nu\arctan\left(\frac{u_5h}{u_5^2h + 2\omega}\right)\end{aligned} \tag{4.71}$$

$$\begin{aligned}\gamma_6(\eta) := \hat{\gamma}_6(v) &= u_6\mu h + \beta h\tau^\alpha|u_6|^\alpha sgn(u_6)\tan\left(\frac{\pi\alpha}{2}\right)\\ &\quad - \nu\arctan\left(\frac{u_6h}{u_6^2h + 2\omega}\right)\end{aligned} \tag{4.72}$$

Let $\hat{\gamma}(z) = (\hat{\gamma}_1(z), \hat{\gamma}_2(z), \hat{\gamma}_3(z), \hat{\gamma}_4(z), \hat{\gamma}_5(z), \hat{\gamma}_6(z))^T$, for $z \in \mathbb{R}^{12}$. Then, we have that

$$\hat{\gamma}^{(1)}(z) = \left(\frac{\partial\hat{\gamma}_j}{\partial z_k}\right)_{1\le j\le 6, 1\le k\le 12}$$

$$\text{and} \ \ \gamma(\eta) = (\gamma_1(\eta), \gamma_2(\eta), \gamma_3(\eta), \gamma_4(\eta), \gamma_5(\eta), \gamma_6(\eta))^T.$$

The partial derivatives of $\hat{\gamma}_j(z), j = 1, 2, 3, 4, 5, 6$ with respect to $z_1, z_2, \ldots z_{12}$ are provided below.

$$
\begin{aligned}
&\frac{\partial \hat{\gamma}_1}{\partial z_1} = \frac{z_1}{(z_1^2+z_2^2)}, \quad \frac{\partial \hat{\gamma}_1}{\partial z_2} = \frac{z_2}{(z_1^2+z_2^2)}, \quad \frac{\partial \hat{\gamma}_1}{\partial z_3} = \ldots = \frac{\partial \hat{\gamma}_1}{\partial z_{12}} = 0,\\
&\frac{\partial \hat{\gamma}_2}{\partial z_1} = 0, \ \frac{\partial \hat{\gamma}_2}{\partial z_2} = 0, \ \frac{\partial \hat{\gamma}_2}{\partial z_3} = \frac{z_3}{(z_3^2+z_4^2)}, \quad \frac{\partial \hat{\gamma}_2}{\partial z_4} = \frac{z_4}{(z_3^2+z_4^2)},\\
&\frac{\partial \hat{\gamma}_2}{\partial z_5} = \ldots = \frac{\partial \hat{\gamma}_2}{\partial z_{12}} = 0,\\
&\frac{\partial \hat{\gamma}_3}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_3}{\partial z_4} = 0, \ \frac{\partial \hat{\gamma}_3}{\partial z_5} = \frac{z_5}{(z_5^2+z_6^2)}, \quad \frac{\partial \hat{\gamma}_3}{\partial z_6} = \frac{z_6}{(z_5^2+z_6^2)},\\
&\frac{\partial \hat{\gamma}_3}{\partial z_7} = \ldots = \frac{\partial \hat{\gamma}_3}{\partial z_{12}} = 0,\\
&\frac{\partial \hat{\gamma}_4}{\partial z_1} = \ldots = \frac{\partial \hat{\gamma}_4}{\partial z_6} = 0, \ \frac{\partial \hat{\gamma}_4}{\partial z_7} = \frac{z_8}{(z_7^2+z_8^2)}, \quad \frac{\partial \hat{\gamma}_4}{\partial z_8} = \frac{z_7}{(z_7^2+z_8^2)},\\
&\frac{\partial \hat{\gamma}_4}{\partial z_9} = \frac{\partial \hat{\gamma}_4}{\partial z_{12}} = 0,\\
&\frac{\partial \hat{\gamma}_5}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_5}{\partial z_8} = 0, \ \frac{\partial \hat{\gamma}_5}{\partial z_{11}} = \frac{\partial \hat{\gamma}_5}{\partial z_{12}} = 0, \quad \frac{\partial \hat{\gamma}_5}{\partial z_9} = \frac{-z_{10}}{(z_9^2+z_{10}^2)},\\
&\frac{\partial \hat{\gamma}_5}{\partial z_{10}} = \frac{z_9}{(z_9^2+z_{10}^2)}, \quad \frac{\partial \hat{\gamma}_5}{\partial z_{11}} = \frac{\partial \hat{\gamma}_5}{\partial z_{12}} = 0,\\
&\frac{\partial \hat{\gamma}_6}{\partial z_1} = \ldots \frac{\partial \hat{\gamma}_6}{\partial z_{10}} = 0, \ \frac{\partial \hat{\gamma}_6}{\partial z_{11}} = \frac{-z_{12}}{z_{11}^2+z_{12}^2}, \quad \frac{\partial \hat{\gamma}_6}{\partial z_{12}} = \frac{z_{11}}{z_{11}^2+z_{12}^2}.
\end{aligned}
$$

Put

$$\Psi_n(\eta) = (\Psi_{1,n}(\eta), \Psi_{2,n}(\eta), \Psi_{3,n}(\eta), \Psi_{4,n}(\eta), \Psi_{5,n}(\eta), \Psi_{6,n}(\eta))^T,$$

where $\Psi_{j,n}(\eta) = \hat{\gamma}_j(V_n) - \gamma_j(\eta),\ j = 1,2,3,4,5,6$. Then, we know that $\hat{\eta}_n$ satisfies $\Psi_n(\hat{\eta}_n) = 0$.

