A COMPUTATIONAL APPROACH TO ANALYZING GLOBAL DYNAMICS

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MOTIVATION

1) Conley's Decomposition Theorem asserts that every dynamical system can be separated into a minimal (chain) recurrent set and its complement on which the dynamics is gradient-like, i.e. there exists a Lyapunov function which is strictly decreasing along orbits. The chain recurrent set can then be divided into components which are partially-ordered (forming a poset) by the existence of connecting orbits between them [1].
2) The distributive lattice structures of attractors and repellers in dynamical systems allows for an algebraic treatment of gradient-like dynamics in general global dynamical systems. The separation of these algebraic structures from underlying topological structure is the basis for the development of algorithms to manipulate those structures [3].
3) The algebraic structure of a finite sublattice of attractors can be captured by lifting it up to the lattice of forward invariant sets or attracting neighborhoods, which are computationally accessible [3].

PROBLEMS AND OBJECTIVES

1. We present an efficient algorithm, utilizing the algebraic structure of lattices of attractors, for constructing piecewise constant Lyapunov functions for dynamics generated by a continuous nonlinear map defined on a compact metric space. We provide a memory efficient data structure for storing nonuniform grids on which the Lyapunov function is defined [1].
2. We provide necessary and sufficient conditions for the existence of the lifting of sublattices of attractors, which are computationally less accessible, to lattices of forward invariant sets and attracting neighborhoods, which are computationally accessible. We will do this starting with the general setting of posets. We will show that the conditions we provide address more general setting than the condition of 'spaciousness' given in [3].
3. We provide an algorithm to check for the existence of such a lift and the algorithmic construction of such a lift, if the lift exists; especially in the context of dynamical systems.

RESULTS 1: COMPUTATION OF LYAPUNOV FUNCTIONS

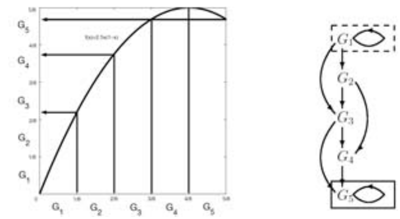


Figure 1: Directed graph representation of the minimal multivalued map $\mathcal{F}: \mathcal{X} \rightrightarrows \mathcal{X}$ for the logistic map $f(x) = 2.5x(1-x)$ on the grid $\mathcal{X} = \{G_1, \ldots, G_5\}$ obtained by dividing the domain $X = [0, 5/8]$ into five equal-length subintervals. By definition, $\mathcal{F}(G) := \{H \in \mathcal{X} | H \cap f(G) \neq \emptyset\}$. The pair $(\{G_1\}, \{G_2\})$ is an attractor-repeller pair.

A continuous function $V_A \colon X \to [0,1]$ that is strictly decreasing for $x \in X \setminus (A \cup A^*)$ and satisfies $V_A(A) = 0$ and $V_A(A^*) = 1$ is called a *Lyapunov function for the attractor repeller pair* (A, A^*). Making use of the fact that $\mathrm{Att}(X, f)$ is at most countable, we call *Lyapunov function for* f to mean any function of the form

$$V_{\mathsf{A}}(x) := \sum_{A_n \in \mathsf{A}} \beta_n V_{A_n}(x) \tag{1}$$

where A is a subset of $\mathrm{Att}(X, f)$, $\beta_n > 0$ and $\sum \beta_n$ is bounded.

Theorem 0.1. Let S be a compact, invariant set, let M be a Morse decomposition for S, labeled by a poset $(\mathsf{P}, \leq)$, and let A be the corresponding sublattice of attractors whose join irreducible elements are A_p. For each $p \in \mathsf{P}$ let $\beta_p > 0$ with $\sum_{p \in \mathsf{P}} \beta_p = 1$ and V_p be a Lyapunov function for the attractor-repeller pair (A_p, A_p^*). Then

$$V = \sum_{p \in \mathsf{P}} \beta_p V_p \tag{2}$$

is a Lyapunov function for $(S, \mathsf{M}, \mathsf{P}, \leq)$.

