ASYMMETRIC INFORMATION IN FADS MODELS IN LÉVY MARKETS

by

Winston S Buckley

A Dissertation Submitted to the Faculty of

The Charles E. Schmidt College of Science

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Florida Atlantic University

Boca Raton, Florida

August 2009

ASYMMETRIC INFORMATION IN FADS MODELS IN LÉVY MARKETS

by

Winston S Buckley

This dissertation was prepared under the direction of the candidate's dissertation advisor, Dr. Hongwei Long, Department of Mathematical Sciences, and it has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

SUPERVISORY COMITTEE:

Hongwei Long

Hongwei Long, Ph.D.
Dissertation Advisor

Fred Richman

Fred Richman, Ph.D.

Lianfen Qian, Ph.D.

Yaozhong Hu, Ph.D.

Spyros S. Magliveras, Ph.D.
Chair, Department of Mathematical Sciences

Gary W. Perry, Ph.D.
Dean, The Charles E. Schmidt College of Science

Barry T. Rosson, Ph.D.
Dean, Graduate College

4/30/2009
Date

Acknowledgements

It was, and still is, the profound respect and admiration for my Advisor, Dr Hongwei Long, that propelled me to undertake this research project under his guidance. I sincerely thank him for his encouragement, understanding and kindness, in addition to the immense motivation and mathematical excitement he invoked in me as a graduate student.

I also sincerely thank the other members of my committee–Professors Hu, Qian and Richman–for their contributions. I am also grateful to many professors in the department for positively impacting my mathematical and professional development. Very special thanks go to professors: Richman (for helping me survive intro algreba), Leanne and Spyros Magliveras, Hoffman, Kalies, Kizlik, Klinger, Lin, Niederhausen, Popova, Steinwandt, and Zhang. I also thank Beth Broer (our tireless secretary) and Emily Cimillo for their daily efforts caring for graduate students. I also thank the Department of Mathematical Sciences and FAU for their financial support.

Very special thanks go to Professor Geoffery Grimmett, Department of Mathematical Sciences, Cambridge University, Professor Bernard Mair, Mathematics Department, University of Florida, and Professor Garth Baker, formerly of Cornell University, for strongly encouraging me to complete the doctorate in mathematics.

I am also indebted to the staff of the International Students and Scholars Services at FAU, especially Ingrid Jones, Diane Campbell and Myrtha Senat. Their invaluable assitance and many interventions, beginning with my late registration in Fall 2005, have helped me to survive this journey. I am also indebted to my peers, some of whom became my close friends. Foremost are Mario Marshall, who invited me to join the program at FAU, Sandun Perera, Madeline Gonzales, Dr Mary Hopkins, Dr Vladimir Bozovic, Djordje Bulj, Cafer Caliskan, Dr Audrey Moore, Shanaz Tiwari, Radleigh Santos, Nicola Pace and Dr Oneil Harris. Many thanks go to Basak Ay, Shanzhen Gao, Samantha McCleod, Kashe Neupane, Nidhi and Nikhil Singhi, Amalya Mihnea, Stephan Grigoriev, Valentine Sobirov, Dr Mark Wess, and others.

I am profoundly indebted to my dear friends Dorothy Graham and Carla Grey; they assisted me selflessly, and in countless ways, as I endured the hardships and challenges of being a doctoral student. I could never have completed this project without their numerous interventions and moral support. I also thank my friends and supporters in Jamaica: Cecile Anderson, Dr Basil Babalol, Dr Garfield Brown, Laverne Baker, Melody Baker, Sister Bernadette, Arnett Campbell, Simone Clifford, Nicollette Deerr, Pauline and Vincent Edwards, Maxine Francis, Sharon James, Carlton Mason, Dr Samuel McDaniel, Bertram Morrison, Cordel Nelson, Donna Nuggent, Hedy Pickersgil, Deon Robinson, Hugh Scott, Kenneth Scott, Vincent Smith, Dr Regina Turner, Maxwell Williams, Mark Wignal, Michael Taffe, and my high school math teacher, Keith Irons.

My deepest gratitude and very special thanks go to my daughter Ingrid, and family in Jamaica for their constant encouragement, patience, and understanding, as they endured the many challenges resulting from my absence. Most importantly, I thank my Holy Father, Jesus Christ, for guiding and supporting me throughout this journey.
–WSB, November 2008.

Abstract

Author: Winston S Buckley

Title: Asymmetric Information in Fads Model in Lévy Markets

Institution: Florida Atlantic University

Dissertation Advisor: Dr. Hongwei Long

Degree: Doctor of Philosophy

Year: 2009

Fads models for stocks under asymmetric information in a purely continuous(GBM) market were first studied by P. Guasoni (2006), where optimal portfolios and maximum expected logarithmic utilities, including asymptotic utilities for the informed and uninformed investors, were presented. We generalized this theory to Lévy markets, where stock prices and the process modeling the fads are allowed to include a jump component, in addition to the usual continuous component. We employ the methods of stochastic calculus and optimization to obtain analogous results to those obtained in the purely continuous market. We approximate optimal portfolios and utilities using the instantaneous centralized and quasi–centralized moments of the stocks percentage returns. We also link the random portfolios of the investors, under asymmetric information to the purely deterministic optimal portfolio, under symmetric information.

Dedication

This research is lovingly dedicated to my deceased grandmother, my wife and best friend, and my high school geography teacher, for the invaluable role each played in my personal, intellectual and professional development.

To: Myra, Monica, and Margaret

"Ability is of little account without opportunity"– Napoleon Bonaparte

Contents

Tables

Chapter 1

Introduction

Fads models for stocks under asymmetric information were first studied by Shiller [47] and Summers [49] in a purely deterministic setting. Guasoni [21] extended this theory to the purely continuous random environment, where stock prices follow Geometric Brownian Motion (GBM). There are two investors trading in the market– the uninformed and informed investors, tagged by the superscripts $i = 0$ and $i = 1$, respectively.

Guasoni [21] gave optimal portfolios and maximum expected logarithmic utilities, including asymptotic utilities for both uninformed and informed investors. He also gave the excess asymptotic utility of the informed investor, which takes the form: $\frac{\lambda}{2}\, p\,(1-p)\,T$, where T is the long run investment horizon, λ is the mean reverting rate of the fads process, and $p \in [0,\, 1]$ is a measure of the lack of fads in the market, where $p = 1$ means the market is 100% free of fads.

In this dissertation, we generalize the theory of fads/mispricing models of stocks under asymmetric information in two important ways.

First, we allow the stock price dynamic to have a jump component driven by a pure jump Lévy process, while the process that models the mispricing/fads remains as a continuous mean-reverting Ornstein–Uhlenbeck process, similar to the one used in Guasoni [21].

Secondly, we allow both the stock price dynamic and the process modelling the fads to have Lévy jumps. In this case the fads process includes a pure jump component driven by a zero mean Lévy process, in addition to its usual continuous mean reverting O–U component. We employ the methods of stochastic calculus and optimization to obtain analogous results as in the purely continuous case studied by Guasoni [21].

Organization

The dissertation is organized as follows:

Chapter 2 reviews purely continuous fads models under asymmetric information where stocks follow GBM, and utility functions are logarithmic. Optimal portfolios and asymptotic (excess) utilities are presented. We review important properties of Lévy processes and general jump processes useful to our research. We also give a brief review of important Lévy processes used in financial applications, such as the celebrated Merton jump diffusion and the Variance Gamma processes. Results for which proofs are provided, are our contribution.

Chapter 3 extends the theory of fads models for stocks under asymmetric inform –ation to the jump case. Jumps are modelled by pure jump Lévy processes, while the fads are represented by a purely continuous mean–reverting O–U process driven by a standard Brownian motion, as in Guasoni [21]. We obtain optimal portfolios and maximum expected logarithmic utilities for both the informed and uninformed investors, including asymptotic excess utility of the form $\frac{\tilde{\lambda}}{2}\, p\,(1\,-\,p)\,T$, which is analogous to the result obtained by Guasoni [21] in the purely continuous case. We also link the random portfolios of the investors to the symmetric, purely deterministic optimal portfolios of Lévy diffusion markets having deterministic market coefficients. We also study the pure jump Lévy market which results when there is no diffusive coefficient.

Chapter 4 generalizes the work done in Chapter 3. In this case, stocks are still subjected to Lévy jumps, but the mispricing/fads process is no longer purely continuous. Instead, we add an innovation–a pure jump component that is driven by a zero mean pure jump Lévy process independent of the Lévy process driving the stock. We solve this model for both investors, obtaining similar optimal portfolios and maximum expected logarithmic utilities, and asymptotics, as in Chapter 3.

Chapter 5 gives a detailed account of specific Lévy markets having diffusive coefficients and deterministic market coefficients μ_t, r_t, and σ_t^2. We study the Kou jump diffusion market; the Variance Gamma market, which is a form of the CGMY market; the Double Poisson and m–double Poisson markets, which are theorectical (toy) models. We obtain optimal portfolios and their approximations based on an indepth study of the instantaneous centralized return moments $M_k, k \in N$, for each process. We show that the optimal portfolios are fixed points of functions and polynomials created from the M_ks, which are dependents of the Lévy measure of the jump process driving the market. We obtain interesting combinatorial identities as a by-product of the instantaneous centralized moments of the Kou [30] jump diffusion model. The combinatorial identities are contained in Appendix A.

Chapter 6 uses quasi–centralized moments $M_k^a = \int_{R_a}(1-e^{-x})^k v(x)\,dx$, $a \in \{+,-\}$, which are always positive, to give an alternative method of approximation of the optimal portfolios and utilities of the investors. The partial objective function $G'(\pi)$ is expressed as the sum of two convergent Taylor series, expanded about $\pi = 0$ and $\pi = 1$, respectively. We truncate these series at a suitable point k, yielding an approximation:

$$G'(\pi) = \Sigma_{j=1}^k M_j^+(1-\pi)^j - \Sigma_{j=1}^k M_j^- \,\pi^j, \pi \in [0,\, 1].$$

Athough this approach uses two independent series expansions to approximate $G'(\pi)$, it does not depend on an integer k, for which $\int_0^\infty (e^{jx}-1)v(x)\,dx \ < \infty$, for each integer $0 \leq j \leq k$, as required when instantaneous centralized moments M_k are used (cf Chapter 5). We therefore propose that this method of approximation is more suitable for the cases where M_k exists only for small values of k, such as $k \leq 4$. A typical case is that of the Kou [30] model for $1 \leq \eta_+ \leq 4$. When M_k exist for $k > 4$, we use the usual instantaneous centarlized moments of returns approximation, with $G'(\pi) = \Sigma_{j=1}^k (-1)^{j-1} M_j \,\pi^j$.

Chapter 7 outlines future research possibilities and presents some numerical findings.

Appendix A presents some combinatorial identities derived from the Kou jump diffusion model, and proofs of the analytic formula for $G'(\pi)$ and $G''(\pi)$.

Appendix B gives a detailed study of the CGMY diffusion market. We give details of the optimal portfolios and maximum expected utilities for this market.

Chapter 2

Review of Continuous Fads Models and Lévy Processes with Applications in Finance

2.1 Part I: Continuous–Time Fads Models

In this chapter, we present an overview of asymmetric information in fads models in a purely continuous random market– that is, in a market where stock prices and fads move continuously, without jumping. Discrete-time fads/mispricing models were first introduced by Shiller [47] in 1981 and by Summers [49] in 1986, as plausible alternatives to the efficient market/constant expected returns assumption (cf Fama [20]).

In 1993, Wang [53] gave a model of intertemporal/continuous–time asset prices under asymmetric information. In this paper, investors have different information concerning the future growth rate of dividends, which satisfies a mean–reverting Ornstein–Uhlenbeck process. Informed investors know the future dividend growth rate, while the uninformed investors do not. All the investors observe current dividend payments and stock prices. The growth rate of dividends determines the rate of appreciation of stock prices, and stock prices changes provide signals about the future growth of dividends. Uninformed investors rationally extract information about the economy from prices, as well as dividends.

It is shown that asymmetry among investors can increase price volatility and negative autocorrelation in returns; that is, we have mean–reverting behavior of stock prices. Thus imperfect information of some investors can cause stock prices to be more volatile than in the symmetric case, when all investors are perfectly informed. In addition, uninformed investors may rationally behave like price chasers, by employing technical analysis for trading.

In 2001, Brunnermeier [11] presented an extensive review of asset pricing under asymmetric information mainly in the discrete setting. He shows how information affects trading activity, and that expected returns depend on the information set or filtration of the investor. These models show that past prices still carry valuable information, which can be exploited using technical/chart analysis, which uses part or all of past prices to predict future prices.

In 2006, Guasoni [21] extended Summers' model to the purely continuous random setting using stochatic calculus. He developed models of stock price evolution for two disjoint classes of investors; the informed and uninformed investors. The informed investor, indexed by $i = 1$, observes both the fundamental and market values of the stock, while the so–called uninformed investor, indexed by $i = 0$, observes market prices only. Both investors have filtrations or information banks $\mathcal{F}^i$, $i \in \{0, 1\}$ with

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F},$$

where $\mathcal{F}$ is a fixed σ–algebra.

The problem of the maximization of expected logarithmic utility from terminal wealth was solved for each investor, and an explicit formula for the asymptotic excess utility of the informed investor is presented. Fads are represented by a mean–reverting Ornstein– Uhlenbeck process U, with reversion rate $\lambda > 0$. The results in this chapter provide the background for the extension to the Lévy market, where stock prices are allowed to jump. We present this generalization in Chapter 3. All proof presented in this, and subsequent chapters, are the author's.

2.2 The Model

The model consists of two assets–a riskless asset **B** called bond, bank account or money market, and a risky asset S called stock. The bond earns a continuously compounded risk–free interst rate r_t, while the stock has total percentage appreciation rate or **expected returns** μ_t, at time $t \in [0, T]$. The stock is subject to volatility $\sigma_t > 0$. The market parameters are μ_t, r_t, σ_t, $t \in [0, T]$, and are assumed to be **deterministic** functions. We have **Standing Assumptions :**

(1) $T > 0$, is the investment horizon; all transactions take place in $[0, T]$.

(2) The market parameters r, μ, σ^2 are Lebesque integrable.

(3) The stock's **Sharpe ratio** or **market price of risk** θ, is square integrable.

(4) The risky asset S lives on a probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ on which is defined two independent standard Brownian motions $W = (W_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$. $\mathcal{F}$ is an sigma–algebra of subsets of $\mathbf{\Omega}$, and $\mathbf{P}$ is the "real–world" probability measure on $\mathcal{F}$.

(5) Fads or mispricings are modelled by the a mean–reverting **Ornstein–Uhlenbeck** (O–U) process $U = (U_t)_{t\geq 0}$ with mean–reversion rate or speed λ.

(6) Informed and uninformed investors are represented by the indices “1”, and “0”, respectively.

2.2.1 The Price Dynamics

The bond $\mathbf{B}$ has price

$$\mathbf{B}_t = \exp\left(\int_0^t r_s ds\right),$$

while the stock S has log returns dynamic

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dY_t, \quad t \in [0, T], \tag{2.2.1}$$

where

$$Y_t = p\, W_t + q\, U_t, \quad p^2 + q^2 = 1,\ p \geq 0,\ q \geq 0, \tag{2.2.2}$$

and

$$dU_t = -\lambda U_t dt + dB_t, \quad U_0 = 0, \quad \lambda > 0. \tag{2.2.3}$$

Applying Itô’s transformation formula to (2.2.1) gives percentage returns dynamic for the stock:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dY_t, \quad t \in [0, T]. \tag{2.2.4}$$

Observe that μ_t is the expected percentage returns on the stock, while $\sigma_t dY_t$ is the **excess** percentage returns. The fads or mispricing process U is a mean–reverting Ornstein–Uhlenbeck process with speed λ, which is the unique solution of the **Langevin stochastic differential equation:** (2.2.3) with explicit solution

$$U_t = U_0\, e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s, \quad t \in [0, T]. \tag{2.2.5}$$

If the speed λ is close to zero, mean reversion is slow and there is a high likelihood of fads, while if $\lambda >> 0$, the mispricing reverts rapidly, thereby reducing any advantages of fads. $100q^2$ **% is the percentage of fads in the market.** Equation (2.2.1) and equivalently, (2.2.4), has unique solution

$$S_t = S_0 \exp\left(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dY_s\right), \quad t \in [0,T]. \tag{2.2.6}$$

By imposing (2.2.3) on (2.2.2), we see that Y is a combination of a martingale W, which represents permanent price shocks, and U the mean–reverting O–U process, which repesents temporary shocks. If $\lambda = 0$ or $q = 0$, (and μ_t and σ_t are constants) we revert to the usual Geometric Brownian Motion (GBM) of Merton [36].

From (2.2.5) it is easy to show that if $U_0 \neq 0$, then

$$\mathbf{E}[U_t] = U_0\, e^{-\lambda t}, \quad \mathbf{E}[U_t^2] = \frac{1 - e^{-2\lambda t}}{2\lambda} + U_0^2\, e^{-2\lambda t}, \quad \mathbf{Var}[U_t] = \frac{1 - e^{-2\lambda t}}{2\lambda}, \tag{2.2.7}$$

with $\mathbf{E}[U_t] \longrightarrow 0$, and $\mathbf{Var}[U_t] \longrightarrow \frac{1}{2\lambda}$, as $t \longrightarrow \infty$. If $U_0 = 0$ then $\mathbf{E}[U_t] = 0$, and from (2.2.5)

$$U_t = \int_0^t e^{-\lambda(t-s)} dB_s, \quad t \in [0,T]. \tag{2.2.8}$$

2.3 Filtrations or Information Flows of Investors

Definition 2.1. *Let $X = (X_t)_{t\geq 0}$ be a process defined on $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$. Its natural filtration $\mathcal{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ is the sub–σ algebra of $\mathcal{F}$ generated by X, and is given by*

$$\mathcal{F}_t^X \triangleq \sigma(X_s : s \leq t) = \{X^{-1}(A) \subset \mathbf{\Omega} : A \in \mathcal{B}(\mathbf{R})\}. \tag{2.3.1}$$

$\mathcal{F}_t^X$ is the information generated by X up to time t. It is the smallest σ–field relative to which X_t is measurable.

2.3.1 Augmentation

We can make $\mathcal{F}_t^X$ right–continuous and complete by augmenting it with $\mathcal{N}$, the $\mathbf{P}$–null sets of $\mathcal{F}$, given by

$$\mathcal{N} = \{A \subset \mathbf{\Omega} : \exists\, B \in \mathcal{F},\; A \subset B,\; \mathbf{P}(B) = 0\}. \tag{2.3.2}$$

The augmented filtration of X is $\sigma\left(\mathcal{F}^X \vee \mathcal{N}\right)$. $\mathcal{F}^X$ is **complete** if it contains the $\mathbf{P}$–null sets of $\mathcal{F}$. This is achieved if $\mathcal{N} \subset \mathcal{F}_0^X$. A filtration $(\mathcal{F}_t)$ is **right continuous** if

$$\mathcal{F}_{t+} = \mathcal{F}_t, \quad \text{where } \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s. \tag{2.3.3}$$

In the sequel, we assume that all filtrations $(\mathcal{F}_t)$ are right–continuous and complete. In this case, we say the filtration satisfies the **usual hypothesis**. Thus $\mathcal{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ will denote the complete right–continuous filtration generated by X on $(\mathbf{\Omega},\, \mathcal{F},\, \mathbf{P})$.

The informed investor observes the pair $(S,\, U)$, while the uninformed investor observes only the stock price S.