We have the following partial derivatives of $\hat{\gamma}_j,\ j = 1,2,\ldots,6$ with respect to parameters ν, ω, α, τ, β and μ:

$$
\begin{aligned}
&\frac{\partial \gamma_1}{\partial \nu} = \log(2\omega) - \frac{1}{2}\log[(u_1^2 h + 2\omega)^2 + u_1^2 h^2],\\
&\frac{\partial \gamma_1}{\partial \omega} = \frac{\nu}{\omega} - \frac{2\nu(u_1^2 h + 2\omega)}{((u_1^2 h + 2\omega)^2 + u_1^2 h^2)}, \quad \frac{\partial \gamma_1}{\partial \alpha} = -h|\tau u_1|^\alpha \log(\tau|u_1|),\\
&\frac{\partial \gamma_1}{\partial \tau} = -\alpha h |u_1|^\alpha \tau^{\alpha-1}, \quad \frac{\partial \gamma_1}{\partial \beta} = \frac{\partial \gamma_1}{\partial \mu} = 0.
\end{aligned}
$$

$$\frac{\partial\gamma_2}{\partial\nu} = \log(2\omega) - \frac{1}{2}\log[(u_2^2h+2\omega)^2 + u_2^2h^2],$$
$$\frac{\partial\gamma_2}{\partial\omega} = \frac{\nu}{\omega} - \frac{2\nu(u_2^2h+2\omega)}{((u_2^2h+2\omega)^2+u_2^2h^2)}, \quad \frac{\partial\gamma_2}{\partial\alpha} = -h|\tau u_2|^\alpha \log(\tau|u_2|)|,$$
$$\frac{\partial\gamma_2}{\partial\tau} = -\alpha h|u_2|^\alpha\tau^{\alpha-1}, \quad \frac{\partial\gamma_2}{\partial\beta} = \frac{\partial\gamma_2}{\partial\mu} = 0.$$

$$\frac{\partial\gamma_3}{\partial\nu} = \log(2\omega) - \frac{1}{2}\log[(u_3^2h+2\omega)^2 + u_3^2h^2],$$
$$\frac{\partial\gamma_3}{\partial\omega} = \frac{\nu}{\omega} - \frac{2\nu(u_3^2h+2\omega)}{((u_3^2h+2\omega)^2+u_3^2h^2)}, \quad \frac{\partial\gamma_3}{\partial\alpha} = -h|\tau u_3|^\alpha \log(\tau|u_3|),$$
$$\frac{\partial\gamma_3}{\partial\tau} = -\alpha h|u_3|^\alpha\tau^{\alpha-1}, \quad \frac{\partial\gamma_3}{\partial\beta} = \frac{\partial\gamma_3}{\partial\mu} = 0.$$

$$\frac{\partial\gamma_4}{\partial\nu} = \log(2\omega) - \frac{1}{2}\log[(u_4^2h+2\omega)^2 + u_4^2h^2],$$
$$\frac{\partial\gamma_4}{\partial\omega} = \frac{\nu}{\omega} - \frac{2\nu(u_4^2h+2\omega)}{((u_4^2h+2\omega)^2+u_4^2h^2)}, \quad \frac{\partial\gamma_4}{\partial\alpha} = -h|\tau u_4|^\alpha \log(\tau|u_4|),$$
$$\frac{\partial\gamma_4}{\partial\tau} = -\alpha h|u_4|^\alpha\tau^{\alpha-1}, \quad \frac{\partial\gamma_4}{\partial\beta} = \frac{\partial\gamma_4}{\partial\mu} = 0.$$

$$\frac{\partial\gamma_5}{\partial\nu} = -\arctan\left(\frac{u_5h}{u_5^2h+2\omega}\right), \quad \frac{\partial\gamma_5}{\partial\omega} = \frac{2\nu u_5h}{(u_5^2h+2\omega)^2+u_5^2h^2},$$
$$\frac{\partial\gamma_5}{\partial\alpha} = h\beta sgn(u_5)\left(\frac{\pi}{2}(\tau|u_5|)^\alpha\sec^2\left(\frac{\pi\alpha}{2}\right) + (\tau|u_5|)^\alpha\tan\left(\frac{\pi\alpha}{2}\right)\log(\tau|u_5|)\right),$$
$$\frac{\partial\gamma_5}{\partial\tau} = h\alpha\beta\tau^{\alpha-1}sgn(u_5)|u_5|^\alpha\tan\left(\frac{\pi\alpha}{2}\right),$$
$$\frac{\partial\gamma_5}{\partial\beta} = h\tau^\alpha|u_5|^\alpha sgn(u_5)\tan\left(\frac{\pi\alpha}{2}\right), \quad \frac{\partial\gamma_5}{\partial\mu} = u_5h.$$

$$\frac{\partial\gamma_6}{\partial\nu} = -\arctan\left(\frac{u_6h}{u_6^2h+2\omega}\right), \quad \frac{\partial\gamma_6}{\partial\omega} = \frac{2\nu u_6h}{(u_6^2h+2\omega)^2+u_6^2h^2},$$
$$\frac{\partial\gamma_6}{\partial\alpha} = h\beta sgn(u_6)\left(\frac{\pi}{2}(\tau|u_6|)^\alpha\sec^2\left(\frac{\pi\alpha}{2}\right) + (\tau|u_6|)^\alpha\tan\left(\frac{\pi\alpha}{2}\right)\log(\tau|u_6|)\right),$$
$$\frac{\partial\gamma_6}{\partial\tau} = h\alpha\beta\tau^{\alpha-1}sgn(u_6)|u_6|^\alpha\tan\left(\frac{\pi\alpha}{2}\right),$$
$$\frac{\partial\gamma_6}{\partial\beta} = h\tau^\alpha|u_6|^\alpha sgn(u_6)\tan\left(\frac{\pi\alpha}{2}\right), \frac{\partial\gamma_6}{\partial\mu} = u_6h.$$