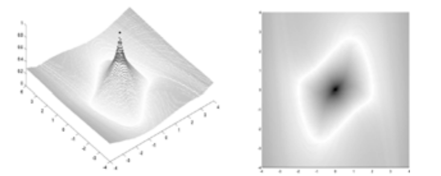


Figure 2: Lyapunov function for the van der Pol ODE $\dot{x} = y$, $\dot{y} = -x + (1-x^2)y$ with a stable periodic orbit and one unstable equilibrium point at $(0,0)$, over $[-4,4] \times [-4,4]$. The grayscale represents the height or the value of the lyapunov function.

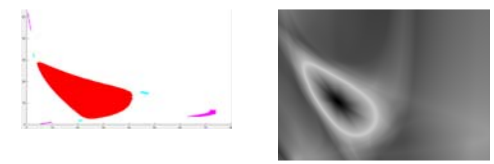
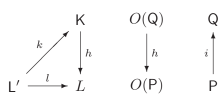


Figure 3: Lyapunov function for two dimensional overcompensatory Leslie model $f(x, \lambda) = ((\theta_1 x_1 + \theta_2 x_2)e^{-\phi(x_1+x_2)}, px_1)$ over $[0, 80] \times [0, 55]$. The grayscale represents the height or the value of the lyapunov function.

RESULT 2: ORDER-EXTENSION THEOREM

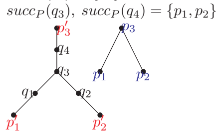
Let K, L be bounded, distributive lattices, let $h : \mathsf{K} \to \mathsf{L}$ and $l : \mathsf{L}' \to \mathsf{L}$ be lattice homomorphisms. We call a lattice homomorphism $k : \mathsf{L}' \to \mathsf{K}$ a lift of l through h, if $h \circ k = l$ i.e. the following diagram commutes:



Definition 0.3 (P-predecessor & P-successor)**.** Let P, Q be finite posets and $i : P \hookrightarrow Q$ be an order embedding. An element $p \in P$ is a *successor in* P, or a *P-successor*, of $q \in Q$ if $i(p) < q$ and whenever $i(p) \leq s < q$ for $s \in i(P)$ we have $i(p) = s$, i.e. there is no other element of $i(P)$ between $i(p)$ and q. Similarly, an element $r \in P$ is a *predecessor in* P, or a *P-predecessor*, of $q \in Q$ if $q < i(r)$ and whenever $q < s \leq i(r)$ for $s \in i(P)$ we have $s = i(r)$, i.e. there is no other element of $i(P)$ between q and $i(r)$. The set of all P-predecessors of q is denoted by $pred_P(q)$, and the set of all P-successors of q is denoted by $succ_P(q)$.

Here, in the following picture representing a poset (or a part of a poset), we have elements $q, q_1, q_2, q_3, q_4 \in Q \setminus i(P)$ and $p_1', p_2', p_3' \subset i(P) \subset Q$ the respective images under the inclusion map $i : P \hookrightarrow Q$ of $p_1, p_2, p_3 \in P$. It is easy to see that:
$pred_P(q_1), pred_P(q_2), pred_P(q_3)$,
$pred_P(q_4) = \{p_3\}$

$succ_P(q_1) = \{p_1\}$
$succ_P(q_2) = \{p_2\}$
$succ_P(q_3)$, $succ_P(q_4) = \{p_1, p_2\}$



Theorem 0.4 (Order-extension Theorem)**.** Let $i : \mathsf{P} \hookrightarrow \mathsf{Q}$ be an order-embedding. Suppose $\sigma : \mathsf{Q} \to \mathsf{P}$ is a surjective extension of $i^{-1} : i(\mathsf{P}) \to P$. Then σ is an order-extension of i^{-1} if and only if σ is order-preserving on every chain of Q contained in $\mathsf{Q} \setminus i(\mathsf{P})$ and every element $q \in \mathsf{Q} \setminus i(\mathsf{P})$ satisfies exactly one of the following conditions (a)-(e):