Definition 2.2 (Filtrations of Investors). *Let $\mathcal{F}^1 = (\mathcal{F}_t^1)_{t\geq 0}$ and $\mathcal{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$ be the filtrations generated by $(S,\, U)$ and S, respectively. That is, for each $t \in [0, T]$*

$$\mathcal{F}_t^1 \triangleq \sigma\left(S_s,\, U_s : s \leq t\right) = \mathcal{F}_t^{S,U}. \tag{2.3.4}$$

$$\mathcal{F}_t^0 \triangleq \sigma\left(S_s : s \leq t\right) = \mathcal{F}_t^{S}. \tag{2.3.5}$$

$\mathcal{F}^1$ and $\mathcal{F}^0$ are the respective information flows of the informed and uninformed investors. Equivalently, since W and B generate S and U for the informed investor, while Y generates S for the uninformed investor, then

$$\mathcal{F}_t^1 \triangleq \sigma\left(W_s,\, B_s : s \leq t\right) = \mathcal{F}_t^{S,U}. \tag{2.3.6}$$

$$\mathcal{F}_t^0 \triangleq \sigma\left(Y_s : s \leq t\right) = \mathcal{F}_t^{Y}. \tag{2.3.7}$$

Clearly

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F} \quad \Longleftrightarrow \mathcal{F}_t^0 \subset \mathcal{F}_t^1, \quad t \in [0, T]. \tag{2.3.8}$$

The market participants can be classified in accordance to their respective information flows. Those with access to $\mathcal{F}^1$, are called informed investors–they observed both the fundamental and market prices of the risky asset.

Those with access to $\mathcal{F}^0$ only, are called uninformed investors–they observe market prices only. These uninformed investors know that there are fads/mispricing in the market but cannot observe them directly. Consequently, these uninformed investors resort to technical analysis for trading strategies. Information asymmetry results if $\mathcal{F}^1 \neq \mathcal{F}^0$. Otherwise, we have a informationally symmetric market in which investors have equal knowledge.

2.4 The Stock Price Dynamic for the Investors

It follows from (2.2.4), that for both investors the general percentage returns for the stock has dynamic

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dY_t. \tag{2.4.1}$$

We will rewrite this dynamic relative to the filtration of each investor.

2.4.1 Price Dynamic for the Informed Investor

From equations (2.2.2) and (2.2.3), we have

$$\begin{aligned} dY_t &= p\,dW_t + q\,dU_t \\ &= p\,dW_t + q\,d(-\lambda U_t dt + dB_t) \\ &= p\,dW_t + q\,dB_t - q\,\lambda U_t dt \\ &= dB_t^1 + \upsilon_t^1 dt, \end{aligned} \tag{2.4.2}$$

where

$$B_t^1 \triangleq p\,W_t + q\,B_t \tag{2.4.3}$$

$$\upsilon_t^1 \triangleq -q\,\lambda U_t. \tag{2.4.4}$$

Substituting (2.4.3) into (2.4.1), yields

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dY_t \\ &= \mu_t dt + \sigma_t (dB_t^1 + \upsilon_t^1 dt) \\ &= (\mu_t + \upsilon_t^1 \sigma_t) dt + \sigma_t dB_t^1 \\ &= \mu_t^1\, dt + \sigma_t dB_t^1, \end{aligned} \tag{2.4.5}$$

where

$$\mu_t^1 \triangleq \mu_t + \upsilon_t^1 \sigma_t. \tag{2.4.6}$$

So under $\mathcal{F}^1$, the informed investor has price dynamic (2.4.5) with price

$$S_t = S_0 \exp\left(\int_0^t (\mu_s^1 - \frac{1}{2}\sigma_s^2) ds + \int_0^t \sigma_s dB_s^1 \right), \quad t \in [0,\, T], \tag{2.4.7}$$

where B^1 is an $\mathcal{F}^1$–Brownian motion given by (2.4.3) and μ_t^1 is given by (2.4.6).

2.4.2 Price Dynamic for the Uninformed Investor

Using the Hitsuda [24] representation of Gaussian processes (see Cheridito [14]), Guasoni [21] obtained $\mathcal{F}^0$–Brownian motion B^0 and process υ^0, such that the dynamic of the uninformed investor is

$$\frac{dS_t}{S_t} = \mu_t^0 dt + \sigma_t dB_t^0, \tag{2.4.8}$$

with price

$$S_t = S_0 \exp\left(\int_0^t (\mu_s^0 - \frac{1}{2}\sigma_s^2) ds + \int_0^t \sigma_s dB_s^0 \right), \quad t \in [0, T], \tag{2.4.9}$$

where

$$\mu_t^0 \triangleq \mu_t + \upsilon_t^0 \sigma_t, \tag{2.4.10}$$

and υ_t^0 and $\gamma(s)$ are defined in the following theorem.

Theorem 2.1 (Guasoni [21], Theorem 2.1). *Let $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ be a probability space on which independent Brownian motions W and B are defined. Let $\mathcal{F}^0 = (\mathcal{F}_t^0)_{t\geq 0} \equiv (\mathcal{F}_t^Y)_{t\geq 0}$ be the filtration generated by Y satisfying the usual hypothesis. Define a function $\psi : [0, \infty) \longrightarrow \mathbf{R}$ by the prescription:*

$$\psi(t) = \int_0^t \gamma(s) ds = -\frac{1}{\lambda} \log\left(\cosh(\lambda p t) + p \sinh(\lambda p t)\right), \tag{2.4.11}$$

where

$$\gamma(s) = \psi'(s) = \frac{1 - p^2}{1 + p \tanh(p\lambda s)} - 1. \tag{2.4.12}$$

Then, we can construct an $\mathcal{F}^0$– Brownian motion $B^0 = (B_t^0)$ on $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ such that *in terms of* $Y_s : s \le t$,

$$B_t^0 = Y_t + \int_0^t \phi_s^0 ds, \tag{2.4.13}$$

where

$$\begin{aligned} \phi_s^0 &\triangleq \lambda(\gamma(s)+1)Y_s + \lambda^2 \int_0^s (\gamma(s)+p^2)e^{\lambda(\psi(s)-\psi(u))}Y_u du \\ &= \lambda \int_0^s e^{\lambda(\psi(s)-\psi(u))}(1+\gamma(s))dY_u. \end{aligned} \tag{2.4.14}$$

The semi–martingale decomposition of Y *under* $\mathcal{F}^0$ *is:*

$$Y_t = B_t^0 + \int_0^t \upsilon_s^0 ds, \tag{2.4.15}$$

where υ_t^0 *is given by*

$$\upsilon_t^0 \triangleq -\lambda \int_0^t e^{-\lambda\,(t-s)}(1+\gamma(s))\,dB_s^0. \tag{2.4.16}$$

Its canonical representation is:

$$Y_t = \int_0^t \left(e^{-\lambda\,(t-s)}(1+\gamma(s)) - \gamma(s)\right)\, dB_s^0. \tag{2.4.17}$$

Remark 2.1. *Observe from equations (2.4.13) and (2.4.15) that* $\phi^0 = -\upsilon^0$, *and hence the drift* $\mu_t^0 = \mu_t + \upsilon_t^0\sigma_t$, *depends on the* ***entire*** *past history of* $Y_u, u \le t$, *and therefore depends on the entire history of the stock price* S_u *up to time* $t \ge u$. *Thus for the uninformed investor, optimal trading involves a knowledge of past prices, which suggest that trading must rely on* ***technical analysis***. $\gamma(u)$ *is the solution of the Cauchy equation:*

$$\gamma'(s) = \lambda(\gamma^2(s) - p^2), \quad \gamma(0) = -p^2. \tag{2.4.18}$$

Since both investors observe the same stock price S, it follows that

$$B_t^0 - B_t^1 = \int_0^t (\upsilon_s^1 - \upsilon_s^0)\ ds.$$

We now prove a useful result for υ_t^0 that will be required in the sequel.

Lemma 2.1. *Let $p \in [0,1]$ and $t \in [0,T]$. Then*

(1) $\mathbf{E}[\upsilon_t^0]^2 = \lambda^2 \int_0^t e^{-2\lambda(t-s)}(1+\gamma(s))^2 ds.$

(2) $\lim_{t\to\infty} \mathbf{E}[\upsilon_t^0]^2 = \frac{\lambda}{2}(1-p)^2 = \frac{\lambda}{2}(1-p)(1+(-1)^{i+1}p), i = 0.$

(3) $\int_0^T \mathbf{E}[\upsilon_t^0]^2 dt \simeq \frac{\lambda}{2}(1-p)^2 T, \quad as \quad T \longrightarrow \infty.$

Proof. (1) By Itô–isometry (cf Oksendal [40]), we have

$$\begin{aligned} \mathbf{E}[\upsilon_t^0]^2 &= \mathbf{E}\left(\lambda \int_0^t e^{-\lambda(t-s)}(1+\gamma(s))dB_s^0\right)^2 = \mathbf{E}\left(\lambda^2 \int_0^t e^{-2\lambda(t-s)}(1+\gamma(s))^2 ds\right) \\ &= \lambda^2 \int_0^t e^{-2\lambda(t-s)}(1+\gamma(s))^2 ds. \end{aligned}$$

(2) Since $\gamma(t)$ is continuous, then by L'Hospital's rule

$$\begin{aligned} \lim_{t\to\infty} \mathbf{E}[\upsilon_t^0]^2 &= \lim_{t\to\infty} \lambda^2 \int_0^t e^{-2\lambda(t-s)}(1+\gamma(s))^2 ds = \lambda^2 \lim_{t\to\infty} \frac{\int_0^t e^{2\lambda s}(1+\gamma(s))^2 ds}{e^{2\lambda t}} \\ &= \lambda^2 \lim_{t\to\infty} \frac{e^{2\lambda t}(1+\gamma(t))^2}{2\lambda e^{2\lambda t}} = \frac{\lambda}{2} \lim_{t\to\infty} (1+\gamma(t))^2 = \frac{\lambda}{2}(1-p)^2, \end{aligned}$$

since

$$\lim_{t\to\infty} \gamma(t) = \lim_{t\to\infty} \frac{1-p^2}{1+p\tanh(p\lambda t)} - 1 = \frac{1-p^2}{1+p} - 1 = -p.$$

(3) $\mathbf{E}[\upsilon_t^0]^2$ is a continuous function of t, so by the Mean Value Theorem

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{E}[\upsilon_t^0]^2 dt = \lim_{t\to\infty}\mathbf{E}[\upsilon_t^0]^2 = \frac{\lambda}{2}\,(1-p)^2.$$

Thus for large values of T, we have $\int_0^T \mathbf{E}[\upsilon_t^0]^2 dt \simeq \frac{\lambda}{2}\,(1-p)^2\,T.$ $\square$

We have an analogous result for the corresponding process υ^1 defined in (2.4.4).

Lemma 2.2. *Let* $p\in[0,1]$, *and* $t\in[0,T]$. *Then*

(1) $\mathbf{E}[\upsilon_t^1]^2 = \frac{\lambda}{2}\,(1-p^2)\,(1-e^{-2\lambda t}).$

(2) $\lim_{t\to\infty}\mathbf{E}[\upsilon_t^1]^2 = \frac{\lambda}{2}\,(1-p^2) = \frac{\lambda}{2}\,(1-p)(1+(-1)^{i+1}p), i=1.$

(3) $\int_0^T \mathbf{E}[\upsilon_t^1]^2 dt \simeq \frac{\lambda}{2}\,(1-p^2)\,T,\quad as\quad T\longrightarrow\infty.$

Proof. (1) Imposing Itô–isometry (cf Oksendal [40]) on U_t yields

$$\begin{aligned}\mathbf{E}[\upsilon_t^1]^2 &= \mathbf{E}(\lambda^2\,q^2\,U_t^2) = \lambda^2\,q^2\mathbf{E}(U_t^2) = \lambda^2\,q^2\left(\int_0^t e^{-\lambda(t-s)}dB_s\right)^2\\ &= \lambda^2\,q^2\int_0^t e^{2\lambda(s-t)}ds = \lambda^2\,q^2\,\frac{(1-e^{-2\lambda t})}{2\lambda}\\ &= \frac{\lambda}{2}q^2\,(1-e^{-2\lambda t}) = \frac{\lambda}{2}(1-p^2)\,(1-e^{-2\lambda t}).\end{aligned}$$

(2) It follows trivially from (1) that if $\lambda\geq 0$ then

$$\lim_{t\to\infty}\mathbf{E}[\upsilon_t^1]^2 = \lim_{t\to\infty}\frac{\lambda}{2}(1-p^2)\,(1-e^{2\lambda t}) = \frac{\lambda}{2}(1-p^2) = \frac{\lambda}{2}(1-p)\,(1+(-1)^{i+1}p),\;\; i=1.$$

(3) $\mathbf{E}[\upsilon_t^1]^2$ is a continuous function of t, so by the Mean Value Theorem

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{E}[\upsilon_t^1]^2 dt = \lim_{t\to\infty}\mathbf{E}[\upsilon_t^1]^2 = \frac{\lambda}{2}\,(1-p^2).$$

Thus for large values of T, we have $\int_0^T \mathbf{E}[\upsilon_t^0]^2 dt \simeq \frac{\lambda}{2}\,(1-p^2)\,T.$ $\square$

We combine Lemmas 2.1 and 2.2 as follows.

Lemma 2.3. *Let $p \in [0,\,1]$, $t \in [0,\,T]$, and $i \in \{0,1\}$.*

Let $\upsilon_t^0 = -\lambda \int_0^t e^{-\lambda(t-s)}(1+\gamma(s))dB_s^0$ and $\upsilon_t^1 = -\lambda\, q\, U_t$ be given in Theorem 2.1, with $p^2+q^2 = 1$. Then

(1) $\mathbf{E}[\upsilon_t^i]^2 = \frac{\lambda}{2}(1-p)(1+(-1)^{i+1}p)$, *as $t \to \infty$.*

(2) $\int_0^T \mathbf{E}[\upsilon_t^i]^2 dt \simeq \frac{\lambda}{2}\,(1-p)(1+(-1)^{i+1}p)\,T$, *as* $T \longrightarrow \infty$.

(3) *As $T \longrightarrow \infty$, the asymptotic excess second moments of the υ^is is*

$$\int_0^T \mathbf{E}[\upsilon_t^1]^2 dt - \int_0^T \mathbf{E}[\upsilon_t^0]^2 dt \simeq \lambda\, p\,(1-p)\,T \leq \frac{\lambda}{4}T.$$

Proof. (1) and (2) follow from from previous lemmas. $p(1-p)$ has a maximum at $p = \frac{1}{2}$, which proves (3). □

2.5 Utility Functions

We assume that each investor has a utility function $\mathbf{U} : (0,\,\infty) \to \mathbf{R}$ for wealth, and that it satisfies the Inada condition.

Definition 2.3 (**Inada Condition**)**.**

A function $\mathbf{U} : (0,\,\infty) \to \mathbf{R}$, satisfies the Inada condition, if it is strictly increasing, strictly concave, continuously differentiable, with .

$$\mathbf{U}'(0) = \lim_{x\downarrow 0} \mathbf{U}'(x) = +\infty, \qquad \mathbf{U}'(\infty) = \lim_{x\to\infty} \mathbf{U}'(x) = 0.$$

$\mathbf{U}(x) = \log x$, the logarithmic utility and $\mathbf{U}_\theta(x) = \frac{x^\theta}{\theta}$, $\theta < 1$, the power utility, satisfy this condition. In the sequel, all utility functions are assumed to be logarithmic.

2.6 Portfolio and Wealth Processes of Investors

Definition 2.4 (**Portfolio Process**). *A portfolio process* $\pi : [0, T] \times \Omega \to \mathbf{R}$, *is an* $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$*–adapted process satisfying*

$$\int_0^T (\pi_t \sigma_t)^2 dt < \infty, \quad \textit{almost surely.} \tag{2.6.1}$$

Although π is a function of (t, ω), in the sequel we keep Ω in the background, and assume that π is primarily a function of time t, where π_t is the proportion of an investor's wealth invested in the stock at time t. The remainder $1 - \pi_t$, is invested in the bond or money market. π is not restricted to $[0, 1]$ for the purely continuous model. That is, we allow short–selling ($\pi < 0$) and borrowing ($\pi > 1$) at the risk–free interest rate. Short–selling occurs when stocks are borrowed and sold, with the proceeds invested in the bond or savings accounts.

Definition 2.5 (**Self–financing**). *A portfolio process* π *is called self–financing if*

$$dV_t = (1 - \pi_t) r_t V_t \, dt + \pi_t V_t \frac{dS_t}{S_t}, \tag{2.6.2}$$

where V_t *is the wealth or value of the holding of stock and bond at time* $t \in [0, T]$.

Thus, for self–financing portfolios, the change in the wealth is due only to the change in prices, **provided** that no money is brought in or taken out by the investor.

The Wealth Process

For a given non–random initial wealth $x > 0$, let $V^{x,\pi} \equiv V^{\pi} \equiv V = (V_t)_{t \geq 0}$ denote the wealth process corresponding to a self–financing portfolio π with $V_0 = x$, and satisfying the stochastic differential equation (2.6.2).

The following result will be useful, so we state it as a lemma:

Lemma 2.4. *Let π be the portfolio process for the stock S with Sharpe ratio*

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}, \quad t \in [0, T], \tag{2.6.3}$$

and percentage returns dynamic given by (2.2.4). Then the wealth process $V = V^\pi$ corresponding to π and initial wealth $x > 0$, has percentage returns dynamic

$$\frac{dV_t}{V_t} = (r_t + \pi_t\sigma_t\theta_t)\,dt + \pi_t\,\sigma_t\,dY_t, \tag{2.6.4}$$

with unique discounted wealth process

$$\widetilde{V}_t = x\ \exp\left(\int_0^t (\pi_s\sigma_s\theta_s - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dY_s\right), \tag{2.6.5}$$

and logarithmic utility

$$\log \widetilde{V}_t = \log x + \frac{1}{2}\int_0^t \theta_s^2\,ds - \frac{1}{2}\int_0^t (\pi_s\,\sigma_s - \theta_s)^2\,ds + \int_0^t \pi_s\,\sigma_s\,dY_s. \tag{2.6.6}$$

Proof. It is clear that

$$\frac{dV_t}{V_t} = (1 - \pi_t)\,r_t\,dt + \pi_t\,\frac{dS_t}{S_t}. \tag{2.6.7}$$

Imposing the stock price dynamic (2.2.4) on the last equation, yields

$$\begin{aligned}
\frac{dV_t}{V_t} &= (1 - \pi_t)r_t\,dt + \pi_t\,\frac{dS_t}{S_t} \\
&= (1 - \pi_t)r_t\,dt + \pi_t\,(\mu_t dt + \sigma_t\,dY_t) \\
&= (r_t + \pi_t(\mu_t - r_t))\,dt + \pi_t\,\sigma_t\,dY_t.
\end{aligned}$$

Thus, we get the percentage returns dynamic of the wealth process

$$\frac{dV_t}{V_t} = (r_t + \pi_t\,\sigma_t\,\theta_t)\,dt + \pi_t\,\sigma_t\,dY_t, \tag{2.6.8}$$

where θ, the stock's Sharpe ratio or markt price of risk is given by (2.6.3). By the Itô formula, the unique solution to (2.6.8) with $V_0 = x$, is the stochastic exponential

$$V_t = x \, \exp\left(\int_0^t r_s \, ds + \int_0^t (\pi_s \, \sigma_s \, \theta_s - \frac{1}{2} \, \pi_s^2 \, \sigma_s^2) \, ds + \int_0^t \pi_s \sigma_s \, dY_s \right), \tag{2.6.9}$$

with discounted wealth process $\widetilde{V}$ given by

$$\widetilde{V}_t \triangleq \exp{(-\int_0^t r_s \, ds)} \, V_t = x \, \exp\left(\int_0^t (\pi_s \, \sigma_s \, \theta_s - \frac{1}{2} \, \pi_s^2 \, \sigma_s^2 \,) ds + \int_0^t \pi_s \, \sigma_s \, dY_s \right).$$

The logarithmic utility of discounted wealth is

$$\begin{aligned} \log \widetilde{V}_t &= \log x + \int_0^t (\pi_s \sigma_s \theta_s - \frac{1}{2} \pi_s^2 \sigma_s^2) ds + \int_0^t \pi_s \sigma_s dY_s \\ &= \log x + \frac{1}{2} \int_0^t \theta_s^2 ds - \frac{1}{2} \int_0^t (\pi_s \sigma_s - \theta_s)^2 ds + \int_0^t \pi_s \sigma_s dY_s . \square \end{aligned}$$

We now appy this result to each investor.