Note that $\nabla_\eta\Psi_n(\eta) = -\nabla_\eta\gamma(\eta)$, where $\nabla_\eta \gamma(\eta) = \begin{pmatrix} \dfrac{\partial\gamma_1(\eta)}{\partial\nu} & \dfrac{\partial\gamma_1(\eta)}{\partial\omega} & \dfrac{\partial\gamma_1(\eta)}{\partial\alpha} & \dfrac{\partial\gamma_1(\eta)}{\partial\tau} & \dfrac{\partial\gamma_1(\eta)}{\partial\beta} & \dfrac{\partial\gamma_1(\eta)}{\partial\mu} \\ \dfrac{\partial\gamma_2(\eta)}{\partial\nu} & \dfrac{\partial\gamma_2(\eta)}{\partial\omega} & \dfrac{\partial\gamma_2(\eta)}{\partial\alpha} & \dfrac{\partial\gamma_2(\eta)}{\partial\tau} & \dfrac{\partial\gamma_2(\eta)}{\partial\beta} & \dfrac{\partial\gamma_21(\eta)}{\partial\mu} \\ \dfrac{\partial\gamma_3(\eta)}{\partial\nu} & \dfrac{\partial\gamma_3(\eta)}{\partial\omega} & \dfrac{\partial\gamma_3(\eta)}{\partial\alpha} & \dfrac{\partial\gamma_3(\eta)}{\partial\tau} & \dfrac{\partial\gamma_3(\eta)}{\partial\beta} & \dfrac{\partial\gamma_3(\eta)}{\partial\mu} \\ \dfrac{\partial\gamma_4(\eta)}{\partial\nu} & \dfrac{\partial\gamma_4(\eta)}{\partial\omega} & \dfrac{\partial\gamma_4(\eta)}{\partial\alpha} & \dfrac{\partial\gamma_4(\eta)}{\partial\tau} & \dfrac{\partial\gamma_4(\eta)}{\partial\beta} & \dfrac{\partial\gamma_4(\eta)}{\partial\mu} \\ \dfrac{\partial\gamma_5(\eta)}{\partial\nu} & \dfrac{\partial\gamma_5(\eta)}{\partial\omega} & \dfrac{\partial\gamma_5(\eta)}{\partial\alpha} & \dfrac{\partial\gamma_5(\eta)}{\partial\tau} & \dfrac{\partial\gamma_5(\eta)}{\partial\beta} & \dfrac{\partial\gamma_5(\eta)}{\partial\mu} \end{pmatrix}$

Let $I(\eta) = \nabla_\eta \gamma(\eta)$. Let $U = (U_1, U_2, \ldots U_{12})^T \sim N(0, \Sigma_{12})$, then we have the following Central Limit Theorem and the proof idea mainly follows Cheng et al. [7].

Theorem 4.3.1. *Let* $U = (U_1, U_2, \ldots, U_{12})^T \sim N(0, \Sigma_{12})$, $V_n = (V_{n1}, V_{n2}, \ldots, V_{n_{12}})^T$, $v = (v_1, v_2, \ldots, v_{12})^T$, *then we have the Central Limit Theorem.*

$$\sqrt{n}(V_n - v) \xrightarrow{d} U \tag{4.73}$$

Proof. For any non-zero vector $a = (a_1, a_2, \ldots, a_{12})^T \in \mathbb{R}^{12}$, we have

$$a^T U \sim N(0, a^T \Sigma_{12} a).$$

Define $H = a^T(g_1, g_2, \ldots, g_{12})^T$, $\bar{H} = H - \mathbb{E}[H(\tilde{Z}_0)] = a^T(\bar{g}_1, \bar{g}_2, \ldots \bar{g}_{12})^T$. By the Central Limit Theorem for ergodic processes by Meyn and Tweedie [39], we have

$$a^T \sqrt{n}(V_n - v) = \sqrt{n} S_n(\bar{H}) \xrightarrow{d} N(0, \sigma_H^2), \tag{4.74}$$

where $\sigma_H^2 = \mathbb{E}[\bar{H}^2(\tilde{Z}_0)] + 2\sum_{k=1}^{\infty} \mathbb{E}[\bar{H}(\tilde{Z}_0)\bar{H}(\tilde{Z}_k)] = a^T \Sigma_{12} a$. Hence, for any nonzero $a \in \mathbb{R}^{12}$, we have $a^T\sqrt{n}(V_n - v) \xrightarrow{d} a^T U$. It follows that $\sqrt{n}(V_n - u) \xrightarrow{d} U$ by Theorem 29.4 of Billingsley [4]. □

The following Central Limit Theorem also holds:

Theorem 4.3.2. *As* $n \to \infty$, *the following convergence is true:*

$$\sqrt{n}\Psi_n(\eta) \xrightarrow{d} \hat{\gamma}^{(1)}(v) U.$$

Proof. From $\sqrt{n}\Psi_n(\eta) = \sqrt{n}(\hat{\gamma}(V_n) - \hat{\gamma}(v))$, we can use Theorem 4.3.1 and the delta method to obtain the result. □

Finally, we have the following main result:

Theorem 4.3.3. *Fix an arbitrary $h > 0$. Denote $\eta = (\nu, \omega, \alpha, \tau, \beta, \mu)$ and $\hat{\eta}_n = (\hat{\nu}_n, \hat{\omega}_n, \hat{\alpha}_n, \hat{\tau}_n, \hat{\beta}_n, \hat{\mu}_n)$, where $\hat{\nu}_n$, $\hat{\omega}_n$, $\hat{\alpha}_n$ $\hat{\tau}_n$, $\hat{\beta}_n$ and $\hat{\mu}_n$ are given by equations* (4.26), (4.28), (4.29), (4.30), (4.40) *and* (4.41) *respectively. Then the following results are true:*

i. $\lim_{n\to\infty} \hat{\eta}_n = \eta$ *almost surely.*

ii. As $n \to \infty$, we have

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{d} N(0, \Sigma_6), \tag{4.75}$$

where $\Sigma_6 = (I(\eta))^{-1}\hat{\gamma}^{(1)}(v)\Sigma_{12}\left(\hat{\gamma}^{(1)}(v)\right)^T \left((I(\eta))^{-1}\right)^T.$

