A CLASS OF RATIONAL SURFACES WITH A NON-RATIONAL SINGULARITY EXPLICITLY GIVEN BY A SINGLE EQUATION

by

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ABSTRACT

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The family of algebraic surfaces X defined by the single equation $z^n = (y - a_1x)\cdots(y - a_nx)(x - 1)$ over an algebraically closed field k of characteristic zero, where $a_1, \ldots, a_n \in k$ are distinct, is studied. It is shown that this is a rational surface with a non-rational singularity at the origin. The ideal class group of the surface is computed. The terms of the Chase-Harrison-Rosenberg seven term exact sequence on the open complement of the ramification locus of $X \to \mathbb{A}^2$ are computed; the Brauer group is also studied in this unramified setting.

The analysis is extended to the surface $\widetilde{X}$ obtained by blowing up X at the origin. The interplay between properties of $\widetilde{X}$, determined in part by the exceptional curve E lying over the origin, and the properties of X is explored. In particular, the implications that these properties have on the Picard group of the surface X are studied.

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CHAPTER 1
PRELIMINARIES

We will begin with some background information about the subjects to be considered, as well as basic concepts, definitions, and theorems that will be applied throughout the remainder of this thesis.

1.1 BACKGROUND

Algebraic geometry is the mathematics of studying geometrical objects through algebraic means, or vice versa. The geometric surfaces studied in this thesis are defined by relatively simple equations, which in turn means that the rings which correspond to them are easy to write down. From this relationship we will look at certain invariants of these rings and surfaces, such as the divisor class group and Brauer group, which would be quite a bit more difficult to do without using both algebra and geometry.

A feature of interest of the class of surfaces to be studied will be the existence of a non-rational singularity on these surfaces which are themselves rational and otherwise uncomplicated. Much work has been done on nonsingular surfaces, or surfaces which contain only rational singularities; see [8] and [21] for examples of such results. There seems, however, to be comparatively much less work that has been done on non-rational singularities, and even more scarce is the study of such singularities on surfaces given by fairly simple equations, which will be the focus of this thesis.

1.2 GEOMETRIC DEFINITIONS

Throughout, k will denote an algebraically closed field. Rings will be assumed to be commutative rings with identity, unless stated otherwise. Geometric definitions and notation largely follow that in [17].

By $\mathbb{A}^d_k$ (or $\mathbb{A}^d$, if k is understood) we denote d-dimensional *affine space* over the field k; that is, $\mathbb{A}^d = \operatorname{Spec}(k[x_1, \ldots, x_d])$, where $\operatorname{Spec}(k[x_1, \ldots, x_d])$ denotes the set of prime ideals of the ring $k[x_1, \ldots, x_d]$. If I is any ideal in the polynomial ring $k[x_1, \ldots, x_d]$, we define $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(k[x_1, \ldots, x_d]) \mid I \subseteq \mathfrak{p}\}$. One can check that the collection $\{V(I) \mid I \subseteq k[x_1, \ldots, x_d] \text{ is an ideal}\}$ satisfies all of the conditions to form the closed sets of a topology on $\operatorname{Spec}(k[x_1, \ldots, x_d]) = \mathbb{A}^d$.

Definition 1.2.1. The topology defined above is called the *Zariski topology* on $\mathbb{A}^d$.

A topology, also called the Zariski topology, may be defined on the set $k^{(d)}$ as well. The closed sets are the sets of the form $\mathcal{Z}(I)$, where I is an ideal of the polynomial ring, and $\mathcal{Z}(I)$ denotes the set of common zeros of the elements of I. Each point of the set $\mathcal{Z}(I)$ corresponds in a natural way to a maximal ideal of $k[x_1, \ldots, x_d]$ (see the following example). These are precisely the maximal ideals of $k[x_1, \ldots, x_d]$ which contain I. Under this identification, we see that the topology on $k^{(d)}$ is precisely the subspace topology on m-Spec $k[x_1, \ldots, x_d]$ induced by the Zariski topology on Spec $k[x_1, \ldots, x_d]$, where m-Spec $k[x_1, \ldots, x_d]$ denotes the set of maximal ideals of the polynomial ring.

Example 1.2.2. Suppose $\mathfrak{m} \in \operatorname{Spec} k[x_1, \ldots, x_d]$ is a maximal ideal. Since k is algebraically closed, Hilbert's Nullstellensatz says that this ideal may be expressed as $\mathfrak{m} = (x_1 - a_1, \ldots, x_d - a_d)$, for some $a_1, \ldots, a_d \in k$. We think of $\mathfrak{m}$ as corresponding to the point $(a_1, \ldots, a_d) \in k^{(d)}$. Notice that this is the only point in $k^{(d)}$ which satisfies the system of equations $x_1 - a_1 = x_2 - a_2 = \cdots = x_d - a_d = 0$.

The maximal ideals $\mathfrak{m}$ are the *closed points* of $\mathbb{A}^d$; they are precisely the points of $\mathbb{A}^d$ satisfying $\overline{\{\mathfrak{m}\}} = \{\mathfrak{m}\}$. These are the points of $\mathbb{A}^d$ that one would intuitively consider the points of a d-dimensional space, as they are in one-to-one correspondence with the set $k^{(d)}$.

Example 1.2.3. Let $f(x, y, z) \in k[x, y, z]$ be an irreducible polynomial. Since the polynomial ring $k[x, y, z]$ is a unique factorization domain, f is also a prime element, from which it follows that (f) is a prime ideal. So (f) is a point in $\mathbb{A}^3 = \operatorname{Spec} k[x, y, z]$. This corresponds geometrically to the surface with equation $f = 0$, denoted by $\mathcal{Z}(f)$ (the *zero set* of f). The points on this surface correspond precisely to those maximal ideals of $k[x, y, z]$ which contain (f). In the more general setup of $\operatorname{Spec} k[x, y, z] = \mathbb{A}^3$, (f) is considered as the *generic point* of the surface $f = 0$, which means that its closure in $\mathbb{A}^3$,

$$\overline{(f)} = V((f)) = \{\mathfrak{p} \in \operatorname{Spec} k[x, y, z] \mid (f) \subseteq \mathfrak{p}\}$$

consists of all the maximal ideals which correspond to the closed points on the surface $f = 0$, as well as all of those prime ideals corresponding to irreducible curves that lie on the surface $f = 0$. Equivalently, the maximal ideals of the ring $k[x, y, z]/(f)$ correspond to the closed points on the surface $\mathcal{Z}(f)$, and the height one primes of this ring correspond to the irreducible curves on $\mathcal{Z}(f)$.

Example 1.2.4. Since $k[x_1, \ldots, x_d]$ is an integral domain, (0) is a prime ideal. Since 0 is contained in every ideal of the polynomial ring, the closure of the zero ideal in $\mathbb{A}^d$ is equal to $\mathbb{A}^d$ itself. We say that (0) is the generic point of $\mathbb{A}^d$. In fact, by definition, every closed set of $\mathbb{A}^d$ is of the form $V(\mathfrak{p})$, for some prime ideal $\mathfrak{p} \in k[x_1, \ldots, x_d]$. It is then clear that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, and that two different prime ideals give two different closed sets. The point $\mathfrak{p}$ of $\mathbb{A}^d$ is called the (unique) generic point of $V(\mathfrak{p})$.

Definition 1.2.5. If $\mathfrak{p}$ is a prime ideal, $V(\mathfrak{p})$ is the *affine algebraic variety* (or *affine variety* for short) associated to $\mathfrak{p}$. It can be shown that $V(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec} k[x_1, \ldots, x_d]/\mathfrak{p}$, which associates the ring $k[x_1, \ldots, x_d]/\mathfrak{p}$ to the affine variety $V(\mathfrak{p})$. This is called the *affine coordinate ring* of the variety $V(\mathfrak{p})$.

Definition 1.2.6. If V is a variety, the *dimension* of V is the supremum of all integers n such that there is a chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ of subvarieties of V.

In general, the height of a prime ideal corresponds to the codimension of the algebraic variety in $\mathbb{A}^d$, and we have $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $V(\mathfrak{q}) \subseteq V(\mathfrak{p})$.

So far, $\mathbb{A}^d$ has been considered only as a topological space. We will apply the following general definitions to $\mathbb{A}^d$ in order to add extra structure.

Definition 1.2.7. Given a topological space X, a *presheaf* on X is a contravariant functor $\mathscr{F} : \mathfrak{Top}(X) \to \mathfrak{C}$, where $\mathfrak{Top}(X)$ is the category of open subsets of X (the morphisms being only the inclusion maps), and $\mathfrak{C}$ is an arbitrary category. Common choices for $\mathfrak{C}$ are the categories of sets, abelian groups, and rings. If $V \subseteq U$ are open subsets of X, the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is called the restriction map and will be denoted by ρ_V^U, and the image of an element $s \in \mathscr{F}(U)$ under this map will sometimes be written as $s|_V$. A *morphism* of sheaves on X, $\mathscr{F} \to \mathscr{G}$, consists of, for each open U, a morphism $\mathscr{F}(U) \to \mathscr{G}(U)$, where we require that these morphisms commute with the restriction maps. $\mathscr{F}$ is a *sheaf* if, in addition, it satisfies the following two properties:

Identity If $\{U_i\}$ is a collection of open subsets of X, and $s, t \in \mathscr{F}\left(\bigcup U_i\right)$ are such that $s|_{U_i} = t|_{U_i}$ for each i, then $s = t$.

Glueability If $\{U_i\}$ is a collection of open subsets of X, and if there are elements $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each pair i, j, then there is an element $s \in \mathscr{F}\left(\bigcup U_i\right)$ such that $s|U_i = s_i$ for each i.

A morphism of sheaves is a morphism of the sheaves when viewed as presheaves.

Remark 1.2.8. The identity and glueability properties may be phrased in category-theoretic terms as follows: a sheaf is a presheaf with the property that for any collection $\{U_i\}$ of open subsets of X, $\mathscr{F}(\bigcup U_i) = \varprojlim_{i,j} \mathscr{F}(U_i \cap U_j)$.

Definition 1.2.9. For a sheaf $\mathscr{F}$ on X, and $x \in X$, the *stalk* of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$, is defined to be $\varinjlim \mathscr{F}(U)$, where the direct limit is taken over all open neighborhoods of x.

Let R now be a commutative ring with identity. As was done for the ring $k[x_1, \ldots, x_d]$ above, we can define the set $\operatorname{Spec} R$ of all prime ideals of R, and endow it with the Zariski topology. For any open set $U \subseteq \operatorname{Spec} R$, define $\mathcal{O}(U)$ to be

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \,\middle|\, s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U, \text{ and } s \text{ is locally constant} \right\} \quad (1.1)$$

It can be shown, as in [17, p. 70] that $\mathcal{O}$ is a sheaf on $\operatorname{Spec} R$. According to [17, Proposition II.2.2], for any $\mathfrak{p} \in \operatorname{Spec} R$ the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$, which is a local ring. We say that the pair $(\operatorname{Spec} R, \mathcal{O})$ is a *locally ringed space*; more generally,

Definition 1.2.10. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space together with a sheaf of rings on X, such that $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$. A *morphism* of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(\varphi, \varphi^{\#})$, where $\varphi : X \to Y$ is a continuous map of topological spaces, and $\varphi^{\#} : \mathcal{O}_Y \to \mathcal{O}_X(\varphi^{-1}(\cdot))$ is a morphism of sheaves on Y. By the universal property of direct limit (see [22, p. 67]), φ induces a map on the stalks of the sheaves. We require each of these maps to be a local homomorphism of local rings.

Remark 1.2.11. For a scheme, we will use the terms "stalk at the point x" and "local ring at x" interchangeably.

Definition 1.2.12. An *affine scheme* is a locally ringed space which is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, for some ring R. A *scheme* is a locally ringed space which is covered by affine open subsets, each of which is isomorphic to an affine scheme.

Example 1.2.13. With this definition, we see that we may view $\mathbb{A}^d$ as a scheme, using the sheaf of rings defined in (1.1). In fact, for any prime ideal $\mathfrak{p} \in \mathbb{A}^d$, the affine variety $V(\mathfrak{p})$ may be viewed as an affine scheme as follows: it is well known that there is a one-to-one correspondence between ideals in $k[x_1, \ldots, x_d]/\mathfrak{p}$ and ideals in $k[x_1, \ldots, x_d]$ which contain $\mathfrak{p}$ (see, for example, [2, Proposition 1.1]), and under this correspondence, prime ideals correspond to prime ideals ([2, p. 3]). Thus, $V(\mathfrak{p}) = \operatorname{Spec}(k[x_1, \ldots, x_d]/\mathfrak{p})$. So, corresponding to $\mathfrak{p}$, we have the affine scheme $\operatorname{Spec}(k[x_1, \ldots, x_d]/\mathfrak{p})$ (where, by abuse of notation, we refer to the scheme by simply stating the underlying topological space, rather than the pair consisting of it and the sheaf of rings on that space).

Later, we shall have need to consider a particular type of scheme, commonly known in geometry as projective space. We briefly describe the construction of a certain type of scheme, an example of which is projective space. For terminology pertaining to graded rings, see [17, p. 9].

Let $S = \bigoplus_{r \geq 0} S_r$ be a graded ring, and $\operatorname{Proj} S$ the set of all homogeneous prime ideals of S that do not contain $S_+ = \bigoplus_{r>0} S_r$. Following the same process as that for the prime spectrum of a ring, we define a topology on $\operatorname{Proj} S$. For any $\mathfrak{p} \in \operatorname{Proj} S$, let $T \subseteq S$ be the multiplicative set of homogeneous elements not contained in $\mathfrak{p}$, the define $S_{(\mathfrak{p})}$ to be the ring of elements of degree zero in the ring of fractions $T^{-1}S$. Mimicking (1.1), for any open $U \subseteq \operatorname{Proj} S$ we let $\mathcal{O}(U)$ denote the set of all functions $s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$ that are locally constant and satisfy $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$. Just as it is above, this can be shown to be a sheaf of rings on $\operatorname{Proj} S$ which turns $(\operatorname{Proj} S, \mathcal{O})$ into a scheme.

Example 1.2.14. The polynomial ring $k[x_0, \ldots, x_d]$ is a graded ring. We define d-dimensional *projective space* over k to be the scheme $\operatorname{Proj} k[x_0, \ldots, x_d]$.

Definition 1.2.15. A point x is a *nonsingular* (or *regular*) point of the scheme X if the local ring $\mathcal{O}_{X,x}$ is a regular local ring; otherwise, x is a *singular* point of X.

Remark 1.2.16. The definition given in [17, p. 31] gives another characterization of a nonsingular point in the case of a closed point P on an affine variety $Y = \mathcal{Z}(f_1, \ldots, f_t) \subseteq \mathbb{A}^d$. Y is nonsingular at P if the rank of the matrix $((\partial f_i/\partial x_j)(P))_{t\times d}$ is $n - r$, where r is the dimension of Y.

1.3 ALGEBRAIC DEFINITIONS

In this section, we focus mainly on the definitions of a few groups which may be associated to either rings or schemes. Generally, there are two definitions; one that may be defined for a ring satisfying a sufficient number of properties, and the other for certain types of schemes. It turns out that, for an affine scheme $X = \operatorname{Spec} T$, the definitions coincide in the sense that the group corresponding to the scheme X is the same as that corresponding to the ring T. This fact will be exploited often.

Definition 1.3.1. An integral domain R is called *normal* if it is integrally closed in its quotient field (some authors define it to mean that the localization of R at each of its prime ideals is integrally closed in its quotient field; it can be shown that the two are equivalent, cf. [7, Proposition 15.49]). A scheme is said to be *normal* if the local ring at each of its points is normal in the sense just defined.

Let R be a normal integral domain, and K its quotient field. Denote by $X_1(R)$ the set of height one prime ideals in R. If $\mathfrak{p} \in X_1(R)$ is a height one prime ideal, then $R_{\mathfrak{p}}$ is a discrete valuation ring by [2, Proposition 9.2]; let $\nu_{\mathfrak{p}}$ be the associated valuation. Let $f \in K^*$. By [29, Corollary VI.10.2], $\nu_{\mathfrak{p}}(f) = 0$ for all but finitely many height one primes $\mathfrak{p}$.

Define the group of *divisors* on R to be $\operatorname{Div}(R) = \bigoplus_{\mathfrak{p} \in X_1(R)} \mathbb{Z}\mathfrak{p}$, the free abelian group generated by symbols corresponding to the height one prime ideals in R. By the previous paragraph, the function $\operatorname{div} : K^* \to \operatorname{Div} R$ given by $f \mapsto \sum \nu_{\mathfrak{p}}(f)\mathfrak{p}$ is well defined. Applying the logarithmic-like properties of valuations as follows

$$\begin{aligned}
\operatorname{div}(fg^{-1}) &= \sum \nu_{\mathfrak{p}}(fg^{-1})\mathfrak{p} \\
&= \sum \left(\nu_{\mathfrak{p}}(f) - \nu_{\mathfrak{p}}(g)\right)\mathfrak{p} \\
&= \sum \nu_{\mathfrak{p}}(f)\mathfrak{p} - \sum \nu_{\mathfrak{p}}(g)\mathfrak{p} \\
&= \operatorname{div}(f) - \operatorname{div}(g)
\end{aligned}$$

shows that div is a group homomorphism. Its image is denoted $\operatorname{Prin}(R)$ and called the subgroup of *principal divisors* of R.

Definition 1.3.2. The quotient group $\operatorname{Div}(R)/\operatorname{Prin}(R)$ is called the *divisor class group* of R, and is denoted $\operatorname{Cl}(R)$.

There is also a definition for the divisor class group of a scheme X which satisfies certain properties. In the case where $X = \operatorname{Spec}(R)$ for a normal integral domain R, the definitions give the same group. We define two of the properties here, for we will see them again.

Definition 1.3.3. A scheme X is *integral* if $\mathcal{O}_X(U)$ is an integral domain, for every open $U \subseteq X$. If there is a finite cover of X of affine open subsets $U_i = \operatorname{Spec}(A_i)$ such that each A_i is a noetherian ring, X is said to be a *noetherian* scheme. It is *regular in codimension one* if all of its local rings $\mathcal{O}_{X,x}$ of (Krull) dimension one are regular local rings. (This terminology stems from the fact that for an affine integral scheme $X = \operatorname{Spec} R$, the dimension of $\mathcal{O}_{X,x}$ is equal to the codimension of $V(x)$ in X.)

Definition 1.3.4. Let X be an integral, separated[1], noetherian scheme which is regular in codimension one. A *prime divisor* on X is a closed integral subscheme of codimension one. The group of *Weil divisors* on X is the free abelian group $\operatorname{Div} X$ generated by symbols corresponding to the prime divisors.

For any prime divisor $Y \subseteq X$, let η be its generic point. Since X is regular in codimension one, $\mathcal{O}_{X,\eta}$ is a regular local ring of Krull dimension one, hence a discrete valuation ring with valuation ν_Y. Let ξ be the generic point of X. Since X is integral, by [17, Exercise II.3.6], the local ring $\mathcal{O}_{X,\xi}$ is a field, called the *function field* of X and denoted $K(X)$. By the same exercise, if $U = \operatorname{Spec} A$ is any affine open subset of X, $K(X)$ is isomorphic to the quotient field of A. Suppose that this U is an affine open set containing η. Then $\mathcal{O}_{X,\eta} = \mathcal{O}_{U,\eta}$ (by, for example, [27, Exercise 5.22(i)]), where the latter ring is a localization of A at a prime ideal. Thus, the quotient field of $\mathcal{O}_{U,\eta}$ is also $K(X)$. It follows that, given $f \in K(X)^*$, it makes sense to compute $\nu_Y(f)$. As above, we define the group homomorphism $\operatorname{div} : K(X)^* \to \operatorname{Div} X$ by $f \mapsto \sum \nu_Y(f)Y$ (where the sum is taken over all prime divisors Y of X) and call the image $\operatorname{Prin} X$, the group of *principal divisors* on X.

Definition 1.3.5. The group $\operatorname{Div}(X)/\operatorname{Prin}(X)$ is called the *divisor class group* of the scheme X, and is denoted $\operatorname{Cl} X$.

We next wish to give the definition of the Picard group of a ring or scheme. This is an important subgroup of the class group, in the case of a noetherian normal integral domain R. To see this, it is convenient to give another characterization of the divisor class group for such a ring. An R-module M is said to be *reflexive* if

[1] See [17, p. 96] for the definition of a separated scheme. All our schemes will be separated.

$M \cong M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$. An R-module F is a *fractional ideal* of R if F is contained in the quotient field of R, and if there exists a nonzero $\alpha \in R$ such that $\alpha F \subseteq R$. It can be shown ([16, Theorem 3.12]) that the set of reflexive fractional ideals of R is a free abelian group with basis $X_1(R)$. Therefore, it is isomorphic to $\mathrm{Div}(R)$. So the divisor class group $\mathrm{Cl}\, R$ may be viewed as the quotient of the group of reflexive fractional ideals of R by the subgroup of principal fractional ideals. Under this association, a principal divisor $\mathrm{div}(f)$ is mapped to the principal fractional ideal Rf.

It can be shown that an R-progenerator (i.e., an R-module which is finitely generated, projective, and a generator) is reflexive. This will give us the means of showing that the Picard group, which will be defined next, is a subgroup of the divisor class group.

In what follows, we need only assume R is a commutative ring with identity. Assume that M is an R-progenerator. If $M^* \otimes_R M \cong R$ (as R-modules), we say that M is an *invertible* R-module (equivalent conditions for invertibility are given in [6, Lemma 1.5.1]). We define a group structure on $\mathrm{Pic}(R)$, the set of isomorphism classes of invertible R-modules, by letting $|M| \cdot |N| = |M \otimes_R N|$. One can check that the class $|R|$ is the identity of this group, that it is abelian, and that for $|M| \in \mathrm{Pic}\, R$, $|M|^{-1} = |M^*|$.

Definition 1.3.6. The set $\mathrm{Pic}\, R$, together with the binary operation defined above, is called the *Picard group* of the ring R. The Picard group is a covariant functor from commutative rings to abelian groups.

Since an element of $\mathrm{Pic}\, R$ is represented by an R-progenerator, in particular, it is also represented by a reflexive module. For a noetherian normal integral domain R, this is the basis for showing that $\mathrm{Pic}\, R$ may be viewed as a subgroup of $\mathrm{Cl}\, R$.

We may also give a scheme-theoretic version of the Picard group. If $(X, \mathcal{O}_X)$ is a scheme, an *invertible sheaf* on X is a locally free $\mathcal{O}_X$-module of rank one (see [17, p. 109] for details). By [17, Proposition II.6.12], the set of isomorphism classes of invertible sheaves on X forms a group.

Definition 1.3.7. This group is called the *Picard group* of the scheme X, and is denoted $\operatorname{Pic} X$.

By [16, Corollary 18.5], if X is a locally noetherian normal scheme, then $\operatorname{Pic} X$ is a subgroup of $\operatorname{Cl} X$.

