A Robust Converse Lyapunov Theorem for Systems with Disturbances Taking Values in a Banach Space

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Background

- A Lyapunov function guarantees stability of a system.
- But on the other hand stability does not imply existence of a Lyapunov function.
- Generally existence depends on the system and and the type of stability.
- Our work extends Converse Lyapunov theory to robustly globally asymptotically control stable systems with disturbances from any Banach space.
- **Importance**: To establish robust stability of a wide class of systems affected by external signals, Applications to systems with delays

General setup

System Σ;

$$\dot{x}(t) = f(x(t), \mu(t), \nu(t)), \qquad x(0) = \xi,$$

- $\mathscr{B}$ an arbitrary Banach space
- $B_{\mathscr{B}}$, $B_{\mathbb{R}^m}$ closed unit balls in $\mathscr{B}$ and $\mathbb{R}^m$

$x(t) \in \mathbb{R}^n$, $\mu(t) \in B_{\mathbb{R}^m}$, $\nu(t) \in B_{\mathscr{B}}$ and $f : \mathbb{R}^n \times B_{\mathbb{R}^m} \times B_{\mathscr{B}} \to \mathbb{R}^n$ with;

1. f is continuous;
2. f is locally Lipschitz on $\xi \in \mathbb{R}^n$, uniformly in (d, v) and
3. $f(0, d, v) = 0$ for all $(d, v) \in B_{\mathscr{B}} \times B_{\mathbb{R}^m}$.

Motivation

- $\theta > 0$ given and $\varphi : [-\theta, T) \to \mathbb{R}^n$, $(T > 0)$
- φ_t defined on $[-\theta, 0]$ by $\varphi_t(s) = \varphi(t + s)$.

An interconnected system affected by delays in the feedback loop:

$$\dot{x}(t) = f(x(t),\ z_t,\ u_1(t)),$$
$$\dot{z}(t) = g(z(t),\ x_t,\ u_2(t)),$$

Treat $z_t \in C[-\theta, 0]$ as an external input. Then

$$\dot{x}(t) = f(x(t), v(t), u(t))$$

has the desired form with $v(t) = z_t$.

rGAS property

The system Σ is said to be *robustly globally asymptotically stable* (rGAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|\phi(t, \xi, \mu, \nu)| \le \beta(|\xi|, t) \qquad \forall, t \ge 0,$$

for all $\xi \in \mathbb{R}^n$, all $\mu \in \mathcal{U}$, and all $\nu \in \Omega$.

Smooth Lyapunov Function

A function $V : \mathbb{R}^n \to \mathbb{R}$

1. V is smooth
2. $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for any $\xi \in \mathbb{R}^n$,
$$\alpha_1(|\xi|) \le V(\xi) \le \alpha_2(|\xi|) \quad \forall \xi;$$
3. $\exists \alpha_3$ continuous, positive definite function such that $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, $\forall d \in B_{\mathbb{R}^m}$ and $\forall v \in B_{\mathscr{B}}$,
$$\mathscr{L}_{f_{d,v}} V(\xi) \le -\alpha_3(|\xi|) \quad \forall (\xi, d, v),$$
where $f_{d,v}(\xi) = f(\xi, d, v)$.

Main Result

The system Σ is rGAS if and only if it admits a smooth Lyapunov function.

Major steps

Sufficiency is a standard Lyapunov type argument. Necessity is the challenging part.

- By Sontag's lemma $\exists \tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ with respect to β.
- Define $V_1 : \mathbb{R}^n \to \mathbb{R}$ by
$$V_1(\xi) = \sup_{t \ge 0, \mu \in \mathcal{U}, \nu \in \Omega} \tilde{\alpha}_1(|\phi(t, \xi, \mu, \nu)|)\, e^t.$$
- Obtain $\forall \xi \in \mathbb{R}^n$, $\tilde{\alpha}_1(|\xi|) \le V_1(\xi) \le \tilde{\alpha}_2(|\xi|)$
- showing that V_1 is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$:This demands the majority of the work due to missing compactness properties.
- $\forall \xi \in \mathbb{R}^n$, $\forall \mu \in \mathcal{U}$, $\forall \nu \in \Omega$, and for all $t \in [0, \infty)$,
$$V_1(\phi(t, \xi, \mu, \nu)) \le V_1(\xi)\, e^{-t}$$