Proof. The proof idea mainly follows Cheng et al. [7].

i. It's clear that each component of $\hat{\eta}_n$ converges to the corresponding component of η almost surely as $n \to \infty$ as discussed in subsections 4.2.2 - 4.2.4. So, $\lim_{n\to\infty} \hat{\eta}_n = \eta$ a.s.

ii. By mean value theorem, we have

$$\Psi_n(\hat{\eta}_n) - \Psi_n(\eta) = \int_0^1 \nabla_\eta \Psi_n(\eta + s(\hat{\eta}_n - \eta))ds.(\hat{\eta}_n - \eta) \tag{4.76}$$

Set $I_n(\eta) = -\int_0^1 \nabla_\eta \Psi_n(\eta + s(\hat{\eta}_n - \eta))ds$, which is invertible. Since $\Psi_n(\hat{\eta}_n) = 0$, we get

$$\sqrt{n}(\hat{\eta}_n - \eta) = (I_n(\eta))^{-1}\sqrt{n}\Psi_n(\eta). \tag{4.77}$$

Since $\hat{\eta}_n \to \eta$ a.s., it follows that $(I(\eta_n))^{-1} \to (I(\eta))^{-1}$ a.s. Hence, from Theorem 4.3.2 and Slutsky's Theorem, we find

$$\sqrt{n}(\hat{\eta}_n - \eta) \xrightarrow{d} (N(0, \Sigma_6).$$

□

4.4 SIMULATION STUDY

In this section, we present simulations to assess the effectiveness of the estimators. We can simulate α-stable random variables as discussed by Janicki and Weron [24] and Weron and Weron [58] as well as Cheng et al. [7]. Let $t_k = kh,\ k = 1, 2, \dots, n$ and $h > 0$ is fixed. Integrating both sides of (4.1) from t_{k-1} to t_k gives us

$$\log(X_{t_k}) = \log(X_{t_{k-1}}) + \mu(t_k - t_{k-1}) - \frac{1}{2}\int_{t_{k-1}}^{t_k} Y_s ds + \int_{t_{k-1}}^{t_k} \sqrt{Y_s} dW_s + \tau \int_{t_{k-1}}^{t_k} dL_s^{\alpha}.$$

Hence

$$\Delta(\log(X_{t_k})) = \log(X_{t_k}) - \log(X_{t_{k-1}}) \approx \mu h + \sqrt{Y_{t_{k-1}}}\Delta W_{t_k} - \frac{1}{2}Y_{t_{k-1}}h + \tau \Delta L_{t_k}^{\alpha}. \quad (4.78)$$

Similarly, integrating both sides of (4.3) from t_{k-1} to t_k gives us

$$\Delta Y_{t_k} = Y_{t_k} - Y_{t_{k-1}} \approx \kappa(m - Y_{t_{k-1}})h + \beta_1\sqrt{Y_{t_{k-1}}}N(0, h). \quad (4.79)$$

Note that the process Y_t in not observed directly, so the estimation of κ is not straightforward. Since we have that $\hat{\mu}_n \to \mu$, $\hat{\tau}_n \to \tau$, $\hat{\alpha}_n \to \alpha$ and $\Delta W_{t_k} \sim N(0, h)$, we now use equation below to obtain the realizations for Y_{t_k}'s:

$$\Delta(\log(X_{t_k})) \approx \hat{\mu}_n h + \sqrt{Y_{t_{k-1}}}\Delta W_{t_k} - \frac{1}{2}Y_{t_{k-1}}h + \hat{\tau}_n \Delta L_{t_k}^{\hat{\alpha}_n}. \quad (4.80)$$

Solving equation (4.80) for $Y_{t_{k-1}}$, we obtain

$$\sqrt{Y_{t_{k-1}}} = \frac{1}{h}\left(\Delta W_{t_k} \pm \sqrt{\Delta W_{t_k}^2 - 2h\left(\Delta(\log(X_{t_k}) - \hat{\mu}_n h - \hat{\tau}_n \Delta L_{t_k}^{\hat{\alpha}_n}\right)}\right). \quad (4.81)$$

Note for some possible sample paths of W_t and L_t^{α}, (4.81) can give negative values. We consider only the sample paths where the right side of (4.81) is positive. We also note that using the characteristic function (4.13) and the simulation procedure explained in Chapter 3, we can obtain the estimators $\hat{\alpha}_n$, $\hat{\tau}_n$, $\hat{m}_n$, $\hat{\beta}_n$ $\hat{\mu}_n$, $\hat{\kappa}_n$ and $\hat{\beta}_{1n}$.

We perform a simulation study to assess the reliability of the proposed estimators. For selected values of α, τ, m, μ, β, κ and β_1, we generate 1000 realizations and simulate samples of sizes $N = 5000, 10000, 15000$, 20000 and 25000. We present the mean and variance of each of the estimators.

The simulation results presented in Table 4.1 shows that variances of estimators decreased while the estimates gets closer to their true values. We choose the true values of the parameters: $\alpha = 1.5,\ \tau = 1,\ m = 2,\ \mu = 2,\ \beta = 0.5,\ \kappa = 1,$ and $\beta_1 = \sqrt{0.5} \approx 0.707$ and use the arguments u_i's: $u_1 = 0.1,\ u_2 = 0.2,\ u_3 = 0.9,\ u_4 = 0.8,\ u_5 = 0.7,$ $u_6 = 0.6$ and the step size $h = 0.2$.

In Table 4.2, we present the parameter estimation for the symmetric α-stable case for the following choice of true values: $\alpha = 1.5,\ \tau = 1,\ m = 2,\ \mu = 2,\ \beta = 0,$ $\kappa = 1,\ \beta_1 = \sqrt{0.5} \approx 0.707$. We choose positive arguments $u_1 = 0.1,\ \ u_2 = 0.2,\ u_3 = 0.3,\ u_4 = 0.4,$ $u_5 = 0.7,$ $u_6 = 0.6$ and the step size $h = 0.25$.