(a) $succ_P(q)$ or $pred_P(q)$ is a singleton;
(b) $|succ_P(q)| > 1$, $|pred_P(q)| = 0$, and $\sigma(q) > s$ for every $s \in succ_P(q)$;
(c) $|succ_P(q)| > 1$, $|pred_P(q)| > 1$, and $s < \sigma(q) < r$ for every $s \in succ_P(q)$, $r \in pred_P(q)$;
(d) $|succ_P(q)| = 0$, $|pred_P(q)| > 1$, and $\sigma(q) < r$ for every $r \in pred_P(q)$;
(e) $succ_P(q) = pred_P(q) = \emptyset$.

Lemma 0.5. For any $q \in Q \setminus i(P)$, $succ_P(q)$ and $pred_P(q)$ are antichains.

Remark 0.6. By the definition of $succ_P(q)$ and $pred_P(q)$, every $q \in Q$, $\sigma(q)$ satisfies $s \leq \sigma(q)$ for all $s \in succ_P(q)$ and $\sigma(q) \leq p$ for all $p \in pred_P(q)$.

ALGORITHMS FOR ORDER-EXTENSION AND LIFT

Assume $succ_P(q) \neq \emptyset$. Then we can check with certainty whether or not such a surjective order extension $\sigma : Q \to P$ of $i^{-1} : i(P) \to P$ exists and if it exists, we will construct it along step by step.
For every $p' \in i(P) \subset Q$, define $\sigma(p') = i^{-1}(p') = p \in P$. Let $q \in Q \setminus i(P)$. Then we check for the following exhaustive cases, in order:

(i) If $|succ_P(q)| = 1$ and $succ_P(q) = \{p\}$, say, then we define $\sigma(q) = p$. Otherwise if $|pred_P(q)| = 1$ and $pred_P(q) = \{p\}$, say, then we define $\sigma(q) = p$.

(ii) Otherwise if $|succ_P(q)| > 1$ and $pred_P(q) = \emptyset$, we test for the non-existence; if the test fails, we continue by assigning 'appropriate' $p \in \mathsf{P}$ (as shown in the following algorithms (7)-(9)) for $\sigma(q)$, otherwise we stop at this point concluding that the *order*-extension doesn't exist.

(iii) Otherwise if $|succ_P(q)| > 1$ and $|pred_P(q)| > 1$, we test for the non-existence; if the test fails, we continue by assigning 'appropriate' $p \in \mathsf{P}$ (as shown in the following algorithm (7)-(9)) for $\sigma(q)$, otherwise we stop at this point concluding that the *order*-extension doesn't exist.

We also provide the proof that the algorithm works. Sample Algorithms:

Algorithm 1 Checks whether the map σ exists for sure or it is inconclusive and requires further analysis

Require: The posets Q and P; $i : P \to Q$; the $PredList$ and $SuccList$ obtained from algorithms 1 and 2.
Ensure: The certainty/uncertainty of the existence of σ

```
for all q ∈ Q\i(P) do
    if |SuccList[q]| = 0 or (|SuccList[q]| ≠ 1 and |PredList[q]| ≠ 1) then
        print INCONCLUSIVE
        EXIT
    end if
end for
print σ exists for sure
```

Algorithm 2 Constructs the map σ after algorithm 5 verifies its existence - SIMPLE CASE

Require: The posets Q and P; $i : P \to Q$; the $PredList$ and $SuccList$ obtained from algorithms 2 and 4; The certainty of the existence of σ from algorithm 3.
Ensure: The map σ.

```
σ ← [−1] ∗ |Q| {Initializing σ as a list of |Q| copies of -1.}
for all q ∈ Q do
    if q ∈ i(P) then
        σ[q] ← q
    else if q ∈ Q\i(P) and |SuccList[q]| = 1 then
        σ[q] ← SuccList[q]
    else
        σ[q] ← PredList[q]
    end if
end for
return σ
```