Notation and Terminology

- $\alpha \in \mathcal{K}_\infty$ if $\alpha : [0, \infty) \to [0, \infty)$, continuous, strictly increasing, $\alpha(0) = 0$ and if $\lim_{r \to \infty} \alpha(r) = \infty$.
- $\beta \in \mathcal{KL}$ if $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous and
  - for every $s \in [0, \infty)$, $\alpha_s(\cdot) = \beta(\cdot, s)$ belongs to class $\mathcal{K}_\infty$, and
  - for every $r \in [0, \infty)$, $\gamma_r(\cdot) = \beta(r, \cdot)$ is decreasing and $\lim_{s \to \infty} \gamma_r(s) = 0$.

For a locally Lipschitz vector field g defined on $\mathbb{R}^n$ and a differentiable function Φ, we follow the convention

$$\mathscr{L}_g \Phi(x) = \lim_{t \to 0} \frac{\Phi(\varphi(t)) - \Phi(x)}{t}$$

where $\varphi(t)$ is the trajectory of $\dot{\varphi}(t) = g(\varphi(t))$, $\varphi(0) = x$. By chain rule, this is the same as $D\Phi(x)g(x)$.

Major Steps..Continued

- establish $\mathscr{L}_{f_{d,v}} V_1(\xi) \le -V_1(\xi)$
- still V_1 is missing smoothness.
- Considered an existing smooth approximation argument and modified for our setup.
- Find an approximation $V_2 : \mathbb{R}^n \to \mathbb{R}$ to V_1 , smooth on $\mathbb{R}^n \setminus \{0\}$ with;
- $\frac{1}{2}\tilde{\alpha}_1(|\xi|) \le V_2(\xi) \le \frac{3}{2}\tilde{\alpha}_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n$
- $\mathscr{L}_{f_{d,v}} V_2(\xi) \le -\frac{1}{2} V_2(\xi)$
- Introduce $\rho \in \mathcal{K}_\infty$, which makes $\rho \circ V_2$ is smooth everywhere (including the origin)
- $V = (\rho \circ V_2)^2$ has all required properties.

Example

Let $0 \le \theta < 1$ and $\mathscr{B} = C([-\theta, 0], \mathbb{R})$ together with the sup norm. Consider the system Σ_1,

$$\dot{x}(t) = -x(t) - x^3(t) + x(t) \int_{-\theta}^{0} [\mu(t)](s)\, ds$$
$$x(0) = \xi.$$

where $\mu : [-\theta, \infty) \to C[-\theta, 0]$ is continuous and $\|\mu\| \le 1$.

Then $V(\xi) = \xi^2/2$ is a smooth Lyapunov function for Σ since,

$$\mathscr{L}_{f_{d,v}} V(\xi) = -\xi^2 - \xi^4 + \xi^2 \int_{-\theta}^{0} v(s)\, ds$$
$$\le -\xi^2 - \xi^4 + |\xi|^2 \theta \|v\| \le -\xi^4$$

Therefore by the main result, Σ_1 is rGAS.

But, why is this example interesting?

If we set $\mu(t) = x_t$, that is, when $[\mu(t)](s) = x(t+s)$, Σ_1 becomes a delayed system (Σ_2) given by;

$$\Sigma_2 : \dot{x}(t) = -x(t) - x^3(t) + x(t) \int_{-\theta}^{0} x(t+s)\, ds$$
$$= -x(t) - x^3(t) + x(t) \int_{t-\theta}^{t} x(s)\, ds.$$

Then Σ_2 is rGASas a specific case of Σ_1

Future Research

Though there has been extensive work on robust stability analysis in the past few decades. Yet not much has been done for systems affected by disturbances/inputs taking values in an infinite dimensional Banach space such as $C[-\theta, 0]$. If possible, extending this result to standard delayed systems (initial states are from $C[-\theta, 0]$) will provide a significant generalization.

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