We also present the graphs of the process $\log(X_t)$ in Fig.4.1, Fig.4.2, Fig.4.3 and Fig.4.4 for specific choice of the parameters as shown. Similarly, Fig.4.5 and Fig.4.6 show multiple sample paths of the process $\log(X_t)$ for specified parameters.

Table 4.1: Numerical values of the estimator for $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0.5$, $\mu = 2$, $\kappa = 1$, $\beta_1 = 0.707$ and $h = 0.2$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.448	1.467	1.471	1.475	1.489
	(0.019)	(0.009)	(0.006)	(0.005)	(0.003)
$\hat{\tau}_n$	0.898	0.944	0.940	0.951	0.973
	(0.008)	(0.004)	(0.003)	(0.002)	(0.001)
$\hat{m}_n$	2.024	2.023	2.020	2.020	2.020
	(0.002)	(0.001	(0.000)	(0.000)	(0.000)
$\hat{\beta}_n$	0.670	0.606	0.613	0.591	0.539
	(0.113)	(0.037)	(0.018)	(0.016)	(0.012)
$\hat{\mu}_n$	2.166	2.102	2.086	2.082	2.035
	(0.136)	(0.028)	(0.015)	(0.011)	(0.005)
$\hat{\kappa}_n$	0.905	0.919	0.928	0.935	0.936
	(0.004)	(0.003)	(0.002)	(0.001)	(0.001)
$\hat{\beta}_{1n}$	0.677	0.683	0.686	0.688	0.690
	(0.0005)	(0.0004)	(0.0003)	(0.0002)	(0.0002)

Table 4.2: Numerical values of the estimator for the symmetric α-stable distribution with $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0$, $\mu = 2$, $\kappa = 1$, $\beta_1 = 0.707$ and $h = 0.25$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.457	1.473	1.484	1.485	1.489
	(0.016)	(0.008)	(0.005)	(0.004	(0.003)
$\hat{\tau}_n$	0.935	0.943	0.984	0.938	0.973
	(0.008)	(0.003	(0.003)	(0.002)	(0.001)
$\hat{m}_n$	2.036	2.024	2.024	2.021	2.018
	(0.002)	(0.0004)	(0.0004)	(0.0004)	(0.0004)
$\hat{\beta}_n$	0.071	0.071	0.062	0.059	0.050
	(0.024)	(0.010)	(0.005)	(0.016)	(0.001)
$\hat{\mu}_n$	2.071	2.059	2.048	2.082	2.040
	(0.034)	(0.008)	(0.004)	(0.011)	(0.003)
$\hat{\kappa}_n$	0.903	0.920	0.936	0.936	0.936
	(0.004)	(0.002)	(0.001)	(0.001)	(0.001)
$\hat{\beta}_{1n}$	0.678	0.684	0.686	0.689	0.690
	(0.0006)	(0.0003)	(0.0002)	(0.0002)	(0.0001)

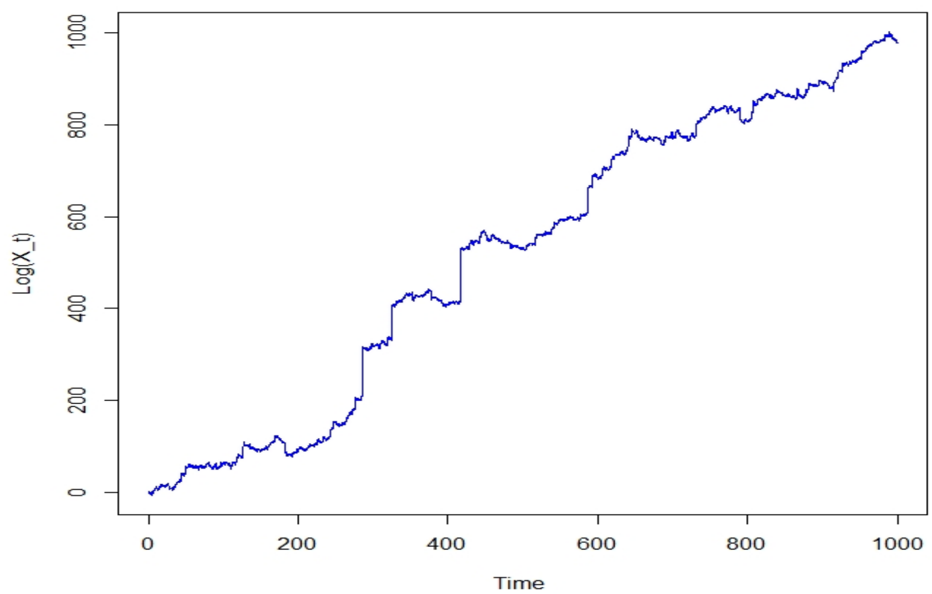


Figure 4.1: Simulation of the process $\log(X_t)$ taking $h = 0.1$, $n = 10000$, $\alpha = 1.5, \beta = 0.5, \tau = 2, \mu = 2$, $m = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.