Theorem 2.2. *Let $i \in \{0, 1\}$ and let r_t be the risk–free interest rate. Let π^i and V^i be the respective portfolio and wealth processes for the i–th investor as a result of investing in the stock S, with Sharpe ratio*

$$\theta_t^i = \frac{\mu_t^i - r_t}{\sigma_t}, \quad \mu_t^i = \mu_t + \upsilon_t^i \, \sigma_t, \;\; t \in [0, \, T], \tag{2.6.10}$$

and percentage returns dynamic driven by $\mathcal{F}^i$–adapted Brownian motion B^i, given by

$$\frac{dS_t}{S_t} = \mu_t^i \, dt + \sigma_t \, dB_t^i . \tag{2.6.11}$$

Then the wealth process $V^i = V^{i, \pi}$ corresponding to π, and initial wealth $x > 0$, has *percentage returns dynamic*

$$\frac{dV_t^i}{V_t^i} = (r_t + \pi_t^i \sigma_t \theta_t^i)\, dt + \pi_t^i\, \sigma_t dB_t^i, \tag{2.6.12}$$

with unique discounted wealth process

$$\widetilde{V}_t^i = x \,\exp\left(\int_0^t (\pi_s^i \sigma_s \theta_s^i - \frac{1}{2}(\pi_s^i)^2 \sigma_s^2) ds + \int_0^t \pi_s^i \sigma_s dB_s^i\right) \tag{2.6.13}$$

and logarithmic utility

$$\log \widetilde{V}_t^i = \log x + \frac{1}{2}\int_0^t (\theta_s^i)^2 ds - \frac{1}{2}\int_0^t (\pi_s^i \sigma_s - \theta_s^i)^2 ds + \int_0^t \pi_s^i \sigma_s dB_s^i. \tag{2.6.14}$$

Proof. The results follow directly from Lemma 2.4 by replacing θ by θ^i, π by π^i and V by V^i. □

2.7 Logarithmic Utility Maximization from Terminal Wealth

In the sequel, each investor is assumed to be rational; that is, the investor is a utility maximizer. Thus, both informed and uninformed investors maximize their respective expected utility from terminal wealth V_T, where T is the investment horizon.

We confine our analysis to the logarithmic utility function $\mathbf{U}(x) = \log x$, as this leads to closed–form tractable solutions. The terminal wealth V_T is represented by its discounted value $\widetilde{V}_T$. We then maximize $\mathbf{E}_\pi \mathbf{U}(\widetilde{V}_T)$ where π is selected from an admissible set $\mathcal{A}(x)$.

Definition 2.6 (**Admissible Portfolio**). *A self–financing portfolio π is admissible if V_t^π is lower bounded for all $t \in [0, T]$. That is, there exists $K > -\infty$ such that* *almost surely,* $V_t^\pi > K$ *for all* $t \in [0,T]$*.-(cf Oksendal [40], page 235)*

Since $x > 0$, we assume that $V_t^\pi > 0$ and therefore $\widetilde{V}_t^\pi > 0$ for all $t \in [0,\, T]$. Thus equivalently, π is admissible if $\widetilde{V}_t^\pi > 0$ for all $t \in [0, T]$. Karatzas and Shreve [29], define an admissible portfolio in terms of the utility function $\mathbf{U}(x)$ by the prescription:

$$\mathbf{E}[\mathbf{U}(\widetilde{V}_t^\pi)]^- < \infty, \tag{2.7.1}$$

where $a^- = \max\{0, -a\}$. Either definition will suffice!

Admissible set

Definition 2.7. *Let $x > 0$ be the initial wealth of the investor. The admissible set $\mathcal{A}(x)$ of this investor is define by*

$$\mathcal{A}(x) = \{\pi : \pi - admissible, S - integrable, \mathcal{F} - predictable\}. \tag{2.7.2}$$

Predictable σ–Algebra

π is $\mathcal{F}$–predictable if it is measurable relative to the predictable sigma–algebra on $[0,\, T] \times \Omega$, which is the sigma–algebra of all **left continuous functions with right limits**(LCRL) on $[0,\, T] \times \Omega$- (see Protter [41] for details). Obviously, if π is admissible it is LCRL; that is, we know the value of π_t just before time t.

2.7.1 Utility Maximization Problem and Optimal Portfolios

For a given utility function $U(\cdot)$ and initial wealth $x > 0$, we maximize the expected utility from (discounted) terminal wealth $\mathbf{E}[\mathbf{U}(\widetilde{V}_t^\pi)]$, over the investors admissible set

$\mathcal{A}(x)$. The value function for this problem is $u(x)$ given by

$$u(x) \stackrel{\triangle}{=} \sup_{\pi \in \mathcal{A}(x)} \mathbf{E}[\mathbf{U}(\widetilde{V}_t^{\pi})], \tag{2.7.3}$$

where it is assumed that $u(x) < \infty$ for all $x > 0$. That is, there is an optimal portfolio $\pi^* \in \mathcal{A}(x)$ such that

$$u(x) = \mathbf{E}[\mathbf{U}(\widetilde{V}_t^{\pi^*})]. \tag{2.7.4}$$

Let $i \in \{0, 1\}$. For the i–th investor, define an admissible set:

$$\mathcal{A}^i(x) = \left\{ \pi : \widetilde{V}_t^{\pi} > 0,\ a.s., S - integrable, \mathcal{F}^i - predictable \right\} \tag{2.7.5}$$

and a utility maximiation problem:

$$\max_{\pi} \left\{ \mathbf{E}[\mathbf{U}(\widetilde{V}_t^{\pi})] : \pi \in \mathcal{A}^i(x) \right\}, \tag{2.7.6}$$

with respective value functions:

$$\begin{aligned} u^i(x) &= \sup_{\pi} \left\{ \mathbf{E}[\mathbf{U}(\widetilde{V}_t^{\pi})] : \pi \in \mathcal{A}^i(x) \right\} \\ &= \mathbf{E}\mathbf{U}(\widetilde{V}_t^{\pi^{*,i}}). \end{aligned} \tag{2.7.7}$$

The logarithmic utility function is used so that an explicit solution of (2.7.7) is obtained(cf Amendinger [25], Imkeller [26], Karatzas & Pikovsky [27]). We now give a slightly modified version of Gausoni's solution to (2.7.7) using the notations developed in preceeding sections.

Theorem 2.3 (**Guasoni [21], Theorem 3.1**). *Let $i \in \{0, 1\}$, $u^i(x)$ be the value function, and $\pi^{*,i}$ the optimal portfolio for the i-th investor that solves (2.7.7), where utility is assumed to be logarithmic and the risk–free interest rate is $r = 0$. Then*

(1) *the optimal portfolio for the i–th investor is:*

$$\pi_t^{*,i} = \frac{\theta_t^i}{\sigma_t} = \frac{\mu_t^i}{\sigma_t^2} = \frac{\mu_t + \upsilon_t^i \sigma_t}{\sigma_t^2}, \quad t \in [0, T], \tag{2.7.8}$$

where θ^i is the Sharpe ratio of the stock for the i–th investor.

(2) *The maximum expected logarithmic utility from terminal wealth for the i–th investor is*

$$\begin{aligned} u^i(x) \equiv u_c^i(x) &= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^i)^2 dt \\ &= \log x + \frac{1}{2}\mathbf{E}\int_0^T \left(\frac{\mu_t + \upsilon_t^i \sigma_t}{\sigma_t}\right)^2 dt. \end{aligned} \tag{2.7.9}$$

(3) *As $T \longrightarrow \infty$, the asymptotic maximum expected logarithmic utility is*

$$u_\infty^i(x) \equiv u_{\infty,c}^i(x) \simeq \log x + \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2}\, dt + \frac{\lambda}{4}(1-p)(1+(-1)^{i+1}p)\, T. \tag{2.7.10}$$

(4) *The excess (additional) asymptotic maximum expected logarithmic utility of the informed investor is*

$$u_\infty^1(x) - u_\infty^0(x) \simeq \frac{\lambda}{2}\, p(1-p)\, T. \tag{2.7.11}$$

Remark 2.2. *In Chapter 3, where jump processes are introduced, $u^i(x)$ and $u_\infty^i(x)$ will be identical to $u_c^i(x)$ and $u_{\infty,c}^i(x)$ respectively, where the subscript "c" denotes continuous. Note that in the foregoing μ_t, the stock's expected percentage rate of return, is identical to the stock's* ***total*** *expected percentage return. In the jump case, μ will become the continuous component of the total expected percentage returns b, where $b = \mu + M_1$; M_1 being the contribution to the returns from the jumps.*

2.8 Part II- A Review of Lévy Processes

Lévy processes are defined as stochastic processes with stationary and independent increments, that start at zero, and are continuous in probability. If $(X_t)_{t\geq 0}$ is a Lévy process, then $X_s - X_t$ is indepentent of the history of the process up to time t if $s > t$. Its law depends only on $s - t$, the elapsed time, and not on s or t exclusively. In this sense, Lévy processes are analogus to linear functions, and indeed, one can view Lévy processes as linear processes.

We give a brief review of the properties of Lévy processes and their applications to finance. We restrict our synopsis to real–valued Lévy processes, which are a special class of infinite divisible stochastic processes. For a complete treatment of Lévy processes and general semi–martingales, the reader may consult Bertoin [7], Sato [45], Applebaum [3], and Protter [41]. For applications to finance, Schoutens [46] and Cont & Tankov [16], are excellent resources. Most of the definitions and results in this section follow Applebaum [3].

2.9 Infinite Divisible Processes

Lévy processes are a special class of infinite divisible stochastic processes. Important examples are: Brownian motion/ Wiener process, Poisson, compound Poisson, Variance Gamma, CGMY, α–stable, Hyperbolic, etc.

Definition 2.8 (**Infinite Divisibility**). *A random variable X on a probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ is infinite divisible, if for each $n \in \mathbf{N}$, there exist independent and identically distributed (iid) random variables $Y_1^n, \ldots, Y_n^n$ such that*

$$X \stackrel{d}{=} Y_1^n + Y_2^n + \cdots + Y_n^n.$$

Definition 2.9 (**Characteristic Function (chf)**). *The characteristic function ϕ_X of the random variable X with Law $\mu_X(dx)$, is defined by*

$$\phi_X(u) \triangleq \mathbf{E}(e^{iuX}) = \int_{\mathbf{R}} e^{iux}\mu_X(dx), \quad i^2 = -1, \ u \in \mathcal{R}.$$

Remark 2.3. *X is infinitely divisible iff for each $n \in \mathbb{N}$, there exist a chf ϕ_n of a random variable X_n, such that $\phi_X(u) = (\phi_n(u))^n$.*

Definition 2.10 (**Lévy Measure**). *Let $v(\cdot)$ be a Borel measure defined on $\mathbf{R} - \{0\}$ such that*

$$\int_{\mathbf{R}-\{0\}} \min(1, x^2) v(dx) < \infty.$$

$v(\cdot)$ is called a Lévy measure on $\mathcal{B}(\mathbf{R} - \{0\})$ and is σ–finite, with $v((-\epsilon, \epsilon)^c) < \infty$ for all $\epsilon > 0$. An equivalent definition of a Lévy measure is any Borel measure $v(\cdot)$ satisfying

$$\int_{\mathbf{R}-\{0\}} \frac{x^2}{1+x^2} v(dx) < \infty.$$

We now state an important result which will be useful in the sequel.

Theorem 2.4 (**Lévy–Khintchine**). *Let $t \geq 0$. The process X_t is infinite divisible iff there exist γ, σ^2 and a Lévy measure v on $\mathbf{R} - \{0\}$, such that for all $u \in \mathcal{R}$,*

$$\phi_{X_t}(u) = e^{t\,\eta(u)}$$

where

$$\eta(u) = i\gamma\, u - \frac{1}{2}\sigma^2\, u^2 + \int_{\mathbf{R}-\{0\}} \left(e^{iux} - 1 - iuxI_{|x|<1}(x)\right) v(dx),$$

is called the ***Lévy exponent, characteristic exponent, or Lévy symbol****, with $\eta(u) = \log \phi_{X_1}(u)$.*

Remark 2.4.

(1) *The reader is directed to Bertoin [7], Sato [45] or Applebaum [3], for a proof.*

(2) *The triple* (γ, σ^2, v) *is called the Lévy triple or characteristic triple of* X_t.

(3) *We list triple for some commom processes:*

Gaussian: γ *is the mean,* σ^2 *is variance/volatility;* $v = 0$.

Poisson: $\gamma = 0$*;* $\sigma^2 = 0$ *;* $v = \lambda \, \delta_1$.

compound Poisson: $\gamma = 0$*;* $\sigma^2 = 0$*;* $v = \lambda \, \mu_Y$.*,*

where λ *is the arrival rate and* μ_Y *is a probability measure on* $\mathbf{R}$.

2.10 Lévy Processes

A Lévy process is any stochastic process that starts at zero, has independent and stationary increments, and is continuous in probability. Let $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a right continuous $\mathbf{P}$–complete filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t \subset \mathcal{F}$ are σ–algebras of subsets of Ω. In the sequel, all processs are assumed to be defined on this filtered space.

Definition 2.11 (**Lévy Process**).

An $\mathcal{F}$*–adapted process* $X = (X_t)_{t \geq 0}$ *is a Lévy process if:*

(1) $X_0 = 0$*, almost surely.*

(2) X *has increments independent of the past; that is* $X_t - X_s$ *is independent of* $\mathcal{F}_s$*, where* $0 \leq s < t < \infty$.

(3) X *has stationary increments; that is* $X_t - X_s$ *has the same distribution as* X_{t-s}*,* $0 \leq s < t < \infty$.

(4) X *is stochastically continuous; that is, for all* $\epsilon > 0$ *and* $s \geq 0$*,*

$$\lim_{t \to s} \mathbf{P}(|X_t - X_s| > \epsilon) = 0.$$

Equivalently, $\lim_{t \to s} X_t = X_s$*, where the limit is taken in probability.*

One may give an alternative definition of a Lévy process without using the filtration.

Definition 2.12 (Intrinsic Lévy Process).

A process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$, is a Lévy process if:

(i) X has independent increments: that is, $X_t - X_s$ is independent of $X_v - X_u$ if $(s,\, t) \cap (u,\, v) = \phi$.

(ii) X has stationary increments; that is, $X_t - X_s$ has the same distribution as $X_v - X_u$ if $t - s = v - u$.

(iii) X_t is continuous in probability: for all $s,\, t \geq 0$, $\lim_{t \to s} X_t = X_s$, in probability.

In the sequel we will assume that all Lévy process are **càdlàg**; that is, are right continuous with left limits (**RCLL**).(cf Theorem 30, Protter [41])

Proposition 2.1. *If X is a Lévy process then X_t is infinite divisible for each $t \geq 0$.*

The characteristic functions of Lévy processes have a simple form, thanks to the Lévy–Khintchine representation.

Theorem 2.5. *Let $t \geq 0$ and $u \in \mathcal{R}$. If X is a Lévy process with triple $(\gamma,\, \sigma^2,\, v)$ then its characteristic function is $\phi_{X_t}(u) = e^{t\, \eta(u)}$, where*

$$\eta(u) = i\gamma\, u - \frac{1}{2}\sigma^2\, u^2 + \int_{\mathbf{R}-\{0\}} \left(e^{iux} - 1 - iux I_{[\,|x|<1\,]}(x)\right) v(dx)$$

is the Lévy exponent of X_1.

2.10.1 Subordinators

Definition 2.13. *A subordinator $\overline{T}$ is a one–dimensional Lévy process that is non–decreasing, almost surely.*

Since a subordinator is non–negative, it can be viewed as a random model of time evolution (or business time, in finance jargon). It has no diffusive component.

Theorem 2.6. *Let* $\overline{T}$ *be a subordinator with triple* $(\gamma,\, 0,\, v)$.

(1) *Its Lévy symbol is*

$$\eta(u) = i\gamma u + \int_0^\infty (e^{iux} - 1)v(dx),$$

where $\gamma \geq 0$, *and the Lévy measure satisfies the requirements*

$$v(-\infty,\, 0) = 0 \text{ and } \int_0^\infty (x \wedge 1)v(dx) < \infty.$$

(2) *If* X *is any Lévy process then* $Z = X(\overline{T})$ *is also a Lévy process.*

2.10.2 Moments of a Lévy Process

Since a Lévy process is càdlàg (RCLL), the only type of discontinuity it posseses is a jump discontinuity. Let $X_{t-} = \lim_{s\uparrow t} X_s$ be the left limit of X at t. The jump of X at time t is given by

$$\Delta X_t \triangleq X_t - X_{t-}.$$

If $\sup_t |\Delta X_t| \leq K < \infty$ almost surely, for some non–random positive constant K, then we say that X has *bounded jumps.* Lévy processes with bounded jumps have finite moments of all orders.

Theorem 2.7. *Let* X *be a Lévy process with bounded jumps. Then for each* $t \geq 0$, $\Delta X_t = 0$ *almost surely; and for each* $n \in \mathbb{N}$,

$$\mathbf{E}(|X_t|^n) < \infty.$$

Theorem 2.8 (Moments and Cumulants of a Lévy Process). *Let* $X = (X_t)_{t\geq 0}$ *be a Lévy process with triple* $(\gamma,\, \sigma^2,\, v)$. *For each* $t \geq 0$ *and* $n \in \mathbf{N}$,

(1) $\quad \mathbf{E}(|X_t|^n) < \infty \quad$ *iff* $\int_{|x|\geq 1} |x|^n v(dx) < \infty.$

(2) $\quad$ *$\phi_t(u)$, the characteristic function of X_t, is of class $\mathcal{C}^n$, and the first n moments of X_t can be computed by differentiation:*

$$\mathbf{E}[X_t^k] = \frac{1}{i^k}\,\frac{\partial^k}{\partial u^k}\,\phi_t(u)|_{u=0},\;\; k = 1, 2, \ldots, n.$$

(3) $\quad$ *The cumulants of X_t, defined by*

$$C_k(X_t) \triangleq \frac{1}{i^k}\,\frac{\partial^k}{\partial u^k}\,\log\phi_t(u)|_{u=0},\;\; k = 1, 2, \ldots, n,$$

are given by

$$C_1(X_t) = \mathbf{E}(X_t) = t\left(\gamma + \int_{|x|\geq 1} x\,v(dx)\right),$$

$$C_2(X_t) = \mathbf{Var}(X_t) = t\left(\sigma^2 + \int_{\mathbf{R}} x^2\,v(dx)\right),$$

$$C_k(X_t) = t\int_{\mathbf{R}-\{0\}} x^k\,v(dx), \quad 3 \leq k \leq n.$$

2.11 Martingales and Semi–martingales

Definition 2.14 (**Martingale**). *A process $X = (X_t)_{t\geq 0}$ defined on a filtered space $(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is called a martingale if the following conditions hold:*

(1) $\quad$ *X_t is $\mathcal{F}_t$–measurable for each $t \geq 0$.*

(2) $\quad$ *X_t is $\mathbf{P}$–integrable; that is, $\int_\Omega |X_t(\omega)|\mathbf{P}(d\omega) < \infty$, a.s..*

(3) $\quad$ *$\mathbf{E}(X_t|\,\mathcal{F}_s) = X_s,\;\; 0 \leq s \leq t.$*

Condition (3) means that the expected value of X at time t, given its current value X_s and all previous values X_u, $u \leq s$, is simply its current value X_s. In addition, $\mathbf{E}(X_t) = E(X_0) = 0$ in the case of Lévy processes. If X is a Lévy process with characteristic exponent $\eta(u)$, then $M_u = (M_u(t))_{t\geq 0}$ defined by $M_u(t) = \exp\left(iuX_t - t\eta(u)\right)$, is a complex martingale relative to $\mathcal{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$, the natural filtration of X.

Tankov [51] showed that if X is a martingale, then the process e^X is also a martingale.

Definition 2.15 (**Finite Variation**).