IMPLEMENTATION WITH $\mathsf{Q} = \mathsf{SCC}(\mathcal{X}, \mathcal{F})$ AND $\mathsf{P} = \mathsf{Rec}(\mathcal{X}, \mathcal{F})$

Here, $\mathsf{SCC}(\mathcal{X}, \mathcal{F})$ = Strongly Connected Components
and $\mathsf{Rec}(\mathcal{X}, \mathcal{F})$ = Recurrent Components (Or, Morse Decomposition) of the Combinatorial dynamical system $\mathcal{F}$.

a) **Leslie Model**: We ran the python code in four different levels of resolutions (or $diam(\mathcal{X})$) for two dimensional version of an overcompensatory Leslie population model given by a map $f : \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$ defined by $f(x, \lambda) = ((\theta_1 x_1 + \theta_2 x_2)e^{-\phi(x_1+x_2)}, px_1)$
Note: It took less than a minute in all the four levels of resolutions to create the poset structures and the lift.

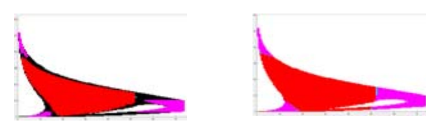


Figure 4: Recurrent components and the lift at the resolution level of 5193 grid elements

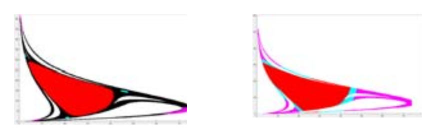


Figure 5: Recurrent components and the lift at the resolution level of 55163 grid elements

b) **An Example with a 'Split'**: We constructed a dynamical system, governed by the following system of ODE, for which an outer approximation can be constructed in such a way that there is a grid element (precisely, a $q \in \mathsf{Q} \setminus i(\mathsf{P})$) that satisfies condition (c) of our main theorem I:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 - x^2 \\ y(1 - y^2) \end{bmatrix}$$

Theorem 0.2 (W.D. Kalies, K. Mischaikow and R.C. Vandervorst)**.** Let K and L be a bounded, distributive lattices, and let $h : \mathsf{K} \to \mathsf{L}$ be a lattice epimorphism. If h is spacious and $h^{-1}(1) = 1$, then every lattice embedding $s : O(\mathsf{P}) \to \mathsf{L}$, with P is finite, admits a lift.

$\forall q \in \mathsf{Q} \setminus i(\mathsf{P})$

- "spaciousness" property and the condition $|pred_P(q)| \leq 1$;
- equivalence of the conditions $h^{-1}(1) = 1$ and $|pred_P(q)| \neq \emptyset$;
- equivalence of the conditions $h^{-1}(0) = 0$ and $|succ_P(q)| \neq \emptyset$;
- "spaciousness" condition is a stronger condition to ask for the lift to exist than necessary, in addition to being "not-so-easy" to visualize. We will illustrate by an example.

CONCLUSION

- Computation of an approximate global Lyapunov function;
- necessary and sufficient conditions for the lift of lattices of attractors;
- algorithm to check the existence and to construct the lift.

REFERENCES

1) H. Ban and W.D. Kalies. A computational approach to Conley's decomposition theorem. 2006.
2) A. Goullet, S. Harker, W.D. Kalies, D. Kasti, and K. Mischaikow. Efficient computation of lyapunov functions for morse decompositions. 2014.
3) B. A Davey and H. A Priestley. Introduction to lattices and order. *Cambridge University Press*, pages xii+298, 2002.
4) W. D. Kalies, K. Mischaikow, and R. C. A. M. VanderVorst. Lattice structures for attractors I & II. 2014.

FUTURE RESEARCH TOPICS

- Lifting of the lattices of Lyapunov functions;
- the case with no finest morse decomposition;
- index filtration, connection matrix.