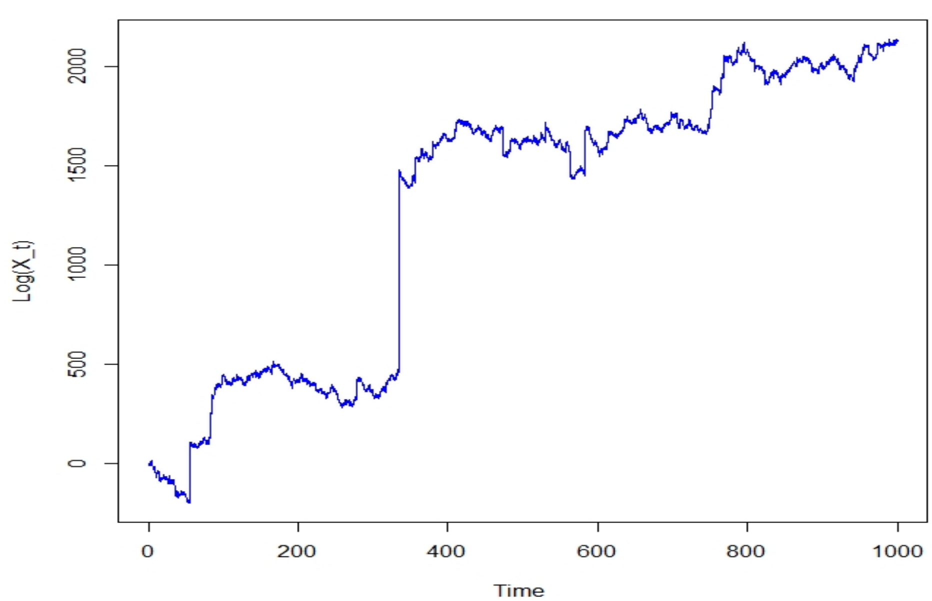


Figure 4.2: Simulation of the process $\log(X_t)$ taking $h = 0.1$, $n = 10000$, $\alpha = 1.7, \beta = 0.5$, $m = 2$, $\tau = 4, \mu = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.

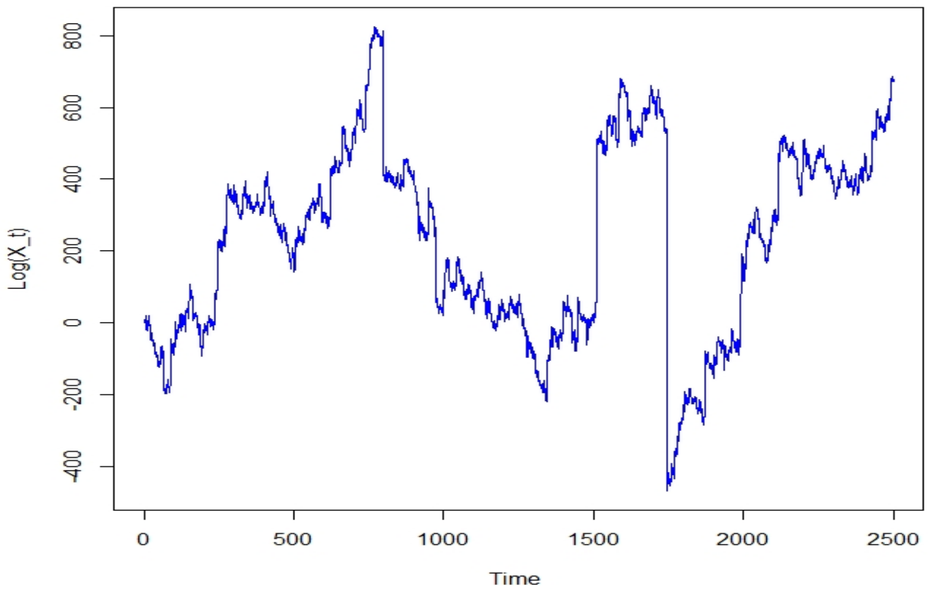


Figure 4.3: Simulation of the process $\log(X_t)$ taking $h = 0.25$, $n = 10000$, $\alpha = 1.5, \beta = 0.5, \tau = 6$, $m = 2$, $\mu = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.

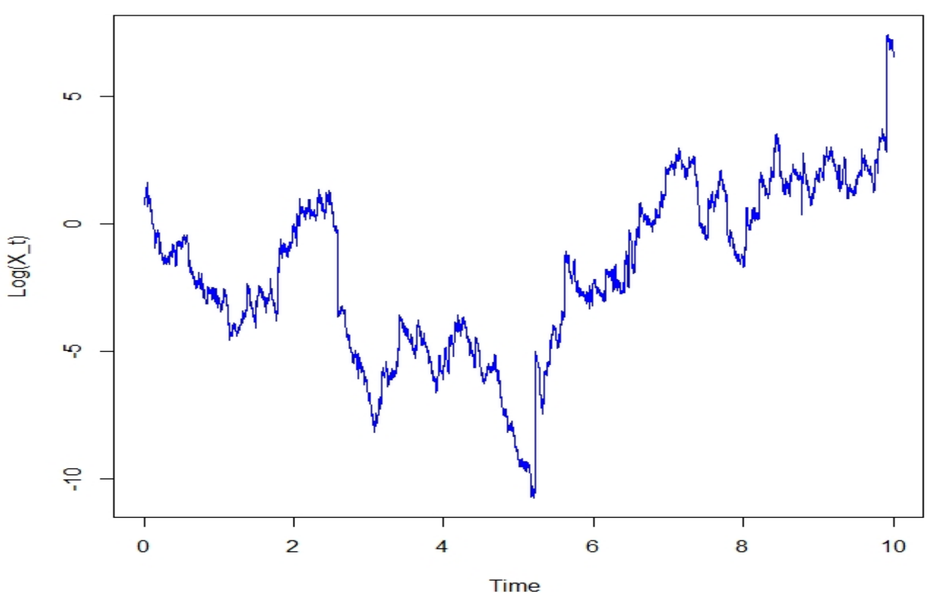


Figure 4.4: Simulation of the process $\log(X_t)$ taking $h = 0.001$, $n = 10000$, $\alpha = 1.5, \beta = 0.5, \tau = 3$, $m = 2$, $\mu = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.