The total variation of a real–valued function $f : [a,\, b] \longrightarrow \mathbf{R}$, *is defined by*

$$TV(f) = \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where π *is a finite partition* $\pi : a = t_0 < t_1 < \cdots, < t_n = b$ *of* $[a,\, b]$.

A function is of finite variation(FV) if $TV(f) < \infty$ *on all compact subsets on* $\mathbf{R}$.

Definition 2.16 (**Finite Variation Process**).

A càdlàg (RCLL) adapted process $A = (A_t)$ *is a finite variation process (FV) if almost surely, the paths of* $A : t \to A_t(\omega),\ \omega \in \Omega$ *are of finite variation on each compact interval of* $[0,\, \infty)$.

Proposition 2.2. *A Lévy process* X *is of finite variation iff its triple* $(\gamma,\, ,\sigma^2,\, v)$ *satisfies* $\sigma^2 = 0$ *and* $\int_{|x|\geq 1} |x| v(dx) < \infty$.

Standard Brownian motion does not have finite variation since $\sigma^2 \neq 0$.

Definition 2.17 (**Semi–martingale**). *A process* $X = (X_t)_{t\geq 0}$ *is a semi–martingale if it is the sum of a martingale* M_t *and a finite variation process,* A_t. *That is,*

$$X_t = M_t + A_t, \quad t \geq 0.$$

Theorem 2.9. *If* $X = (X_t)_{t\geq 0}$ *is a Lévy process, it is a semi–martingale. That is,* $X_t = M_t + A_t$ *where* M_t *is a martingale with bounded jumps and* $M_t \in L^p$, $p \geq 1$, *and* A_t *has paths of finite variation on compact subsets of* $\mathbf{R}$.

Definition 2.18 (Quadratic Variation). *The quadratic variation process of a semi–martingale X, is the adapted cadlag process $[X,\, X]_t$ defined by the prescription:*

$$[X,\, X]_t = \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}} \right)^2,$$

where π is any partition: $\pi : 0 = t_0 < t_1 < \cdots, < t_n = t$ of $[0,\, t]$, with mesh size

$$\|\pi\| = max_{1 \leq i \leq n} |t_i - t_{i-1}|.$$

Remark 2.5. *$[X,\, X]_t$ is an increasing process with jumps that are linked to the jumps of X via the formula:*

$$\Delta[X,\, X]_t = (\Delta X_t)^2, \;\; t \geq 0.$$

Theorem 2.10 (Properties of Quadratic Variation). *Let X be a semi–martingale.*

(1) *If X is continuous and has paths of finite variation, then $[X,\, X] = 0$.*

(2) *If X is a martingale and $[X,\, X] = 0$, then $X = X_0$, almost surely.*

(3) *If $X^c = (X_t^c)_{t \geq 0}$ is the continuous part of X, then*

$$[X,\, X]_t = [X^c,\, X^c]_t + \sum_{0 < s \leq t} (\Delta X_s)^2.$$

(4) *If X is a Lévy process with charcteristic triple $(\gamma,\, \sigma^2,\, v)$, then*

$$\mathbf{E}[X,\, X]_t = t \left(\sigma^2 + \int_{\mathbf{R}} x^2 \, v(dx) \right).$$

Remark 2.6. *Note that if σ_s is càdlàg, then the process $X_t = \int_0^t \sigma_s \, dB_s$ is a martingale with quadratic variation $[X,\, X]_t = \int_0^t \sigma_s^2 \, ds$.*

Definition 2.19 (Pure Jump Process). *X is a pure jump process if $[X^c,\, X^c] = 0$,* *almost surely. That is,*

$$[X,\, X]_t = \sum_{0<s\le t} (\Delta X_s)^2 = \int_0^t \int_{\mathbf{R}} x^2\, N(ds,\, dx),$$

where N is a Poisson random measure that counts the jumps of X.

Pure jump processes have no diffusive component; that is $\sigma^2 = 0$.

2.12 Random Measures and Poisson Integrals

Random measures are important integrators in stochastic integrals.

Definition 2.20 (**Random Measure**).

Let $(\mathcal{S},\, \mathcal{A})$ be a measurable space and let $(\mathbf{\Omega},\, \mathcal{F},\, \mathbf{P})$ be a probability space. A random measure $\mathcal{M}$ on $(\mathcal{S},\, \mathcal{A})$ is a collection of random variables $\{\mathcal{M}(A) : A \in \mathcal{A}\}$, such that:

(1) $\mathcal{M}(\emptyset) = 0$.

(2) $\mathcal{M}$ *is sigma–additive: For any sequence $\{A_n : n \in \mathcal{N}\}$ of mutually disjoint $\mathcal{A}$–sets, almost surely*

$$\mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{M}(A_n).$$

(3) *Independently scattered property: For each $n \in \mathcal{N}$, and disjoint family of $\mathcal{A}$–sets $\{A_1, \ldots, A_n\}$, the random variables $\mathcal{M}(A_1), \ldots, \mathcal{M}(A_n)$ are independent.*

Definition 2.21 (**Poisson Random Measure**). *A random measure $\mathcal{M}$ is a Poisson random measure if $\mathcal{M}(A)$ has a Poisson distribition for all $A \in \mathcal{A}$ with $\mathcal{M}(A) < \infty$, almost surely. It has compensator measure $\lambda(A) = \mathbf{E}\mathcal{M}(A)$.*

One can construct the compensated Poisson random measure $\widetilde{\mathcal{M}}$, which is a martingale, by subtracting the compensator or intensity measure from $\mathcal{M}$; that is,

$$\widetilde{\mathcal{M}}(A) = \mathcal{M}(A) - \lambda(A).$$

Lévy Measure

Let X be a Lévy process. Instead of examiming the jump size $\Delta X_s = X_s - X_{s-}$, we count the jumps of a particular size in a fixed interval $[0,\, t]$. For each $A \in \mathcal{B}(\mathbf{R} - \{0\})$ with $0 \notin \overline{A}$, let

$$N(t,\, A) = \#\{s \leq t : \Delta X_s \in A, \Delta X_s \neq 0\} = \sum_{0<s\leq t} I_A(\Delta X_s).$$

$N(t,\, A)$ is the number of jumps of size in the set A up to time t, and is a random counting measure with

$$\mathbf{E}[N(t,\, A)] = \int_{\Omega} N(t,\, A)(\omega)\mathbf{P}(d\omega).$$

We give an alternative, but equivalent definition of the Lévy measure of a process X.

Definition 2.22 (**Lévy Measure**). *The Lévy measure $v(\cdot)$ of the process X is defined as*

$$v(A) = \mathbf{E}[N(1, A)] = \sum_{0<s\leq 1} \mathbf{E}\, I_A(\Delta X_s),$$

where $A \in \mathcal{B}(\mathbf{R} - \{0\})$ with $0 \notin \overline{A}$. It is the average number of jumps per unit time in any bounded Borel set A.

Theorem 2.11. *Let X be a Lévy process on the filtered space $(\mathbf{\Omega},\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0}\,,\, \mathbf{P})$ and $A \in \mathcal{B}(\mathbf{R} - \{0\})$. Let $t \geq 0$.*

(1) *If A is bounded below; that is, $0 \notin A$, then $N(t, A)$ is a Poisson process with intensity $v(A) = \mathbf{E}[N(1, A)]$.*

(2) *If $A_1,\, A_2, \ldots,\, A_m \in \mathcal{B}(\mathbf{R} - \{0\})$ are disjoint Borel sets, then the random variables*

$$N(t,\, A_1),\, ,N(t,\, A_2),\, \ldots,\, N(t,\, A_m) \quad \textit{are independent .}$$

(3) $(\widetilde{N}(t,\, A)_{t\geq 0})$ *is a martingale–valued random measure, where*

$$\widetilde{N}(t,\, A) = N(t,\, A) - t\, v(A).$$

Poisson Integrals

Definition 2.23 (**Poisson Integral**). *Let f be a Borel measurable function from $\mathbf{R} \to \mathbf{R}$ and let $A \in \mathcal{B}(\mathbf{R} - \{0\})$ be bounded below. For each $t > 0, \omega \in \Omega$, we define the Poisson integral of f as the random finite sum:*

$$\int_A f(x) N(t,\, dx) = \sum_{x\in A} f(x)\, N(t,\, \{x\})(\omega) = \sum_{0<s\leq t} f(\Delta X_s) I_A(\Delta X_s).$$

Remark 2.7. $\int_A x\, N(t,\, dx) = \sum_{0<s\leq t} \Delta X_s\, I_A(\Delta X_s)$ *is the sum of all jumps in A up to time t.*

Theorem 2.12. *Let X be a Lévy process with counting measure N, and let f be a real–valued Borel measurable function on $\mathbf{R} - \{0\}$.*
(1) *For each $t \geq 0$ and $A \in \mathcal{B}(\mathbf{R}-\{0\})$, bounded below, the process $Y_t = \int_A f(x) N(t,\, dx)$ has a **compound Poisson** process with characteristic function*

$$\phi_{Y_t}(u) = \exp\,(t \int_A (e^{iux} - 1) v_f(dx)),$$

where $v_f = v \circ f^{-1}$.
(2)*If $f\, I_A$ is v–integrable; that is, $f \in L(A,\, v)$, then*

$$\mathbf{E}\left(\int_A f(x) N(t,\, dx)\right) = t \int_A f(x) v(dx).$$

(3) *If* $f\, I_A$ *is* v*–square integrable; that is,* $f^2 \in L(A,\, v)$, *then*

$$\mathbf{Var}\left(\int_A f(x)N(t,\, dx)\right) = t\int_A f^2(x)\, v(dx).$$

Definition 2.24 (**Poisson Stochastic Integral**). *Let* $N(ds,\, dx)$ *be a Poisson random measure on* $\mathbf{R}^+ \times (\mathbf{R} - \{0\})$ *with intensity measure* $ds\, v(dx)$, *where* $v(\cdot)$ *is its Lévy measure. Let* $A \in \mathcal{B}(\mathbf{R} - \{0\})$ *be bounded below and let* $X_t = \int_A x\, N(t,\, dx)$ *be the compound Poisson process that counts the jumps of* X *in* A*. Let* f *be a predictble function on* $\mathbf{R}^+ \times (\mathbf{R} - \{0\}) \times \Omega$*. The Poisson integral of* f *with respect to* N *is the random sum (also a compound Poisson process)*

$$\int_0^t \int_A f(s,\, x)\, N(ds,\, dx) \triangleq \sum_{0<s\leq t} f(s,\, \Delta X_s) I_A(\Delta X_s).$$

If f *is* v*–integrable on* A*, we define the compensated Poisson integral of* f *to be*

$$\int_0^t \int_A f(s,\, x)\, \widetilde{N}(ds,\, dx) \triangleq \int_0^t \int_A f(s,\, x)\, N(ds,\, dx) - \int_0^t \int_A f(s,\, x)\, v(dx)ds.$$

We have a similar result to the last.

Theorem 2.13. *Let* X *be a Lévy process with counting measure* N*, and* f *a real–valued Borel measurable function on* $\mathbf{R}^+ \times (\mathbf{R} - \{0\}) \times \Omega$.

(1) *For each* $t \geq 0$ *and* $A \in \mathcal{B}(\mathbf{R} - \{0\})$, *and predictable function* f *on* $\mathbf{R}^+ \times (\mathbf{R} - \{0\}) \times \Omega$,

$$\int_0^t \int_A f(s,\, x)\, N(ds,\, dx),$$

is a compound Poisson process.

(2) *If* $f\, I_A$ *is* v*–integrable; that is,* $f \in L(A,\, v)$, *then*

$$\mathbf{E}\left(\int_0^t \int_A f(s,\, x)N(ds,\, dx)\right) = \int_0^t \int_A f(s,\, x)v(dx)ds.$$

(3) *If* $f\, I_A$ *is* v*–square integrable; that is,* $f^2 \in L(A,\, v)$*, then the following process is a martingale:*

$$\int_0^t \int_A f(s,\, x)\, \widetilde{N}(ds,\, dx).$$

2.13 Lévy–Itô Decomposition

Definition 2.25 (**Compensated Poisson Process**)**.** *Let* $N = (N_t)_{t\geq 0}$ *be a Poisson process with intensity* λ*. That is,* $\mathbf{P}(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},\ k = 0, 1, 2, \cdots, \infty.$
The compensated Poisson process $\widetilde{N} = (\widetilde{N}_t)_{t\geq 0}$ *is the martingale given by*

$$\widetilde{N}_t = N_t - \lambda\, t.$$

We now state an extremely important result for Lévy processes; see Bertoin [7], or Applebaum [3], Theorem 2.4.16, for a proof.

Theorem 2.14 (**Lévy–Itô Decomposition**)**.** *Let* X *be a Lévy process. Then there exist* $\gamma,\, \sigma^2 > 0$*, standard Brownian motion* B_t*, and independent Poisson random measure* N *on* $\mathbf{R}^+ \times (\mathbf{R} - \{0\})$*, such that for each* $t \geq 0$

$$X_t = \gamma\, t + \sigma\, B_t + \int_{|x|<1} x\, \widetilde{N}(t,\, dx) + \int_{|x|\geq 1} x\, N(t,\, dx),$$

where

$$\gamma = \mathbf{E}\left(X_1 - \int_{|x|\geq 1} x\, N(1,\, dx)\right),$$

and $v(\cdot)$ *is the Lévy measure of* X *with triple* $(\gamma,\, \sigma^2,\, v)$*.*

The Lévy–Khintchine formula now follows directly from the Lévy–Itô decomposition theorem. We restate it as

Corollary 2.1. *If* X *is a Lévy process with triple* $(\gamma,\, \sigma^2,\, v)$*, then its characteristic* *function* $\phi_{X_t}(u)$, $u \in \mathcal{R}$, *is*

$$\phi_{X_t}(u) = \exp\left(t\left\{i\gamma\, u - \frac{1}{2}\sigma^2\, u^2 + \int_{\mathbf{R}-\{0\}} \left(e^{iux} - 1 - iuxI_{|x|<1}(x)\right) v(dx)\right\}\right).$$

Remark 2.8.

(1) *The process* $\left(\int_{|x|\le 1} x\, \widetilde{N}(t,\, dx)\right)_{t\ge 0}$ *is the compensated sum of "**small**" jumps and is a martingale.*

(2) *The process* $\left(\int_{|x|\ge 1} x\, N(t,\, dx)\right)_{t\ge 0}$ *is the sum of "**large**" jumps, and is a compound Poisson process.*

A direct consequence of the Lévy–Itô decomposition is:

Theorem 2.15. *Let* $X = (X_t)_{t\ge 0}$ *be a Lévy process with triple* $(\gamma,\, \sigma^2,\, v)$. *Then* X *is a linear combination of a Brownian motion* B, *an independent compound Poisson process, and a finite variation process. That is, for each* $t \ge 0$

$$X_t = \gamma_0\, t + \sigma\, B_t + \int_{-\infty}^{\infty} x\, N(t,\, dx) = X_t^c + \int_{-\infty}^{\infty} x\, N(t,\, dx),$$

where $N(t,\cdot)$ *is a Poisson process independent of* B, *and* $\gamma_0 = \gamma - \int_{|x|<1} x\, v(dx)$ *is the* ***drift*** *of the process and* X^c *is the continuous part of* X.

Remark 2.9.

(1) *The process* $Y_1(t) = X_t - \int_{|x|\ge 1} x\, N(t,\, dx)$ *has bounded jumps and consequently, has moments of all orders. However,* $\int_{|x|\ge 1} x\, N(t,\, dx)$ *may have no finite moments, as in the case when* X *is* α*–stable,* $0 < \alpha < 1$.

(2) *Every pure jump process* X *has decomposition*

$$X_t = \gamma_0\, t + \int_{-\infty}^{\infty} x\, N(t,\, dx) = \gamma_0\, t + \sum_{0<s\le t} \Delta X_s.$$

2.14 Itô Formula

The Itô formula allows Lévy processes and other general semi–martingales to be transformed by smooth ($\mathcal{C}^2$) functions. For a proof , see Protter [41], Theorem 32.

Theorem 2.16 (**Itô–Formula**). *Let X be a semi–martingale and let f be a real $\mathcal{C}^2$ function. Then $f(X)$ is again a semi–martingale, with*

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X^c,\, X^c]_s$$

$$+ \sum_{0\le s\le t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

In differential form, we get

$$df(X_s) = f'(X_{s-})dX_s + \frac{1}{2}f''(X_{s-})d[X^c,\, X^c]_s + \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

We apply Theorem 2.16 directly to Lévy processes (cf Tankov [51]).

Theorem 2.17 (**Itô Formula for Scalar Lévy Processes**). *Let $X = (X_t)_{t\ge 0}$ be a Lévy process with triple $(\gamma,\, \sigma^2,\, v)$ and $f : \mathbf{R} \to \mathbf{R}$ be a $\mathcal{C}^2$ function. Then*

$$f(X_t) = f(0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t \sigma^2 f''(X_{s-})ds$$

$$+ \sum_{0\le s\le t} \{f(X_{s-} + \Delta X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

2.14.1 Stochastic Exponential

The Itô change of variable formula applied to simple, non–trivial, stochastic differential equations, yield the Doleans–Dade or Stochastic Exponential of a semi–martingale X (cf Protter [41], Theorem 37, of Chapter II). This result will be used extensively in the sequel.

Theorem 2.18. *Let X be a semi–martingale with $X_0 = 0$. There exists a unique càdlàg semi–martingale Z that satisfies the equation*

$$Z_t = 1 + \int_0^t Z_{s-} dX_s \Longleftrightarrow dZ_t = Z_{t-}\, dX_t, \;\; Z_0 = 1,$$

given by any one of the equivalent formulas:

(1) $\qquad Z_t = Z_0 \exp\left\{X_t - \frac{1}{2}[X,\, X]_t\right\} \Pi_{0<s\leq t}(1+\Delta X_s)\, e^{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2}.$

(2) $\qquad Z_t = Z_0 \exp\left\{X_t - \frac{1}{2}[X^c,\, X^c]_t\right\} \Pi_{0<s\leq t}(1+\Delta X_s)\, e^{-\Delta X_s}.$

(3) $\qquad Z_t = Z_0 \exp\left\{X_t^c - \frac{1}{2}[X^c,\, X^c]_t\right\} \Pi_{0<s\leq t}(1+\Delta X_s).$

Definition 2.26 (Stochastic Exponential).

For a semi–martingale X with $X_0 = 0$, the stochastic exponential of X denoted by $\mathcal{E}(X)$, is the unique semi–martingale Z, that is the solution of the equation:

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

Equivalently, $\mathcal{E}(X)$ is the unique solution of the SDE

$$dZ_t = Z_{t-}\, dX_t, \;\; Z_0 = 1.$$

Tankov [51] proved that the stochastic exponential of a Lévy process that is a martingale, is also a martingale. This is an important martingale preserving property that is encapsulated in the following theorem.

Theorem 2.19 (Martingale Preserving Property).

(1) *If X is a Lévy process and a martingale then $\mathcal{E}(X)$, its Doleans–Dade (Stochastic) exponential, is also a martingale.*

(2) *X is a martingale if and only if $\int_{|x|\geq 1} |x| v(dx) < \infty$ and $\gamma + \int_{|x|\geq 1} x\, v(dx) = 0$, where X has triple $(\gamma,\, \sigma^2,\, v)$.*

2.15 Part-III: Lévy Processes in Finance

The most famous continuous–time financial model is the celebrated Black–Scholes model (cf [10], [37], [38]), which uses the normal distribution to fit the log returns of a stock price S that has dynamic:

$$d(\log S_t) = (\mu - \frac{1}{2}\sigma^2)\, dt + \sigma\, dB_t.$$

This is equivalent to the percentage returns model:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t.$$

One of the main problems with the Black–Scholes [10] model is that empirical studies (cf Akgiray and Booth [2]) prove that log returns of stock/indices are not normally distributed, and indeed, the log returns of most financial assets do not follow a normal law. They are actually skewed and have kurtosis higher than that of the normal distribution. That is, they exhibit the so called **leptokurtic** feature–having higher peaks and thicker tails. New models were therefore required.