The final definitions we wish to make are in regard to a third group, the Brauer group. Let R be a commutative ring with identity, and A an R-algebra, with structure homomorphism $\theta : R \to Z(A)$ (the center of A). The algebra A is *central* over R if $R \cong Z(A)$ under θ. (Note, in particular, that this makes θ injective. Since R is assumed to have an identity, we must have $\operatorname{ann}_R(A) = \{0\}$. This forces a central R-algebra to be faithful as an R-module.) It is *separable* over R if A is projective as a left $A \otimes_R A^o$-module (where A^o denotes the opposite ring of A). ($A \otimes_R A^o$ is called the *enveloping algebra* of A, and denoted A^e.)

Definition 1.3.8. An R-algebra A is called *Azumaya* over R if it is central and separable over R.

Remark 1.3.9. There are many equivalent conditions for an R-algebra to be separable, as well as equivalent definitions of Azumaya. See [6, Theorem 2.1.1 and Theorem 2.3.4] for some of these conditions; if any are used later, they will be detailed.

If P is an R-progenerator, it can be shown [6, Proposition 2.4.1] that the endomorphism ring $\operatorname{End}_R(P) = \operatorname{Hom}_R(P, P)$ is an Azumaya R-algebra. We say that two Azumaya R-algebras A, B are *Brauer equivalent* if there exist R-progenerators P, Q

such that $A \otimes_R \operatorname{End}_R(P) \cong B \otimes_R \operatorname{End}_R(Q)$. It can be shown that this is an equivalence relation on the set of all Azumaya R-algebras. For Azumaya algebras A, B, define $[A] \cdot [B] = [A \otimes_R B]$. This can be seen to give the set of Brauer equivalence classes the structure of an abelian group.

Definition 1.3.10. The set of Brauer equivalence classes of Azumaya R-algebras, together with the operation defined above, forms a group, called the *Brauer group* of the ring R, and denoted $\mathrm{B}(R)$. The equivalence class consisting of endomorphism rings of R-progenerators is the identity element of this group, and the inverse of a class $[A]$ is the class $[A^o]$. A homomorphism of commutative rings $R \to S$ induces a mapping $\mathrm{B}(R) \to \mathrm{B}(S)$ defined by $[A] \mapsto [A \otimes_R S]$. The kernel of this homomorphism of groups is called the *relative Brauer group* and is denoted $\mathrm{B}(S/R)$. The Brauer group is a covariant functor from commutative rings to abelian groups.

Example 1.3.11. The Brauer group of a field F is parameterized by the finite dimensional central F-division algebras. This is because any Azumaya algebra over F is simple (has no nontrivial two-sided ideals), and the Wedderburn–Artin Theorem [18, Theorem IX.1.14] says that every (left artinian) simple ring is of the form $\operatorname{Hom}_D(V, V)$, for some finite dimensional vector space over a division ring D.

Over an arbitrary commutative ring R, an Azumaya algebra need not be simple, so Wedderburn–Artin does not apply. From the point of view of Morita equivalence, the Brauer group is the group of equivalence classes of "invertible" R-algebras, where invertible is taken to mean that $A \otimes_R A^o$ is isomorphic as an R-algebra to the endomorphism ring of an R-progenerator (see [20, Théorème III.5.1]).

Definition 1.3.12. Let X be a scheme. An *Azumaya algebra* on X is an $\mathcal{O}_X$-algebra $\mathscr{A}$ such that, for each $x \in X$, the stalk $\mathscr{A}_x$ is an Azumaya algebra over the ring $\mathcal{O}_{X,x}$, in the sense defined in Definition 1.3.8. Two Azumaya $\mathcal{O}_X$-algebras $\mathscr{A}$ and

$\mathscr{B}$ are *equivalent* if there exist locally free $\mathcal{O}_X$-modules $\mathscr{P}$ and $\mathscr{Q}$ such that $\mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathscr{P}) \cong \mathscr{B} \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathscr{Q})$. The *Brauer group* of X is the group of equivalence classes of Azumaya algebras on X with operation induced by tensor product of sheaves.

CHAPTER 2

THE SURFACE $z^n = (y - a_1x)\cdots(y - a_nx)(x-1)$

Let $n \geq 3$ be an integer. Throughout, k denotes an algebraically closed field of characteristic zero. Let $a_1, \ldots, a_n \in k$ such that $a_i \neq a_j$ if $i \neq j$. Denote by T the ring $k[x,y,z]/\left(z^n - (y-a_1x)\cdots(y-a_nx)(x-1)\right)$. We wish to study the surface $X = \operatorname{Spec} T$. When we refer to a "point" of X, we will usually mean a closed point of the scheme X. These points are in one-to-one correspondence with the set

$$\begin{aligned}\mathcal{Z}&\left(z^n - (y-a_1x)\cdots(y-a_nx)(x-1)\right)\\ &= \left\{(x,y,z) \in k^{(3)} \mid z^n - (y-a_1x)\cdots(y-a_nx)(x-1) = 0\right\},\end{aligned}$$

as is explained in Example 1.2.2. To simplify notation in places, we will let $\ell_i = y - a_ix$ for $1 \leq i \leq n$, and let $f(x,y) = \ell_1\ell_2\cdots\ell_n = (y-a_1x)\cdots(y-a_nx)$. Sometimes, $x-1$ will be written as ℓ_{n+1} for convenience. With these simplifications, we rewrite the ring $T = k[x,y,z]/(z^n - \ell_1\cdots\ell_n\ell_{n+1}) = k[x,y,z]/(z^n - f(x,y)(x-1))$.

From now on, when referring to "the surface X", we shall generally mean the variety $\mathcal{Z}(z^n - f(x,y)(x-1))$; many properties of the scheme are determined by working with the associated algebraic variety.

2.1 THE DIVISOR CLASS GROUP OF X

Lemma 2.1.1. *The polynomial $p(x,y,z) = z^n - (y-a_1x)\cdots(y-a_nx)(x-1)$ is irreducible (in $k[x,y,z]$).*

Proof. This is a consequence of Eisenstein's Criterion (see [18, Theorem III.6.15]).

View p as being a polynomial in z with coefficients in the unique factorization domain $k[x, y]$. Then the "constant term" is completely factored into irreducibles. The irreducible element $x - 1 \in k[x, y]$ divides the constant term, but its square does not; furthermore, $x-1$ does not divide the leading coefficient of p (which is 1). Eisenstein's criterion then says that $p(x, y, z)$ is irreducible in $k(x, y)[z]$, and since p is a primitive polynomial, it is in fact irreducible in $k[x, y, z]$. □

Next, we want to show that X is a normal surface. For this, it will be helpful to know what the singular closed points $\operatorname{Sing}(X)$ of X are. A more general result is proved in Lemma 2.1.2 and Lemma 2.1.3.

Lemma 2.1.2. *Suppose that a hypersurface $Y \subseteq \mathbb{A}_k^{d+1}$ is defined by $Y = \mathcal{Z}(z^n - g(x_1, \ldots, x_d))$, for some $n \geq 2$. If $\mathcal{Z}(g)$ is considered as a hypersurface in the same affine space $\mathbb{A}^{d+1}$, then $\operatorname{Sing} Y = \operatorname{Sing}(\mathcal{Z}(g)) \cap \mathcal{Z}(z)$.*

Proof. Let $p(x_1, \ldots, x_d, z) = z^n - g(x_1, \ldots, x_d)$. Then $\partial p/\partial z = nz^{n-1}$, which equals zero if and only if $z = 0$. This at once forces $\operatorname{Sing} Y \subseteq \mathcal{Z}(z)$. On the other hand, for any $1 \leq i \leq d$, $\partial p/\partial x_i = -\partial g/\partial x_i$. Since $Y \cap \mathcal{Z}(z) = \mathcal{Z}(g) \cap \mathcal{Z}(z)$, the result follows. □

Lemma 2.1.3. *Suppose $g = g_1 \cdots g_t$ in the polynomial ring $k[x_1, \ldots, x_d]$. Then the set of singular points of $G = \mathcal{Z}(g)$ consists precisely of the singularities of each of the $G_j = \mathcal{Z}(g_j)$, as well as all of the points where two or more of the G_j intersect.*

Proof. For each $1 \leq i \leq d$,

$$\frac{\partial g}{\partial x_i} = \sum_{j=1}^{t} \left(\frac{\partial g_j}{\partial x_i} \prod_{\substack{k=1 \\ k \neq j}}^{t} g_k \right). \tag{2.1}$$

If $P \in \operatorname{Sing} G_m$ for some m, then $g_m(P) = 0$ and $(\partial g_m/\partial x_i)(P) = 0$ for each i. However, for a fixed m and for every i, we see from equation (2.1) that every term of $\partial g/\partial x_i$ contains either g_m or $\partial g_m/\partial x_i$. It follows that $(\partial g/\partial x_i)(P) = 0$ for each i, and hence $P \in \operatorname{Sing} G$.

Suppose now that $Q \in G_m \cap G_n$, for some $1 \leq m < n \leq t$. Equation (2.1) shows that every term of $\partial g/\partial x_i$ involves all but one of the g_j. In particular, every term contains either g_i or g_j (or both) as a factor, and therefore $(\partial g/\partial x_i)(Q) = 0$. Thus, $Q \in \operatorname{Sing} G$.

It remains to show that we have found all of the singular points of G. Say $P \in \operatorname{Sing} G$; certainly then $P \in G$, so that $P \in G_m$ for at least one m. If also $P \in G_n$ for some $m \neq n$, we are done. Otherwise, note that $g_m(P) = 0$ and $(\partial g/\partial x_i)(P) = 0$ for each i. This reduces equation (2.1) to $0 = (\partial g_m/\partial x_i)(P) \prod_{\substack{k=1 \\ k \neq j}}^{t} g_k(P)$. Since $P \notin G_k$ for $k \neq m$ by assumption, this forces $(\partial g_m/\partial x_i)(P) = 0$ for each i, and hence $P \in \operatorname{Sing} G_m$. □

Corollary 2.1.4. $\operatorname{Sing} X = \{(0,0,0)\} \cup \{(1, a_i, 0) \mid 1 \leq i \leq n\}$, *which is precisely the singular locus of the curve* $\mathcal{Z}(\ell_1 \cdots \ell_{n+1})$ *in the plane* $\mathcal{Z}(z)$. □

Proposition 2.1.5. *The surface* X *is normal.*

Proof. Hartshorne [17, Proposition II.8.23] gives criteria for a scheme to be normal: it need only be shown that X is a locally complete intersection subscheme of $\mathbb{A}^3$ which is regular in codimension one. The fact that X is a hypersurface (is defined by a single, irreducible equation) means that X is a complete intersection (it is the "intersection of one hypersurface"). By Definition 1.3.3, to show that X is regular in codimension one, we need to check that the local rings of dimension one of X are regular. Equivalently, any subvariety of X of codimension one (i.e., a curve) contains a nonsingular (closed) point. This is equivalent because if a curve on X, which corresponds to a height one prime ideal $\mathfrak{p}$ in T, contains a point (corresponding to a maximal ideal $\mathfrak{m}$ of T) which is a nonsingular point of X, we get $\mathfrak{p} \subseteq \mathfrak{m}$. Then the local ring $\mathcal{O}_{X,\mathfrak{p}} = T_{\mathfrak{p}}$ is a localization of $\mathcal{O}_{X,\mathfrak{m}} = T_{\mathfrak{m}}$. But a localization of a regular local ring is still regular [9, Corollary 19.14].

Corollary 2.1.4 showed that the singular locus of the surface X is finite. As any curve defined over an algebraically closed field must contain an infinite number of points, it is clear that any curve on X must contain a nonsingular point (in fact infinitely many nonsingular points). So X is regular in codimension one, and the proof is complete. □

Corollary 2.1.6. *The ring T is normal (integrally closed).*

Proof. Combining Proposition 2.1.5 and Definition 1.3.1 shows that every local ring of the scheme X is integrally closed. Every point $\mathfrak{p} \in X = \operatorname{Spec} T$ is a prime ideal of the ring T, and the local ring of X at $\mathfrak{p}$ is $\mathcal{O}_{X,\mathfrak{p}} \cong T_{\mathfrak{p}}$. Thus, every localization of T at a prime ideal is integrally closed. Again referring to 1.3.1 finishes the proof. □

Next, we will show that the surface X is rational; that is, X is birationally equivalent to $\mathbb{A}^2$. In [17, Corollary I.4.5], it is shown that two varieties are birationally equivalent if and only if they have isomorphic (as k-algebras) function fields (defined in the paragraph following Definition 1.3.4 in Section 1.3). Since X and $\mathbb{A}^2$ are both affine, it is enough to check that $\mathcal{O}_X(X) = T$ and $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) = k[v, w]$ (for some indeterminates v and w) have isomorphic quotient fields.

The following lemma will prove useful.

Lemma 2.1.7. *A surjective ring homomorphism between integral domains of the same (finite) Krull dimension must be injective (and therefore an isomorphism).*

Proof. Suppose $D \to E$ is a surjective homomorphism of integral domains of the same dimension, with kernel K. By the first isomorphism theorem, $E \cong D/K$. So any chain of prime ideals in E corresponds to a chain of primes of the same length in D, each of which contains K. But since D is an integral domain, (0) is a prime ideal in D. If $K \neq 0$, then we may extend any chain of primes in D so obtained from E by putting (0) at the beginning. This would force D have a larger Krull dimension than E, which would be a contradiction. So $K = 0$, and $D \to E$ is injective. □

Proposition 2.1.8. *$K(X) \cong k(v, w)$, for some indeterminates v, w. Furthermore, the surface X is rational.*

Proof. By the discussion in the preceding paragraph, the second statement immediately follows from the first. To show that $K(X)$ is isomorphic to a field of rational functions in two indeterminates, we will explicitly construct an isomorphism between a localization of T and a localization of $k[v, w]$. The construction will be carried out in a few steps.

Start by going to the localization $T[x^{-1}]$. Define the ring $D = \frac{k[x,v,w]}{(w^n - f(1,v)(x-1))}$ (recall the polynomial f is defined as $f(x, y) = (y - a_1 x) \cdots (y - a_n x)$, so that $f(1, v) = (v - a_1) \cdots (v - a_n)$), and define a map

$$\alpha : T[x^{-1}] \to D[x^{-1}]$$

by

$$x \mapsto x$$

$$y \mapsto xv$$

$$z \mapsto xw.$$

Certainly $x \mapsto x, y \mapsto xv, z \mapsto xw$ defines a ring homomorphism (in fact a k-algebra homomorphism) on $k[x, y, z]$. To check that α gives a well-defined homomorphism on $T[x^{-1}]$, one needs to apply the universal mapping property of localizations [18, Theorem III.4.5] and check that $z^n - f(x,y)(x-1)$ maps to zero:

$$\begin{aligned} z^n - f(x,y)(x-1) &\mapsto (xw)^n - f(x,xv)(x-1) \\ &= x^n w^n - x^n f(1,v)(x-1) \text{ } (f \text{ is a form of degree } n) \\ &= x^n(w^n - f(1,v)(x-1)), \end{aligned}$$

which is clearly zero in $D[x^{-1}]$. We claim that α is an isomorphism. The invertibility of x forces α to be surjective: $\alpha(y/x) = v, \alpha(z/x) = w$, and $\alpha(x) = x$. To show that α is injective, we look at the Krull dimensions of $T[x^{-1}]$ and $D[x^{-1}]$. Inverting x on either side of α did not change the Krull dimension of that ring (there exist maximal chains of prime ideals avoiding x). Both T and $\frac{k[x,v,w]}{(w^n - f(1,v)(x-1))}$ were formed by taking the quotient of a polynomial ring in three indeterminates by a height one prime ideal; thus, each has Krull dimension 2, from which it follows that both $T[x^{-1}]$ and $D[x^{-1}]$ have dimension two.

By Lemma 2.1.7, α is an isomorphism. We wish to extend α by further localizing. We want to be able to solve for one of the indeterminates in the equation $w^n - f(1,v)(x-1) = 0$. If $f(1,v)$ is inverted, then it becomes possible to solve for x. So, localize $D[x^{-1}]$ to the extension $D\left[\frac{1}{xf(1,v)}\right]$. For α to remain an isomorphism, we need to invert the corresponding elements in $T[x^{-1}]$. As was computed above, $\alpha(f(x,y)) = x^n f(1,v)$, that is, $\alpha(x^{-n}f(x,y)) = f(1,v)$. Since x is already a unit in $T[x^{-1}]$, it is enough to invert $f(x,y)$. After doing so, we obtain a map (which we will also call α)

$$\alpha : T\left[\frac{1}{xf(x,y)}\right] \to D\left[\frac{1}{xf(1,v)}\right].$$

The proof that this map is a well-defined isomorphism is the same as for the original map. In $D\left[\frac{1}{xf(1,v)}\right]$, the equation $w^n - f(1,v)(x-1) = 0$ can be solved for x:

$$x = \frac{w^n + f(1,v)}{f(1,v)}. \tag{2.2}$$

Use this to define a homomorphism; define

$$\beta : D\left[\frac{1}{xf(1,v)}\right] \to k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]$$

by

$$\begin{aligned} v &\mapsto v \\ w &\mapsto w \\ x &\mapsto \frac{w^n + f(1,v)}{f(1,v)}. \end{aligned}$$

We invert $w^n + f(1,v)$ on the right hand side to ensure that the universal property of localizations applies, since this is the image of x. This map is an isomorphism by Lemma 2.1.7. But the ring $k\left[v, w, \frac{1}{f(1,v)(w^n+f(1,v))}\right]$ is a localization of the polynomial ring $k[v, w]$; in particular, this means that its quotient field is $k(v, w)$. Thus, the quotient field of $T\left[\frac{1}{xf(x,y)}\right]$, and hence that of T, is isomorphic to $k(v, w)$ as well. Since the quotient field of T is isomorphic to $K(X)$, the proposition is proven. □

Next, we wish to compute the class group $\mathrm{Cl}(T)$ of T. A version of Nagata's sequence [10, Theorem 1.1] will be used. Phrased in terms of rings, the sequence says that if R is a noetherian normal integral domain, and $g \in R$ is a nonzero nonunit with $\mathrm{div}(g) = v_1\mathfrak{p}_1 + \cdots + v_m\mathfrak{p}_m$, then the sequence

$$1 \to R^* \to R[g^{-1}]^* \xrightarrow{\mathrm{div}} \bigoplus_{i=1}^{m} \mathbb{Z} \cdot \mathfrak{p}_i \to \mathrm{Cl}(R) \to \mathrm{Cl}(R[g^{-1}]) \to 0 \qquad (2.3)$$

is exact, where $\mathrm{div} : R[g^{-1}]^* \to \bigoplus_{i=1}^{m} \mathbb{Z} \cdot \mathfrak{p}_i$ is the map defined in Section 1.3. We will apply this sequence with the ring T, and with $g = xf(x,y)$[1]. This g is chosen to simplify the sequence; note that $T\left[\frac{1}{xf(x,y)}\right]$ is a unique factorization domain, since Proposition 2.1.8 showed that it is isomorphic to $k\left[v, w, \frac{1}{f(1,v)(w^n+f(1,v))}\right]$ (a localization of a UFD is again a UFD [18, Exercise III.4.6]). By [16, Proposition 6.1], a

[1]That $xf(x,y)$ is not a unit in T is explained in the proof of 2.1.10.

noetherian integral domain is factorial if and only if its ideal class group is trivial. Thus, $\mathrm{Cl}\left(T\left[\frac{1}{xf(x,y)}\right]\right) = 0$. With this information, Nagata's sequence (2.3) becomes

$$1 \to T^* \to T\left[\frac{1}{xf(x,y)}\right]^* \xrightarrow{\text{div}} \bigoplus_{\mathfrak{p}} \mathbb{Z}\cdot\mathfrak{p} \to \mathrm{Cl}(T) \to 0, \tag{2.4}$$

where the summation ranges over all prime ideals which appear in $\mathrm{div}(xf(x,y))$. Since (2.4) is exact, $\bigoplus_{\mathfrak{p}} \mathbb{Z}\cdot\mathfrak{p} \to \mathrm{Cl}(T)$ is surjective. By the first isomorphism theorem, $\mathrm{Cl}(T)$ is isomorphic to $\bigoplus_{\mathfrak{p}} \mathbb{Z}\cdot\mathfrak{p}$ modulo the kernel of $\bigoplus_{\mathfrak{p}} \mathbb{Z}\cdot\mathfrak{p} \to \mathrm{Cl}(T)$. Again by exactness, this latter group is equal to the image of $\mathrm{div}(\cdot)$. The first thing to be done is to compute $T\left[\frac{1}{xf(x,y)}\right]^*$.

Lemma 2.1.9. $T\left[\frac{1}{xf(x,y)}\right]^* = k^* \times \langle x\rangle \times \langle y - a_1x\rangle \times \cdots \times \langle y - a_nx\rangle$.

Proof. The isomorphism of k-algebras $\beta\alpha : T\left[\frac{1}{xf(x,y)}\right] \cong k\left[v, w, \frac{1}{f(1,v)(w^n+f(1,v))}\right]$ from the proof of Proposition 2.1.8 induces an isomorphism on the groups of units of these two rings. Since the polynomial ring $k[v,w]$ is a unique factorization domain with group of units k^*, the units of $k\left[v, w, \frac{1}{f(1,v)(w^n+f(1,v))}\right]$ are precisely those elements that may be formed from the irreducible elements that have been inverted:

$$k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]^* = k^* \times \langle v - a_1\rangle \times \cdots \times \langle v - a_n\rangle \times \langle w^n + f(1,v)\rangle. \tag{2.5}$$

The linear polynomials $v - a_i, 1 \leq u \leq n$, are the irreducible factors of $f(1,v)$. The other polynomial which has been inverted, $w^n + f(1,v)$, is already irreducible (for example, by applying Eisenstein's criterion with the irreducible element $v - a_1 \in k[v]$, similarly to how the criterion was applied in Lemma 2.1.1). We will use the isomorphism (2.5) to prove that

$$T\left[\frac{1}{xf(x,y)}\right]^* = k^* \times \langle x\rangle \times \langle y - a_1x\rangle \times \cdots \times \langle y - a_nx\rangle. \tag{2.6}$$

That the right hand side of (2.6) is contained in the left hand side is obvious; each generator is one of the elements whose inverse has been specifically added to T. For simplicity, let H denote the subgroup $k^* \times \langle x \rangle \times \langle y - a_1 x \rangle \times \cdots \times \langle y - a_n x \rangle$ of $T\left[\frac{1}{xf(x,y)}\right]^*$. To show $T\left[\frac{1}{xf(x,y)}\right]^* \subseteq H$, it is enough to show that $\beta\alpha\left(T\left[\frac{1}{xf(x,y)}\right]^*\right) = k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]^* \subseteq \beta\alpha(H)$. Since $\beta\alpha$ is a k-algebra isomorphism, the fact that the scalars on either side correspond to each other under this map is trivial. What needs to be shown is that each of $v - a_1, v - a_2, \ldots, v - a_n, w^n + f(1,v)$ is in $\beta\alpha(H)$.