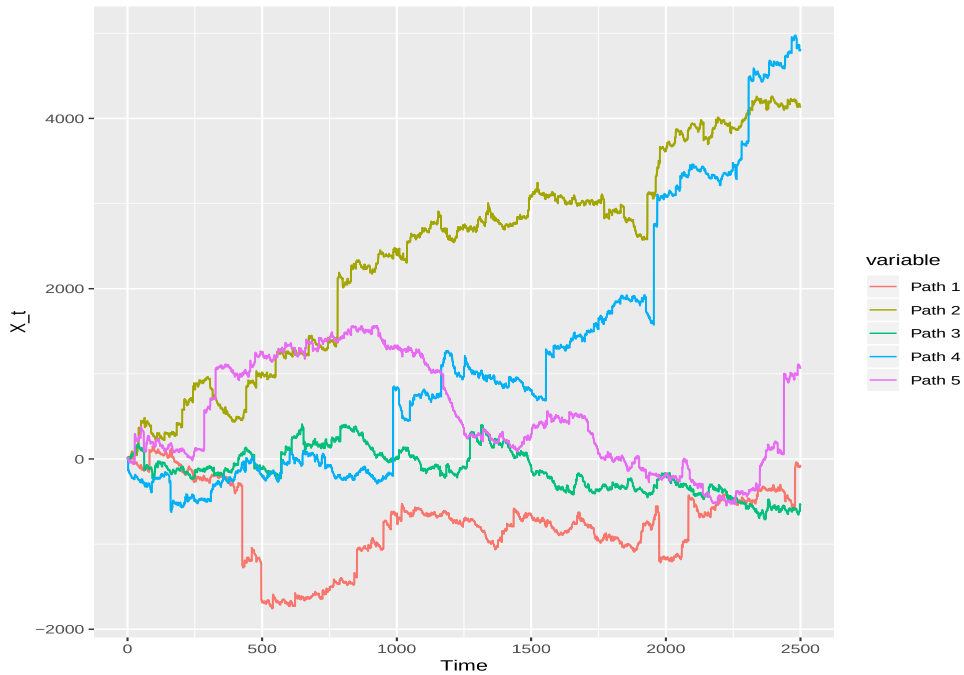
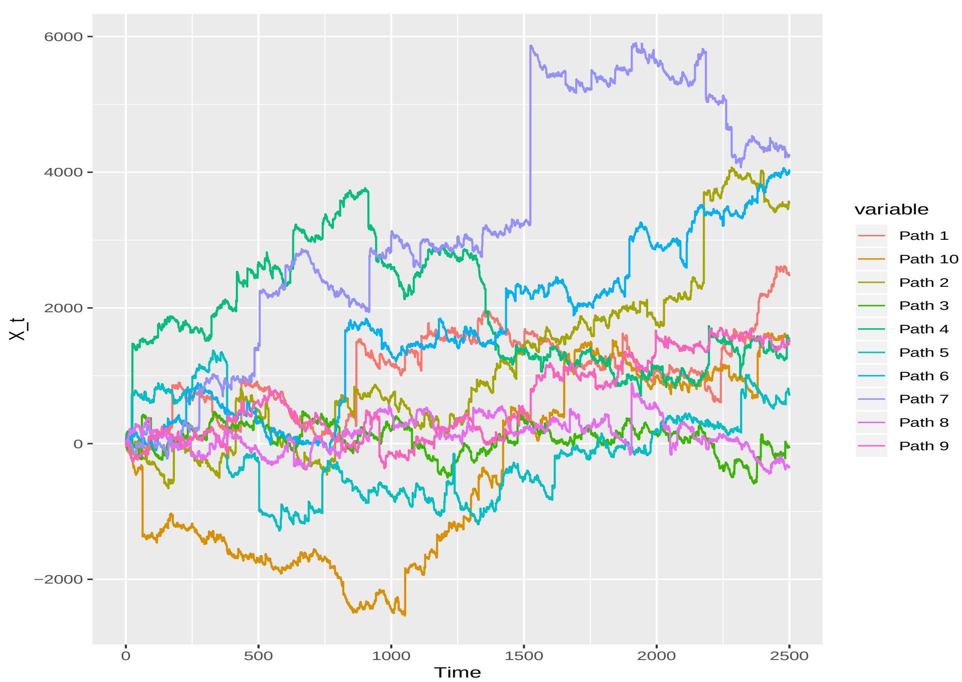


Figure 4.5: Simulated paths of the process $\log(X_t)$ taking $h = 0.25$, $n = 10000$, $\alpha = 1.5, \beta = 0.5$, $\tau = 7$, $m = 2$, $\mu = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.



pro

Figure 4.6: Simulated paths of the process $\log(X_t)$ taking $h = 0.25$, $n = 10000$, $\alpha = 1.5, \beta = 0.5$, $m = 2$, $\tau = 10, \mu = 2$, $\beta_1 = 0.5$ and $\kappa = 1$.

4.4.1 Comparison of Estimators for Different Choices of Arguments u_i's

In this subsection, we obtain the results for the estimators for the different choices of u_i's. We choose the true values: $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0.5$, $\mu = 2$, $\kappa = 1$, $\beta_1 = \sqrt{0.5} \approx 0.707$. For these values of α, τ, m, β, μ, κ, and β_1, we generate 1000 realizations and simulate samples of sizes: $N = 5000, N = 10000, N = 15000, N = 20000, N = 25000$. We present the average and variance of the estimators: $\hat{\alpha}_n$, $\hat{\tau}_n$, $\hat{m}_n$, $\hat{\beta}_n$, $\hat{\mu}_n$, $\hat{\kappa}_n$, and $\hat{\beta}_{1n}$. For this purpose, we choose the following sets of u_i's and perform the simulation. We fix the step-size $h = 0.25$.

Table 4.3: Choices of u_i's

Choice	u_1	u_2	u_3	u_4	u_5	u_6
1	0.25	1	0.75	0.8	0.7	0.6
2	0.50	1	0.75	0.8	0.7	1.4
3	0.03	1	0.75	0.8	1.3	1.4

From the estimators presented in Tables 4.4, 4.5, and 4.6, we see that all the estimators converge to the true parameter values regardless of the choices of u_i's.