To be useful in finance, a process must be able to represent jumps, skewness, excess kurtosis, be infinite divisible, with independent and stationary increments. Lévy processes have all these desirable properties, and in the late 1980s Lévy models were proposed for modelling financial data. Examples of such processes are: compound Poisson, Kou and Merton Jump diffusions; Variance Gamma (VG), Normal Inverse Guassian (NIG), CGMY (Carr, Geman, Madan & Yor), Tempered Stable, Generalized Hyperbolic processes. There are many others.

Madan and Seneta [35] proposed a Lévy process with VG increments. The Hyperbolic model was proposed by Eberlein and Keller [18], while Barndorff–Nielson [4] proposed the NIG process. The Generalized Hyperbolic process, of which the above– mentioned are examples, was developed by Eberlein and Prause [19]. The CGMY model was introduced by Carr et al [12], while Schoutens [46] developed the Meixner model. The reader is directed to Schoutens [46] for details of these and other models.

2.15.1 Exponential Lévy Market

Instead of modelling the log returns with a Brownian motion with drift, as in the classical Black–Scholes model of asset price, we replace it with a Lévy process $X = (X_t)_{t\geq 0}$. Our market consist of one riskless asset B, called bond or savings account, which earns the risk–free continuously compounded interest rate r, and one risky asset S, called stock. The model for this market, called an exponential Lévy market, is:

$$\mathbf{B}_t = e^{r\,t}, \quad S_t = S_0 e^{X_t},$$

where X_t is a Lévy process on a filtered probability space $(\mathbf{\Omega},\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0}\,,\, \mathbf{P})$. The log returns $\log(\frac{S_{t+s}}{S_t})$ of this model follow the distribution of the increments $(X_{t+s} - X_t)$, of length s of the Lévy process X.

2.16 Exponential Lévy Models in Finance

Financial models with jumps are of two basic types– jump diffusion/finite activity models and infinite activity models. It is called a finite activity model if $\int_{\mathbf{R}} v(dx)$ is finite, and infinite activity, otherwise.

2.16.1 Jump–Diffusion/ Finite Activity Models

A Lévy process of the jump–diffusion type has the form:

$$X_t = \gamma\, t + \sigma\, B_t + \sum_{i=1}^{N_t} Y_i,$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, that counts the jumps of X. Y_i are jump sizes that are iid random variables with law F_Y. The Lévy triple of X is $(\gamma,\, \sigma^2,\, \lambda\, F_Y)$. Log prices are linear combination of a Brownian motion B_t, with drift, and a compound Poisson process. The jumps represent rare events, such as market crashes and large sell–offs. There are a finite number of jumps in any finite time interval $[0,\, t]$, and hence the reason why it is called a finite activity process.

Merton Model

The Merton [38] model is the first model of this type found in the literature. Jumps in the log–price Y are assumed to have a Gaussian distribution with mean m and variance δ^2, with $Y_i \sim N(m,\, \delta^2)$. The Lévy measure of X is

$$v(dx) = \lambda\, F_Y(dx) = \frac{\lambda}{\sqrt{2\,\pi\,\delta^2}}\, e^{-\frac{1}{2}\left(\frac{x-m}{\delta}\right)^2} dx.$$

This model has heavier tails ($kurtosis > 3$) than the Gaussian distribution.

Kou Model

The Kou [30] jump–diffusion model has log jump size Y distributed according to an asymmetric Laplace law. Its Lévy density is:

$$v_{kou}(x) = p\,\lambda_+\, e^{-\lambda_+\, x} I_{\{x>0\}}(x) + (1-p)\,\lambda_- e^{-\lambda_-\,|x|} I_{\{x<0\}}(x),$$

where $\lambda_{\pm} > 0$ are parameters governing the decay of the tails for the distribution of positive and negative jump sizes. $p \in [0, 1]$ is the probability of an upward jump. The probability distribution of returns in this model has semi–heavy tails.

The symmetric Kou model (cf Lewis [33]) is a double–exponential model with Poisson–driven jumps, with Lévy density:

$$v_0(x) = \frac{1}{2\,\eta}\, e^{-\frac{|x-\kappa|}{\eta}}, \quad 0 < \eta < 1.$$

Because of the memoryless property of the exponential random variables, the Kou model allows for tractable option pricing formulas, which is a big advantage over the Merton model.

Dirac Model

This is a simple jump–diffusion model, which has only one possible jump amplitide x_0. Its Lévy density is the Dirac measure δ_{x_0}, where

$$v(dx) = \delta_{x_0}(dx).$$

For all jump–diffusion models, $\int_{\mathbf{R}} v(dx) < \infty$, and hence are finite activity models.

2.16.2 Infinite Activity Models

These models exhibit an infinite number of small jumps in every finite time interval, and are called infinite activity or infinite intensity models and so, $\int_{\mathbf{R}} v(dx) = \infty$. Many of these models can be constructed via Brownian subordination; that is, by taking a Brownian motion with drift $(\gamma\, t + \sigma\, B_t)$ and sampling it at random times (

business time) of a Lévy subordinator $\overline{T}$, thereby producing a new Lévy process

$$X_t = \gamma\, \overline{T_t} + \sigma\, B_{\overline{T_t}}.$$

Examples are the Variance Gamma (VG) and Normal Inverse Guassian(NIG) processes. Others are specified by their Lévy densities; for example, CGMY and Tempered Stable processes.

Variance Gamma (VG)

The VG process is built by subordinating a Brownian motion with drift, by a Gamma suborninator. It's a finite variation process with infinite, but relatively low activity of small jumps. It can also be expressed as the difference of two independent Gamma processes; that is,

$$X_t^{VG} = \Gamma_t^1 - \Gamma_t^2.$$

Its Lévy density is

$$v(x) = \frac{C}{|x|}\, e^{-\lambda_-\, |x|} I_{\{x<0\}}(x) + \frac{C}{x}\, e^{-\lambda_+\, x} I_{\{x>0\}}(x),$$

where

$$C = \frac{1}{\kappa} \quad \text{and} \quad \lambda_\pm = \frac{\sqrt{\theta^2 + \frac{2\,\sigma^2}{\kappa}} \pm \theta}{\sigma^2}.$$

Carr, Geman, Madan, Yor (CGMY) Process

The CGMY process is named after its creators, Carr et al [12]. It depends on four parameters C, M, G, Y, with Lévy density

$$v(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x),$$

where $C, M, G > 0$ and $Y < 2$, but $Y \neq 1$. The behaviour of this process is controlled by the stability parameter Y. If $Y < 0$, the paths have finite jumps in any finite interval; that is, the process has finite activity. If $Y \geq 0$, the process exhibits infinitely many jumps in any finite time interval; that is, it has infinite activity. If $Y = 0$, the CGMY reduces to the VG process. If $Y \in (1, 2)$, the process is of infinite variation. The CGMY process was built by Carr, Geman, Madan and Yor [12], by adding the extra parameter Y into the VG process, which allows for finite or infinite activity, and finite or infinite variation.

Tempered Stable Process (TS)

The Tempered stable process generalizes the CGMY process, and is obtained by taking a 1–dimensional stable process and multiplying its Lévy measure by a decreasing exponential on each half of the real axis. After this exponential softening, the small jumps keep their initial stable–like behaviour, while the large jumps become much less violent. It has Lévy density

$$v_{TS}(x) = \frac{C_-}{|x|^{1+\alpha}}\, e^{-\lambda_-\,|x|} I_{\{x<0\}}(x) + \frac{C_+}{x^{1+\alpha}}\, e^{-\lambda_+\,x} I_{\{x>0\}}(x),$$

where $C_\pm > 0$, $\lambda_\pm > 0$, and $\alpha < 2$, where $\alpha \notin \{0, 1\}$ is the stability parameter (cf Cont [16]).

Chapter 3

Asymmetric Information in Continuous Fads Models in Lévy Markets

3.1 Introduction

We develop a theory of fads models under asymmetric information in a general Lévy market X where the fads/mispricing are modeled by a continuous Ornstein–Uhlenbeck process U. As in the purely continuous case reviewed in Chapter 2, there are two investor classes consisting of informed and uninformed investors. The informed investor indexed by $i = 1$, has knowledge of both the stock's price and its fundamental or true value. Consequently, the informed investor has knowledge of the mispricing or fads in the stock price at each time t, in the investment period $[0, T]$.

The uninformed investor, indexed by $i = 0$, has knowledge of the stock price only. Although this investor is aware of the existence of fads, they cannot be observed directly. This investor therefore resorts to the use of technical analysis to assist in trading. Parameters and other characteristics of informed and uninformed investors, are indexed by "1" and "0", respectively.

The market is driven by a Brownian motion and a general pure jump Lévy process with triple $(\gamma, 0, v)$, where $\gamma = \int_{|x|\leq 1} xv(dx)$. The market consist of two assets. A bond $\mathbf{B}$ with price

$$\mathbf{B}_t = \exp\left(\int_0^t r_s \, ds\right), \tag{3.1.1}$$

where r_t is the continuously compounded risk–free interest rate, T is the investment horizon and $t \in [0, T]$. There is also a risky asset S called stock. The stock price is viewed by investors in disjoint classes, populated by uninformed and informed investors, and have individual dynamics relative to these investors. Investors have filtrations $\mathcal{H}_t^i$ contained in $\mathcal{F}$, defined in Chapter 2, where

$$\mathcal{H}_t^0 \subset \mathcal{H}_t^1 \subset \mathcal{F}, \; t \in [0, T].$$

All random objects are defined on a filtered probability space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{H}^i, \mathbf{P})$.
We begin by developing the model for the informed investor.

3.2 The Model

Although each investor observes the same stock price S, its dynamic depends on the filtration of the observer. For the informed investor the stock has log returns dynamic:

$$\begin{aligned}
d(\log S_t) &= (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dY_t + dX_t, \; t \in [0, T], && (3.2.1)\\
Y_t &= p\,W_t + q\,U_t, \;\; p^2 + q^2 = 1, \; p \geq 0, \; q \geq 0, && (3.2.2)\\
dU_t &= -\lambda\,U_t\,dt + dB_t, \; \lambda > 0, \; U_0 = 0, && (3.2.3)\\
X_t &= \int_0^t \int_{\mathbf{R}} xN(ds, dx), && (3.2.4)
\end{aligned}$$

where W and B are independent standard Brownian motions independent of X, while $U = (U_t)$ is a mean–reverting **Ornstein–Uhlenbeck** process with rate λ, and X is a pure jump Lévy process having a σ-finite Lévy measure v on $\mathcal{B}(\mathbf{R} - \{0\})$, with triple $(\gamma, 0, v)$ where $\gamma = \int_{|x|<1} xv(dx)$. N is a Poisson random measure on $\mathbf{R}_+ \times \mathcal{B}(\mathbf{R} - \{0\})$ that is linked to the stock. It counts the jumps of X in the time interval $(0,\, t)$.

The returns of the stock has three components; a continuous component $\mu_t^* = \mu_t - \frac{1}{2}\sigma_t^2$, a diffusive component $\sigma_t Y_t$ which is random, and a discontinuous component dX_t, which is also random. The continuous component of the stock's return μ_t^* and the volatility σ_t, are assumed to be deterministic functions with

$$\sigma = \lim_{t\to\infty} \sigma_t = \sigma_\infty > 0. \tag{3.2.5}$$

In practice σ_t will be taken to be a constant and so (3.2.5) will hold. The process $Y = (Y_t)_{t\geq 0}$, the continuous random component of excess return, and the mean–reverting O–U process $U = (U_t)_{t\geq 0}$ representing the fads, are defined exactly as in Guasoni [21], where U satisfies the Langevin stochastic differential equation (3.2.3). This admits a unique solution

$$U_t = U_0\, e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s = \int_0^t e^{-\lambda(t-s)} dB_s, \tag{3.2.6}$$

with

$$\mathbf{E}U_t = 0, \quad \text{and} \quad \mathbf{E}U_t^2 = \mathbf{Var}(U_t) = \frac{1}{2\lambda}(1 - e^{-2\lambda t}). \tag{3.2.7}$$

By the Lévy–Itô decomposition theorem (cf Theorem 2.14), if X has triple $(\gamma, 0, v)$

with $\gamma = \int_{|x|<1} x\, v(dx)$, it has a representation:

$$\begin{aligned}
X_t &= \gamma t + \int_{|x|<1} x\widetilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx), \\
&= \gamma t - t\int_{|x|<1} xv(dx) + \int_{\mathbf{R}} xN(t,dx), \\
&= t\left(\gamma - \int_{|x|<1} xv(dx)\right) + \int_{\mathbf{R}} xN(t,dx), \\
&= \int_{\mathbf{R}} xN(t,dx),
\end{aligned} \tag{3.2.8}$$

whence $dX_t = \int_{\mathbf{R}} xN(dt,dx)$, and equation (3.2.1) now becomes

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dY_t + \int_{\mathbf{R}} xN(dt,dx),\ t \in [0,\, T] \tag{3.2.9}$$

Applying Itô's formula (cf Theorem 2.14) to (3.2.9) yields percentage returns:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dY_t + \int_{\mathbf{R}} (e^x - 1)N(dt,dx),\ t \in [0,\, T]. \tag{3.2.10}$$

Under the compensated Poisson random measure $\widetilde{N}(t,A) \triangleq N(t,A) - tv(A)$, $A \in \mathcal{B}(\mathbf{R} - \{0\})$, we get (3.2.10) in **semi–martingale** from:

$$\frac{dS_t}{S_t} = b_t dt + \sigma_t dY_t + \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt,dx),\ t \in [0,\, T], \tag{3.2.11}$$

where

$$b_t = \mu_t + \int_{\mathbf{R}} (e^x - 1)v(dx) = \mu_t + M_1, \tag{3.2.12}$$

is the **total** expected instantaneous returns of the stock, and x is the log jump amplitude of the process X.

3.2.1 Filtration of the Informed Investor

Recall that all random objects for the i–th investor live on a filtered space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{H}^i, \mathbf{P})$. For the informed investor, we take $\mathcal{H}^1 = (\mathcal{H}_t^1)_{t\geq 0}$ to be the completed filtration generated by W, B and X, augmented by the $\mathbf{P}$–null sets of $\mathcal{F}$. That is, the filtration of the informed investor is

$$\mathcal{H}_t^1 \triangleq \sigma\left(W_s, B_s, X_s : s \leq t\right) \vee \sigma\left(\mathcal{N}\right) = \mathcal{F}_t^1 \vee \sigma(X_s : s \leq t), \tag{3.2.13}$$

where

$$\mathcal{N} = \{D \subset \Omega : \exists\, A \in \mathcal{F}, D \subset A, \mathbf{P}(A) = 0\}$$

and $\mathcal{F}_t^1 = \sigma\left(W_s, B_s : s \leq t\right) \vee \sigma\left(\mathcal{N}\right)$ is previously given in Chapter 2.

For the informed investor, $\mathcal{H}_t^1$ is equal to $\sigma(S_u,\, U_u : u \leq t)$ or $\sigma(W_s,\, B_s,\, X_s : s \leq t)$.

3.3 Asset Price Dynamic for the Informed Investor

Define a proces Z by the following prescription: for each $t \in [0,\, T]$, set

$$Z_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dY_s + \int_0^t \int_{\mathbf{R}} (e^x - 1) N(ds, dx). \tag{3.3.1}$$

Then Z is a semi–martingale with

$$dZ_t = b_t dt + \sigma_t dY_t + \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt, dx), \tag{3.3.2}$$

where b_t is the total expected returns on the stock. Equation (3.2.11) can now be written as

$$dS_t = S_t\, dZ_t, \;\; Z_0 = 0. \tag{3.3.3}$$

By Theorem 2.18, S is the **Stochastic or Dolean–Dades Exponential** of Z. Explicitly, (cf Protter [41], Theorem 37, of Chapter II)

$$\begin{aligned} S_t &= S_0 \exp\left(Z_t - \frac{1}{2}[Z^c,\, Z^c]_t\right) \Pi_{0\le s\le t}(1+\Delta Z_s)e^{-\Delta Z_s} \\ &= S_0 \exp\left(Z_t^c - \frac{1}{2}[Z^c,\, Z^c]_t\right) \Pi_{0\le s\le t}(1+\Delta Z_s), \end{aligned} \tag{3.3.4}$$

where

$$Z_t^c = \int_0^t \mu_s ds + \int_0^t \sigma_s dY_s \tag{3.3.5}$$

is the **continuous** part of Z, having **quadratic variation**

$$[Z^c, Z^c]_t = \int_0^t \sigma_s^2 d[Y,Y]_s,\ \ t\in[0,\, T]. \tag{3.3.6}$$

Define the martingale J_t by the prescription:

$$J_t \triangleq \int_0^t \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(ds, dx). \tag{3.3.7}$$

J is a pure jump process with $\mathbf{E}J_t = 0,\ \ t\in[0,T]$. We have the following:

Lemma 3.1. *Let Y and J be give by (3.2.2) and (3.3.7), respectively and*

$$Z_t = \int_0^t b_s ds + \int_0^t \sigma_s dY_s + J_t,\ \ t\in[0,\, T]. \tag{3.3.8}$$

Then

$$[Z^c, Z^c]_t = \int_0^t \sigma_s^2 ds,\ \ t\in[0,\, T] \tag{3.3.9}$$

Proof.

$$[Z^c, Z^c]_s = \int_0^t \sigma_s^2 d[Y,Y]_s,$$

and for each $s \in [0,\, T]$, B_s, W_s and U_s are independent. Thus

$$\begin{aligned}
[Y\,,Y]_s &= [p\,W + q\,U\,, p\,W + q\,U]_s \\
&= [pW\,,pW]_s + [pW\,,qU]_s + [qU\,,pW]_s + [qU\,,qU]_s \\
&= p^2[W\,,W]_s + pq[W\,,U]_s + qp[U\,,W]_s + q^2[U\,,U]_s \\
&= p^2 s + 0 + 0 + q^2 s \\
&= (p^2+q^2)s = s. \qquad \square
\end{aligned}$$

From the definition of Poisson random integrals (cf Applebaum [3]),

$$\begin{aligned}
J_t &\triangleq \int_0^t \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(ds, dx) \\
&= \sum_{0\le s\le t} (e^{\Delta X_s} - 1) - t\int_{\mathbf{R}} (e^x - 1)v(dx),
\end{aligned}$$

whence

$$\Delta J_s = e^{\Delta X_s} - 1 = \Delta Z_s, \tag{3.3.10}$$

which ensures that

$$\Delta J_t \ge -1, \text{ almost surely.}$$

This is a necessary requirement to obtain non-negative stock prices. From (3.3.4) and Lemma 3.1, we get

$$\begin{aligned}
S_t &= S_0 \,\exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dY_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right)\Pi_{0<s\le t}(1+\Delta J_s) \\
&= S_0 \,\exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dY_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right)\Pi_{0<s\le t}\, e^{\Delta X_s} \\
&= S_0 \,\exp\left(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dY_s + X_t\right), \qquad (3.3.11)
\end{aligned}$$

where X is the driving Lévy process in (3.2.1) and (3.2.8).

We have the following result for the stock price from the perspective of the informed investor:

Lemma 3.2. *There exists a Brownian motion $B^1 = (B^1_t)_{t\geq 0}$ adapted to $\mathcal{H}^1$, the filtration of the informed investor, such that the stock S with dynamic (3.2.1), has percentage returns dynamic given by*

$$\frac{dS_t}{S_t} = \mu^1_t dt + \sigma_t dB^1_t + \int_{\mathbf{R}} (e^x - 1) N(dt, dx),\ t \in [0, T], \tag{3.3.12}$$

and price

$$S_t = S_0 \exp\left(\int_0^t (\mu^1_s - \frac{1}{2}\,\sigma^2_s) ds + \int_0^t \sigma_s\, dB^1_s + X_t \right), \tag{3.3.13}$$

where W and B are independent $\mathcal{H}^1$–adapted Brownian motions.