The preimage of $v - a_i$ $(i = 1, 2, \ldots, n)$ is $\frac{y}{x} - a_i$. Written as one fraction, this is equal to $\frac{y - a_i x}{x}$, which is certainly in H. This takes care of every generator of $k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]^*$ except for $w^n + f(1,v)$. We claim that the preimage under $\beta\alpha$ of this element is $\frac{f(x,y)}{x^{n-1}}$, which is in H. The calculation can be done, making use of the result already obtained in this paragraph:

$$\begin{aligned}
\beta\alpha\left(\frac{f(x,y)}{x^{n-1}}\right) &= \beta\alpha(x) \cdot \beta\alpha\left(\frac{f(x,y)}{x^n}\right) \\
&= \beta(x) \cdot \beta\alpha\left(f\left(1, \frac{y}{x}\right)\right) \\
&= \beta(x) \cdot \beta(f(1,v)) \\
&= \frac{w^n + f(1,v)}{f(1,v)} \cdot f(1,v) \\
&= w^n + f(1,v).
\end{aligned}$$

Thus, each generator of $k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]^*$ is in $\beta\alpha(H)$. This suffices to prove the containment $T\left[\frac{1}{xf(x,y)}\right]^* \subseteq H$, which in turn finishes the proof of the lemma. □

Knowing the group of units of $T\left[\frac{1}{xf(x,y)}\right]$, together with Nagata's sequence (2.4), is enough information to be able to compute the divisor class group $\mathrm{Cl}(T)$ of T.

Theorem 2.1.10. $\mathrm{Cl}(T) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)} \oplus \mathbb{Z}^{(n-1)}$. *Generators and relations for the group are given by* $\mathrm{Cl}(T) = \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}_1 \ldots, \mathfrak{q}_{n-1} \mid n\mathfrak{p}_1 \sim \cdots \sim n\mathfrak{p}_n \sim 0 \rangle$, *where* $\mathfrak{p}_i = (y - a_i x, z)$ *and* $\mathfrak{q}_i = (x, y - \zeta^{2i-1} z)$ *are prime ideals of* T *for* $1 \leq i \leq n$ *(*$\mathfrak{q}_n$ *is defined but not required for a basis of* $\mathrm{Cl}(T)$*). As divisors on* X, $\mathrm{div}(y - a_i x) = n\mathfrak{p}_i, 1 \leq i \leq n$ *and* $\mathrm{div}(x) = \mathfrak{q}_1 + \mathfrak{q}_2 + \cdots + \mathfrak{q}_n$.

Proof. Because Nagata's sequence (2.4) is exact, $\mathrm{Cl}(T) \cong \left(\bigoplus_{\mathfrak{p}} \mathbb{Z} \cdot \mathfrak{p}\right) / \mathrm{im}(\mathrm{div})$, where the summation is over all height one prime ideals of T appearing in the divisors of any element of $T\left[\frac{1}{xf(x,y)}\right]^*$. Notice that any element of k^* must map to 0 under the divisor map div; since any nonzero scalar is already a unit in T, it cannot be contained in any height one prime ideal of T. Thus, im(div) will be generated by the images of the elements $x, y - a_1x, \ldots, y - a_nx$.

First, fix i between 1 and n, and consider the element $y - a_ix$. Notice that this element is not a unit in T, since $T/(y - a_ix) \cong k[x, z]/(z^n)$; if $y - a_ix$ were a unit in T, it would generate the unit ideal, and the quotient ring would be trivial. Let $\mathfrak{p}$ denote a minimal prime ideal in T containing $y - a_ix$. By Krull's Hauptidealsatz [17, Theorem I.1.1A], $\mathfrak{p}$ necessarily has height one. Because $z^n = f(x, y)(x - 1)$ in T and $\mathfrak{p}$ is prime, $\mathfrak{p}$ necessarily contains z, and so $(y - a_ix, z) \subseteq \mathfrak{p}$. Now,

$$\frac{T}{(y - a_ix, z)} \cong \frac{k[x, y, z]}{(y - a_ix, z, z^n - f\ell_{n+1})} \overset{z \mapsto 0}{\cong} \frac{k[x, y]}{(y - a_ix, f\ell_{n+1})} \overset{y \mapsto a_ix}{\cong} k[x];$$

as this last ring is an integral domain, the isomorphism shows that the ideal $(y - a_ix, z)$ is in fact prime itself. Since $\mathfrak{p}$ is a minimal prime ideal containing $y - a_ix$, this forces $\mathfrak{p} = (y - a_ix, z)$.

So there are n prime ideals corresponding to the n linear polynomials $y - a_ix$. Let

$$\mathfrak{p}_i = (y - a_ix, z), i = 1, 2, \ldots, n; \tag{2.7}$$

this notation will remain fixed. These really are n distinct ideals; if $y - a_jx \in \mathfrak{p}_i$ for some $j \neq i$, then $\frac{1}{a_j - a_i}[(y - a_ix) - (y - a_jx)] = x$ is in $\mathfrak{p}_i$ as well. But this would mean that $\mathfrak{p}_i \supseteq (x, y - a_ix, z)$, the latter of which is a maximal ideal in T (see [17, Example I.1.4.4]). This is obviously impossible (maximal ideals of T have height two, for example), proving that none of the $\mathfrak{p}_i$ contain any of the factors of $f(x, y)$ other than the one that appears in its generating set.

Now, to compute $\operatorname{div}(y - a_i x)$; since $\mathfrak{p}_i$ is the only height one prime ideal which contains this element, $\operatorname{div}(y - a_i x) = \nu_{\mathfrak{p}_i}(y - a_i x) \cdot \mathfrak{p}_i$. The valuation $\nu_{\mathfrak{p}_i}(y - a_i x)$ is determined by going to the localization $T_{\mathfrak{p}_i}$, determining a generator g of the maximal ideal $\mathfrak{p}_i T_{\mathfrak{p}_i}$ (it is necessarily principal since $T_{\mathfrak{p}_i}$ is a discrete valuation ring, by [2, Proposition 9.2]), and finding the unique $r \in \mathbb{Z}$ such that $y - a_i x = u g^r$ in $T_{\mathfrak{p}_i}$, for some unit $u \in T_{\mathfrak{p}_i}$. The integer r will necessarily be positive, since $y - a_i x \in T \setminus T^*$. As was shown above, $y - a_j x \notin \mathfrak{p}_i$ for $j \neq i$; this means that each of these elements become units in the localization $T_{\mathfrak{p}_i}$. The element $x - 1$ also becomes a unit; it cannot be in the ideal $\mathfrak{p}_i$ because, as before, containing all three of $x - 1, y - a_i x$, and z would force $\mathfrak{p}_i$ to contain a maximal ideal. It follows that in the ring $T_{\mathfrak{p}_i}$, the equation $z^n = (y - a_1 x) \cdots (y - a_n x)(x - 1)$ can be solved for $y - a_i x$:

$$y - a_i x = \underbrace{\frac{y - a_i x}{f(x, y)(x - 1)}}_{\in T^*_{\mathfrak{p}_i}} \cdot z^n. \tag{2.8}$$

Since $\mathfrak{p}_i$ is generated by $y - a_i x$ and z in T, those two elements will still generate the ideal $\mathfrak{p}_i T_{\mathfrak{p}_i}$; however, (2.8) shows that only z is necessary. Therefore, $\mathfrak{p}_i T_{\mathfrak{p}_i} = z \cdot T_{\mathfrak{p}_i}$, and $\nu_{\mathfrak{p}_i}(y - a_i x) = n$. Thus, for each $i = 1, 2, \ldots, n$, the divisor of $y - a_i x$ is

$$\operatorname{div}(y - a_i x) = n \mathfrak{p}_i. \tag{2.9}$$

It remains to compute $\operatorname{div}(x)$. This is somewhat more complicated than the above computations, owing to the fact that x itself does not appear as a factor of $f(x, y)(x - 1)$. Again, we note that x cannot be a unit in T: $T/(x) \cong k[y, z]/(y^n + z^n)$ is certainly not the trivial ring. However, this isomorphism is useful in finding the minimal prime ideals containing x in T; there is a one-to-one correspondence between such ideals and the minimal prime ideals of the ring $T/(x) \cong k[y, z]/(y^n + z^n)$. Because $y^n + z^n$ is a homogeneous polynomial in two variables, it can be factored into n linear factors. The minimal prime ideals of the ring $k[y,z]/(y^n+z^n)$ will correspond to these linear factors.

Let ζ be a generator of $\mu_{2n}(k)$, the group of $2n^{\text{th}}$ roots of unity in k. Note that, since k is algebraically closed, $|\mu_{2n}(k)| = 2n$ and $\mu_{2n}(k) \cong \mathbb{Z}_{2n}$. To factor $y^n + z^n$, dehomogenize with respect to z to get the polynomial y^n+1. The roots of this polynomial consist of the n distinct elements $\zeta, \zeta^3, \ldots, \zeta^{2n-1}$ in k (since ζ is a primitive $2n^{\text{th}}$ root of unity, $\zeta^n = -1$). By the Remainder Theorem [18, Corollary III.6.3], y^n+1 factors as $(y-\zeta)(y-\zeta^3)\cdots(y-\zeta^{2n-1})$. Homogenize with respect to z:

$$\begin{aligned} z^n\left(\left(\frac{y}{z}\right)^n+1\right) &= z^n\left(\frac{y}{z}-\zeta\right)\left(\frac{y}{z}-\zeta^3\right)\cdots\left(\frac{y}{z}-\zeta^{2n-1}\right) \\ y^n+z^n &= (y-\zeta z)(y-\zeta^3 z)\cdots(y-\zeta^{2n-1}z). \end{aligned}$$

Hence, in the ring $k[y,z]/(y^n+z^n)$, $(y-\zeta z)(y-\zeta^3 z)\cdots(y-\zeta^{2n-1}z) = 0$. Since 0 is in every ideal, it follows that any minimal prime of $k[y,z]/(y^n+z^n)$ contains at least one of the linear polynomials $y-\zeta^{2i-1}z, i=1,2,\ldots,n$. Conversely, forming the quotient by the ideal generated by any one of these,

$$\frac{k[y,z]/(y^n+z^n)}{(y-\zeta^{2i-1}z)} \cong k[y,z]/(y-\zeta^{2i-1}z) \overset{y\mapsto\zeta^{2i-1}z}{\cong} k[z],$$

where the final ring is an integral domain. So each of these n polynomials actually generates a prime ideal in $k[y,z]/(y^n+z^n)$; thus, this ring has precisely n minimal prime ideals, corresponding to these n polynomials (the n ideals are distinct, as any ideal containing two of these polynomials necessarily contains both y and z). Under the homomorphism $T \to k[y,z]/(y^n+z^n)$, the prime ideal $(y-\zeta^{2i-1}z)$ pulls back to the (necessarily prime) ideal $(x, y-\zeta^{2i-1}z)$ in T. We will call these ideals

$$\mathfrak{q}_i = (x, y-\zeta^{2i-1}z), i=1,2,\ldots,n, \tag{2.10}$$

and we have shown that these are the height one prime ideals in T which contain x.

Next, we must compute the valuations $\nu_{\mathfrak{q}_i}(x)$, for each i. Fix any i from 1 to n. We claim that $\nu_{\mathfrak{q}_i}(x) = 1$, or equivalently that x generates the maximal ideal of the localization $T_{\mathfrak{q}_i}$. To prove this, it is enough to show that in $T_{\mathfrak{q}_i}$, $y - \zeta^{2i-1}z$ is a multiple of x. In order to show this, we must write the defining equation

$$z^n = f(x,y)(x-1) \tag{2.11}$$

of T in a more convenient form. Distribute $f(x,y)$ on the right hand side of equation (2.11) to get

$$z^n = f(x,y)x - f(x,y). \tag{2.12}$$

Recall that $f(x,y) = (y - a_1 x) \cdots (y - a_n x)$. Separating off the y^n term of the expansion of this product, and factoring an x out of the remaining terms, write $f(x,y) = y^n + xg(x,y)$. Substituting this into (2.12),

$$z^n = f(x,y)x - y^n - xg(x,y) \tag{2.13}$$

$$y^n + z^n = (f(x,y) - g(x,y))x \tag{2.14}$$

$$(y - \zeta z)(y - \zeta^3 z) \cdots (y - \zeta^{2n-1} z) = (f(x,y) - g(x,y))x. \tag{2.15}$$

Since none of the factors on the left hand side of (2.15), other than $y - \zeta^{2i-1}z$, are in $\mathfrak{q}_i$, they all become units in the localization $T_{\mathfrak{q}_i}$. So, in $T_{\mathfrak{q}_i}$, equation (2.15) can be solved for $y - \zeta^{2i-1}z$:

$$y - \zeta^{2i-1}z = \frac{y - \zeta^{2i-1}z}{y^n + z^n}(f(x,y) - g(x,y))x. \tag{2.16}$$

It follows that x is a local parameter for the discrete valuation ring $T_{\mathfrak{q}_i}$, and $\nu_{\mathfrak{q}_i}(x) = 1$. Thus,

$$\operatorname{div}(x) = \mathfrak{q}_1 + \mathfrak{q}_2 + \cdots + \mathfrak{q}_n. \tag{2.17}$$

Now that all of the prime ideals that are necessary have been identified, Nagata's sequence (2.4) can be written as

$$1 \to T^* \to T\left[\frac{1}{xf(x,y)}\right]^* \xrightarrow{\text{div}} \bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{p}_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{q}_i \to \text{Cl}(T) \to 0. \tag{2.18}$$

There is finally enough information to compute the divisor class group $\text{Cl}(T)$ of T. Using the divisors of the generators of $T\left[\frac{1}{xf(x,y)}\right]^*$ computed in (2.9) and (2.17), and the exact sequence (2.18), $\text{Cl}(T)$ is computed as

$$\begin{aligned}\text{Cl}(T) &\cong \frac{\bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{p}_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{q}_i}{\text{im}(\text{div})} \\ &= \frac{\bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{p}_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{q}_i}{\langle n\mathfrak{p}_1, \ldots, n\mathfrak{p}_n, \mathfrak{q}_1 + \cdots + \mathfrak{q}_n \rangle} \\ &\cong \frac{\bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{p}_i}{\langle n\mathfrak{p}_1, \ldots, n\mathfrak{p}_n \rangle} \oplus \frac{\bigoplus_{i=1}^{n} \mathbb{Z}\mathfrak{q}_i}{\langle \mathfrak{q}_1 + \cdots + \mathfrak{q}_n \rangle} \\ &\cong (\mathbb{Z}/n\mathbb{Z})^{(n)} \oplus \mathbb{Z}^{(n-1)}.\end{aligned}$$

This completes the proof. □

Corollary 2.1.11. $T^* = k^*$.

Proof. Use the exactness of sequence (2.18). The group T^* is equal to its own image in $T\left[\frac{1}{xf(x,y)}\right]^*$, which is equal to the kernel of div by exactness. Certainly k^* is in the kernel of the divisor map. In the proof of the previous theorem, it is shown that not only are the generators of $T\left[\frac{1}{xf(x,y)}\right]^*$ not in the kernel of div, but their images are all linearly independent (over $\mathbb{Z}$) of each other. This means that no combination of the generators can possibly map to zero under div, and so $\ker(\text{div}) = k^*$. □

The surface $X = \mathcal{Z}(z^n - f(x,y)(x-1))$ can be viewed as a cyclic cover of degree n of $\mathbb{A}^2$. That is, a map $X \to \mathbb{A}^2$ can be defined which simply projects a point from the surface X onto the affine plane $\mathbb{A}^2$. Since nonzero elements of k have n distinct n^{th} roots, this map is n-to-one everywhere except those points which satisfy

$f(x,y)(x-1) = 0$. Viewing $f(x,y)(x-1)$ as a polynomial in $k[x,y]$ and its zero set $\mathcal{Z}(f(x,y)(x-1))$ as a subset of the affine plane $\mathbb{A}^2$, we see that $X \to \mathbb{A}^2$ is one-to-one precisely at those points of X which lie over $\mathcal{Z}(f(x,y)(x-1))$. We say that the map $X \to \mathbb{A}^2$ *ramifies* on the set $\mathcal{Z}(f(x,y)(x-1))$. The fact that $X \to \mathbb{A}^2$ is ramified prevents the extension of rings $\mathcal{O}_X(X)/\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$ (that is, $T/k[x,y]$) from being separable; in particular, according to [6, Proposition 3.1.2], $T/k[x,y]$ is not Galois.

We will attempt to get a separable extension of rings by going to the open set where $X \to \mathbb{A}^2$ is unramified; that is, by removing the points on X where $f(x,y)(x-1) = 0$. By [17, Lemma I.4.2] (and the fact that forming a quotient ring and localization are interchangeable), $X \setminus \mathcal{Z}(f(x,y)(x-1))$ is an affine variety, and has coordinate ring $\mathcal{O}_{\mathbb{A}^2}(X \setminus \mathcal{Z}(f(x,y)(x-1))) = T\left[\frac{1}{f(x,y)(x-1)}\right] = T[z^{-1}]$. Denote this ring by S. By the same lemma, the open subset $\mathbb{A}^2 \setminus \mathcal{Z}(f(x,y)(x-1))$ of the affine plane has coordinate ring $k[x,y]\left[\frac{1}{f(x,y)(x-1)}\right]$; call this ring R. We have constructed a diagram of rings

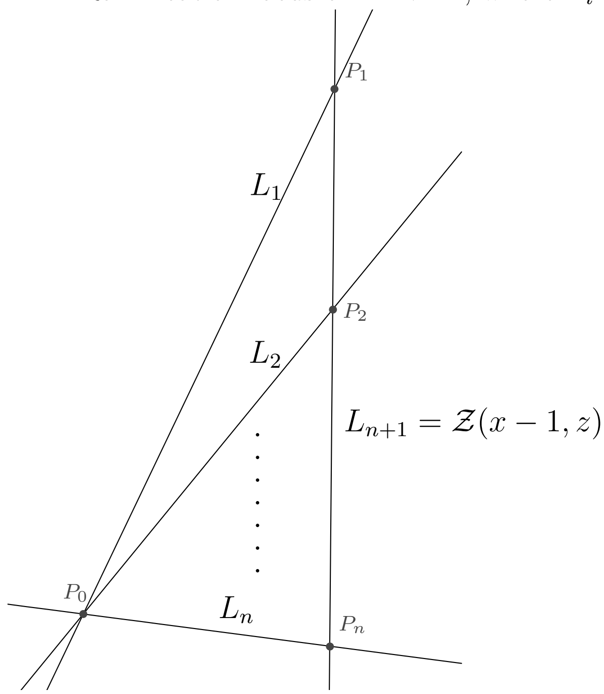
$$\begin{array}{ccc}
 & & S = T[z^{-1}] \\
 & \nearrow & \uparrow \\
T = \frac{k[x,y,z]}{(z^n - f(x,y)(x-1))} & & \\
\uparrow & & R = k[x,y]\left[\frac{1}{f(x,y)(x-1)}\right] \\
 & \nearrow & \\
A = k[x,y] & &
\end{array}$$

where each arrow represents inclusion.

Lemma 2.1.12. *The ring S is a Galois extension of R of degree n, with Galois group $G = \langle \sigma : z \mapsto \omega z \rangle \cong \mathbb{Z}/n\mathbb{Z}$, where $\omega = \zeta^2$ is a primitive n^{th} root of unity.*

Proof. By [6, Proposition 3.1.2], S is a Galois extension of R with Galois group G if

Figure 2.1: Ramification locus of $X \to \mathbb{A}^2$, where $L_i = \mathcal{Z}(\mathfrak{p}_i)$.



and only if the following three conditions are satisfied:

1. $S^G = R$, where $S^G = \{s \in S \mid \rho(s) = s \,\forall\, \rho \in G\}$

2. For each nonzero idempotent $e \in S$ and each pair $\rho \neq \tau \in G$, there exists $s \in S$ with $\rho(s)e \neq \tau(s)e$

3. S is a separable R-algebra.

Since S is a localization of T, S is an integral domain. As such, S cannot have any nontrivial idempotents, for a nontrivial idempotent must also be a zero divisor. So condition number 2 is automatically satisfied. To show that S/R is separable and prove condition 3, one can show through a tedious calculation that the element $\frac{1}{n}\sum_{i=0}^{n-1}\left(z^i \otimes \frac{1}{z^i}\right) \in S \otimes_R S$ is a separability idempotent.

For 1, certainly $S^G \supseteq R$. For the other containment, notice that S is free of rank n as an R-algebra, with free basis $1, z, z^2, \ldots, z^{n-1}$. An arbitrary element of S can be written as $s = r_0 + r_1 z + \cdots + r_{n-1} z^{n-1}$, for some $r_0, \ldots, r_{n-1} \in R$. Suppose such an element s is in S^G. Then $\sigma(s) = s$, or

$$r_0 + r_1 \omega z + \cdots + r_{n-1} \omega^{n-1} z^{n-1} = r_0 + r_1 z + \cdots + r_{n-1} z^{n-1}.$$

Because $1, z, \ldots, z^{n-1}$ is a basis, uniqueness of representation says that $r_i \omega^i = r_i, i = 1, 2, \ldots, n-1$. This implies that either $r_i = 0$ for each $1 \leq i \leq n-1$, or $\omega^i = 1$ for some $1 \leq i \leq n-1$. But this latter case is impossible since ω is a primitive n^{th} root of unity. Thus, $r_i = 0$ for each $1 \leq i \leq n-1$, which says that $s = r_0$ is an element of R. This proves $S^G \subseteq R$.

Thus, the extension of rings S/R satisfies all three of the conditions listed above, and S is therefore a Galois extension of R. □

The Galois extension S/R, as well as the group G itself, are interesting to look at. To begin, notice that the map σ can be defined on T just as easily as it can be defined on S. An action of the group G on $\mathrm{Cl}(T)$ can be defined, simply by extending the map $\sigma : z \mapsto \omega z$ to the elements of the ideals which generate $\mathrm{Cl}(T)$. The ideals $\mathfrak{p}_i = (y - a_i x, z)$ are each fixed by this action, because all multiples of z are already contained in this ideal. That is, $\sigma \mathfrak{p}_i = \mathfrak{p}_i, i = 1, 2, \ldots, n$. The ideals $\mathfrak{q}_i = (x, y - \zeta^{2i-1} z)$ are cyclically permuted by σ: applying σ to the generator $y - \zeta^{2i-1} z$ gives

$$\begin{aligned} \sigma(y - \zeta^{2i-1} z) &= y - \zeta^{2i-1} \omega z \\ &= y - \zeta^{2i-1} \zeta^2 z \\ &= y - \zeta^{2i+1} z \\ &= y - \zeta^{2(i+1)-1} z, \end{aligned}$$

which is the generator for the ideal $\mathfrak{q}_{i+1}$. However, since only $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_{n-1}$ are needed to form a basis of $\mathrm{Cl}(T)$, the action $\sigma \mathfrak{q}_{n-1} = \mathfrak{q}_n$ will usually be expressed as $\sigma \mathfrak{q}_{n-1} \sim -\mathfrak{q}_1 - \mathfrak{q}_2 - \cdots - \mathfrak{q}_{n-1}$.