Table 4.4: Numerical values of the estimators for $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0.5$, $\mu = 2$, $\kappa = 1$, $\beta_1 = \sqrt{0.5} \approx 0.707$, $h = 0.25$ and $u_1 = 0.25$, $u_2 = 1$, $u_3 = 0.75$, $u_4 = 0.8$, $u_5 = 0.7$, and $u_6 = 0.6$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.474	1.491	1.493	1.498	1.499
	(0.002)	(0.001)	(0.001)	(0.001)	(0.001)
$\hat{\tau}_n$	0.959	0.985	0.986	0.998	0.999
	(0.002)	(0.001)	(0.001)	(0.001)	(0.0001)
$\hat{m}_n$	1.866	1.890	1.93	1.955	1.960
	(0.020)	(0.011)	(0.007)	(0.003)	(0.001)
$\hat{\beta}_n$	0.568	0.563	0.562	0.553	0.532
	(0.024)	(0.011)	(0.007)	(0.006)	(0.004)
$\hat{\mu}_n$	1.988	1.979	1.974	1.970	1.990
	(0.018)	(0.003)	(0.006)	(0.005)	(0.004)
$\hat{\kappa}_n$	1.000	1.001	1.001	1.013	1.012
	(0.007)	(0.006)	(0.006)	(0.005)	(0.005)
$\hat{\beta}_{1n}$	0.691	0.693	0.694	0.696	0.697
	(0.001)	(0.001)	(0.001)	(0.001)	(0.0001)

Table 4.5: Numerical values of the estimators for $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0.5$, $\mu = 2$, $\kappa = 1$, $\beta_1 = \sqrt{0.5} \approx 0.707$, $h = 0.25$ and $u_1 = 0.5$, $u_2 = 1$, $u_3 = 0.75$, $u_4 = 0.8$, $u_5 = 0.7$, and $u_6 = 1.4$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.476	1.478	1.494	1.497	1.498
	(0.002)	(0.001)	(0.0006)	(0.005)	(0.0004)
$\hat{\tau}_n$	0.964	0.949	0.994	0.993	0.998
	(0.001)	(0.001)	(0.0004)	(0.0003)	(0.0003)
$\hat{m}_n$	1.889	1.890	1.894	1.961	1.983
	(0.021)	(0.013)	(0.011)	(0.005)	(0.001)
$\hat{\beta}_n$	0.567	0.574	0.550	0.543	0.532
	(0.019)	(0.009)	(0.006)	(0.006)	(0.003)
$\hat{\mu}_n$	1.992	2.001	1.979	1.988	1.989
	(0.021)	(0.011)	(0.006)	(0.005)	(0.004)
$\hat{\kappa}_n$	0.997	0.992	0.998	0.999	1.005
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
$\hat{\beta}_{1n}$	0.688	0.689	0.690	0.690	0.693
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)

Table 4.6: Numerical values of the estimators for $\alpha = 1.5$, $\tau = 1$, $m = 2$, $\beta = 0.5$, $\mu = 2$, $\kappa = 1$, $\beta_1 = \sqrt{0.5} \approx 0.707$, $h = 0.25$ and $u_1 = 0.03$, $u_2 = 1$, $u_3 = 0.75$, $u_4 = 0.8$, $u_5 = 1.3$, and $u_6 = 1.4$

Estimators	N=5000	N=10000	N=15000	N=20000	N=25000
$\hat{\alpha}_n$	1.440	1.474	1.487	1.490	1.498
	(0.006)	(0.003)	(0.002)	(0.001)	(0.001)
$\hat{\tau}_n$	0.922	0.983	0.992	0.999	1.013
	(0.030)	(0.016)	(0.012)	(0.009)	(0.008)
$\hat{m}_n$	1.853	1.847	1.846	1.890	1.955
	(0.003)	(0.002)	(0.002)	(0.001)	(0.0001)
$\hat{\beta}_n$	0.775	0.679	0.670	0.653	0.549
	(0.128)	(0.0039)	(0.028)	(0.020)	(0.004)
$\hat{\mu}_n$	2.177	2.095	2.071	2.061	1.987
	(0.077)	(0.026)	(0.016)	(0.013)	(0.003)
$\hat{\kappa}_n$	0.9791	0.964	0.960	0.971	0.982
	(0.006)	(0.004)	(0.004)	(0.004)	(0.003)
$\hat{\beta}_{1n}$	0.682	0.676	0.674	0.672	0.696
	(0.001)	(0.001)	(0.001)	(0.001)	(0.0001)

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

5.1 CONCLUSIONS

We successfully used the sample characteristic function approach to estimate parameters of alpha-stable geometric Lévy processes in Chapter 3 for the constant volatility case. We express our gratitude to Press [48] for introducing the characteristic function approach to estimate parameters of stable distributions. We have also discussed the consistency and the asymptotic behavior of the proposed estimators and developed a central limit theorem. Then we extended this approach to a model with stochastic volatility in Chapter 4. We chose the SDE driven by the Lévy noise where the stochastic volatility is the CIR model. The estimation of parameters became much more difficult due to the stochastic volatility. We utilized the stationary distribution of the CIR model to estimate parameters. We have also discussed the consistency and asymptotic properties of the proposed parameters. Finally, we performed simulation study to assess the validity of the estimators.

5.2 FUTURE RESEARCH

We note that the primary closed-form based estimators by Press [48] use the empirical characteristic function. One of the major drawbacks of the approach in Press [48] is that there is no rule for the selection of arguments of the characteristic function. We can find a number of attempts for finding the optimal arguments of the characteristic function approach by many authors including Knight and Satchel [27], Kogon and Williams [28] and Krutto [31]. We will try to find the optimal u_i's in both models introduced in Chapter 3 and Chapter 4 to estimate parameters. We hope these findings are very useful for future research in many applications in the areas of finance, insurance and climate change.

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