Proof. Set $\mu^1_t = \mu_t + \upsilon^1_t\,\sigma_t$, $\upsilon^1_t = -\lambda q\, U_t$, $B^1_t \overset{\triangle}{=} p\, W_t + q\, B_t$, and $p^2 + q^2 = 1$.
For the informed investor $Y_t = p\, W_t + q\, U_t$ and from (3.2.3)

$$\begin{aligned} dY_t &= p\, dW_t + q\, dU_t \\ &= p\, dW_t + q\,(-\lambda dU_t dt + dB_t) \\ &= -\lambda\, p\, U_t dt + p\, dW_t + q\, dB_t \\ &= -\lambda\, p\, U_t dt + d(p\, W_t + q\, B_t), \end{aligned}$$

whence

$$dY_t = \upsilon^1_t\, dt + dB^1_t. \tag{3.3.14}$$

Since W and B are $\mathcal{F}^1$–adapted, so is $B^1 = p\, W + q\, B$. Since $\mathcal{F}^1 \subset \mathcal{H}^1$, then B^1 is $\mathcal{H}^1$–adapted. Moreover

$$[B^1, B^1]_t = [p\, W + q\, B\,, p\, W + q\, B]_t = p^2\,[W\,, W]_t + 2p\, q\,[W\,, B]_t + q^2\,[B\,, B]_t$$

$$= p^2 t + 0 + q^2 t = (p^2 + q^2)t = t.$$

By the Lévy Characterization Theorem (cf Applebaum[3] or Protter [41]), B^1 is Brownian motion. Imposing (3.3.14) on equation (3.3.11) yields

$$\begin{aligned}
S_t &= S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s\, dY_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + X_t\right) \\
&= S_0 \exp\left(\int_0^t \mu_s ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s (v_s^1 + dB_s^1) ds + X_t\right) \\
&= S_0 \exp\left(\int_0^t (\mu_s + v_s^1 \sigma_s) ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s\, dB_s^1 + X_t\right) \\
&= S_0 \exp\left(\int_0^t \mu_s^1 ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s\, dB_s^1 + X_t\right). \qquad \square
\end{aligned}$$

3.3.1 Portfolio and Wealth Processes of the Informed Investor

Recall that all random objects for the i–th investor live on a filtered space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{H}^i, \mathbf{P})$.

Definition 3.1 (**Portfolio Process**). *The process $\pi \equiv \pi^1 : [0,\, T] \longrightarrow \mathbf{R}$ is called the portfolio process of the informed investor, if π_t is $\mathcal{H}_t^1$–adapted for each $\omega \in \Omega$ and $\mathbf{E}\int_0^T (\pi_t \sigma_t)^2 dt < \infty$.*

Note that π_t is really $\pi_t(\omega), \omega \in \Omega$ and hence, is a random process. π_t^1 is the proportion of the wealth of the informed investor that is invested in the risky asset (the stock) at time $t \in [0, T]$. The remainder $1 - \pi_t^1$, is invested in the bond or money market. Where it is clear, we drop the superscript "1" and simply use π for the portfolio process.

The Wealth Process

The wealth process for the informed investor is $V^{1,\pi,x} : [0,\, T] \longrightarrow \mathbf{R}$ where $V_t^{1,\pi,x}$ is the value of the portfolio consisting of the stock and bond at time t, when π_t is invested in the stock. The initial capital is $x > 0$. For brevity, we denote this process by $V^{1,\pi} = (V_t^{1,\pi})$ or simply $V^1 = (V_t^1)$, $t \in [0, T]$ when the context is clear.

Admissible Portfolio

π is called an admissible portfolio if $V_t^\pi > 0$ almost surely, for all $t \in [0,\, T]$.

3.4 The Dynamic of the Wealth Process for the Informed Investor

Let V_t^1 be the wealth of the informed investor at time t resulting from the investment of π_t in the stock. Assume that the bond earns continuously compounded risk–free interest rate r_t. Let n_t be the number of stocks in the portfolio at time t. Then $\pi_t = \frac{n_t S_{t-}}{V_{t-}^1}$, where V_{t-}^1 is the value of the portfolio just before time t. It follows that

$$\begin{aligned} dV_t^1 &= (1-\pi_t)V_{t-}^1\, r_t\, dt + n_t dS_t \\ &= (1-\pi_t)V_{t-}^1\, r_t\, dt + \pi_t\, V_{t-}^1\, \frac{dS_t}{S_{t-}}. \end{aligned} \tag{3.4.1}$$

Therefore

$$\frac{dV_t^1}{V_{t-}^1} = (1-\pi_t)\, r_t\, dt + \pi_t \frac{dS_t}{S_{t-}}. \tag{3.4.2}$$

We have the following result for the wealth dynamic of the informed investor.

Theorem 3.1. *If the stock's percentage returns dynamic for the informed investor is*

$$\frac{dS_t}{S_t} = b_t\, dt + \sigma_t\, dY_t + dJ_t, \;\; t \in [0,\, T], . \tag{3.4.3}$$

wher J is given by (3.3.7), then the wealth process V^1 has dynamic

$$\frac{dV_t^1}{V_{t-}^1} = (M_1\pi_t + \pi_t\,\sigma_t\,\theta_t^1 + r_t)\,dt + \pi_t\,\sigma_t\,dB_t^1 + \pi_t dJ_t, \tag{3.4.4}$$

and discounted value $\widetilde{V}_t^1 = V_t^1 \exp\left(-\int_0^t r_s ds\right)$ *given by*

$$\begin{aligned}\widetilde{V}_t^1 &= V_0^1 \exp\left(\int_0^t (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^1\right)\Pi_{0\le s\le t}(1+\pi_s\Delta J_s), \quad (3.4.5)\\ &= V_0^1 \exp\left(\int_0^t (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^1\right)\Pi_{0\le s\le t}(1+\pi_s(e^{\Delta X_s}-1)).\end{aligned}$$

Proof. From equations (3.2.11) and (3.3.7), we have

$$\frac{dS_t}{S_t} = b_t\,dt + \sigma_t\,dY_t + dJ_t,\ \ t \in [0,\,T]. \tag{3.4.6}$$

Thus

$$\begin{aligned}\frac{dV_t^1}{V_{t-}^1} &= (1-\pi_t)r_t dt + \pi_t(b_t dt + \sigma_t dY_t + dJ_t)\\ &= ((1-\pi_t)r_t + \pi_t b_t)dt + \pi_t\sigma_t dY_t + \pi_t dJ_t\\ &= ((b_t - r_t)\pi_t + r_t)dt + \pi_t\sigma_t dY_t + \pi_t dJ_t. \qquad (3.4.7)\end{aligned}$$

From (3.3.14), with $b_t = \mu_t + \int_{\mathbf{R}}(e^x-1)v(dx) = \mu_t + M_1$, we get

$$\begin{aligned}\frac{dV_t^1}{V_{t-}^1} &= ((b_t - r_t)\pi_t + r_t)dt + \pi_t\sigma_t(dB_t^1 - \lambda\,q\,U_t\,dt) + \pi_t dJ_t\\ &= ((b_t - r_t - \lambda\,q\,U_t\sigma_t)\pi_t + r_t)dt + \pi_t\sigma_t dB_t^1 + \pi_t dJ_t\\ &= ((M_1 + \mu_t - r_t - \lambda\,q\,U_t\sigma_t)\pi_t + r_t)dt + \pi_t\sigma_t dB_t^1 + \pi_t dJ_t\\ &= ((M_1 + \mu_t^1 - r_t)\pi_t + r_t)dt + \pi_t\sigma_t dB_t^1 + \pi_t dJ_t\\ &= (M_1\pi_t + \pi_t\sigma_t\theta_t^1 + r_t)dt + \pi_t\sigma_t dB_t^1 + \pi_t dJ_t.\end{aligned}$$

By applying Theorem 2.18 on Stochastic/Doleans exponentials to the previously established equation (3.4.4), we get (3.4.5). $\square$

From the foregoing, we may effectively set the interest rate r to be zero, and use the discounted wealth process $\widetilde{V}^1$ instead of the wealth process V^1, to analyse the utility from terminal wealth. Thus V_T is equivalent to the discounted terminal wealth $\widetilde{V}_T$. Conseqently, we will maximize the utility from terminal wealth using the discounted terminal wealth:

$$\widetilde{V}_T^1 = V_0^1 \exp\left(\int_0^T (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB_s^1\right)\Pi_{0\leq s\leq T}(1+\pi_s\Delta J_s), \quad (3.4.8)$$

where

$$\Delta J_s = e^{\Delta X_s} - 1 \text{ and } \Delta X_s = \int_{\mathbf{R}} xN(ds, dx).$$

3.5 Maximization of Logarithmic Utility from Terminal Wealth for the Informed Investor

To obtain tractable solutions, we assume that all investors have logarithmic utility function $u(x) = \log x$.

From (3.4.8), with $V_0^1 = x$ and $\theta^1 = \frac{\mu^1 - r}{\sigma}$, we get

$$\begin{aligned}
u(\widetilde{V}_T^1) &\equiv \log \widetilde{V}_T^1 \\
&= \log V_0^1 + \int_0^T (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB_s^1 + \sum_{0\leq s\leq T}\log(1+\pi_s\Delta J_s), \\
&= \log V_0^1 + \int_0^T (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB_s^1 \\
&\quad + \sum_{0\leq s\leq T}\log(1+\pi_s(e^{\Delta X_s} - 1)), \qquad (3.5.1)
\end{aligned}$$

where ΔX_s is the log jump amplitude at time $s \leq t$, given by

$$\Delta X_s \triangleq \log(1 + \Delta J_s) \Leftrightarrow \Delta J_s = e^{\Delta X_s} - 1. \tag{3.5.2}$$

Let

$$G(\pi) = \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))v(dx). \tag{3.5.3}$$

We now have the following:

Theorem 3.2. *The expected logarithmic utility from discounted terminal wealth $\widetilde{V}_T^1$ for the informed investor is given by*

$$\mathbf{E}u(\widetilde{V}_T^{1,\pi}) = \log V_0 + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^1)^2 dt + \mathbf{E}\int_0^T f^1(\pi_t)dt, \tag{3.5.4}$$

where

$$f^1(\pi_t) = -\frac{1}{2}(\pi_t \, \sigma_t - \theta_t^1)^2 + G(\pi_t). \tag{3.5.5}$$

Proof. Let $N(t, A)$ be the Poisson random measure on $\mathbf{R}_+ \times \mathcal{B}(\mathbf{R} - \{0\})$ that counts the number of jumps of X up to time t in the lower bounded set $A \in \mathcal{B}(\mathbf{R} - \{0\})$; that is, $0 \notin \overline{A}$ (cf Applebaum [3]). Then

$$N(t, A) = \#\{0 < s \leq t : \Delta X_s \in A\}, \tag{3.5.6}$$

with Lévy measure $v(A) = \mathbf{E}N(1, A)$. By Definition 2.25, we have

$$\sum_{0 \leq s \leq T} \log(1+\pi_s \Delta J_s) = \sum_{0 \leq s \leq T} \log(1+\pi_s(e^{\Delta X_s}-1)) = \int_0^T \int_{\mathbf{R}} \log(1+\pi_s(e^x-1))N(ds, dx). \tag{3.5.7}$$

Since $\log(1 + \pi_s(e^x - 1))$ vanishes near $x = 0$, then by Theorem 2.13 we have

$$\mathbf{E}\int_0^T \int_{\mathbf{R}} \log(1+\pi_s(e^x-1))N(ds, dx) = \mathbf{E}\int_0^T \int_{\mathbf{R}} \log(1+\pi_s(e^x-1))v(dx)ds. \tag{3.5.8}$$

Taking expectation of (3.5.1) yields,

$$
\begin{aligned}
\mathbf{E}u(\widetilde{V}_T^{1,\pi}) &\equiv \mathbf{E}\log \widetilde{V}_T^{1,\pi} \\
&= \log V_0^1 + \mathbf{E}\int_0^T (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \mathbf{E}\int_0^T \pi_s\sigma_s dB_s^1 \\
&\quad + \mathbf{E}\int_0^T \int_{\mathbf{R}} \log(1+\pi_s(e^x-1))N(ds,dx), \\
&= \log V_0^1 + \mathbf{E}\int_0^T (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds \\
&\quad + \mathbf{E}\int_0^T \int_{\mathbf{R}} \log(1+\pi_s(e^x-1))v(dx)ds. \qquad (3.5.9)
\end{aligned}
$$

Note that

$$(\pi_t\sigma_t\theta_t^1 - \frac{1}{2}\pi_t^2\sigma_t^2) = \frac{1}{2}(\theta_t^1)^2 - \frac{1}{2}(\pi_t\sigma_t - \theta_t^1)^2. \qquad (3.5.10)$$

Therefore, we have

$$\mathbf{E}u(\widetilde{V}_T^{1,\pi}) = \log V_0^1 + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^1)^2 dt - \frac{1}{2}\mathbf{E}\int_0^T (\pi_t\sigma_t - \theta_t^1)^2 dt + \mathbf{E}\int_0^T G(\pi_t)dt. \quad \square$$

Admissible Set for Informed Investor

Let $V_0^1 = x$. We seek a portfolio process $\pi = (\pi_t)_{t\geq 0}$ in an admissible set $\mathcal{A}_1(x)$ defined by

$$\mathcal{A}_1(x) = \{\pi;\, \widetilde{V}_t^{1,\pi} > 0, \pi \ \textit{is}\, \mathcal{H}^1 - \textit{predictable}, S - \textit{integrable}\}. \qquad (3.5.11)$$

π is predictable if it is measurable with respect to the predictable σ–algebra on $[0,\, T] \times \Omega$, which is the σ–algebra of all LCRL functions on $[0,\, T] \times \Omega$.

The optimal portfolio for the informed investor is $\pi^* \in \mathcal{A}_1(x)$ such that

$$\mathbf{E}(\log \widetilde{V}_T^{1,\pi^*}) = \max_{\pi\in\mathcal{A}_1(x)} \mathbf{E}\log \widetilde{V}_T^{1,\pi}. \qquad (3.5.12)$$

That is,

$$\pi^* = \arg \max_{\pi \in \mathcal{A}_1(x)} \mathbf{E} \log \widetilde{V}_T^{1,\pi}. \tag{3.5.13}$$

Since $\theta_t \equiv \theta_t^1$ is independent of π_t, we see from (3.5.4) that $\mathbf{E} \log \widetilde{V}_T^{1,\pi}$ is maximized iff $\mathbf{E} \int_0^T f^1(\pi_t^1) dt$ is maximized. That is, if $f(\pi)$ is maximized on the admissible set $\mathcal{A}_1(x)$. This approach is similar to the optimization method used by Imkeller and Amendinger [25], Imkeller and Ankirchner [26], Liu, Longstaff and Pan [34], and yields the same optimal as the H-J-B approach.

Lemma 3.3. *If $G(\cdot)$ is twice differentiable then $f^1(\pi)$ is strictly concave on $\mathbf{R}$, with unique maximum at $\pi^{*,1}$, such that for each $t \in [0, T]$*

$$\pi_t^{*,1} = \frac{\theta_t^1}{\sigma_t} + \frac{G'(\pi_t^{*,1})}{\sigma_t^2}. \tag{3.5.14}$$

Proof. Let $f \equiv f^1$ and assume that $G''(\pi)$ exists on $\mathbf{R}$. From (3.5.5),

$$f(\pi_t) = -\frac{1}{2}(\pi_t \sigma_t - \theta_t)^2 + G(\pi_t),$$

whence

$$f'(\pi_t) = -\sigma_t(\pi_t \sigma_t - \theta_t) + G'(\pi_t) = -\sigma(\pi_t \sigma_t - \theta_t) + \int_{\mathbf{R}} \frac{(e^x - 1)v(dx)}{1 + \pi_t(e^x - 1)}$$

and

$$f''(\pi_t) = -\sigma_t^2 + G''(\pi_t) = -\sigma_t^2 - \int_{\mathbf{R}} \frac{(e^x - 1)^2 v(dx)}{(1 + \pi_t(e^x - 1))^2} < 0.$$

Thus f is strictly concave on $\mathbf{R}$, and therefore admits a unique maximum π_t^*, where $f'(\pi_t^*) = 0$. Thus

$$\pi_t^* \sigma_t - \theta_t = \frac{G'(\pi_t^*)}{\sigma_t} \Leftrightarrow \pi_t^* = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t^*)}{\sigma_t^2}$$

The result follows from the fact that, for each $t \in [0, T]$,

$$\max_{\pi_t \in \mathbf{R}} f(\pi_t) = f(\pi_t^*). \qquad \square$$

Theorem 3.3. *Assume that $G(\pi)$ is twice differentiable with respect to π. For the informed investor, the maximum expected logarithmic utility from terminal wealth is given by*

$$\begin{aligned} u^1(x) &\equiv \max_{\pi \in \mathcal{A}_1(x)} \mathbf{E} \log \widetilde{V}_T^{1,\pi} \\ &= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^1)^2 dt + \mathbf{E}\int_0^T f^1(\pi_t^{*,1}) dt, \end{aligned} \tag{3.5.15}$$

provided

$$\pi_t^{*,1} = \frac{\theta_t^1}{\sigma_t} + \frac{G'(\pi_t^{*,1})}{\sigma_t^2} \in \mathcal{A}_1(x). \tag{3.5.16}$$

Proof. Since $G''(\pi)$ exists, then by Lemma 3.3 we have an optimal portfolio π^* given by

$$\pi_t^{*,1} = \frac{\theta_t^1}{\sigma_t} + \frac{G'(\pi_t^{*,1})}{\sigma_t^2},$$

and

$$max_{\pi_t \in \mathbf{R}} f^1(\pi) = f^1(\pi_t^{*,1}).$$

Assume that $\pi^{*,1} \in \mathcal{A}_1(x)$. Then $\mathbf{E}\int_0^T f^1(\pi_t)dt \leq \mathbf{E}\int_0^T f^1(\pi_t^{*,1})dt$, whence

$$max_{\pi_t \in \mathcal{A}_1(x)} \mathbf{E}\int_0^T f^1(\pi_t)dt = \mathbf{E}\int_0^T f^1(\pi_t^{*,1})dt.$$

Therefore by Theorem 3.2 with $V_0^1 = x$, we get

$$\begin{aligned} u^1(x) &\equiv \max_{\pi \in \mathcal{A}_1(x)} \mathbf{E} \log \widetilde{V}_T^{1,\pi} \\ &= \log x + \frac{1}{2} \mathbf{E} \int_0^T (\theta_t^1)^2 dt + max_\pi \mathbf{E} \int_0^T f^1(\pi_t^1) dt, \\ &= \log x + \frac{1}{2} \mathbf{E} \int_0^T (\theta_t^1)^2 dt + \mathbf{E} \int_0^T f^1(\pi_t^{*,1}) dt. \quad \square \end{aligned}$$

Remark 3.1. *We achieve optimality if G is twice differentiable. That is if $\frac{(e^x-1)^2}{(1+\pi(e^x-1))^2}$ is v–integrable. Under what condition(s) is*

$$G''(\pi) = - \int_{\mathbf{R}} \frac{(e^x-1)^2 v(dx)}{(1+\pi(e^x-1))^2} \; < \; \infty\,?$$

If we restrict π to the interval $[0,1)$, then $G''(\pi) < \infty$ whenever $\int_{\mathbf{R}} (e^x-1)^2 v(dx) < \infty$. If $\pi = 1$, then $G''(\pi)$ exists if $\int_{\mathbf{R}} (e^{-x}-1)^2 v(dx) < \infty$. Therefore, if no short–selling ($\pi < 0$) or borrowing ($\pi > 1$) from the bank account is allowed, and $(e^{\pm x} - 1)^2$ is v-integrable, then Lemma 3.3 and Theorem 3.3 hold.