Proposition 2.1.13. *The cohomology groups of G with coefficients in* $\mathrm{Cl}(T)$ *are*

$$H^r(G, \mathrm{Cl}(T)) \cong \begin{cases} (\mathbb{Z}/n\mathbb{Z})^{(n)}, & \text{if } r \text{ is even} \\ (\mathbb{Z}/n\mathbb{Z})^{(n+1)}, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Define the maps $D = \sigma - 1$ and $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1}$, where σ is defined as in the discussion prior to this proposition. Since G is a cyclic group, [27, Theorem 9.27] says that, for an integer $r \geq 1$, the cohomology groups are as follows:

$$H^0(G, \mathrm{Cl}\, T) = \mathrm{Cl}(T)^G, \tag{2.19}$$

$$H^{2r-1}(G, \mathrm{Cl}\, T) = {}_N\mathrm{Cl}(T)/D(\mathrm{Cl}\, T), \text{ and} \tag{2.20}$$

$$H^{2r}(G, \mathrm{Cl}\, T) = \mathrm{Cl}(T)^G/N(\mathrm{Cl}\, T), \tag{2.21}$$

where $\mathrm{Cl}(T)^G = \{\mathfrak{D} \in \mathrm{Cl}(T) \mid \rho\mathfrak{D} \sim \mathfrak{D}, \text{ for all } \rho \in G\}$ is the subgroup of $\mathrm{Cl}(T)$ fixed by G, and ${}_N\mathrm{Cl}(T) = \{\mathfrak{D} \in \mathrm{Cl}(T) \mid N(\mathfrak{D}) \sim 0\}$ is the subgroup annihilated by N. We need to compute these various subgroups and images of $\mathrm{Cl}(T)$.

It is clear that $\langle \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle \subseteq \mathrm{Cl}(T)^G$ since each $\mathfrak{p}_i$ is fixed by σ. For convenience, we use the matrix defined by the action of σ on $\mathrm{Cl}(T)$:

$$\mathbf{M} := \mathfrak{M}(\sigma, \mathrm{Cl}\, T) = \left(\begin{array}{cccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & -1 \end{array}\right), \tag{2.22}$$

or more simply,

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & \mathcal{C}(x^{n-1}+x^{n-2}+\cdots+x+1) \end{array}\right), \tag{2.23}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{0}$ is the all zero matrix, and $\mathcal{C}(x^{n-1} + x^{n-2} + \cdots + x + 1)$ the companion matrix of the polynomial $x^{n-1} + x^{n-2} + \cdots + x + 1 := c(x)$. Finding an element of $\mathrm{Cl}(T)^G$ is equivalent to finding an eigenvector of the matrix $\mathbf{M}$ corresponding to the eigenvalue 1. But the eigenvectors and eigenvalues of a block diagonal matrix are exactly the combined set of eigenvectors and eigenvectors of the individual blocks (where the vectors have zeros appended to them in the correct places to make the multiplication well-defined). Since the characteristic polynomial of $\mathcal{C}(x^{n-1} + x^{n-2} + \cdots + x + 1)$ is $x^{n-1} + x^{n-2} + \cdots + x + 1$ [7, Lemma 12.19(1)], which does not have 1 as a root, this companion matrix does not have an eigenvalue of 1. It follows that any element of $\mathrm{Cl}(T)$ which is fixed by G does not involve any of the $\mathfrak{q}_i$. Therefore, $\mathrm{Cl}(T)^G = \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$. By (2.19), $H^0(G, \mathrm{Cl}\, T) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$.

Next, we compute ${}_N\mathrm{Cl}(T)$. Starting with $\mathbf{M}$, we can find the matrix for N acting on $\mathrm{Cl}(T)$. Since $\mathbf{M}$ is block diagonal, the matrix for N can be computed piecewise:

$$\begin{aligned}
\mathfrak{M}(N, \mathrm{Cl}\, T) &= \mathbf{I}_{2n-1} + \mathbf{M} + \cdots + \mathbf{M}^{n-1} \\
&= \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{n-1} \end{array}\right) + \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & \mathcal{C}(c(x)) \end{array}\right) \\
&\qquad + \cdots + \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & \mathcal{C}(c(x)) \end{array}\right)^{n-1} \\
&= \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{n-1} \end{array}\right) + \left(\begin{array}{c|c} \mathbf{I}_n & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & \mathcal{C}(c(x)) \end{array}\right)
\end{aligned}$$

$$
+\cdots+\left(\begin{array}{c|c} (\mathbf{I}_n)^{n-1} & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & [\mathcal{C}(c(x))]^{n-1} \end{array}\right)
$$

$$
\begin{aligned}
&= \left(\begin{array}{c|c} \mathbf{I}_n+\mathbf{I}_n+\cdots+(\mathbf{I}_n)^{n-1} & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & \mathbf{I}_{n-1}+\mathcal{C}(c(x))+\cdots+[\mathcal{C}(c(x))]^{n-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} n\mathbf{I}_n & \mathbf{0}_{n\times(n-1)} \\ \hline \mathbf{0}_{(n-1)\times n} & c(\mathcal{C}(c(x))) \end{array}\right) \\
&= \mathbf{0}_{(2n-1)\times(2n-1)},
\end{aligned} \tag{2.24}
$$

where the last equality is because $n\mathfrak{p}_i \sim 0$ for each i, and the fact that any matrix satisfies its own characteristic polynomial (since by the Cayley-Hamilton Theorem [7, Proposition 12.20(2)], the characteristic polynomial of a matrix is always divisible by the minimal polynomial of that same matrix). It follows that N acts as the zero map on $\mathrm{Cl}(T)$, and therefore ${}_N\mathrm{Cl}(T) = \mathrm{Cl}(T)$.

Since N acts as the zero map, the previous paragraph also proves that $N(\mathrm{Cl}\,T) = 0$. There is now enough information to compute the even cohomology groups (of degree larger than zero); (2.21) gives $H^{2r}(G, \mathrm{Cl}\,T) = \mathrm{Cl}(T)^G/N(\mathrm{Cl}\,T) = \mathrm{Cl}(T)^G \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$.

Only the odd cohomology groups are left to determine, and by (2.20), we just need to find $D(\mathrm{Cl}\,T)$. The action of D on the generators of $\mathrm{Cl}(T)$ is

$$D(\mathfrak{p}_i) = \mathfrak{p}_i - \mathfrak{p}_i = 0, \text{ for } i = 1, 2, \ldots, n; \tag{2.25}$$

$$D(\mathfrak{q}_i) = \mathfrak{q}_{i+1} - \mathfrak{q}_i, \text{ for } i = 1, 2, \ldots, n-2; \text{ and} \tag{2.26}$$

$$D(\mathfrak{q}_{n-1}) = \mathfrak{q}_n - \mathfrak{q}_{n-1} \sim -\mathfrak{q}_1 - \mathfrak{q}_2 - \ldots - \mathfrak{q}_{n-2} - 2\mathfrak{q}_{n-1}. \tag{2.27}$$

Therefore, $D(\mathrm{Cl}\,T) = \langle \mathfrak{q}_2 - \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1} - \mathfrak{q}_{n-2}, -\mathfrak{q}_1 - \mathfrak{q}_2 - \ldots - \mathfrak{q}_{n-2} - 2\mathfrak{q}_{n-1}\rangle$. Thus, the odd cohomology groups are given by

$$\begin{aligned}H^{2r-1}(G, \operatorname{Cl} T) &= {}_N\operatorname{Cl}(T)/D(\operatorname{Cl} T) \\ &= \operatorname{Cl}(T)/\langle \mathfrak{q}_2 - \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1} - \mathfrak{q}_{n-2}, -\mathfrak{q}_1 - \mathfrak{q}_2 - \ldots - \mathfrak{q}_{n-2} - 2\mathfrak{q}_{n-1}\rangle.\end{aligned}$$

Looking at all but the final generator, we see that in $H^{2r-1}(G, \operatorname{Cl} T)$, the relations $\mathfrak{q}_1 \sim \mathfrak{q}_2, \mathfrak{q}_2 \sim \mathfrak{q}_3, \ldots, \mathfrak{q}_{n-2} \sim \mathfrak{q}_{n-1}$ hold. That is, all of the generators $\mathfrak{q}_i$, $1 \leq i \leq n-1$, become equivalent in the quotient group. This implies

$$\begin{aligned}H^{2r-1}(G, \operatorname{Cl} T) &\cong \operatorname{Cl}(T)/\langle \mathfrak{q}_2, \ldots, \mathfrak{q}_{n-1}, -n\mathfrak{q}_1\rangle \\ &\cong \operatorname{Cl}(T)/\langle \mathfrak{q}_2, \ldots, \mathfrak{q}_{n-1}, n\mathfrak{q}_1\rangle \\ &\cong \frac{\frac{\bigoplus_{i=1}^n \mathbb{Z}\mathfrak{p}_i}{\langle n\mathfrak{p}_1, \ldots, n\mathfrak{p}_n\rangle} \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\mathfrak{q}_i}{\langle \mathfrak{q}_2, \ldots, \mathfrak{q}_{n-1}, n\mathfrak{q}_1\rangle} \\ &\cong \frac{\bigoplus_{i=1}^n \mathbb{Z}\mathfrak{p}_i}{\langle n\mathfrak{p}_1, \ldots, n\mathfrak{p}_n\rangle} \oplus \frac{\bigoplus_{i=1}^{n-1} \mathbb{Z}\mathfrak{q}_i}{\langle \mathfrak{q}_2, \ldots, \mathfrak{q}_{n-1}, n\mathfrak{q}_1\rangle} \\ &\cong (\mathbb{Z}/n\mathbb{Z})^{(n)} \oplus \frac{\mathbb{Z}\mathfrak{q}_1}{\langle n\mathfrak{q}_1\rangle} \oplus \bigoplus_{i=2}^{n-1} \frac{\mathbb{Z}\mathfrak{q}_i}{\langle \mathfrak{q}_i\rangle} \\ &\cong (\mathbb{Z}/n\mathbb{Z})^{(n)} \oplus \mathbb{Z}/n\mathbb{Z} \oplus 0^{(n-2)} \\ &\cong (\mathbb{Z}/n\mathbb{Z})^{(n+1)}.\end{aligned}$$

This completes the proof. □

We will now concern ourselves with properties of the varieties $\operatorname{Spec} S$ and $\operatorname{Spec} R$, and the Galois extension S/R. We begin with the divisor class group, as we did for the surface X. The fact that S/R is Galois, or equivalently $\operatorname{Spec} S \to \operatorname{Spec} R$ is unramified, will allow us to compute the terms of the so-called Chase-Harrison-Rosenberg seven term exact sequence, which is not possible for the cyclic cover $X \to \mathbb{A}^2$; see Section 2.2 for more details. The following theorem is the first step in computing that sequence; note that S is normal, since it equals $T[z^{-1}]$, with T normal [2, Proposition 5.12].

Theorem 2.1.14. $\mathrm{Cl}(S) \cong \mathbb{Z}^{(n-1)}$, *with the prime ideals* $\mathfrak{q}_1 S, \mathfrak{q}_2 S, \ldots, \mathfrak{q}_{n-1} S$ *forming a basis.*

Proof. Recall that the ring S is defined as $S = T[z^{-1}] = T\left[\frac{1}{f(x,y)(x-1)}\right]$, where $f(x,y) = (y - a_1 x)\cdots(y - a_n x)$ and $T = k[x,y,z]/(z^n - f(x,y)(x-1))$. To prove the theorem, we will use Nagata's sequence (2.3) on S and one of its localizations; however, we will begin with the ring $T\left[\frac{1}{xf(x,y)}\right]$ which, in the proof of Proposition 2.1.8 was shown to be isomorphic under the map $\beta\alpha$ to the ring $k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]$, which is a unique factorization domain. Notice that this ring looks quite similar to the localization of T which defines S; the only difference is that it inverts x where S inverts $x - 1$. That is, inverting $x - 1$ in the ring $T\left[\frac{1}{xf(x,y)}\right]$ would result in the ring $T\left[\frac{1}{xf(x,y)(x-1)}\right] = S[x^{-1}]$.

Let's extend the isomorphism $\beta\alpha$ to the ring $T\left[\frac{1}{xf(x,y)(x-1)}\right]$. The image of $x - 1$ under $\beta\alpha$ is

$$\begin{aligned}
\beta\alpha(x-1) &= \beta(\alpha(x-1)) \\
&= \beta(x-1) \\
&= \frac{w^n + f(1,v)}{f(1,v)} - 1 \\
&= \frac{w^n}{f(1,v)}.
\end{aligned}$$

Inverting this element in $k\left[v, w, \frac{1}{f(1,v)(w^n + f(1,v))}\right]$ is equivalent to inverting w, since $f(1,v)$ is already a unit in this ring. It follows that $S[x^{-1}] = T\left[\frac{1}{xf(x,y)(x-1)}\right] \cong k\left[v, w, \frac{1}{wf(1,v)(w^n + f(1,v))}\right]$, which is still a UFD; denote the latter ring by $\mathscr{S}$, and the isomorphism by $\overline{\beta\alpha}$. Thus, the divisor class group $\mathrm{Cl}(S[x^{-1}])$ is equal to 0.

As with any localization, there is a one-to-one correspondence between the set of prime ideals in S, and those prime ideals in T which don't contain z. None of the prime ideals $\mathfrak{q}_i$ in T contain z, so they all remain nontrivial in S. It follows that the ideals $\mathfrak{q}_i S, i = 1, 2, \ldots, n$, are precisely the height one prime ideals in S which contain x. So Nagata's sequence applied to S and the localization $S[x^{-1}]$ is

$$1 \to S^* \to S[x^{-1}]^* \xrightarrow{\text{div}} \bigoplus_{i=1}^{n} \mathbb{Z} \cdot \mathfrak{q}_i S \to \text{Cl}(S) \to 0, \tag{2.28}$$

where the last term is $0 = \text{Cl}(S[x^{-1}])$. By the argument in the previous paragraph, $S[x^{-1}]^* \cong \mathscr{S}^*$. The group of units of $k\left[v, w, \frac{1}{wf(1,v)(w^n+f(1,v))}\right]$ requires no work; it is

$$\mathscr{S}^* = k^* \times \langle v - a_1 \rangle \times \cdots \times \langle v - a_n \rangle \times \langle w^n + f(1, v) \rangle \times \langle w \rangle.$$

Extending Lemma 2.1.9 to the current situation shows that the subgroup $k^* \times \langle v - a_1 \rangle \times \cdots \times \langle v - a_n \rangle \times \langle w^n + f(1, v) \rangle$ of $\mathscr{S}^*$ is isomorphic to the subgroup $k^* \times \langle x \rangle \times \langle y - a_1 x \rangle \times \cdots \times \langle y - a_n x \rangle$ of $S[x^{-1}]^*$. Denote these subgroups by $\mathscr{H}$ and H, respectively.

Since $z \in S[x^{-1}]^*$, $H \times \langle z \rangle \subseteq S[x^{-1}]^*$. Hence $\overline{\beta\alpha}(H \times \langle z \rangle) = \mathscr{H} \times \langle \frac{w(w^n+f(1,v))}{f(1,v)} \rangle \subseteq \mathscr{S}^*$. On the other hand, the elements $f(1, v)$, and $w^n + f(1, v)$ are already in the subgroup $\mathscr{H}$ of $\mathscr{S}^*$. This implies that $\mathscr{H} \times \langle \frac{w(w^n+f(1,v))}{f(1,v)} \rangle$ must also contain w, and since it is a group, it contains the subgroup $\langle w \rangle$ generated by w. Thus, $\mathscr{S}^* = \overline{\beta\alpha}(H \times \langle z \rangle)$, implying that $S[x^{-1}]^* = H \times \langle z \rangle$.

Since $k^* \times \langle y - a_1 x \rangle \times \cdots \times \langle y - a_n x \rangle \times \langle z \rangle \subseteq S^*$, the exactness of Nagata's sequence implies that the image of div in (2.28) is generated by $\text{div}(x)$ which, as in (2.17), is equal to $\mathfrak{q}_1 S + \cdots + \mathfrak{q}_n S$. Combining this once more with the exactness of (2.28) proves $\text{Cl}(S) \cong \frac{\bigoplus_{i=1}^n \mathbb{Z} \cdot \mathfrak{q}_i S}{\text{im(div)}} = \frac{\bigoplus_{i=1}^n \mathbb{Z} \cdot \mathfrak{q}_i S}{\langle \mathfrak{q}_1 S + \cdots + \mathfrak{q}_n S \rangle} \cong \mathbb{Z}^{(n-1)}$. □

Corollary 2.1.15. $\text{Pic}(S) = \text{Cl}(S) \cong \mathbb{Z}^{(n-1)}$.

Proof. Recall that, geometrically, S corresponds to the open subset of X obtained by removing all those points on X which lie over the curve $f(x, y)(x - 1) = 0$ in $\mathbb{A}^2$. That is, $S = \mathcal{O}_{\mathbb{A}^2}(X \setminus \mathcal{Z}(f(x, y)(x - 1)))$. In Corollary 2.1.4, it was shown that every singularity of this variety is in the zero set of $f(x, y)(x - 1)$. Hence, $X \setminus \mathcal{Z}(f(x, y)(x -$

1)) is nonsingular. By Definition 1.2.15, every local ring of $X \setminus \mathcal{Z}(f(x,y)(x-1))$ is a regular local ring (hence a UFD, by the Auslander-Buchsbaum Theorem [23, (19.A), Theorem 48]). By [16, Corollary V.18.5], $\operatorname{Pic}(S) = \operatorname{Cl}(S)$. □

Corollary 2.1.16. $S^* = k^* \times \langle y - a_1 x\rangle \times \cdots \times \langle y - a_n x\rangle \times \langle z\rangle$.

Proof. The proof of Theorem 2.1.14 shows that $S[x^{-1}]^* = k^* \times \langle x\rangle \times \langle y - a_1 x\rangle \times \cdots \times \langle y - a_n x\rangle \times \langle z\rangle$. The exactness of (2.28) says that S^* is equal to the kernel of div. But the ideals $\mathfrak{q}_i S$ contain none of the elements $y - a_1 x, \ldots, y - a_n x, z$. Thus, these elements necessarily map to zero under div; together with k^*, they generate ker(div). □

2.2 THE CHASE-HARRISON-ROSENBERG SEVEN TERM EXACT SEQUENCE

With the groups $\operatorname{Pic}(S)$ and S^* now known, we may begin to compute the terms in the seven term exact sequence of Chase, Harrison, and Rosenberg. As stated in [4, Corollary 5.5], if R is a commutative ring and S is a Galois extension of R with Galois group G, then there is an exact sequence

$$\begin{aligned}1 \to H^1(G, S^*) \to \operatorname{Pic}(R) \to \operatorname{Pic}(S)^G \to \\ H^2(G, S^*) \to \operatorname{B}(S/R) \to H^1(G, \operatorname{Pic}(S)) \to H^3(G, S^*). \quad (2.29)\end{aligned}$$

Theorem 4.1.1 of [6] provides detailed descriptions of the constructions of each of the homomorphisms (except $H^1(G, \operatorname{Pic}(S)) \to H^3(G, S^*)$); we shall not in general be concerned with the explicit maps here.

The rings R and S that have been discussed here satisfy the conditions to be able to apply sequence (2.29), as shown in Lemma 2.1.12.

Lemma 2.2.1. $\operatorname{Pic}(R) = \operatorname{Cl}(R) = 0$.

Proof. Certainly the ring $R = k[x,y][f(x,y)^{-1}(x-1)^{-1}]$ is a unique factorization domain. By [23, (19.A), Theorem 47], every prime ideal of R of height one is principal. By definition, $\mathrm{Div}(R)$ is the free abelian group on the set of height one prime ideals of R. The cited theorem implies that $\mathrm{Div}(R) = \mathrm{Prin}(R)$, or equivalently, $\mathrm{Cl}(R) = 0$. Since $\mathrm{Pic}(R)$ is a subgroup of $\mathrm{Cl}(R)$, the proof follows. □

Corollary 2.2.2. $H^1(G, S^*) = H^3(G, S^*) = 0$.

Proof. That the first cohomology group $H^1(G, S^*)$ is trivial follows at once from the previous lemma and the exactness of sequence (2.29). Since G is cyclic, [27, Theorem 9.27] may be applied to show that all odd cohomology groups are the same. □

Lemma 2.2.3. $\mathrm{Pic}(S)^G = 0$.

Proof. By Corollary 2.1.15, $\mathrm{Pic}(S) = \mathrm{Cl}(S)$. Recall that in the proof of Proposition 2.1.13, it was shown that $\mathrm{Cl}(T)^G = \langle \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$; in particular, the subgroup of $\mathrm{Cl}(T)$ generated by the height one primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}$ containing x has trivial intersection with the fixed subgroup $\mathrm{Cl}(T)^G$ of $\mathrm{Cl}(T)$. There is an obvious (non-canonical) isomorphism $\mathrm{Cl}(S) = \langle \mathfrak{q}_1 S, \ldots, \mathfrak{q}_{n-1} S \rangle \to \langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1} \rangle \leq \mathrm{Cl}(T)$ preserving the action of G, and in light of this correspondence it is clear that $\mathrm{Cl}(S)^G = 0$. □

Lemma 2.2.4. $H^2(G, S^*) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$.

Proof. Again we refer to the formula for the cohomology of a cyclic group given in [27, Theorem 9.27]. The formula says that $H^2(G, S^*) = (S^*)^G / N(S^*)$, where the norm map is defined as the function $N = 1 \cdot \sigma \cdots \sigma^{n-1}$ (where N is written multiplicatively because S^* is a multiplicative group). Corollary 2.1.16 showed that $S^* = k^* \times \langle y - a_1 x \rangle \times \cdots \times \langle y - a_n x \rangle \times \langle z \rangle$. Clearly, $\sigma : z \mapsto \omega z$ fixes all generators of S^* except z itself. However, notice that $\sigma(z^{kn}) = (\sigma(z)^n)^k = ((\omega z)^n)^k = (\omega^n z^n)^k = z^{nk}$, for any $k \in \mathbb{Z}$. Since $\omega^m \neq 1$ whenever $n \nmid m$, we also see that z^m is not fixed by σ for such integers m. We conclude that the subgroup of S^* fixed by σ is $(S^*)^G = k^* \times \langle y - a_1 x\rangle \times \cdots \times \langle y - a_n x\rangle \times \langle z^n\rangle$.

Next, we compute the image of S^* under the map N. First, for any $\alpha \in k^*$, $N(\alpha) = \alpha^n$; since k is algebraically closed, $N(k^*) = k^*$. On any generator of S^* that is fixed by σ, the norm map is simply the n^{th} power map. It turns out that the same is (almost) true of $N(z)$:

$$\begin{aligned} N(z) &= (1 \cdot \sigma \cdots \sigma^{n-1})(z) \\ &= z \cdot \sigma(z) \cdots \sigma^{n-1} \\ &= z \cdot z\omega \cdots z\omega^{n-1} \\ &= \omega\omega^2 \cdots \omega^{n-1} z^n \\ &= \begin{cases} z^n, & \text{if } n \text{ is odd} \\ \omega^{\frac{n}{2}} z^n = -z^n, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

So, if n is odd, one easily sees that $N(S^*) = (S^*)^n$. However, this is also true if n is even; in this case, consider the subgroup $H = k^* \times \langle -z^n\rangle$ of $N(S^*)$. Since $-1 \in k^*$, the element $(-1)(-1)z^n = z^n$ is in H. So $\langle z^n\rangle \subseteq H$, and so $k^* \cdot \langle z^n\rangle \subseteq H$. Because this group contains $-z^n$, it is clear that $k^* \cdot \langle z^n\rangle = H$. It is also clear, however, that both k^* and $\langle z^n\rangle$ are normal subgroups of H (H is abelian) with trivial intersection (z is subject only to the relation $z^n = f(x, y)(x - 1)$ in S; no power of z can equal a scalar from k). This forces $H = k^* \times \langle z^n\rangle$. Thus, as in the previous case, $N(S^*) = (S^*)^n$.

Finally, the formula for the cohomology group $H^2(G, S^*)$ says that $H^2(G, S^*) = (S^*)^G/N(S^*) = \frac{k^* \times \langle y - a_1 x\rangle \times \cdots \times \langle y - a_n x\rangle \times \langle z^n\rangle}{(S^*)^n} \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$. ∎

Lemma 2.2.5. $H^1(G, \operatorname{Pic} S) \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. Apply [27, Theorem 9.27]. As in Lemma 2.2.3, we refer to the calculations done in Proposition 2.1.13. The work done there shows that ${}_N\mathrm{Pic}(S) = \mathrm{Pic}(S)$ and $D(\mathrm{Pic}\, S) = \langle \mathfrak{q}_2 S - \mathfrak{q}_1 S, \ldots, \mathfrak{q}_{n-1} S - \mathfrak{q}_{n-2} S, -\mathfrak{q}_1 S - \mathfrak{q}_2 S - \ldots - \mathfrak{q}_{n-2} S - 2\mathfrak{q}_{n-1} S\rangle$. In the quotient group $\mathrm{Pic}(S)/D(\mathrm{Pic}\, S)$, the relations defined by the generators of $D(\mathrm{Pic}\, S)$ force each of $\mathfrak{q}_1 S, \mathfrak{q}_2 S, \ldots, \mathfrak{q}_{n-1} S$ to become equivalent, as well as forcing this common element to have order n. As this is the extent of the effect of the relations, we conclude $\mathrm{Pic}(S)/D(\mathrm{Pic}\, S) \cong \mathbb{Z}/n\mathbb{Z}$. □

Theorem 2.2.6. $\mathrm{B}(S/R) \cong (\mathbb{Z}/n\mathbb{Z})^{(n+1)}$.

Proof. The computations done so far reduce the nontrivial terms of the Chase-Harrison-Rosenberg exact sequence (2.29) to the short exact sequence

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{(n)} \to \mathrm{B}(S/R) \to \mathbb{Z}/n\mathbb{Z} \to 0. \tag{2.30}$$

By Lemma 2.2.7, which follows, we may view sequence (2.30) as an exact sequence of $\mathbb{Z}/n\mathbb{Z}$-modules. Since $\mathbb{Z}/n\mathbb{Z}$ is free (hence projective) as a module over itself, (2.30) is split exact, which implies that $\mathrm{B}(S/R) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)} \oplus \mathbb{Z}/n\mathbb{Z}$. □

Lemma 2.2.7. *The relative Brauer group* $\mathrm{B}(S/R)$ *is annihilated by* n*; in other words,* $n \cdot \mathrm{B}(S/R) = 0$.

Proof. Let $K = k(x, y)$ be the quotient field of the rings A and R, and let $L = K[z]/(z^n - f(x, y)(x - 1))$ be the quotient field of T and S. For this lemma, denote by H the image of the group homomorphism $\mathrm{B}(R) \to \mathrm{B}(S)$. Consider the following diagram of groups with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{B}(S/R) & \xrightarrow{i} & \mathrm{B}(R) & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b\ \text{1−1} & & \downarrow c & & \\ 0 & \longrightarrow & \mathrm{B}(L/K) & \longrightarrow & \mathrm{B}(K) & \longrightarrow & \mathrm{B}(L) & & \end{array} \tag{2.31}$$

The map b in the above diagram is injective because R is a regular ring [24, Example III.2.22]. The Snake Lemma [27, Corollary 6.12] says that there is an exact sequence

$$0 \to \ker a \xrightarrow{i^*} \ker b \to \ker c, \tag{2.32}$$

where the maps are simply restrictions to subgroups of the maps in the top row of (2.31). (The further terms of sequence (2.32) involving cokernels will not be necessary here.) Since $\ker b = 0$, the exactness of sequence (2.32) forces $\ker a = 0$. That is, the map $\mathrm{B}(S/R) \to \mathrm{B}(L/K)$ is injective, and using this map we may view $\mathrm{B}(S/R)$ as a subgroup of $\mathrm{B}(L/K)$.

The extension L/K is Galois with group G because $L^G = K$ [6, Corollary 3.1.4]. By the Crossed Product Theorem [26, Theorem (29.12)], $\mathrm{B}(L/K) \cong H^2(G, L^*)$. Since G is a finite group of order n, the cohomology group $H^2(G, L^*)$ is annihilated by n [7, Corollary 17.27]. Since we are viewing $\mathrm{B}(S/R)$ as a subgroup of $\mathrm{B}(L/K)$, this proves $n \cdot \mathrm{B}(S/R) = 0$. □

This completes the determination of the terms of the Chase-Harrison-Rosenberg exact sequence (2.29).

2.3 BRAUER GROUPS AND SYMBOL ALGEBRAS OVER *X*

In Theorem 2.2.6, it was shown that the relative Brauer group $\mathrm{B}(S/R)$ is isomorphic, as an abstract group, to $(\mathbb{Z}/n\mathbb{Z})^{(n+1)}$. In this section, we will compute a more explicit representation of this group, as well as computing the Brauer groups of the rings R and S. Many of the elements of these groups may be expressed as symbol algebras; we begin with the definition of these.

Definition 2.3.1. Let R be a commutative ring with identity. Let $a, b \in R^*$, and $r \geq 2$ an integer. Fix a primitive r^{th} root of unity μ in R (here we will use ω, as has been used above, when $r = n$). The *symbol algebra* $(a, b)_r$ is the associative R-algebra generated by two elements u and v subject to the relations $u^r = a, v^r = b$, and $vu = \mu uv$.

Remark 2.3.2. The symbol algebra $(a, b)_r$ can be viewed as a cyclic algebra, in the terminology of [26, §30]. It can be shown that, using the notation found there, $(a, b)_r \cong (R[\sqrt[r]{b}]/R, \sigma, a)$. While Reiner's book only discusses crossed product algebras over fields, the construction may be extended to Galois extensions of rings.

Note that the algebra given in this definition may be written down whether or not a and b are units in R; in that case, we get an associative R-algebra which becomes an Azumaya algebra over the quotient field of R. However, the symbol algebra $(a, b)_r$ is an Azumaya R-algebra if both $a, b \in R^*$. This is a result of an application of the so-called cup product map

$$\smile : H^1(R, \mu_r) \times H^1(R, \mu_r) \to H^2(R, \mu_r) \tag{2.33}$$

[24, p. 172] followed by the map $H^2(R, \mu_r) \to {}_r\mathrm{B}(R)$ induced by the long exact sequence of cohomology associated to the Kummer sequence

$$1 \to \mu_r \to \mathbb{G}_m \xrightarrow{r} \mathbb{G}_m \to 1 \tag{2.34}$$

[24, p. 66]. Since we will only use the composition of these maps, as opposed to using either individually, we will also refer to the composition as the cup product and denote it by $\smile$. In the sequences above, μ_r is the sheaf of r^{th} roots of unity, and $\mathbb{G}_m$ the sheaf of multiplicative units. The cohomology group $H^1(R, \mu_r)$ classifies the cyclic Galois extensions of R of degree r.

We turn our attention back to the Galois extension of rings S/R. The ring S may be viewed as $S = R\left[\sqrt[n]{f(x, y)(x-1)}\right]$. Fixing $[S]$ on one side of the cup product map (2.33) and composing with $H^2(R, \mu_n) \to {}_n\mathrm{B}(R)$ produces a group homomorphism

$$(\cdot) \smile [S] : H^1(R, \mu_n) \to {}_n\mathrm{B}(R) \tag{2.35}$$

which, for a cyclic Galois extension $\mathfrak{S} = R[\sqrt[n]{g}]$, is given by $[\mathfrak{S}] \mapsto (g, f(x, y)(x-1))_n$. The image of the map (2.35) is denoted by $\mathrm{B}^{\smile}(S/R)$. This is actually equal to another group that we have already seen; the construction given in [6, p. 121] shows that the map $\alpha_4 : H^2(G, S^*) \to \mathrm{B}(S/R)$ is defined by sending a unit $a \in R^*$ to the cyclic algebra $(S/R, \sigma, a)$ which, by Remark 2.3.2, is isomorphic to $(a, (x-1)f)_n$. Thus $\mathrm{B}^{\smile}(S/R)$ agrees with the image of $H^2(G, S^*)$ under α_4. But Lemma 2.2.3 showed that α_4 is injective, and we conclude

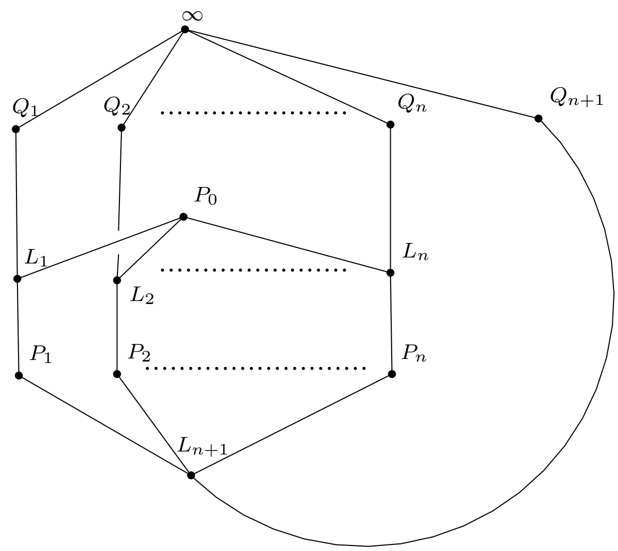
Lemma 2.3.3. $\mathrm{B}^{\smile}(S/R) \cong H^2(G, S^*) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$. □

Since any symbol algebra in $\mathrm{B}^{\smile}(S/R)$ has $f(x,y)(x-1)$ in its right-hand side, every element of this subgroup is split by S; that is, $(g, f(x,y)(x-1))_n \otimes_R S$ is Brauer equivalent to S (by [6, Theorem 2.5.5], since S is a maximal commutative subalgebra of $(g, f(x,y)(x-1))_n \otimes_R S$). So there is a chain of subgroups $\mathrm{B}^{\smile}(S/R) \leq \mathrm{B}(S/R) \leq {}_n\mathrm{B}(R)$, where the latter containment was shown in Lemma 2.2.7. We compute each of these groups by writing down explicit generators; in particular, we will confirm the result of Theorem 2.2.6. We begin with

Proposition 2.3.4. $\mathrm{B}(R) \cong (\mathbb{Q}/\mathbb{Z})^{(2n-1)}$. *Furthermore, for $r \geq 2$, the r-torsion subgroup ${}_r\mathrm{B}(R)$ of $\mathrm{B}(R)$ has as a free $\mathbb{Z}/r\mathbb{Z}$-basis the symbol algebras $\{(\ell_1, \ell_j)_r \mid 2 \leq j \leq n+1\} \cup \{(\ell_i, \ell_{n+1})_r \mid 2 \leq i \leq n\}$.*

Proof. The main tool in the computation will be [12, Theorem 4]. Use Figure 2.1 to define a graph Γ_R, and define the point Q_i to be the point where L_i intersects the line at infinity. The vertex set $V(\Gamma_R)$ of the graph consists of one vertex for each of the lines $L_1, \cdots, L_{n+1}$, and the line at infinity, as well as one vertex for each of the points $P_i, Q_{i+1}, 0 \leq i \leq n$. In projective coordinates, the points are $P_0 = [0 : 0 : 1], P_i = [1 : a_i : 1]$, and $Q_i = [1 : a_i : 0], Q_{n+1} = [0 : 1 : 0]$. The graph is bipartite, the vertex set being divided into those vertices corresponding to lines and those corresponding to points. The edge $\{P_j, L_i\}$ is in the edge set $E(\Gamma_R)$ if and only if the point P_j is on line L_i (and similarly with the points Q_j). The graph

Figure 2.2: The graph Γ_R.



Γ_R is oriented by setting the positive end of an edge to be the end connecting to a line, and the negative end to be the one incident to a vertex corresponding to a point. It is the assertion of [12, Theorem 4] that $\mathrm{B}(R)$ is a free $\mathbb{Q}/\mathbb{Z}$-module of rank $|E(\Gamma_R)| - |V(\Gamma_R)| + 1$.

This makes the first claim of this theorem very straightforward. A simple count shows that Γ_R has $e = 5n+2$ edges and $v = 3n+4$ vertices, so that $e - v + 1 = (5n+2) - (3n+4) + 1 = 2n - 1$, proving that assertion. Since the r-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, and torsion commutes with direct sum, the r-torsion subgroup of $\mathrm{B}(R)$ is ${}_r\mathrm{B}(R) \cong (\mathbb{Z}/r\mathbb{Z})^{(2n-1)}$. It remains to show that the set claimed in the proposition is indeed a basis.

The theorem we are relying on also explicitly describes the elements and relations which make up ${}_r\mathrm{B}(R)$. The cup product map $\smile: H^1(R, \mu_r) \times H^1(R, \mu_r) \to {}_r\mathrm{B}(R)$ is a surjection, so that the set $\{(\ell_i, \ell_j)_r \mid 1 \leq i < j \leq n+1\}$ generates ${}_r\mathrm{B}(R)$. The relations in ${}_r\mathrm{B}(R)$ are given by the generators of the kernel of $\smile$:

$$(\ell_i, \ell_j)_r (\ell_j, \ell_t)_r (\ell_i, \ell_t)_r^{-1} \sim 1, 1 \leq i, j, t \leq n. \tag{2.36}$$

It is immediate that $\{(\ell_1, \ell_j)_r \mid 2 \leq j \leq n+1\} \cup \{(\ell_i, \ell_{n+1})_r \mid 2 \leq i \leq n\}$ is an independent set; none of the algebras $(\ell_i, \ell_{n+1})_r$ are involved in any relation in ${}_r\mathrm{B}(R)$, and the other members of this set may only form the trivial relation $(\ell_1, \ell_t)_r(\ell_i, \ell_t)_r^{-1} \sim 1$. To show that this set is a basis, we show that every element of the generating set described above may be expressed in terms of these elements. This set is actually already a subset of that generating set; we are missing only the elements $(\ell_i, \ell_j)_r$ for $2 \leq i < j \leq n$. Notice that, using the relations (2.36) in ${}_r\mathrm{B}(R)$, for any $1 \leq i \neq j \leq n$ we can write

$$\begin{aligned} (\ell_i, \ell_j)_r &\sim (\ell_i, \ell_1)_r(\ell_1, \ell_j)_r \\ &\sim (\ell_1, \ell_i)_r^{-1}(\ell_1, \ell_j)_r, \end{aligned} \tag{2.37}$$

which is in the subgroup generated by the set $\{(\ell_1, \ell_j)_r \mid 2 \leq j \leq n+1\} \cup \{(\ell_i, \ell_{n+1})_r \mid 2 \leq i \leq n\}$ Therefore this is a generating set and, in particular, a basis. □

Proposition 2.3.5. *The set $\{(\ell_i, f\ell_{n+1})_n \mid 1 \leq i \leq n\}$ is a free $\mathbb{Z}/n\mathbb{Z}$-basis for $\mathrm{B}^{\smile}(S/R)$.*

Proof. Since by definition $\mathrm{B}^{\smile}(S/R)$ is the image of the map $(\cdot) \smile [S] : H^1(R, \mu_n) \to {}_n\mathrm{B}(R)$, it is generated by the classes of $(\ell_i, f\ell_{n+1})_n, 1 \leq i \leq n+1$, in ${}_n\mathrm{B}(R)$. Of course, this is one too many elements to be a basis. We will prove the proposition by writing these generators in terms of the basis for ${}_n\mathrm{B}(R)$ determined in Proposition 2.3.4.

Using the bilinearity property of symbol algebras as elements of $\mathrm{B}(R)$, write

$$(\ell_1, f\ell_{n+1})_n = (\ell_1, \ell_1 \cdots \ell_{n+1})_n \sim (\ell_1, \ell_1)_n \cdots (\ell_1, \ell_{n+1}) \sim \prod_{j=2}^{n+1} (\ell_1, \ell_j)_n, \tag{2.38}$$

where the final equivalence is due to the fact that the symbol algebra $(a, a)_r$ is split (Brauer equivalent to R), for any $a \in R^*$ and $r \geq 2$.

The other expansions will make use of (2.37). For $2 \le i \le n$,

$$
\begin{aligned}
(\ell_i, f\ell_{n+1})_n &\sim \prod_{\substack{j=1\\ j\neq i}}^{n+1} (\ell_i, \ell_j)_n \\
&\sim (\ell_1, \ell_i)_n^{-1} \prod_{\substack{j=2\\ j\neq i}}^{n} \Big[(\ell_1, \ell_i)_n^{-1} (\ell_1, \ell_j)_n\Big] (\ell_i, \ell_{n+1})_n \\
&\sim (\ell_1, \ell_i)_n^{-(n-1)} \prod_{\substack{j=2\\ j\neq i}}^{n} \Big[(\ell_1, \ell_j)_n\Big] (\ell_i, \ell_{n+1})_n.
\end{aligned}
\tag{2.39}
$$

For any i between 1 and n, the expansion of $(\ell_i, f\ell_{n+1})_n$ in terms of this basis contains $(\ell_i, \ell_{n+1})_n$, while $(\ell_j, f\ell_{n+1})_n$ does not, if $j \neq i$. It follows that $\{(\ell_i, f\ell_{n+1})_n \mid 1 \le i \le n\}$ is linearly independent in ${}_n\mathrm{B}(R)$, and therefore independent in $\mathrm{B}^{\smile}(S/R)$. To show these are a generating set, it is enough to show that $(\ell_{n+1}, f\ell_{n+1})_n$ can be expressed in terms of these n elements. This is accomplished through the calculation

$$
\begin{aligned}
\prod_{i=1}^{n} (\ell_i, f\ell_{n+1})_n^{-1} \sim \left[\prod_{i=1}^{n} (\ell_i, f\ell_{n+1})_n\right]^{-1} \sim (f, f\ell_{n+1})_n^{-1} &\sim (f\ell_{n+1}, f)_n \\
&\sim (f, f)_n (\ell_{n+1}, f)_n \\
&\sim (\ell_{n+1}, f)_n \\
&\sim (\ell_{n+1}, f)_n (\ell_{n+1}, \ell_{n+1})_n \\
&\sim (\ell_{n+1}, f\ell_{n+1})_n.
\end{aligned}
$$

We conclude that $\{(\ell_i, f\ell_{n+1})_n \mid 1 \le i \le n\}$ is a basis for $\mathrm{B}^{\smile}(S/R)$. □

The proof of Theorem 2.2.6 showed that $\mathrm{B}(S/R) \cong H^2(G, S^*) \oplus H^1(G, \operatorname{Pic} S) \cong \mathrm{B}^{\smile}(S/R) \oplus H^1(G, \operatorname{Pic} S)$. Given the result of the previous proposition, computing a basis for the relative Brauer group $\mathrm{B}(S/R)$ reduces to finding a generator for the cyclic group $H^1(G, \operatorname{Pic} S)$. This will not be an "obvious" symbol algebra over R; however, extension of scalars to the ring $R[x^{-1}]$ will make this Azumaya algebra into an element of the subgroup defined by the cup product. Let Y be the quasi-affine surface $\operatorname{Spec} R$, so that $K(Y) = K(\mathbb{A}^2) = k(x, y) = K$. There is an exact sequence

$$1 \to \mathrm{B}(Y) \to \mathrm{B}(K) \xrightarrow{\text{ram}} \bigoplus_{C \in Y_1} H^1(K(C), \mathbb{Q}/\mathbb{Z}), \tag{2.40}$$

where the summation is over all prime divisors C on Y and the map ram is called the ramification map (see, for example, [1, p. 86]). The image of this map agrees with the *tame symbol*; given a prime divisor C, the image of the map $\mathrm{B}(K) \to H^1(K(C), \mathbb{Q}/\mathbb{Z})$ on a symbol algebra $(a, b)_r \in \mathrm{B}(K)$ is the cyclic Galois extension of $K(C)$ obtained by adjoining the r^{th} root of

$$(-1)^{\nu_C(a)\nu_C(b)} a^{\nu_C(b)} b^{-\nu_C(a)}, \tag{2.41}$$

where a, b are viewed as functions on C. The image of the symbol algebra $(a, b)_r$ in $H^1(K(C), \mathbb{Q}/\mathbb{Z})$ is the trivial element if and only if the Galois extension of fields $K(C)\left[\sqrt[r]{(-1)^{\nu_C(a)\nu_C(b)} a^{\nu_C(b)} b^{-\nu_C(a)}}\right]/K(C)$ is trivial; by definition, this means that $K(C)\left[\sqrt[r]{a^{\nu_C(b)} b^{-\nu_C(a)}}\right]$ is isomorphic to a direct sum of r copies of $K(C)$, which is equivalent to saying that the element $a^{\nu_C(b)} b^{-\nu_C(a)}$ is already an r^{th} power in $K(C)$. Notice that if $\nu_C(a) = \nu_C(b) = 0$, then the tame symbol is trivially an r^{th} power, so that $(a, b)_r$ is unramified along C. It follows that the set of divisors where $(a, b)_r$ ramifies, the *ramification locus* of $(a, b)_r$, is a subset of the union of the divisors of a and b. An algebra in the kernel of the map ram of (2.40) is said to be *unramified* over Y. By the exactness of that sequence, a symbol algebra is unramified over Y if and only if it is the image of an Azumaya algebra defined over R; that is, it is equal to $\mathfrak{A} \otimes_R K$, for some R-Azumaya algebra $\mathfrak{A}$.

Consider the symbol algebra $(x, f\ell_{n+1})_n$ defined over K, and note that it actually defines a symbol algebra over $R[x^{-1}]$. We claim that this symbol algebra is the image of an Azumaya R-algebra that generates the cyclic group $H^1(G, \operatorname{Pic} S)$, and wish to explicitly write down that algebra. To start,

Lemma 2.3.6. *The symbol algebra* $\Lambda = (x, f\ell_{n+1})_n$ *is unramified over* $Y = \mathrm{Spec}(R)$. *Therefore, it is the image of some Azumaya* R*-algebra* $\mathfrak{A}$ *under the natural map* $\mathrm{B}(R) \to \mathrm{B}(R[x^{-1}])$.