3.6 Asset Price Dynamic for Uninformed Investor

The uninformed investor observes only the the stock price S and does not have any information on the mispricing process U, although the investor may be aware that fads exist. Uninformed investors and all characteristics related thereto, are represented by the index $i = 0$. These investors have filtration $\mathcal{H}^0 = (\mathcal{H}_t^0)_{t \geq 0}$ defined below. Let $X = (X_t)_{t \geq 0}$ be the pure jump Lévy process that drives the stock price. Define the process Y on $(\Omega, \mathcal{F}, \mathcal{H}^0, \mathbf{P})$ by the prescription:

$$Y_t \;=\; p\, W_t + q\, U_t \tag{3.6.1}$$

where W, U, Y, X are defined by equations(3.2.2)–(3.2.4). X and Y are independent, since X, W, U are independent. We take the filtration of the uninformed investor as

$$\mathcal{H}_t^0 = \mathcal{F}_t^0 \vee \sigma(X_u : u \leq t),\ t \in [0,\, T], \tag{3.6.2}$$

where $\mathcal{F}_t^0 = \sigma(Y_s : s \leq t)$ is given by Guasoni [21], Theorem 2.1. Clearly,

$$\mathcal{F}_t^0 \subset \mathcal{H}_t^0 \subset \mathcal{H}_t^1 \subset \mathcal{F}. \tag{3.6.3}$$

As in the case of the informed investor, the uninformed investor has stock price dynamic (cf equations (3.2.1),(3.2.11) and (3.2.12)), given by

$$\frac{dS_t}{S_t} = b_t\, dt + \sigma_t\, dY_t + dJ_t,\ t \in [0,\, T], \tag{3.6.4}$$

where

$$J_t = \int_0^t \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(ds, dx).$$

Lemma 3.4. *There exist an $\mathcal{H}^0$–Brownian motion B^0 and a process ϕ_t^0 adapted to $\mathcal{H}_t^0$, such that for each $t \in [0, T]$*

$$B_t^0 = Y_t + \int_0^t \phi_s^0 ds, \tag{3.6.5}$$

$$\phi_s^0 = -\upsilon_s^0 = \lambda \int_0^s e^{-\lambda(s-u)}(1 + \gamma(u))dB_u^0, \tag{3.6.6}$$

where λ is the mean reversion rate of the O–U process U.

Proof. $Y_t = p\, W_t + q\, U_t$ is a Gaussian process which is adapted to $\mathcal{F}_t^0$, and hence adapted to $\mathcal{H}_t^0$, since it is contained therein. By Guasoni [21] Theorem 2.1.1, using Hitsuda [24] representation of Gaussian processes, there exists an $\mathcal{F}^0$–adapted and hence, $\mathcal{H}^0$–adapted Brownian motion B^0 and process ϕ^0, given by (3.6.5) and (3.6.6).

$\square$

Equivalently,

$$Y_t = B_t^0 - \int_0^t \phi_s^0 ds = B_t^0 + \int_0^t \upsilon_s^0 ds, \tag{3.6.7}$$

whence

$$dY_t = dB_t^0 + \upsilon_t^0 dt. \tag{3.6.8}$$

Under this transformation, the stock price dynamic (3.6.4) becomes

$$\begin{aligned}
\frac{dS_t}{S_t} &= b_t\, dt + \sigma_t\, (dB_t^0 + \upsilon_t^0 dt) + dJ_t \\
&= (b_t + \sigma_t\, \upsilon_t^0)dt + \sigma_t\, dB_t^0 + dJ_t \\
&= b_t^0 dt + \sigma_t\, dB_t^0 + dJ_t \\
&= b_t^0 dt + \sigma_t dB_t^0 + \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt, dx) \\
&= (\mu_t^0 + M_1)dt + \sigma_t dB_t^0 + \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt, dx).
\end{aligned}$$

We summarize this result in the following theorem.

Theorem 3.4. *The percentage returns dynamic of the uninformed investor is*

$$\frac{dS_t}{S_t} = b_t^0\, dt + \sigma_t\, dB_t^0 + \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt, dx), \tag{3.6.9}$$

with price

$$S_t = S_0 \exp\left(\int_0^t \mu_s^0 ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dB_s^0 + X_t\right), \tag{3.6.10}$$

where $b_t^0 = b_t + \sigma_t \upsilon_t^0$*,* $\mu_t^0 = \mu_t + \sigma_t \upsilon_t^0$*,* $b_t = \mu_t + M_1$*, and* $M_1 = \int_{\mathbf{R}} (e^x - 1)\upsilon(dx)$*.*

Proof. Equation (3.6.10) follows directly from (3.6.9) by the application of Theorem

2.18 on stochastic exponentials. $\square$

Remark 3.2. *We see from the last result and Lemma 3.2, that relative to the filtration* $\mathcal{H}^i$, $i = 0,\, 1$, *the stock price is*

$$S_t = S_0 \, \exp\left(\int_0^t \mu_s^i ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dB_s^i + X_t\right), \tag{3.6.11}$$

where $X_t = \int_{\mathbf{R}} xN(t, dx)$ *and* $\mu_t^i = \mu_t + \sigma_t \upsilon_t^i,\;\; i = 0, 1,$ *and* υ^i *and* B^i *are previously defined.*

3.6.1 The Wealth Process for the Uninformed Investor

The wealth process for the uninformed investor is $V^{0,\pi,x} : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}$, where $V_t^{0,\pi,x}$ is the value of the portfolio (of stock and bond) at time t. π_t is the proportion of wealth invested in the stock. $x > 0$ is the initial capital, i.e., $V_0^0 = x$. For brevity, we will denote this process by $V^0 = (V_t^{0,\pi,x})_{t \geq 0}$.

Admissible Set for Uninformed Investor

The admissible set for the uninformed investor is denoted by $\mathcal{A}_0(x)$ where

$$\begin{aligned} \mathcal{A}_0(x) &= \{\pi^0 : V_t^{0,\pi^0,x} > 0;\; \pi^0 \; is\, \mathcal{H}^0 - predictable, S - integrable\,\} \\ &= \{\pi^0 : \widetilde{V}_t^{0,\pi^0,x} > 0;\; \pi^0 \; is\, \mathcal{H}^0 - predictable, S - integrable\,\}. \end{aligned} \tag{3.6.12}$$

A process $\pi = \pi^0$ is called admissible if it is a member of $\mathcal{A}_0(x)$. Here $\widetilde{V}^0$ is the discounted wealth process of V^0, and π is predictable if it is measurable with respect to the predictable sigma–algebra on $[0,\, T] \times \Omega$, the sigma–algebra of LCRL (left continuous with right limits) functions on $[0,\, T] \times \Omega$.

Dynamics of the Wealth Process for the Uninformed Investor

Let $V_t \equiv V_t^0$ be the wealth process of the uninformed investor at time t as a result of investing $\pi_t \equiv \pi_t^0$ in the stock. Assume that the bond earns the same continuously compounded deterministic risk–free interest rate r_t, as the informed investor. Let n_t^0 be the number of stocks in the portfolio at the time t. Then $\pi_t^0 = \frac{n_t^0 S_{t-}}{V_{t-}^0}$, where V_{t-}^0 is the value of the portfolio just before time t.

We have the following result for the wealth dynamic of the uninformed investor.

Theorem 3.5. *If the stock's percentage return dynamic for the informed investor is*

$$\frac{dS_t}{S_t} = b_t^0\, dt + \sigma_t\, dY_t + dJ_t,\ \ t \in [0,\, T], \tag{3.6.13}$$

where J is given by (3.3.7), then the wealth process V^0 has dynamic

$$\frac{dV_t^0}{V_{t-}^0} = (k\pi_t + \pi_t\, \sigma_t\, \theta_t^0 + r_t)dt + \pi_t\, \sigma_t\, dB_t^0 + \pi_t\, dJ_t, \tag{3.6.14}$$

and discounted value $\widetilde{V}_t^0 = V_t^0 \exp\left(-\int_0^t r_s ds\right)$, given by

$$\widetilde{V}_t^0 = V_0^0 \exp\left(\int_0^t (\pi_s\sigma_s\theta_s^0 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^0\right) \Pi_{0\le s\le t}(1 + \pi_s \Delta J_s), \tag{3.6.15}$$

where B^0 and υ^0 are given by Lemma 3.4 and Theorem 3.4.

Proof. Since all portfolios are assume to be self–financing, then

$$\begin{aligned} dV_t^0 &= (1 - \pi_t)\, V_{t-}^0\, r_t\, dt + n_t^0\, dS_t, \\ &= (1 - \pi_t)\, V_{t-}^0\, r_t\, dt + \pi_t\, V_{t-}^0\, \frac{dS_t}{S_{t-}}. \end{aligned} \tag{3.6.16}$$

Therefore

$$\frac{dV_t^0}{V_{t-}^0} = (1-\pi_t)\, r_t\, dt + \pi_t\, \frac{dS_t}{S_{t-}}. \tag{3.6.17}$$

From Theorem 3.4, with $J_t = \int_0^t \int_{\mathbf{R}} (e^x - 1)\widetilde{N}(dt, dx)$, we have percentage returns

$$\frac{dS_t}{S_t} = b_t^0\, dt + \sigma_t\, dB_t^0 + dJ_t, \;\; t \in [0,\, T]. \tag{3.6.18}$$

Using the fact that $b_t^0 = \mu_t^0 + \int_{\mathbf{R}} (e^x - 1) v(dx) = \mu_t^0 + M_1$, we get

$$\begin{aligned} \frac{dV_t^0}{V_{t-}^0} &= (1-\pi_t)\, r_t dt + \pi_t\, (b_t^0\, dt + \sigma_t\, dB_t^0 + dJ_t) \\ &= ((1-\pi_t)\, r_t + \pi_t\, b_t^0)\, dt + \pi_t\, \sigma_t\, dB_t^0 + \pi_t\, dJ_t \\ &= ((b_t^0 - r_t)\, \pi_t + r_t)\, dt + \pi_t\, \sigma_t\, dB_t^0 + \pi_t\, dJ_t \\ &= (M_1 \pi_t + \pi_t\, \sigma_t\, \theta_t^0 + r_t) dt + \pi_t\, \sigma_t\, dB_t^0 + \pi_t\, dJ_t, \end{aligned} \tag{3.6.19}$$

where the stock's Sharpe ratio or market price of risk is θ_t^0. Equation (3.6.14) was established above, and applying Theorem 2.18 on stochastic exponentials yields (3.6.15). □

Remark 3.3. *As with the model for the informed investor, we may effectively set the interest rate r_t to be zero, by using the discounted wealth process $\widetilde{V}^0$ in the sequel. In this case, the terminal wealth V_T^0 is equivalent to the discounted terminal wealth $\widetilde{V}_T^0$.*

3.7 Maximization of Logarithmic Utility from Terminal Wealth for the Uninformed Investor

We proceed almost identically to the approcah used for the informed investor. We assume that all uninformed investors have logarithmic utility function $u(x) = \log x$.

Let

$$G(\pi) = \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))v(dx).$$

Theorem 3.6. *The expected logarithmic utility from terminal wealth $\widetilde{V}_T^0$ for the uninformed investor is given by*

$$\mathbf{E}u(\widetilde{V}_T^0) = \log V_0^0 + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^0)^2 dt + \mathbf{E}\int_0^T f^0(\pi_t^0)dt, \tag{3.7.1}$$

where

$$f^0(\pi_t^0) = -\frac{1}{2}\,(\pi_t^0\,\sigma_t - \theta_t^0)^2 + G(\pi_t^0). \tag{3.7.2}$$

Proof. For convenience, we let $\pi_t = \pi_t^0$. From (3.6.15), with $V_0^0 = x$ we get

$$\begin{aligned} u(\widetilde{V}_T^0) &\equiv \log \widetilde{V}_T^0 \\ &= \log V_0^0 + \int_0^T (\pi_s\sigma_s\theta_s^0 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB_s^0 + \sum_{0\le s\le T}\log(1+\pi_s\Delta J_s), \\ &= \log V_0^0 + \int_0^T (\pi_s\sigma_s\theta_s^0 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB_s^0 \\ &\quad + \sum_{0\le s\le T}\log(1+\pi_s(e^{\Delta X_s}-1)), \end{aligned} \tag{3.7.3}$$

where ΔX_s is the log jump amplitude at time $s \le t$, given by

$$\Delta X_s \stackrel{\triangle}{=} \log(1 + \Delta J_s) \Leftrightarrow \Delta J_s = e^{\Delta X_s} - 1. \tag{3.7.4}$$

By Theorem 2.13, we have

$$\sum_{0\le s\le T}\log(1+\pi_s\Delta J_s) = \sum_{0\le s\le T}\log(1+\pi_s(e^{\Delta X_s}-1)) = \int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^x-1))N(ds,dx),$$

and

$$\mathbf{E}\int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^x-1))N(ds,dx)=\mathbf{E}\int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^x-1))v(dx)ds.$$

Taking expectation of (3.7.3) yields,

$$\begin{aligned}
\mathbf{E}u(\widetilde{V}_T^0) &\equiv \mathbf{E}\log\widetilde{V}_T^0 \\
&= \log V_0^0+\mathbf{E}\int_0^T(\pi_s\sigma_s\theta_s^0-\frac{1}{2}\pi_s^2\sigma_s^2)ds+\mathbf{E}\int_0^T\pi_s\sigma_s dB_s^0 \\
&\quad +\mathbf{E}\int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^x-1))N(ds,dx), \\
&= \log V_0^0+\mathbf{E}\int_0^T(\pi_s\sigma_s\theta_s^0-\frac{1}{2}\pi_s^2\sigma_s^2)ds \\
&\quad +\mathbf{E}\int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^x-1))v(dx)ds.
\end{aligned} \tag{3.7.5}$$

Now

$$(\pi_t\sigma_t\theta_t^0-\frac{1}{2}\pi_t^2\sigma_t^2)=\frac{1}{2}(\theta_t^0)^2-\frac{1}{2}(\pi_t\sigma_t-\theta_t^0)^2. \tag{3.7.6}$$

By adding the superscript "0", equations (3.7.2) and (3.7.6) yield

$$\mathbf{E}u(\widetilde{V}_T^0)=\log V_0^0+\frac{1}{2}\mathbf{E}\int_0^T(\theta_t^0)^2dt+\mathbf{E}\int_0^T f^0(\pi_t^0)dt. \qquad \square$$

3.8 Optimal Portfolio for the Uninformed Investor

The optimal portfolio for the informed investor is $\pi\equiv\pi^*\in\mathcal{A}_0(x)$ such that

$$\mathbf{E}(\log\widetilde{V}_T^{0,\pi^{0,*}})=\max_{\pi\in\mathcal{A}_0(x)}\mathbf{E}\log\widetilde{V}_T^{0,\pi}. \tag{3.8.1}$$

That is,

$$\pi^{0,*} = arg \max_{\pi \in \mathcal{A}_0(x)} \mathbf{E} \log \widetilde{V}_T^{0,\pi}. \tag{3.8.2}$$

Since $\theta_t^0 \equiv \theta_t$ is independent of π_t we see that $\mathbf{E}(\log \widetilde{V}_T^{0,\pi^{0,*}})$ is maximized iff $\mathbf{E}\int_0^T f^0(\pi_t^0)dt$ is maximized. That is, if $f^0(\pi)$ is maximized on the admissible set $\mathcal{A}_0(x)$. Since this is the same procedure used for the informed investor, we have the following:

Lemma 3.5. *If G is twice differentiable then $f^0(\pi)$ is strictly concave on $\mathbf{R}$, and for each $t \in [0, T]$, there is an unique optimal $\pi_t^{0,*}$, where*

$$\pi_t^{0,*} = \frac{\theta_t^0}{\sigma_t} + \frac{G'(\pi_t^{0,*})}{\sigma_t^2}. \tag{3.8.3}$$

Proof. The proof is essentially the same as that of Lemma 3.3 with $i = 0$, and is omitted. □

We have the following major result as a consequence of the foregoing.

Theorem 3.7. *Assume that $G(\pi)$ is twice differentiable. For the uninformed investor with initial wealth $x > 0$, the maximum expected logarithmic utility from terminal wealth is given by $u^0(x)$ where*

$$u^0(x) \equiv \max_{\pi \in \mathcal{A}_0(x)} \mathbf{E} \log \widetilde{V}_T^{0,\pi} = \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^0)^2 dt + \mathbf{E}\int_0^T f^0(\pi_t^{0,*})dt, \tag{3.8.4}$$

provided

$$\pi_t^{0,*} = \frac{\theta_t^0}{\sigma_t} + \frac{G'(\pi_t^{0,*})}{\sigma_t^2} \in \mathcal{A}_0(x). \tag{3.8.5}$$

Proof. Since $G''(\pi)$ exists, then by Lemma 3.5 we have an optimal portfolio π^* given by

$$\pi_t^{0,*} = \frac{\theta_t^0}{\sigma_t} + \frac{G'(\pi_t^{0,*})}{\sigma_t^2}$$

and

$$max_{\pi_t \in \mathbf{R}} f^0(\pi) = f^0(\pi_t^*).$$

Assume that $\pi^{0,*} \in \mathcal{A}_0(x)$. Then

$$\mathbf{E}\int_0^T f^0(\pi_t)dt \leq \mathbf{E}\int_0^T f^0(\pi_t^*)dt,$$

whence

$$max_{\pi_t \in \mathcal{A}_0(x)} \mathbf{E}\int_0^T f^0(\pi_t)dt = \mathbf{E}\int_0^T f^0(\pi_t^*)dt.$$

Therefore, by Theorem 3.6 with $V_0^0 = x$, we get

$$\begin{aligned} u^0(x) &\equiv \max_{\pi \in \mathcal{A}_0(x)} \mathbf{E}\log \widetilde{V}_T^{0,\pi} \\ &= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^0)^2 dt + max_{\pi \in \mathcal{A}_1(x)} \mathbf{E}\int_0^T f^0(\pi_t)dt, \\ &= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^0)^2 dt + \mathbf{E}\int_0^T f^0(\pi_t^{0,*})dt, \end{aligned}$$

provided

$$\pi_t^{0,*} = \frac{\theta_t^0}{\sigma_t} + \frac{G'(\pi_t^{0,*})}{\sigma_t^2} \in \mathcal{A}_0(x). \qquad \square$$

Note that the above proof was obtained from the proof of Theorem 3.3 by replacing the superscript “1”, by “0”.

Remark 3.4. *As in the case of the informed investor, we achieve optimality if G is twice differentiable. That is if $(e^{\pm x} - 1)^2$ is v–integrable.*

Remark 3.5. *In the sequel, we assume that $\pi \in [0, 1]$ and $\int_{\mathbf{R}} (e^{\pm x} - 1)^2 v(dx) < \infty$* ***unless*** *$G(\pi) = \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))v(dx)$ can be explicitly computed. This special case holds when the jumps are driven by linear combinations of Poisson processes, in which case the restriction $\pi \in [0, 1]$, is relaxed. For most cases, however, such* *as markets driven by Variance Gamma, CGMY, Merton and Kou Jump–diffusion processes, we resort to this restriction to ensure optimality of portfolio processes for both informed and uninformed investors.*

We now combine the results for both investors in one general theorem.

Theorem 3.8. *Let* $\pi \in [0,\, 1]$. *If* $\int_{\mathbf{R}}(e^{\pm x}-1)^2 v(dx) < \infty$, *then*

(1) $G''(\pi) < 0$.

(2) *Let* $i \in \{0, 1\}$. *For the i–th investor, there is an unique optimal portfolio* $\pi^{i,*} \in [0, 1]$ *for the stock with dynamic (3.2.1), given by*

$$\pi_t^{i,*} = \frac{\theta_t^i}{\sigma_t} + \frac{G'(\pi_t^{i,*})}{\sigma_t^2},$$

provided $\pi^{i,*} \in [0, 1] \subset \mathcal{A}_i(x)$.

(3) *The maximum expected logarithmic utility from terminal wealth for the i–th investor, having* $x > 0$ *of initial capital, is*

$$u^i(x) = \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^i)^2 dt + \mathbf{E}\int_0^T f^i(\pi_t^{i,*})dt,$$

where $f^i(\pi)$ *is given by (3.5.5) and (3.7.2).*

We now break down this result in terms of continuous and discrete parts, in line with the Merton's GBM model. The Merton [36] optimal portfolio π^*_{Mer} for a stock with GBM dynamic

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t,$$

is

$$\pi^*_{Mer}(t) = \frac{\theta_t}{\sigma_t} = \frac{\mu_t - r_t}{\sigma_t}.$$

We introduce some useful objects in the following definition.