Proof. It is enough to prove the first claim, and for this it suffices to show that the ramification locus of Λ is contained in $\mathbb{A}^2 \backslash Y$; that is, Λ may only ramify along divisors contained in $\mathcal{Z}(f\ell_{n+1})$. So the question comes down to showing that Λ is unramified along $C = \mathcal{Z}(x)$. Clearly $\nu_{(x)}(x) = 1$, and since the linear polynomials $\ell_1, \dots, \ell_{n+1}$ are not contained in the prime ideal (x) of A, they become units in the localization $k[x, y]_{(x)}$, so that the valuation of each is zero. So the tame symbol is

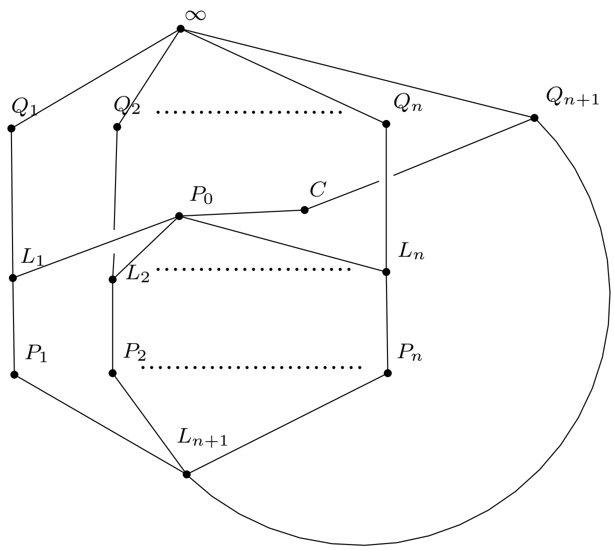
$$\begin{aligned}(-1)^{\nu_{(x)}(x)\nu_{(x)}(f\ell_{n+1})} x^{\nu_{(x)}(f\ell_{n+1})} (f\ell_{n+1})^{-\nu_{(x)}(x)} &= (-1)^{1\cdot 0} x^0 (f\ell_{n+1})^{-1} \\ &= 1 \cdot 0^0 \cdot (-y^n)^{-1} \\ &= -y^{-n},\end{aligned} \tag{2.42}$$

where in the second line we are viewing everything as a function on $C = \mathcal{Z}(x)$, and for the purposes of computing the tame symbol we take $0^0 = 1$. This element is clearly an n^{th} power in $K(C)$, which implies that Λ is unramified along C. Therefore the ramification locus of Λ is contained in $\mathcal{Z}(f\ell_{n+1})$, and hence Λ is unramified over $\mathbb{A}^2 \setminus \mathcal{Z}(f\ell_{n+1}) = Y$. □

Lemma 2.3.7. *The symbol algebra* $\Lambda = (x, f\ell_{n+1})_n$ *has order* n *in* $\mathrm{B}(R[x^{-1}])$.

Proof. It needs only be shown that the order of Λ cannot be smaller than n. We will make use of [13, Theorem 2.1], which gives a map from the n-torsion subgroup ${}_n\mathrm{B}(R[x^{-1}])$ of the Brauer group to the cycle space $H_1(\Gamma_{R[x^{-1}]}, \mathbb{Z}/n\mathbb{Z})$ of the graph $\Gamma_{R[x^{-1}]}$, which is defined in a similar manner to the graph in Figure 2.2. The vertex set of $\Gamma_{R[x^{-1}]}$ is equal to $V(\Gamma_R) \cup \{C\}$, where $C = \mathcal{Z}(x)$, and the edge set is $E(\Gamma_R) \cup \{\{P_0, C\}, \{Q_{n+1}, C\}\}$. The map ${}_n\mathrm{B}(R[x^{-1}]) \to H_1(\Gamma_{R[x^{-1}]}, \mathbb{Z}/n\mathbb{Z})$ describes how to

Figure 2.3: The graph $\Gamma_{R[x^{-1}]}$.



weight the edges of $\Gamma_{R[x^{-1}]}$ to produce an element of the cycle space. We need to look at the projective versions of the functions; that is, we want to consider the symbol algebra $(\frac{x}{t}, \frac{f\ell_{n+1}}{t^{n+1}})_n$. The weights are expressed in terms of the intersection divisors $C.(L_1 \cdots L_{n+1}), C.((n+1)\infty)$, and $\infty.(L_1 \cdots L_{n+1})$. The intersection divisor $C.(L_1 \cdots L_{n+1})$ is expanded to $\sum_{i=1}^{n+1} C.L_i$, and $C.L_i = 1P_0$ for $1 \leq i \leq n$, while $C.L_{n+1} = 1Q_{n+1}$. This means that, for each i between 1 and n, the edge $\{P_0, C\}$ is assigned a weight of 1 and the edge $\{P_0, L_i\}$ a -1, while $C.L_{n+1} = 1Q_{n+1}$ assigns weights of 1 to $\{Q_{n+1}, C\}$ and -1 to $\{Q_{n+1}, L_{n+1}\}$. Multiple weights assigned to the same edge are summed, so that the total weight so far on the edge $\{P_0, C\}$ is $n \equiv 0$, where we are reducing modulo n. The other intersection divisors are $C.((n+1)\infty) = (n+1)Q_{n+1} = Q_{n+1}$ and $\infty.(L_1 \cdots L_{n+1}) = Q_1 + \cdots + Q_{n+1}$.

We actually have enough information to complete the proof without writing down the entire cycle. Consider, for example, the edge $\{P_0, L_1\}$ of $\Gamma_{R[x^{-1}]}$. The only contribution to its weight is the -1 due to the intersection number $(C, L_1)_{P_0}$. This forces the cycle we are building to have order n in $H_1(\Gamma_{R[x^{-1}]})$, regardless of the weights on any of the other edges. Since we now have a group homomorphism that sends Λ to an element of order n, we conclude that Λ has order a multiple of n in $\mathrm{B}(R[x^{-1}])$. $\square$

With this result, it is enough to construct an Azumaya R-algebra $\mathfrak{A}$ which is split by S such that $\mathfrak{A} \otimes_R R[x^{-1}] \sim \Lambda$. Such an element would necessarily have order n in $\mathrm{B}(S/R)$, but it wouldn't be in the subgroup $\mathrm{B}^{\smile}(S/R)$ because x is not a unit in R. The construction will be discussed briefly; for more detail, see [15].

Recall that the ideal $\mathfrak{q}_1$ of T is defined as $\mathfrak{q}_1 = (x, y - \zeta z)$, and that $\mathrm{div}(x) = \mathfrak{q}_1 + \sigma\mathfrak{q}_1 + \cdots + \sigma^{n-1}\mathfrak{q}_1$. Recall also from Section 1.3 that the group $\mathrm{Cl}(T)$ is isomorphic to the class group of reflexive fractional ideals of T; under this isomorphism, the equation $\mathrm{div}(x) = \mathfrak{q}_1 + \cdots + \mathfrak{q}_n = \mathfrak{q}_1 + \sigma\mathfrak{q}_1 + \cdots + \sigma^{n-1}\mathfrak{q}_1$ becomes

$$Tx = T : (T : \mathfrak{q}_1\sigma(\mathfrak{q}_1)\cdots\sigma^{n-1}(\mathfrak{q}_1)). \tag{2.43}$$

Use this identity in constructing an A-algebra. To simplify notation, refer to the ideal $\mathfrak{q}_1$ as I during this construction. Let ${}_1I_\sigma$ denote the left $T \otimes_A T$-module with action defined on simple tensors as $(a \otimes b) \cdot d = a\sigma(b)d$. For $0 \leq i \leq n-1$, let $\Delta_{\sigma^i}(I) = T^i({}_1I_\sigma) = {}_1I_\sigma^{\otimes i} = {}_1I_\sigma \otimes_T \cdots \otimes_T {}_1I_\sigma$ (where $\Delta_{\sigma^0}(I) = T^0({}_1I_\sigma)$ is taken to be the ring T). This is a left $T \otimes_A T$-module under the action $(a \otimes b) \cdot (d_1 \otimes \cdots \otimes d_i) = a\sigma^i(b) \cdot (d_1 \otimes \cdots \otimes d_i) = a\sigma^i(b)d_1 \otimes d_2 \otimes \cdots \otimes d_i$. Then

$$\Delta(I) = \bigoplus_{i=0}^{n-1} \Delta_{\sigma^i}(I) \tag{2.44}$$

has a left $T \otimes_A T$-module structure induced by that of its summands. In particular, $\Delta(I)$ is an A-module, and we define a multiplication rule to turn it into an A-algebra (it won't be a T-algebra because T will not be contained in its center). If $i + j < n$, define $f_{i,j} : \Delta_{\sigma^i}(I) \otimes_T \Delta_{\sigma^j}(I) \to \Delta_{\sigma^{i+j}}(I)$ to be the natural map:

$$f_{i,j}((d_1 \otimes \cdots \otimes d_i) \otimes (e_1 \otimes \cdots \otimes e_j)) = d_1 \otimes \cdots \otimes d_i \otimes e_1 \otimes \cdots \otimes e_j. \tag{2.45}$$

(If necessary, given a T-module M, we identify $T \otimes_T M$ with M under the natural map $t \otimes m \mapsto tm$.) Now, suppose that $i + j \geq n$, and let $h = n - i$. In this case, define $f_{i,j} : \Delta_{\sigma^i}(I) \otimes_T \Delta_{\sigma^j}(I) \to \Delta_{\sigma^{i+j-n}}(I)$ by the following rule:

$$f_{i,j}((d_1 \otimes \cdots \otimes d_i) \otimes (e_1 \otimes \cdots \otimes e_j)) = d_1\sigma(d_2) \cdots \sigma^{n-1}(e_h)x^{-1} \otimes e_{h+1} \otimes \cdots \otimes e_j. \quad (2.46)$$

The functions $f_{i,j}$ turn $\Delta(I)$ into a G-graded A-algebra which contains $T = \Delta_{\sigma^0}(I)$ as a commutative subalgebra. The set $\Delta_{\sigma^0}(I) + \Delta_{\sigma^1}(I) = T + {}_1I_\sigma$ contains a generating set for $\Delta(I)$ as an A-algebra. In fact, since x and $y - \zeta z$ generate I as a T-module, and T is generated by 1 and z as an A-algebra, $\Delta(I)$ is generated by the elements $1, z \in T = \Delta_{\sigma^0}(I)$ and $x, y - \zeta z \in {}_1I_\sigma = \Delta_{\sigma^1}(I)$. In the language of [19], $\Delta(I)$ is a generalized crossed product algebra with $\{f_{i,j}\}$ as a factor set. In particular, it is an Azumaya A-algebra, and by extension of scalars, $\Delta(I) \otimes_A R$ is Azumaya over R. Since I extends to a projective module over S (by Corollary 2.1.15), $(\Delta(I) \otimes_A R) \otimes_R S$ is a trivial Azumaya S-algebra. Thus, $[\Delta(I) \otimes_A R] \in \mathrm{B}(S/R)$. To prove that this is the Azumaya R-algebra $\mathfrak{A}$ we have been looking for, we need only show

Proposition 2.3.8. *The Azumaya R-algebra $\Delta(I) \otimes_A R$ is Brauer equivalent to $\Lambda = (x, f\ell_{n+1})_n$ when scalars are extended to K. In fact, $(\Delta(I) \otimes_A R) \otimes_R R[x^{-1}] \sim (x, f\ell_{n+1})_n$.*

Proof. The second claim follows immediately from the first; the fact that $R[x^{-1}]$ is regular implies that $\mathrm{B}(K/R[x^{-1}])$ is trivial. The ideal $I \subseteq T$ becomes a rank one projective module (invertible ideal) in S by Corollary 2.1.15. Since S is free of rank n as an R-module, this says that I has rank n over R as well. This implies that $\Delta(I) \otimes_A R$ is a direct sum of n modules which each have rank n, and so $\Delta(I) \otimes_A R$ is an R-module of rank n^2. Tensoring up to K, we see that $\Delta(I) \otimes_A K$ is an n^2-dimensional K-vector space. This matches the dimension of $(x, f\ell_{n+1})_n$ over K.

In $\Delta(I)$, a straightforward computation shows $(z, 0, \ldots, 0)^n = (f\ell_{n+1}, 0, \ldots, 0) \in T$ and $(0, x, 0, \ldots, 0)^n = (x^{n-1}, 0 \ldots, 0) \in T$. We may express the symbol algebra Λ in terms of these elements. Using the bilinear and antisymmetry properties of symbol algebras, write

$$\Lambda = (x, f\ell_{n+1})_n \sim (f\ell_{n+1}, x)_n^{-1} \sim (f\ell_{n+1}, x)_n^{n-1} \sim (f\ell_{n+1}, x^{n-1})_n. \tag{2.47}$$

Let u, v be the generators for the symbol algebra $(f\ell_{n+1}, x^{n-1})_n$ described in Definition 2.3.1. Since u and v are n^{th} roots of $f\ell_{n+1}$ and x^{n-1} respectively, the map

$$\begin{aligned} (f\ell_{n+1}, x^{n-1})_n &\to \Delta(I) \otimes_A K \\ (z, 0, 0, \ldots, 0) &\mapsto u \\ (0, x, 0, \ldots, 0) &\mapsto v \end{aligned}$$

is a well-defined K-algebra homomorphism. Since by [28, Lemma 2.1] an Azumaya algebra over a field is a simple ring, the map described above must in fact be injective. Finally, an injective map between vector spaces of the same dimension is also surjective, and so we have an isomorphism of K-algebras. Thus, $\Delta(I) \otimes_A K$ is Brauer equivalent to Λ, and the two represent the same element of $\mathrm{B}(K)$. □

Summarizing, we have shown that the (class of the) Azumaya R-algebra $\Delta(I) \otimes_A R$ generates the cyclic group $H^1(G, \operatorname{Pic} S)$, and therefore a free $\mathbb{Z}/n\mathbb{Z}$-basis for $\mathrm{B}(S/R)$ is $\{(\ell_i, f\ell_{n+1})_n \mid 1 \leq i \leq n\} \cup \{\Delta(I) \otimes_A R\}$. We would like to point out that while $\Delta(I) \otimes_A R$ is not an element of $\mathrm{B}^{\smile}(S/R)$, the fact that it's in $\mathrm{B}(S/R) \leq \mathrm{B}(R)$ implies that it must be Brauer equivalent to a product of symbol algebras. By computing images of elements in $H_1(\Gamma_{R[x^{-1}]}, \mathbb{Z}/n\mathbb{Z})$, one can show

Theorem 2.3.9. *The Azumaya R-algebra $\Delta(I) \otimes_A R$ is Brauer equivalent to the symbol algebra $(\ell_1, \ell_2 \cdots \ell_n)_n$.* □

The Brauer group of the ring S may also be computed. The calculation is not nearly as straightforward as that for the ring R. It will be useful to assume that in the polynomial $f(x,y) = (y - a_1x)\cdots(y - a_nx)$, none of the a_i are zero. If necessary this may be achieved through an affine change of coordinates. The strategy will be to first compute the Brauer group of the ring $S[x^{-1}]$, which was shown in the proof of Theorem 2.1.14 to be isomorphic to the unique factorization domain $k\left[v, w, \frac{1}{wf(1,v)(w^n+f(1,v))}\right]$. Since $w^n + f(1,v)$ is not linear, we may not immediately proceed as in Proposition 2.3.4. However, [13] provides tools to deal with this situation. For reference, we quote [13, Corollary 3.2], which we will soon make use of.

Lemma 2.3.10. *Let Y be an affine surface and D_1, D_2 curves on Y with no common irreducible component. Then*

$$0 \to \mathrm{B}(Y) \to \mathrm{B}(Y \setminus D_1) \oplus \mathrm{B}(Y \setminus D_2) \to \mathrm{B}(Y \setminus (D_1 \cup D_2)) \to (\mathbb{Q}/\mathbb{Z})^{(d)} \to 0 \quad (2.48)$$

is exact, where d is the number of points in $D_1 \cap D_2$ (not counting multiplicities). □

Proposition 2.3.11. $\mathrm{B}(S[x^{-1}]) \cong (\mathbb{Q}/\mathbb{Z})^{(n^2+1)}$.

Proof. We will make use of the isomorphism $S[x^{-1}] \cong k\left[v, w, \frac{1}{wf(1,v)(w^n+f(1,v))}\right]$. Apply sequence (2.48) with $Y = \mathbb{A}^2$, $D_1 = \mathcal{Z}(wf(1,v))$, and $D_2 = \mathcal{Z}(w^n + f(1,v))$. Then $\mathrm{B}(Y) = \mathrm{B}(\mathbb{A}^2) = 0$, $\mathrm{B}(Y \setminus D_1) \cong \mathrm{B}\left(k\left[v, w, \frac{1}{wf(1,v)}\right]\right)$, and $\mathrm{B}(Y \setminus (D_1 \cup D_2)) \cong \mathrm{B}(S[x^{-1}])$. The intersection of the curves D_1 and D_2 consists precisely of those points where $v = a_i, 1 \le i \le n$, and $w = 0$. Thus, $d = |D_1 \cap D_2| = n$. As in Proposition 2.3.4, [12, Theorem 4] applies to the ring $k\left[v, w, \frac{1}{wf(1,v)}\right]$, and one shows that $\mathrm{B}(Y \setminus D_1) \cong (\mathbb{Q}/\mathbb{Z})^{(n)}$.

It remains to compute $\mathrm{B}(\mathbb{A}^2 \setminus \mathcal{Z}(w^n + f(1,v)))$. Since $\mathbb{A}^2$ is affine and μ is a torsion sheaf, [25, Theorem 15.1] says that $H^3(\mathbb{A}^2, \mu) = 0$, and so [13, Lemma 0.1] says that $\mathrm{B}(\mathbb{A}^2 \setminus D_2) \cong H^3_{D_2}(\mathbb{A}^2, \mu)$. Combining this with [13, Corollary 1.3], we have $\mathrm{B}(\mathbb{A}^2 \setminus D_2) \cong H^1(\widetilde{D}_2, \mathbb{Q}/\mathbb{Z}) \oplus H_1(\Gamma, \mu(-1))$, where $\widetilde{D}_2$ denotes normalization, and Γ is the graph constructed the same way as every graph we have already defined, this time over the ring $k[v, w, (w^n + f(1, v))^{-1}]$. Since $w^n + f(1, v)$ is irreducible, the only vertices of the graph Γ are those corresponding to $w^n + f(1, v)$ itself, the projective line at infinity, and one vertex for each point of intersection of these two curves. If we homogenize with respect to a variable t, then the completed curve $\widetilde{D}_2$ has equation $w^n + f(t, v) = 0$, and the intersection with the line at infinity is given by $w^n + v^n = 0$. This equation has exactly n distinct solutions, and each of these points of intersection has multiplicity 1 by Bézout's Theorem [17, Corollary I.7.8]. There are therefore n vertices in Γ corresponding to these intersection points, each of which has valence 2. So Γ has $2n$ edges and $n + 2$ vertices, so that $H_1(\Gamma, \mu(-1))$ is free over $\mathbb{Q}/\mathbb{Z}$ of rank $2n - (n + 2) + 1 = n - 1$.

We still need the rank of $H^1(\widetilde{D}_2, \mathbb{Q}/\mathbb{Z})$. The theory of abelian varieties implies that this group has rank $2g$, where g is the genus of $\widetilde{D}_2 : w^n + f(t, v) = 0$. This curve can be viewed as a cyclic cover of the projective line $\mathbb{P}^1$ of degree n which ramifies at the n points $[v : t : w] = [a_i : 1 : 0]$, the ramification index at each of these points being n. Thus, the genus g of this curve is given by the Riemann-Hurwitz formula [17, Corollary IV.2.4] and is equal to

$$g = \frac{(n-1)(n-2)}{2}. \tag{2.49}$$

We conclude that $\mathrm{B}(\mathbb{A}^2 \setminus D_2)$ has rank $2 \cdot \frac{(n-1)(n-2)}{2} + (n-1) = (n-1)^2 = n^2 - 2n + 1$.

Finally, putting this information into (2.48) gives the split exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{(n^2-n+1)} \to \mathrm{B}(S[x^{-1}]) \to (\mathbb{Q}/\mathbb{Z})^{(n)} \to 0, \tag{2.50}$$

which forces $\mathrm{B}(S[x^{-1}])$ to have rank $n^2 + 1$. □

Theorem 2.3.12. $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{(n^2-n+1)}$.

Proof. Applying [13, Lemma 0.1] to the variety $\mathrm{Spec}(S)$ and the closed subset $\mathcal{Z}(x)$ produces a short exact sequence

$$0 \to \mathrm{B}(S) \to \mathrm{B}(S[x^{-1}]) \to H^3_{\mathcal{Z}(x)}(\mathrm{Spec}\, S, \mu) \to 0. \tag{2.51}$$

Since the subset $\mathcal{Z}(x)$ of $\mathrm{Spec}\, S$ is nonsingular, $\emptyset = \mathrm{Sing}(\mathcal{Z}(x)) = \mathrm{Sing}(\mathrm{Sing}(\mathcal{Z}(x)))$, and [12, Theorem 1] implies that $H^3_{\mathcal{Z}(x)}(\mathrm{Spec}\, S, \mu) \cong H^1(\mathcal{Z}(x) \setminus \mathrm{Sing}(\mathcal{Z}(x)), \mu) = H^1(\mathcal{Z}(x), \mu)$. The subset of $\mathrm{Spec}(S)$ where $x = 0$ is defined by the equations $y^n + z^n = 0, z \neq 0$, which is a disjoint union of n algebraic tori, and so its ring of regular functions is isomorphic to $\bigoplus_{i=1}^n k[x, x^{-1}]$. Since cohomology commutes with the direct sum,

$$H^1(\mathcal{Z}(x), \mu) \cong H^1\left(\bigoplus_{i=1}^n k[x, x^{-1}], \mu\right) \cong \bigoplus_{i=1}^n H^1(k[x, x^{-1}], \mu) \cong (\mathbb{Q}/\mathbb{Z})^{(n)}. \tag{2.52}$$

Using this and the result of Proposition 2.3.11, sequence (2.51) becomes the split exact sequence

$$0 \to \mathrm{B}(S) \to (\mathbb{Q}/\mathbb{Z})^{(n^2+1)} \to (\mathbb{Q}/\mathbb{Z})^{(n)} \to 0, \tag{2.53}$$

which forces $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{(n^2-n+1)}$. □

CHAPTER 3

THE NON-RATIONAL SINGULARITY ON X

We will now shift our attention to the singularity at the origin on the surface X. For reference, the $n+1$ singularities of X are listed in Corollary 2.1.4. The singularities at the points $P_i = (1, a_i, 0)$ are well studied; in the language of [21], they are rational double points of type A_{n-1}. Each is analytically isomorphic to the singularity at the origin of the surface $\mathcal{Z}(z^n - xy)$.

However, the singularity at the origin of X is different. Unlike the other singularities, this one is *non-rational*, which will soon be made precise. Each of the singularities on the line $x = 1$ is only a point on one generator of $\mathrm{Cl}(X)$, while *every* generator of the divisor class group of X passes through the origin. This makes it relatively easy to determine the class groups of the local rings of X at each of the rational singularities (each is cyclic of order n, generated by $\mathfrak{p}_i T_{\mathfrak{p}_i}$), but it is more difficult to do the same at the origin.

3.1 THE BLOWING-UP $\widetilde{X}$

To define a non-rational singularity, we must first discuss the idea of blowing up a variety at a point. For simplicity, we will only describe the blowing-up of X at the origin $P_0 = O$. Continue to assume that none of the a_i are zero. Following [17, pp. 28-29], we first define the blowing-up of affine space $\mathbb{A}^3$. If we continue to let x, y, z denote the affine coordinates and let u, v, w be coordinates for $\mathbb{P}^2$, then the blowing-up of $\mathbb{A}^3$ is the closed subset $\widetilde{\mathbb{A}^3}$ of $\mathbb{A}^3 \times \mathbb{P}^2$ defined by the equations

$$xv = yu, xw = zu, yw = zv. \tag{3.1}$$

Let $\pi : \widetilde{\mathbb{A}^3} \to \mathbb{A}^3$ denote the restriction of the projection mapping $\mathbb{A}^3 \times \mathbb{P}^2 \to \mathbb{A}^3$. The equations (3.1) imply that, if $O \neq P \in \mathbb{A}^3$, the fiber $\pi^{-1}(P)$ contains exactly one point. Therefore π induces an isomorphism $\widetilde{\mathbb{A}^3} \setminus \pi^{-1}(O) \cong \mathbb{A}^3 \setminus \{O\}$. Over the origin O, the blowing-up equations give no restrictions on u, v, w at all. So the fiber of π which lies over O is isomorphic to the projective plane $\mathbb{P}^2$ with projective coordinates u, v, w.

For the affine variety $X \subseteq \mathbb{A}^3$, the blowing-up at O is defined by taking a restriction of the map π defined above. The *strict transform* $\widetilde{X}$ of X is defined to be the Zariski closure of $\pi^{-1}(X \setminus \{O\})$ inside of $\widetilde{\mathbb{A}^3}$. As above, the restriction $\pi : \widetilde{X} \to X$ induces an isomorphism $\widetilde{X} \setminus \pi^{-1}(O) \cong X \setminus \{O\}$. The *exceptional curve* of the blowing-up is the subset E of the strict blowing-up $\widetilde{X}$ that lies over the origin O in $\mathbb{A}^3$ (in X). We can now define a non-rational singularity, but leave the proof that the singularity we are currently working on is non-rational for a little later (see Theorem 3.1.5).

Definition 3.1.1. The singularity O of the variety X is said to be *rational* if it can be resolved by a finite sequence of blowings up, and the exceptional curve is a tree of irreducible curves, each of which is isomorphic to the projective line $\mathbb{P}^1$.

Remark 3.1.2. The projective line $\mathbb{P}^1$ has genus zero. Since genus is invariant under isomorphism, a sufficient condition that a singularity be *non-rational* is that, after resolving the singularity, at least one irreducible component of the exceptional curve has positive genus.

It is difficult to study the blowing-up variety $\widetilde{X}$, and exceptional curve E, directly. Instead, we use the fact [17, Corollary I.2.3] that the projective plane $\mathbb{P}^2$ is covered by the open sets where u, v, and w are nonzero respectively, and each of these open sets is isomorphic to the affine plane $\mathbb{A}^2$. The open subsets of $\widetilde{X}$ where u, v, w are nonzero, denoted $\widetilde{X}_u, \widetilde{X}_v$, and $\widetilde{X}_w$ respectively, are affine varieties and may be studied in that setting. When $u \neq 0$, equations (3.1) may be written in the form

$$x\frac{v}{u} = y, x\frac{w}{u} = z, yw = zv; \tag{3.2}$$

notice that the third equation is redundant. Making the substitutions (3.2) in the equation defining X gives

$$\begin{aligned} \left(x\frac{w}{u}\right)^n &= f\left(x, x\frac{v}{u}\right)(x-1) \\ x^n\left(\frac{w}{u}\right)^n &= x^n f\left(1, \frac{v}{u}\right)(x-1), \end{aligned} \tag{3.3}$$

where we have used the fact that f is a homogeneous polynomial of degree n. The subset of $\widetilde{\mathbb{A}^3}$ defined by this equation has two clear components. Notice that setting $x = 0$ in equations (3.2) and (3.3) forces $y = z = 0$ while leaving u, v, w free (except, of course, that u still cannot be zero). This is simply the subset of the projective plane $\mathbb{P}^2$ defined by $u \neq 0$, and includes no information about the original variety X. This is not the strict transform of X. The other component of (3.3), given by the equation

$$\left(\frac{w}{u}\right)^n = \left(\frac{v}{u} - a_1\right)\cdots\left(\frac{v}{u} - a_n\right)(x-1), \tag{3.4}$$

is precisely the open affine subset of the strict blowing-up $\widetilde{X}$ defined by $u \neq 0$; that is, (3.4) is the defining equation for $\widetilde{X}_u$. The exceptional curve E may also be studied in affine pieces; a defining equation is determined simply by allowing x, y, and z to be zero. On this open set, it is enough to set $x = 0$, and the open affine subset of E where u is nonzero is given by

$$E_u : \left(\frac{w}{u}\right)^n = -\left(\frac{v}{u} - a_1\right)\cdots\left(\frac{v}{u} - a_n\right). \tag{3.5}$$

The derivation for the defining equations on the other two open sets is similar to that just carried out, and we list the equations here:

$$\widetilde{X}_v : \left(\frac{w}{v}\right)^n = \left(1 - a_1\frac{u}{v}\right)\cdots\left(1 - a_n\frac{u}{v}\right)\left(y\frac{u}{v} - 1\right) \tag{3.6}$$

$$E_v : \left(\frac{w}{v}\right)^n = -\left(1 - a_1\frac{u}{v}\right)\cdots\left(1 - a_n\frac{u}{v}\right) \tag{3.7}$$

$$\widetilde{X}_w : 1 = \left(\frac{v}{w} - a_1\frac{u}{w}\right)\cdots\left(\frac{v}{w} - a_n\frac{u}{w}\right)\left(z\frac{u}{w} - 1\right) \tag{3.8}$$

$$E_w : 1 = -\left(\frac{v}{w} - a_1\frac{u}{w}\right)\cdots\left(\frac{v}{w} - a_n\frac{u}{w}\right). \tag{3.9}$$

Lemma 3.1.3. $\widetilde{X} = \widetilde{X}_u \cup \widetilde{X}_v$.

Proof. As discussed above, it is certainly true that $\widetilde{X} = \widetilde{X}_u \cup \widetilde{X}_v \cup \widetilde{X}_w$. Therefore, it is enough to show that $\widetilde{X}_w \subseteq \widetilde{X}_u \cup \widetilde{X}_v$, or equivalently, $\left[\widetilde{X} \setminus \left(\widetilde{X}_u \cup \widetilde{X}_v\right)\right] \cap \widetilde{X}_w = \emptyset$. An element of $\widetilde{X}$ which is not in $\widetilde{X}_u$ or $\widetilde{X}_v$ must have $u = v = 0$. By the equivalence relation on $\mathbb{P}^2$, $[0 : 0 : 1]$ is the only possible point in $\widetilde{X} \setminus \left(\widetilde{X}_u \cup \widetilde{X}_v\right)$. However, looking at the equation (3.8) defining $\widetilde{X}_w$, it is clear that $[0 : 0 : 1]$ is not a point on this affine subset. Hence $\widetilde{X}_u \cup \widetilde{X}_v$ does contain every point of the blowing-up $\widetilde{X}$. □

Corollary 3.1.4. $E = E_u \cup E_v$. □

From Corollary 3.1.4, we can see precisely what the exceptional curve E is. Homogenizing either (3.5) or (3.7) gives us an equation for the irreducible nonsingular projective plane curve E:

$$E : w^n = -(v - a_1u)\cdots(v - a_nu). \tag{3.10}$$

Using (3.10), the curve E can be viewed as a cover of the projective line $\mathbb{P}^1$ of degree n which ramifies at n points, with ramification index n at each of those points. By the Riemann-Hurwitz formula [17, Corollary IV.2.4], this curve has genus $\frac{(n-1)(n-2)}{2}$. Recall that n is assumed to be at least 3, and so this genus is at least 1. According to [17, Example IV.1.3.5], a complete nonsingular curve is rational if and only if it has genus 0. It follows that E is not rational, and we have shown

Theorem 3.1.5. *The singularity on X at the origin O is non-rational.* □

We would next like to compute the divisor class group of $\widetilde{X}$ and see how it relates to $\mathrm{Cl}(X)$. Since the origin $O \in X$ is a single point, $\mathrm{Cl}(X) \cong \mathrm{Cl}(X \setminus O)$. But under the blowing-up π, $X \setminus O$ is isomorphic to $\widetilde{X} \setminus E$, and from this we see $\mathrm{Cl}(X) \cong \mathrm{Cl}(\widetilde{X} \setminus E)$. Since E is an irreducible curve, the only prime divisor of $\widetilde{X}$ supported on E is E itself. So $\mathrm{Cl}(\widetilde{X})$ is generated by the divisors on $\widetilde{X}$ which lie over the generators of $\mathrm{Cl}(X)$ together with E.

Denote the lines on X which generate the divisor class group by $L_i = \mathcal{Z}(\mathfrak{p}_i)$ and $C_i = \mathcal{Z}(\mathfrak{q}_i)$. Use tildes to denote the strict transform of a divisor under $\pi : \widetilde{X} \to X$. Also important will be the divisor on X where $y = 0$; it is irreducible and given by

$$Y : z^n = (-1)^n a_1 \cdots a_n x^n (x-1). \tag{3.11}$$

For simplicity, denote $\widetilde{X}_u \cap \widetilde{X}_v$ by $\widetilde{X}_{uv}$.

Lemma 3.1.6. $\widetilde{X}_{uv} = \widetilde{X} \setminus \left(\widetilde{C}_1 + \cdots + \widetilde{C}_n + \widetilde{Y}\right)$.

Proof. The open subset $\widetilde{X}_{uv}$ is obtained by removing all points of $\widetilde{X}$ where either u or v is zero. We will compute the divisor of v/u on the affine open subset $\widetilde{X}_u$, and the divisor of u/v on $\widetilde{X}_v$. Since there is no point with $u = v = 0$, this will account for all points that have been removed from $\widetilde{X}$; putting the results together will complete the proof.

Setting $v/u = 0$ in the defining equation $\left(\frac{w}{u}\right)^n = \left(\frac{v}{u} - a_1\right) \cdots \left(\frac{v}{u} - a_n\right)(x-1)$ for $\widetilde{X}_u$ gives us the equation

$$\left(\frac{w}{u}\right)^n = (-1)^n a_1 \cdots a_n (x-1). \tag{3.12}$$

We claim that this is precisely the defining equation of $\widetilde{Y}$. This follows immediately from the form of the blowing-up equations in (3.2); setting $z = \frac{xw}{u}$ in the defining equation for Y results in the equation

$$x^n \left(\frac{w}{u}\right)^n = (-1)^n a_1 \cdots a_n x^n (x-1), \tag{3.13}$$

where again the $x = 0$ component refers to E in its entirety. Canceling x^n from both sides of (3.13) produces the equation (3.12), and hence this is precisely $\widetilde{Y}$. Solving equation (3.4) for $\left(\frac{w}{u}\right)^n - (-1)^n a_1 \cdots a_n (x-1)$ shows that this expression is actually a multiple of v/u in $\mathcal{O}(\widetilde{X}_u)$, and so v/u has valuation 1 at this divisor. It follows that the divisor of v/u on the open set $\widetilde{X}_u$, which will be denoted by $\mathrm{div}_u(v/u)$, is

$$\mathrm{div}_u(v/u) = \widetilde{Y}. \tag{3.14}$$

Next, we set $u/v = 0$ in $\left(\frac{w}{v}\right)^n = \left(1 - a_1 \frac{u}{v}\right) \cdots \left(1 - a_n \frac{u}{v}\right) \left(y\frac{u}{v} - 1\right)$; the equation becomes

$$\left(\frac{w}{v}\right)^n = -1, \tag{3.15}$$

or $(w/v)^n + 1 = 0$, which factors as

$$\left(\frac{w}{v} - \zeta\right)\left(\frac{w}{v} - \zeta^3\right) \cdots \left(\frac{w}{v} - \zeta^{2n-1}\right) = 0. \tag{3.16}$$

The n components of this are precisely the lines $\widetilde{C}_i, 1 \le i \le n$. Since equation (3.6) can be rearranged to show that $(w/v)^n + 1$ is a multiple of u/v, it follows that u/v has valuation 1 at each of these n divisors, and so

$$\mathrm{div}_v(u/v) = \widetilde{C}_1 + \cdots + \widetilde{C}_n. \tag{3.17}$$

Combining the results in (3.14) and (3.17) shows that the open set $\widetilde{X}_{uv}$ is obtained by removing precisely the curve $\widetilde{Y}$ and the lines $\widetilde{C}_1, \ldots, \widetilde{C}_n$ from $\widetilde{X}$. $\square$

The affine coordinate ring (or ring of regular functions) of $\widetilde{X}_{uv}$ is

$$\mathcal{O}(\widetilde{X}_{uv}) = \mathcal{O}(\widetilde{X}_u)\left[\left(\frac{v}{u}\right)^{-1}\right] = \frac{k\,[x, v/u, w/u]}{((w/u)^n - f(1, v/u)(x-1))}\left[\frac{u}{v}\right], \tag{3.18}$$

obtained by inverting v/u in the ring $\mathcal{O}(\widetilde{X}_u)$, which is equivalent to removing those points of $\widetilde{X}_u$ where $v = 0$. It is a straightforward matter to solve the equation $(w/u)^n = f(1, v/u)(x-1)$ for x (after inverting $f(1, v/u)$), and the corresponding map

$$\begin{aligned} \mathcal{O}(\widetilde{X}_{uv})[f(1, v/u)^{-1}] &\to k\left[v/u, w/u, \frac{1}{(v/u)f(1, v/u)}\right] \\ v/u &\mapsto v/u \\ w/u &\mapsto w/u \\ x &\mapsto \frac{(w/u)^n + f(1, v/u)}{f(1, v/u)} \end{aligned} \tag{3.19}$$

is an isomorphism. Using this, we can see that the group of units of the ring $\mathcal{O}(\widetilde{X}_{uv})[f(1, v/u)^{-1}]$ is

$$\mathcal{O}(\widetilde{X}_{uv})[f(1, v/u)^{-1}]^* = k^* \times \left\langle \frac{u}{v} \right\rangle \times \left\langle \frac{v}{u} - a_1 \right\rangle \times \cdots \times \left\langle \frac{v}{u} - a_n \right\rangle. \tag{3.20}$$

The zero set of the line $\frac{v}{u} - a_i$ is $\widetilde{L}_i, 1 \le i \le n$. If we let $D = \widetilde{C}_1 + \cdots + \widetilde{C}_n + \widetilde{L}_1 + \cdots + \widetilde{L}_n + \widetilde{Y}$ for brevity, it follows that the ring $\mathcal{O}(\widetilde{X}_{uv})[f(1, v/u)^{-1}]$ is precisely the ring of regular functions of $\widetilde{X} \setminus D$.

Theorem 3.1.7. $\mathrm{Cl}(\widetilde{X}) \cong \mathbb{Z}^{(n)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-1)}$. *A basis for* $\mathrm{Cl}(\widetilde{X})$ *is obtained by taking as a basis for the torsion-free part* $\widetilde{C}_1, \ldots, \widetilde{C}_{n-1}, \widetilde{L}_1$*, and then* $\widetilde{L}_1 - \widetilde{L}_2, \ldots, \widetilde{L}_1 - \widetilde{L}_n$ *for the torsion subgroup.*

Proof. Since $\mathcal{O}(\widetilde{X} \setminus D)$ is a unique factorization domain, the class group of $\widetilde{X} \setminus D$ is trivial. So Nagata's sequence applied to $\widetilde{X}$ and the open subset $\widetilde{X} \setminus D$ takes the form

$$1 \to \mathbb{G}_m(\widetilde{X}) \to \mathbb{G}_m(\widetilde{X} \setminus D) \xrightarrow{\text{div}} \bigoplus \mathbb{Z}\widetilde{C}_i \oplus \bigoplus \mathbb{Z}\widetilde{L}_i \oplus \mathbb{Z}\widetilde{Y} \to \mathrm{Cl}(\widetilde{X}) \to 0. \tag{3.21}$$

The group of units on $\widetilde{X}$ is trivial; consider the following portion of the Nagata sequence applied to $\widetilde{X}$ and $\widetilde{X} \setminus E$:

$$1 \to \mathbb{G}_m(\widetilde{X}) \to \mathbb{G}_m(\widetilde{X} \setminus E) \to \mathbb{Z} \cdot E \to \mathrm{Cl}(\widetilde{X}). \tag{3.22}$$

Since no nonzero multiple of E may be a principal divisor, the map $\mathbb{Z}E \to \mathrm{Cl}(\widetilde{X})$ is injective. By exactness of (3.22), $\mathbb{G}_m(\widetilde{X}) \cong \mathbb{G}_m(\widetilde{X} \setminus E)$. But $\mathbb{G}_m(\widetilde{X} \setminus E) \cong \mathbb{G}_m(X \setminus O) \cong \mathbb{G}_m(X) = T^* = k^*$.

Hence, the map div of sequence (3.21) is injective on the nontrivial units. In proving Lemma 3.1.6, we have already seen that

$$\mathrm{div}(u/v) = \widetilde{C}_1 + \cdots + \widetilde{C}_n - \widetilde{Y}. \tag{3.23}$$

We must compute the divisor of $\frac{v}{u} - a_i, 1 \leq i \leq n$. On the open set $\widetilde{X}_u$, setting $\frac{v}{u} - a_i = 0$ forces $(w/u)^n = 0$, and so the divisor of $\frac{v}{u} - a_i$ on this open set is $n\widetilde{L}_i$. On $\widetilde{X}_v$, look at $1/\left(\frac{v}{u} - a_i\right) = \frac{u}{v}\left(1 - a_i\frac{u}{v}\right)^{-1}$. Since we have already seen that the divisor of u/v on this set is $\sum \widetilde{C}_i$, the divisor of $\frac{v}{u} - a_i$ is $n\widetilde{L}_i - \sum \widetilde{C}_i$. Combining these results,

$$\mathrm{div}\left(\frac{v}{u} - a_i\right) = n\widetilde{L}_i - \widetilde{C}_1 - \cdots - \widetilde{C}_n, 1 \leq i \leq n. \tag{3.24}$$

Using (3.23) and (3.24), the matrix of the map div is

$$\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}. \tag{3.25}$$

The invariant factors of this matrix are $1, 1, \overbrace{n, \ldots, n}^{n-1}, \overbrace{0, \ldots, 0}^{n}$, which implies that the cokernel of div (which, by the exactness of (3.21) and the first isomorphism theorem, is isomorphic to $\mathrm{Cl}(\widetilde{X})$) is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{(n-1)} \oplus \mathbb{Z}^{(n)}$. $\square$

Remark 3.1.8. It is interesting to note that, on $\widetilde{X}_u$, the line ℓ_1 factors as $x\left(\frac{v}{u} - a_1\right)$ (and similarly on the other open set), so that $\mathrm{div}(\ell_1) = n\widetilde{L}_1 + E$. Therefore, in $\mathrm{Cl}(\widetilde{X})$, $E \sim -n\widetilde{L}_1$. In particular, E does not generate a direct summand of $\mathrm{Cl}(\widetilde{X})$ (it is n times a generator of a direct summand of infinite order).

For convenience, we gather some results on principal divisors on $\widetilde{X}$ obtained in the proof of the preceding theorem. These provide important relations in the group $\mathrm{Cl}(\widetilde{X})$.

Lemma 3.1.9. *As divisors on $\widetilde{X}$,*

$$\begin{aligned} \mathrm{div}(u/v) &= \widetilde{C}_1 + \cdots + \widetilde{C}_n - \widetilde{Y}, \\ \mathrm{div}\left(\frac{v}{u} - a_i\right) &= n\widetilde{L}_i - \widetilde{C}_1 - \cdots - \widetilde{C}_n, 1 \leq i \leq n, \text{ and} \\ \mathrm{div}(\ell_i) &= n\widetilde{L}_i + E, 1 \leq i \leq n. \end{aligned}$$

In particular, $E \sim -n\widetilde{L}_i \sim -\widetilde{C}_1 - \cdots - \widetilde{C}_n \sim -\widetilde{Y}$ *in* $\mathrm{Cl}(\widetilde{X})$. $\square$

3.2 THE PICARD GROUP OF *X*

We would now like to consider a question which has, until now, been ignored: what is the Picard group of the surface X (equivalently, of the ring T)? The divisor class group of both T and S, as well as the Picard group of S, were computed in Theorems 2.1.10 and 2.1.14, and Corollary 2.1.15, respectively. The computation of $\mathrm{Pic}(S)$ required no extra work, as the fact that $\mathrm{Spec}(S)$ is nonsingular was enough to force $\mathrm{Pic}(S) = \mathrm{Cl}(S)$. However, the surface X enjoys no such property. Instead, we use the divisor class groups of the blowing-up $\widetilde{X}$ and the exceptional curve E to say something about $\mathrm{Pic}(X)$. Before looking at these groups, we can still say that $\mathrm{Pic}(X)$ is a torsion-free group.

Lemma 3.2.1. *$\mathrm{Pic}(X)$ is torsion-free. In fact, $\mathrm{Pic}(X) \subseteq \langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1} \rangle \leq \mathrm{Cl}(X)$.*

Proof. The singular points of X are $P_0 = O, P_1, \ldots, P_n$. According to [5, Corollary 2(c)], there is an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i}) \to H^2(X, \mathbb{G}_m) \to \bigoplus_{i=0}^{n} H^2(\mathcal{O}_{X,P_i}, \mathbb{G}_m). \tag{3.26}$$

The divisor class groups of the local rings at the singularities other than O are all isomorphic. Since each is a singularity of type A_{n-1}, the class group of the completed local ring $\widehat{\mathcal{O}}_{X,P_i}, 1 \leq i \leq n$, is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and is generated by the ideal corresponding to $\mathfrak{p}_i$ in the completion. By Mori's Theorem [16, Corollary 6.12], the map $\mathrm{Cl}(\mathcal{O}_{X,P_i}) \to \mathrm{Cl}(\widehat{\mathcal{O}}_{X,P_i})$ is one-to one. But since the ideal $\mathfrak{p}_i$ is already in $\mathrm{Cl}(\mathcal{O}_{X,P_i})$, this map must also be onto, and hence $\mathrm{Cl}(\mathcal{O}_{X,P_i}) \cong \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq n$.

For $1 \leq i, j \leq n$, the line $L_j = \mathcal{Z}(\mathfrak{p}_j)$ passes through the point P_i if and only if $i = j$. Therefore, under the natural map $\mathrm{Cl}(X) \to \mathrm{Cl}(\mathcal{O}_{X,P_i})$, $\mathfrak{p}_j$ maps to the identity element if $i \neq j$. It follows that under the map $\mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i})$ of (3.26), the ideals $\mathfrak{p}_j, 1 \leq j \leq n$ map to linearly independent elements (up to n-torsion) regardless of their images in $\mathrm{Cl}(\mathcal{O}_{X,O})$. Note also that the lines $C_i = \mathcal{Z}(\mathfrak{q}_i)$ pass only through the singularity of X at the origin; these lines do not meet the points $P_j, 1 \leq j \leq n$. So under $\mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i})$ they each map to an $(n+1)$-tuple of the form $(\mathfrak{q}_i, 0, 0, \ldots, 0)$. Thus, an arbitrary element $\sum r_i \mathfrak{p}_i + \sum s_j \mathfrak{q}_j$ of $\mathrm{Cl}(X)$ maps under $\mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i})$ to $(\sum r_i \mathfrak{p}_i + \sum s_j \mathfrak{q}_j, r_1 \mathfrak{p}_1, r_2 \mathfrak{p}_2, \ldots, r_n \mathfrak{p}_n)$. So it is necessary that any element of the kernel of $\mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i})$ have $r_1 \equiv r_2 \equiv \cdots \equiv r_n \equiv 0 \pmod{n}$; that is, it must be in the subgroup of $\mathrm{Cl}(X)$

generated by $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}$. By the exactness of (3.26), we are done. □

Each of the divisors $\widetilde{C}_i, \widetilde{L}_i$ intersect the exceptional curve E in precisely one point. Recalling that E is a curve in $\mathbb{P}^2 = \operatorname{Proj} k[u, v, w]$, the intersections are the points $p_i := \widetilde{L}_i \cap E = [1/a_i : 1 : 0]$ and $q_i := \widetilde{C}_i \cap E = [0 : 1 : \zeta^{2n-2i+1}]$. We can then define a natural map $\mathrm{Cl}(\widetilde{X}) \to \mathrm{Cl}(E)$ by sending $\widetilde{L}_i \mapsto p_i, \widetilde{C}_i \mapsto q_i$, and extending by linearity. We need to know what the subgroup of $\mathrm{Cl}(E)$ generated by $p_1, \ldots, p_n, q_1, \ldots, q_n$ looks like. Unfortunately, we need an extra assumption; namely, that $\mathbb{G}_m(E_u) = k^*$. At least one set of sufficient conditions is known to guarantee this. If $k = \mathbb{C}$, n is an odd prime, and $a_1, \ldots, a_n$ are chosen to be sufficiently general, then this condition will be true (see [11, Proposition 3.6]).

Remark 3.2.2. The map $\mathrm{Cl}(\widetilde{X}) \to \mathrm{Cl}(E)$ described above is well-defined because it can be viewed as a map of Picard groups, using the fact that $\mathrm{Pic}(\cdot)$ is a functor. Since $\mathrm{Sing}(\widetilde{X})$ is finite, $\mathrm{Cl}(\widetilde{X}) = \mathrm{Cl}(\widetilde{X} \setminus \mathrm{Sing}(\widetilde{X})) = \mathrm{Pic}(\widetilde{X} \setminus \mathrm{Sing}(\widetilde{X}))$. Since E is nonsingular, $\mathrm{Cl}(E) = \mathrm{Pic}(E)$. Finally, since none of the singularities of $\widetilde{X}$ lie on E, we have $E \subseteq \widetilde{X} \setminus \mathrm{Sing}(\widetilde{X})$, so that the map $\mathrm{Pic}(\widetilde{X} \setminus \mathrm{Sing}(\widetilde{X})) \to \mathrm{Pic}(E)$ is well-defined.

Proposition 3.2.3. *Suppose $\mathbb{G}_m(E_u) = k^*$. The points $p_1, \ldots, p_n, q_1, \ldots, q_n$ on E generate a subgroup of $\mathrm{Cl}(E)$ which is isomorphic to $\mathbb{Z}^{(n)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-2)}$. The torsion-free summand has as a basis $q_1, \ldots, q_{n-1}, p_1$, and a basis for the torsion summand is $p_1 - p_2, \ldots, p_1 - p_{n-1}$.*

Proof. Recall that the projective plane curve E is defined by the equation $w^n = -(v - a_1 u) \cdots (v - a_n u)$. Note that the points p_i are exactly the points on E where $w = 0$, and the q_i are those points where $u = 0$. Dehomogenizing with respect to u, the affine curve $E_u : \left(\frac{w}{u}\right)^n = -\left(\frac{v}{u} - a_1\right) \cdots \left(\frac{v}{u} - a_n\right)$ can be viewed as a degree n cyclic cover of $\mathbb{A}^1 = \operatorname{Spec} k[v/u]$ which ramifies at the points where $w/u = 0$; these are precisely the points p_i. Let $\pi : E_u \to \mathbb{A}^1$ be the corresponding morphism (which is trivially finite), and for $1 \leq i \leq n$, let $\pi(p_i) = \bar{p}_i$. Then $\bar{p}_i$ is the point of $\mathbb{A}^1$ where $v/u = a_i$.

A definition given in [17, p. 137] describes a map $\pi^* : \mathrm{Div}(\mathbb{A}^1) \to \mathrm{Div}(E_u)$. Under this map, $\pi^*(\bar{p}_i) = np_i$. Let $E_{uw} = E_u \cap E_w$. Then there is a commutative diagram

$$\begin{array}{ccccc} & & H^1(\mathbb{A}^1, \mu_n) & \longrightarrow & H^1(\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}, \mu_n) \\ & & \downarrow & & \downarrow \pi^* \\ 0 & \longrightarrow & H^1(E_u, \mu_n) & \xrightarrow{\rho} & H^1(E_{uw}, \mu_n) \end{array} \tag{3.27}$$

with an exact bottom row. By [11, Proposition 3.10], the kernel of π^* is cyclic of order n and is generated by the class represented by the cyclic covering $\pi : E_u \to \mathbb{A}^1$ of order n (recall that $H^1(\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}, \mu_n)$ classifies the cyclic degree n Galois covers of $\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}$). Since $\mathbb{A}^1$ is simply connected, $H^1(\mathbb{A}^1, \mu) = 0$ [25, Proposition 14.13]. On the affine line $\mathbb{A}^1$, removing a single point is equivalent to inverting the corresponding line in $k[v/u]$. The group of units on $\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}$ is $k^* \times \langle \frac{v}{u} - a_1 \rangle \times \cdots \times \langle \frac{v}{u} - a_n \rangle$. Because finding a cyclic Galois extension of $\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}$ is equivalent to adjoining the n^{th} root of a unit, this implies that $H^1(\mathbb{A}^1 \setminus \{\bar{p}_1, \ldots, \bar{p}_n\}, \mu) \cong (\mathbb{Z}/n\mathbb{Z})^{(n)}$. This forces the image of π^* in $H^1(E_{uw}, \mu_n)$ to be generated by $p_1, \ldots, p_n$ and have rank $n - 1$ over $\mathbb{Z}/n\mathbb{Z}$.

By [11, Proposition 3.11], the image of π^* in diagram (3.27) is contained in the image of ρ. The Nagata sequence for the open subset $E_{uw} \subseteq E_u$ is

$$1 \to k^* \to \mathbb{G}_m(E_{uw}) \xrightarrow{\mathrm{div}} \bigoplus_{i=1}^{n} \mathbb{Z} \cdot p_i \xrightarrow{\chi} \mathrm{Cl}(E_u) \to \mathrm{Cl}(E_{uw}) \to 0. \tag{3.28}$$

The functions w/u and $\frac{v - a_1 u}{u}, \ldots, \frac{v - a_n u}{u}$ are invertible on E_{uw}. Computing divisors in $\mathrm{Div}(E)$,

$$\mathrm{div}(w/u) = p_1 + \cdots + p_n - q_1 - \cdots - q_n \tag{3.29}$$

$$\mathrm{div}\left(\frac{v - a_i u}{u}\right) = np_i - q_1 - \cdots - q_n. \tag{3.30}$$

The points q_i do not exist on E_u (they are the "points at infinity" on this open set). So equations (3.29) and (3.30) imply the relations $p_1 + \cdots + p_n \sim 0$ and $np_i \sim 0$ in $\mathrm{Cl}(E_u)$. Thus, the image of χ in $\mathrm{Cl}(E_u)$ is a $\mathbb{Z}/n\mathbb{Z}$-module ((3.30) forces $\mathrm{im}(\chi)$ to be n-torsion) which can be generated by $n-1$ or fewer elements. Under the assumption $\mathbb{G}_m(E_u) = k^*$, the Kummer sequence for E_u is

$$k^* \xrightarrow{n} k^* \to H^1(E_u, \mu_n) \to \mathrm{Cl}(E_u) \xrightarrow{n} \mathrm{Cl}(E_u). \tag{3.31}$$

This sequence shows that the kernel of multiplication by n on $\mathrm{Cl}(E_u)$ is exactly $H^1(E_u, \mu_n)$. In particular, $\mathrm{im}(\chi) \subseteq H^1(E_u, \mu_n)$; since the image of π^* in $H^1(E_u, \mu_n)$ is already generated by the p_i, these two subgroups must be equal. This shows that $\mathrm{im}(\chi)$ has rank exactly $n-1$ over $\mathbb{Z}/n\mathbb{Z}$. In (3.28), this forces the group of units of E_{uw} to have rank n, and $\frac{w}{u}, \frac{v-a_1u}{u}, \ldots, \frac{v-a_nu}{u}$ to form a basis of $\mathbb{G}_m(E_{uw})/k^*$.

Finally, consider Nagata's sequence for the open subset $E_{uw} \subseteq E$: it is

$$1 \to k^* \to \mathbb{G}_m(E_{uw}) \xrightarrow{\mathrm{div}} \bigoplus_{i=1}^{n} \mathbb{Z} \cdot p_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z} \cdot q_i \to \mathrm{Cl}(E) \to \mathrm{Cl}(E_{uw}) \to 0, \tag{3.32}$$

where the sequence begins with k^* because the group of units on a projective variety is trivial [17, Theorem I.3.4(a)]. A basis for $\mathbb{G}_m(E_{uw})$ has been computed, so the map div is given by formulas (3.29) and (3.30). As in (3.25), we may write down the matrix of this map, and we compute the invariant factors to be 1 (2 times), n ($n-2$ times), and 0 (n times). $\square$

Lemma 3.2.4. *On the exceptional curve E,* $\mathrm{div}(w/u) = p_1 + \cdots + p_n - q_1 - \cdots - q_n$ *and* $\mathrm{div}\left(\frac{v-a_iu}{u}\right) = np_i - q_1 - \cdots - q_n$. $\square$

Let $U = X \setminus \mathcal{Z}(x-1)$, the open subset of the surface X obtained by removing all of those points where $x = 1$. Since every singularity of X except that at the origin lies on the line $x = 1$, the only singularity of U is at O. Since $O \in U$, $\mathcal{O}_{U,O} = \mathcal{O}_{X,O}$. Each of the lines C_i, L_i goes through the origin, which means that they each represent a class in the divisor class group of X, U, and $\mathcal{O}_{X,O}$, and in fact, the class group of any of these is generated by the C_i and L_i. Under the map $\operatorname{Cl}\widetilde{X} \to \operatorname{Cl} E$, the divisor E maps to $-np_1$ by Lemma 3.1.9. So we may define a map from the class group of any of these three to $\operatorname{Cl}(E)/\langle np_1\rangle$ by sending L_i to p_i and C_i to q_i. The image of any of these groups inside of $\operatorname{Cl}(E)$ is the same; all are equal to the image of $\operatorname{Cl}(\widetilde{X}) \to \operatorname{Cl}(E) \to \operatorname{Cl}(E)/\langle np_1\rangle$. Denote the image of $\operatorname{Cl}(\widetilde{X})$ in $\operatorname{Cl}(E)$ by H; by construction, this is the same subgroup of $\operatorname{Cl}(E)$ determined in Proposition 3.2.3.

Lemma 3.2.5. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\cdot E & \longrightarrow & \operatorname{Cl}(\widetilde{X}) & \longrightarrow & \operatorname{Cl}(X) & & \\
 & & \downarrow{=} & & \downarrow{\text{onto}} & & \downarrow{\text{onto}} & & \\
0 & \longrightarrow & \mathbb{Z}\cdot E & \longrightarrow & \operatorname{Cl}(\widetilde{U}) & \longrightarrow & \operatorname{Cl}(U) & \overset{\text{onto}}{\searrow} & \\
 & & \downarrow{=} & & \downarrow & & \downarrow{\varphi} & & \\
0 & \longrightarrow & \mathbb{Z}\cdot np_1 & \longrightarrow & \operatorname{Cl}(E) & \longrightarrow & \operatorname{Cl}(E)/\langle np_1\rangle & \hookleftarrow & H/\langle np_1\rangle
\end{array} \tag{3.33}$$

where in the first column we have used the fact that E maps to $-np_1$ in $\operatorname{Cl}(E)$ *(see Remark 3.1.8). The vertical maps* $\operatorname{Cl}(\widetilde{X}) \to \operatorname{Cl}(\widetilde{U})$ *and* $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ *are the maps of Nagata's sequence, and the horizontal maps are the natural maps. The group* $H/\langle np_1\rangle \leq \operatorname{Cl}(E)/\langle np_1\rangle$ *is the image of φ.* □

On the surface X, the divisor of the function z is $\operatorname{div}(z) = \mathfrak{p}_1 + \cdots + \mathfrak{p}_{n+1}$, where $\mathfrak{p}_{n+1} = (x-1, z)$ is the ideal of the line $L_{n+1} = \mathcal{Z}(x-1, z)$. So the ideal $(x-1, z)$ could be taken as a basis element of the n-torsion subgroup of $\operatorname{Cl}(X)$; for example, $\mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_{n+1}$ is a basis. With this observation, the ideal class group of $\operatorname{Cl}(U)$ is obtained by simply losing one direct summand from $\operatorname{Cl}(X)$, and therefore $\operatorname{Cl}(U) \cong \mathbb{Z}^{(n-1)} \oplus (\mathbb{Z}/n\mathbb{Z})^{(n-1)}$.

Theorem 3.2.6. *If* $\mathbb{G}_m(E_u) = k^*$, *then* $\operatorname{Pic}(X) = 0$.

Proof. By Proposition 3.2.3 and the fact that $\mathbb{Z}p_1$ is a torsion-free direct summand of H, $\mathrm{Cl}(U)$ is isomorphic to $H/\langle np_1\rangle$ as abelian groups. Since $\varphi : \mathrm{Cl}(U) \to H/\langle np_1\rangle$ is onto, φ is an isomorphism. Then the commutative triangle

$$\begin{array}{ccc} \mathrm{Cl}(U) & \xrightarrow{\cong} & H/\langle np_1\rangle \\ {\scriptstyle\text{onto}}\big\downarrow & \nearrow{\scriptstyle\text{onto}} & \\ \mathrm{Cl}(\mathcal{O}_{X,O}) & & \end{array} \tag{3.34}$$

implies that each of the three maps must be an isomorphism, and in particular, $\mathcal{O}_{U,O} = \mathcal{O}_{X,O} \cong \mathrm{Cl}(U)$. The map $\mathrm{Cl}(X) \to \mathrm{Cl}(U) \cong \mathrm{Cl}(\mathcal{O}_{X,O})$ is one-to-one when restricted to the subgroup $\langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}\rangle$. So no element of $\langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}\rangle$ can be in the kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(\mathcal{O}_{X,O})$, and since any linear combination of the $\mathfrak{q}_i$ maps to zero at every other component anyway, we can conclude that no element of $\langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}\rangle$ can be in the kernel of $\mathrm{Cl}(X) \to \bigoplus_{i=0}^{n} \mathrm{Cl}(\mathcal{O}_{X,P_i})$. Thus, $\mathrm{Pic}(X) \cap \langle \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}\rangle$ is trivial. Apply Lemma 3.2.1. □

Corollary 3.2.7. *If $k = \mathbb{C}$, n is an odd prime, and the a_i are chosen to be sufficiently general, then the Picard group of X is trivial.*

Proof. By [11, Proposition 3.6], the hypothesis of Theorem 3.2.6 is satisfied. □

Remark 3.2.8. It is interesting that choosing the a_i to be sufficiently general (among other conditions) is enough to guarantee that $\mathrm{Pic}(X)$ is trivial. The Picard group of X is always contained in the subgroup of $\mathrm{Cl}(X)$ generated by the lines C_i which lie over $x = 0$ in $\mathbb{A}^2$. However, the equations for these lines actually do not depend on the choice of a_i at all. So, at first glance, it seems that the a_i should have nothing to do with the Picard group of X. As we have seen, it turns out that they are very much related.

CHAPTER 4

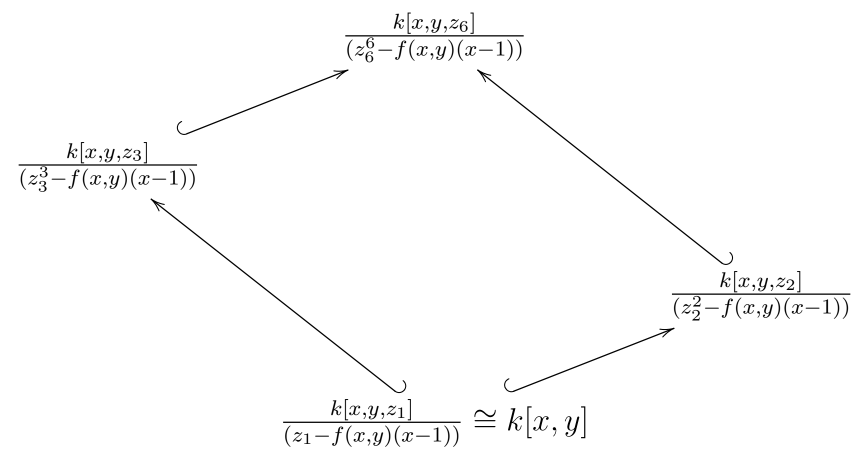
FUTURE WORK

There are many questions still to be answered about this family of surfaces. A few have already been considered in joint work with Dr. Timothy Ford. For example, we have shown that the surface defined by $z^n = (y^n - x^n)(x - 1)$, which can be viewed as being in our family of surfaces by taking the a_i to be the n^{th} roots of unity in k, has nontrivial Picard group [14]. This shows that the Picard group of X really does depend on the choice of the elements $a_i \in k$.

In [3], we were able to show that the Brauer group of the local ring $\mathcal{O}_{X,O}$ of X at the origin contains a subgroup isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(2g)}$, where $g \geq 1$ is the genus of the exceptional curve E. This subgroup consists of Azumaya algebras that are *generically trivial*; that is, they are in the kernel of the natural mapping to $\mathrm{B}(K)$. This answers a question originally posed by Grothendieck about constructing a surface with nontrivial relative Brauer group. However, we do not yet have a description of just what the Azumaya algebras in $\mathrm{B}(K/\mathcal{O}_{X,O})$ look like. It would be interesting to know explicitly how to construct these algebras.

A question that is currently wide open is whether or not there is anything functorial about the embedding of the non-rational singularity in the rational surface X. That is, is it possible to come up with a procedure that may be iterated, producing a rational surface with finitely many non-rational singularities? So far, we have not even written down a surface resembling ours which contains two such singularities.

I have also recently become interested in extending the definition of this family of surfaces, and considering diagrams of morphisms between them (equivalently, of ring homomorphisms). That is, consider the surfaces defined by $z^n = (y - a_1x)\cdots(y - a_mx)(x-1)$, where $n \geq 2, m \geq 3$, and $n \mid m$. Many of the calculations performed in this dissertation can be performed on this surface as well, with the added difficulty that if $m \neq n$, then blowing up this surface at the origin produces a surface which is not normal, requiring an extra step of normalization. However, in some sense, the information about these more general surfaces is "contained" in the information about the surface studied in this thesis. For example, say $m = 6$, with divisors $1, 2, 3, 6$. We easily construct a diagram of rings



where we identify $z_d = z_6^{6/d}$ for each divisor d of 6. What, if any, special properties such diagrams may have is unknown.

BIBLIOGRAPHY

[1] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[3] Djordje N. Bulj, Timothy J. Ford, and Drake M. Harmon. Generically trivial Azumaya algebras on a rational surface with a non-rational singularity. to appear, 2012.

[4] S. U. Chase, D. K. Harrison, and Alex Rosenberg. Galois theory and Galois cohomology of commutative rings. *Mem. Amer. Math. Soc. No.*, 52:15–33, 1965.

[5] F. R. DeMeyer and T. J. Ford. Nontrivial, locally trivial Azumaya algebras. In *Azumaya algebras, actions, and modules (Bloomington, IN, 1990)*, volume 124 of *Contemp. Math.*, pages 39–49. Amer. Math. Soc., Providence, RI, 1992.

[6] F. R. DeMeyer and Edward Ingraham. *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.

[7] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.

[8] Alan H. Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math. (2)*, 25(1-2):131–163, 1979.

[9] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[10] T. J. Ford. The Brauer group of an affine cone. *J. Pure Appl. Algebra*, 155(1):29–40, 2001.

[11] Timothy J. Ford. The group of units on an affine variety. preprint, available at math.fau.edu/ford/publications.

[12] Timothy J. Ford. On the Brauer group of $k[x_1, \cdots, x_n, 1/f]$. *J. Algebra*, 122(2):410–424, 1989.

[13] Timothy J. Ford. On the Brauer group of a localization. *J. Algebra*, 147(2):365–378, 1992.

[14] Timothy J. Ford and Drake M. Harmon. The Brauer group of an affine rational surface with a non-rational singularity. Under review, 2012.

[15] Timothy J. Ford and Drake M. Harmon. The relative Brauer group and generalized cross products for a cyclic covering of affine space. Under review, 2012.

[16] Robert M. Fossum. *The divisor class group of a Krull domain.* Springer-Verlag, New York, 1973.

[17] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977.

[18] Thomas W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1980. Reprint of the 1974 original.

[19] Teruo Kanzaki. On generalized crossed product and Brauer group. *Osaka J. Math.*, 5:175–188, 1968.

[20] Max-Albert Knus and Manuel Ojanguren. *Théorie de la descente et algèbres d'Azumaya.* Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 389.

[21] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969.

[22] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[23] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series.* Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[24] James S. Milne. *Étale cohomology.* Princeton University Press, Princeton, N.J., 1980.

[25] James S. Milne. Lectures on etale cohomology (v2.20), 2012. Available at www.jmilne.org/math/.

[26] I. Reiner. *Maximal orders.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1975. London Mathematical Society Monographs, No. 5.

[27] Joseph J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[28] David J. Saltman. *Lectures on division algebras*, volume 94 of *CBMS Regional Conference Series in Mathematics*. Published by American Mathematical Society, Providence, RI, 1999.

[29] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vols. I and II*. Springer-Verlag, New York, 1975.