Definition 3.2. *Let $t \in [0,\, T]$. Define the following objects by the prescriptions:*

$$\pi^i_{t,\,c} \;\triangleq\; \frac{\theta^i}{\sigma_t} = \frac{\mu^i_t - r_t}{\sigma_t} = \pi^i_{Mer}(t). \tag{3.8.6}$$

$$\pi^i_{t,\,d} \;\triangleq\; \frac{G'(\pi^i_t)}{\sigma^2_t}. \tag{3.8.7}$$

$$u^i_{T,\,c}(x) \;\triangleq\; \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta^i_t)^2 dt. \tag{3.8.8}$$

$$u^i_{T,\,d}(x) \;\triangleq\; \mathbf{E}\int_0^T f^i(\pi^i_t) dt. \tag{3.8.9}$$

Remark 3.6. *$\pi^i_{t,\,c}$ and $\pi^i_{t,\,d}$ are the continuous and discrete components, respectively, of the portfolio π_t.*

Similarly, $u^i_{T,\,c}$ and $u^i_{T,\,d}$ are the continuous and discrete components of the maximum expected logarithmic utility based on an optimal portfolio π wit investment horizon T. The processes $\pi^i_{t,\,c}$ and $\pi^i_{t,\,d}$ are assumed to be adapted to their filtrations $\mathcal{H}^i$, $i \in \{0, 1\}$. In the sequel, we fix the interest rate $r = 0$. In this case, the discounted wealth process and the wealth process are identical.

We now restate Theorem 3.8 in an equivalent form:

Theorem 3.9. *Let $\pi \in [0,\, 1]$ and $\int_{\mathbf{R}}(e^{\pm x} - 1)^2 v(dx) < \infty$. Let $\in \{0, 1\}$.*

(1) *For both the informed and uninformed investors, there is an unique optimal portfolio $\pi^{*,\,i} \in \mathcal{A}_i(x)$, such that*

$$\pi^{*,\,i}_t = \pi^{*,\,i}_{t,\,c} + \pi^{*,\,i}_{t,\,d} \equiv \pi^{*,\,i}_{Mer} + \pi^{*,\,i}_{t,\,d}. \tag{3.8.10}$$

That is,

$$\pi^{*,\,i} = \pi^{*,\,i}_c + \pi^{*,\,i}_d \equiv \pi^{*,\,i}_{Mer} + \pi^{*,\,i}_d, \tag{3.8.11}$$

where π^*_c is the Merton optimal and π^*_d is the excess stock holding resulting from the jumps.

(2) *The maximum expected logarithmic utility from terminal wealth for each*

investor, having $x > 0$ in initial wealth, is given by

$$u^i(x) \equiv u_T^i(x) = u_{T,c}^i(x) + u_{T,d}^i(x). \tag{3.8.12}$$

That is,

$$u(x) \equiv u_T(x) = u_{T,c}(x) + u_{T,d}(x), \tag{3.8.13}$$

where $u_{T,c}(x)$ is the maximum expected logarithmic utility from terminal wealth for the purely continuous Merton case, with optimal portfolio π_c^*. $u_{T,d}(x)$ is the excess utility resulting from the jumps.

Proof. This follows directly from definition 3.2 and Theorem 3.8. □

3.8.1 Asymptotic Utilities of Investors

Let $u_T^i(x)$ be the maximum expected logarithmic utility of the i–th investor resulting from an optimal portfolio $\pi^{*,i}$. Assume that the risk–free interest rate $r = 0$. Then the stock's Sharpe ratio for the i–th investor is

$$\theta^i = \frac{\mu^i}{\sigma} = \frac{\mu + \upsilon^i \sigma}{\sigma} = \frac{\mu}{\sigma} + \upsilon^i. \tag{3.8.14}$$

We now restate Theorem 1.4, in Guasoni [21] using the new notations above.

Theorem 3.10. *Let $x > 0$ be the initial wealth of the investors and let $i \in \{0, 1\}$.*
(1) *The optimal portfolio and maximum expected logarithmic utility from terminal wealth for the i–th investor in the purely continuous Merton market are given by:*

$$\pi_c^{*,i} \equiv \pi_{Mer}^{*,i} = \frac{\theta^i}{\sigma} = \frac{\mu}{\sigma^2} + \frac{\upsilon^i}{\sigma}, \tag{3.8.15}$$

and

$$\begin{aligned} u^i_{T,\,c}(x) &= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta^i_t)^2 dt \\ &= \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{1}{2}\int_0^T \mathbf{E}(\upsilon^i_t)^2 dt. \end{aligned} \tag{3.8.16}$$

(2) *As $T \longrightarrow \infty$, we have asymptotic utilities:*

$$u^i_{\infty,\,c}(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda\,T}{4}(1-p)(1+(-1)^{i+1}p),\ p\in[0,\,1]. \tag{3.8.17}$$

Proof. See Theorem 1.4 in Guasoni [21]. □

We now have asymptotic results for the jump case.

Theorem 3.11. *As $T \longrightarrow \infty$, the maximal expected logarithmic utilities from terminal wealth for the i–th investor $i = 0,\,1$, is $u^i_\infty(x)$ given by*

$$u^i_\infty(x) = u^i_{\infty,\,c}(x) + u^i_{\infty,\,d}(x), \tag{3.8.18}$$

where $p \in [0,\,1]$, and

$$u^i_{\infty,\,c}(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda\,T}{4}(1-p)(1+(-1)^{i+1}p), \tag{3.8.19}$$

$$u^i_{\infty,\,d}(x) \sim T\,\phi_{\pi^*,\,i}(i), \quad \textit{and} \quad \phi_{\pi^*,\,i}(i) \sim \lim_{t\to\infty} \mathbf{E} f^i(\pi^{*,\,i}_t). \tag{3.8.20}$$

Proof. Let $i \in \{0,1\}$ and $T \longrightarrow \infty$. From Theorem 3.10 we have

$$u^i_{T,\,c} \to u^i_{\infty,\,c}(x) = \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda\,T}{4}(1-p)(1+(-1)^{i+1}p).$$

Since $f^i(\pi)$ is continuous, then by L'Hospital's rule

$$\lim_{T\to\infty}\frac{u^i_{T,d}(x)}{T} = \lim_{T\to\infty}\frac{1}{T}\,\mathbf{E}\int_0^T f^i(\pi_t^{*,i})dt = \lim_{t\to\infty}\mathbf{E}f^i(\pi_t^{*,i}) = \phi_{\pi^{*,i}}(i).$$

Thus $u^i_{T,d}(x) \sim u^i_{\infty\, d}(x) = T\,\phi_{\pi^{*,i}}(i)$, which completes the proof. □

Corollary 3.1.

(1) *Let $x > 0$ be the initial capital of the investors and $i = 0, 1$. As $T \longrightarrow \infty$, the maximal expected logarithmic utilities from terminal wealth for the i–th investor, is*

$$u^i_{\infty}(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda T}{4}(1-p)(1+(-1)^{i+1}p) + T\,\phi_{\pi^{*,i}}(i). \quad (3.8.21)$$

(2) *Moreover, the asymptotic excess utility of the informed investor is*

$$u^1_{\infty}(x) - u^0_{\infty}(x) \sim \frac{\lambda T}{2}p\,(1-p) + T\,(\phi_{\pi_t^{*,1}}(1) - \phi_{\pi_t^{*,0}}(0)). \quad (3.8.22)$$

Proof. Part (1) follows directly from Theorem 3.11, while Part(2) follows from Part(1) by subtraction. □

Remark 3.7. *Clearly, if there are no jumps $G \equiv 0$ because the Lévy measure $v \equiv 0$, and we revert to the purely continuous GBM case studied by Guasoni [21].*
In the sequel, we give an explicit formula for $\phi_{\pi_t^{*,i}}(i)$ after studying the excess stock holdings of the investors.

3.8.2 Excess Stock Holdings of Investors

Let $G(\pi) = \int_{\mathbf{R}} \log(1+\pi(e^x-1))v(dx)$ with $\pi \in [0,1]$ and $\int_{\mathbf{R}}(e^{\pm x}-1)^2 v(dx) < \infty$. Then $G''(\pi) < 0$. Let $i \in \{0,1\}$. Define the Merton optimal for each investor by:

$$\alpha_i \triangleq \frac{\theta_t^i}{\sigma_t} \equiv \pi^{*,i}_{t,c} \equiv \pi^{*,i}_{Mer}(t). \quad (3.8.23)$$

The following important result is presented without the superscript "i".

Theorem 3.12. *Assume that* $\pi \in [0, 1]$ *and* $\int_{\mathbf{R}}(e^{\pm x} - 1)^2 v(dx) < \infty$. *Let*

$$\alpha \triangleq \frac{\theta}{\sigma} \equiv \pi_c^* \equiv \pi_{Mer}^*$$

be the Merton optimal for each investor, where θ *is the stock's Sharpe ratio. There exists* $\psi_\alpha \in (\alpha, 1)$ *such that the optimal portfolio* π^* *that maximizes the expected logarithmic utility from terminal wealth is*

$$\pi^* = \alpha + \frac{G'(\alpha)}{\sigma^2 + |G''(\psi_\alpha)|}. \tag{3.8.24}$$

Proof. Set $\alpha = \frac{\theta}{\sigma}$. Dropping the superscript "i", and subscript "t", the optimal portfolio π^* for any investor is given by Theorem 3.8, as

$$\pi^* = \frac{\theta}{\sigma} + \frac{G'(\pi*)}{\sigma^2} = \alpha + \frac{G'(\pi*)}{\sigma^2}.$$

By the Mean Value Theorem there exists ψ_α between π^* and α, such that

$$G'(\pi^*) = G'(\alpha) + (\pi^* - \alpha)G''(\psi_\alpha).$$

Thus

$$\pi^* - \alpha = \frac{G'(\pi*)}{\sigma^2} = \frac{1}{\sigma^2}(G'(\alpha) + (\pi^* - \alpha)G''(\psi_\alpha).$$

Rearranging yields

$$\pi^* = \alpha + \frac{G'(\alpha)}{\sigma^2 - G''(\psi_\alpha)} = \alpha + \frac{G'(\alpha)}{\sigma^2 + |G''(\psi_\alpha)|}. \tag{3.8.25}$$

□

An immediate consequence of the last result is

Corollary 3.2. *For each investor, the excess stock holding over the Merton optimal* $\alpha = \frac{\theta}{\sigma}$, *is*

$$\pi_d^* = \frac{G'(\alpha)}{\sigma^2 + |G''(\psi_\alpha)|}. \tag{3.8.26}$$

Remark 3.8. *Observe that we have a positive or negative excess over the Merton optimal in lock-step with the sign of* $G'(\alpha)$.

3.9 Optimal Portfolios and Utilities Under Quadratic Approximation of $G(\alpha)$

In this section, we derive some useful formulas based on the assumption that $G(\alpha)$ is approximated by a Taylor expansion built from the constants M_1 and M_2, defined below. While this is a crude approximation, it leads to very nice consequences. In Chapter 5, we will turn to higher degree polynomial approximations of G, using higher order moments. In Chapter 6, we approximate $G(\cdot)$ using two infinite series expanded about $\pi = 0$ and $\pi = 1$, respectively.

Let $v(\cdot)$ be the Lévy measure of the jump process, and for each $\alpha \in [0, 1]$, set

$$G(\alpha) = \int_{\mathbf{R}} \log(1 + \alpha(e^x - 1))v(dx).$$

Standing Assumption:

There exists $k \geq 3$, such that $\int_{\mathbf{R}}(e^{\pm x} - 1)^k v(dx) < \infty$. This ensures that $G^{(k)}(\alpha)$, the k–th derivative of $G(\alpha)$, exists on $[0, 1]$. In particular, $G''(\pi) < 0$.

Instantaneous Centralized Moments of Returns

We introduce some extremely useful objects linked to the Lévy measure that will be instrumental in computing approximations.

Definition 3.3. *Let* $j \in \{1, 2, \ldots, k\}$*. Define the* j*–th instantaneous centralized moment of returns for the stock with dynamic (3.2.1) by the prescription:*

$$M_j \triangleq \int_{\mathbf{R}} (e^x - 1)^j v(dx). \tag{3.9.1}$$

M_j is well defined because $\int_{\mathbf{R}} (e^{\pm x} - 1)^j v(dx) < \infty$, and

$$G^{(j)}(0) = (-1)^{j-1}(j-1)! \int_{\mathbf{R}} (e^x - 1)^j v(dx) = (-1)^{j-1}(j-1)! M_j, \tag{3.9.2}$$

whence

$$M_j = (-1)^{j-1} \frac{G^{(j)}(0)}{(j-1)!}. \tag{3.9.3}$$

We now assume that $k = 3$, whence $\int_{\mathbf{R}} (e^{\pm x} - 1)^3 v(dx) < \infty$, and so $G'''(\alpha)$ exists on [0,1]. Expanding $G(\alpha)$ about $\alpha = 0$, yields

$$G(\alpha) = G(0) + \alpha G'(0) + \frac{1}{2}\alpha^2 G''(0) + \frac{1}{3}\alpha^3 G'''(\psi_\alpha), \;\; \psi_\alpha \in (0, \alpha). \tag{3.9.4}$$

With error term $R_3(\alpha) = \frac{1}{3}\alpha^3 G'''(\psi_\alpha)$, we have quadratic approximation of $G(\alpha)$ given by

$$\begin{aligned} G(\alpha) &= \alpha G'(0) + \frac{1}{2}\alpha^2 G''(0) + R_3(\alpha) \\ &= M_1 \alpha - \frac{1}{2} M_2 \alpha^2 + R_3(\alpha). \end{aligned} \tag{3.9.5}$$

Thus

$$G(\alpha) \approx M_1 \alpha - \frac{1}{2} M_2 \alpha^2, \tag{3.9.6}$$

whence

$$G'(\alpha) \approx M_1 - M_2\alpha \quad \text{and} \quad G''(\alpha) \approx -M_2. \tag{3.9.7}$$

Thus for all $\alpha \in (0, 1)$ the optimal portfolio becomes (cf Theorem 3.12)

$$\pi^* - \alpha = \frac{G'(\alpha)}{\sigma^2 - G''(\psi_\alpha)} \approx \frac{M_1 - M_2\alpha}{\sigma^2 + M_2}. \tag{3.9.8}$$

Expanding $G(\pi^*)$ about α, there exists $\eta_\alpha \in (\alpha, 1)$ such that

$$G(\pi^*) = G(\alpha) + (\pi^* - \alpha)G'(\alpha) + \frac{1}{2}(\pi^* - \alpha)^2 G''(\eta_\alpha) \approx G(\alpha) + (\pi^* - \alpha)G'(\alpha) - \frac{1}{2}\, M_2(\pi^* - \alpha)^2.$$

Thus

$$\pi_d^* = \pi^* - \alpha = \frac{G'(\alpha)}{\sigma^2 - G''(\psi_\alpha)} \approx \frac{M_1 - M_2\alpha}{\sigma^2 + M_2}.$$

We now compute the integrand $f(\pi^*)$ in the excess utility formula $\mathbf{E}\int_0^T f(\pi_t^*)dt$, which is define by equation (3.5.5) or (3.7.2). Under quadratic approximation

$$\begin{aligned} f(\pi^*) &= G(\pi^*) - \frac{1}{2}\sigma^2(\pi^* - \alpha)^2 \\ &\approx G(\alpha) + G'(\alpha)\pi_d^* - \frac{1}{2}\, M_2(\pi_d^*)^2 - \frac{1}{2}\sigma^2(\pi_d^*)^2 \\ &= G(\alpha) + (\pi_d^*)^2(\sigma^2 + M_2) + \frac{1}{2}((-M_2 - \sigma^2))(\pi_d^*)^2 \\ &= G(\alpha) + \frac{1}{2}\,(\pi_d^*)^2(\sigma^2 + M_2), \end{aligned}$$

whence

$$\begin{aligned}
f(\pi^*) &= M_1\alpha - \frac{1}{2}M_2\alpha^2 + \frac{1}{2}\frac{(M_1 - M_2\alpha)^2}{(\sigma^2 + M_2)} \\
&= \frac{(2M_1\alpha - M_2\alpha^2)(\sigma^2 + M_2) + (M_1 - M_2\alpha)^2}{2(\sigma^2 + M_2)} \\
&= \frac{2M_1\alpha\sigma^2 - M_2\alpha^2\sigma^2 + 2M_2M_1\alpha - M_2^2\alpha^2 + M_1^2 - 2M_1M_2\alpha + M_2^2\sigma^2}{2(\sigma^2 + M_2)} \\
&= \frac{-M_2\alpha^2\sigma^2 + 2M_1\alpha\sigma^2 + M_1^2}{2(\sigma^2 + M_2)} \\
&= \frac{-M_2\theta^2 + 2M_1\theta\sigma + M_1^2}{2(\sigma^2 + M_2)}, \quad \alpha = \frac{\theta}{\sigma}.
\end{aligned}$$

Define the functions A, B, C of t, σ, M_1, M_2, by the prescriptions:

$$A \triangleq \frac{-M_2}{2(\sigma^2 + M_2)} \tag{3.9.9}$$

$$B \triangleq \frac{2M_1\sigma}{2(\sigma^2 + M_2)} \tag{3.9.10}$$

$$C \triangleq \frac{M_1^2}{2(\sigma^2 + M_2)}. \tag{3.9.11}$$

Then

$$\begin{aligned}
f(\pi^*) &= \frac{-M_2\theta^2 + 2M_1\theta\sigma + M_1^2}{2(\sigma^2 + M_2)}, \\
&= A\theta^2 + B\theta + C, \\
&\triangleq Q(\theta) \equiv Q(\theta : \sigma, M_1, M_2).
\end{aligned} \tag{3.9.12}$$

The foregoing leads to the following important result.

Theorem 3.13. *Let $G(\alpha)$ be defined on $[0, 1]$. The jump component of the maximum expected utility for the i–th investor resulting from quadratic approximation of G is*

$$u^i_{T,d}(x) \approx \mathbf{E}\int_0^T Q\left(\theta^i_t : \sigma_t, M_1, M_2\right) dt, \;\; i \in \{0, 1\}, \tag{3.9.13}$$

where θ is the stock's Sharpe ratio, A, B, C given by (3.9.9) and $Q(\theta) = A\theta^2 + B\theta + C$. The optimal portfolio for each investor is

$$\pi^* \approx \alpha + \frac{M_1 - M_2\alpha}{\sigma^2 + M_2}. \tag{3.9.14}$$

Corollary 3.3. *Let $i \in \{0, 1\}$. Under quadratic approximation of $G(\cdot)$, the jump component of the maximum expected utility from terminal wealth for the i–th investor is given by*

$$u^i_{T,d}(x) \approx \mathbf{E} \int_0^T Q\left(\frac{\mu_t}{\sigma_t} : \sigma_t,\ M_1,\ M_2\right) dt + \int_0^T A_t\, \mathbf{E}(\upsilon^i_t)^2 dt, \tag{3.9.15}$$

where υ^i is defined in Lemma 3.4 and Theorem 3.4.

Proof. By Theorem 3.13, the excess optimal utility due to the jumps for the i–th investor, where $i \in \{0, 1\}$, is given by

$$\begin{aligned} u^i_{T,d}(x) &\triangleq \mathbf{E} \int_0^T f^i(\pi^i_t) dt \approx \mathbf{E} \int_0^T Q\left(\theta^i_t; \sigma_t, M_1, M_2\right) dt \\ &= \mathbf{E} \int_0^T \left(A_t(\theta^i_t)^2 + B_t\theta^i_t + C_t\right) dt \\ &= \int_0^T \left(A_t\mathbf{E}(\theta^i_t)^2 + B_t\mathbf{E}\theta^i_t + C_t\right) dt. \end{aligned} \tag{3.9.16}$$

Since

$$\theta^i_t = \frac{\mu_t}{\sigma_t} = \frac{\mu_t + \upsilon^i_t\sigma_t}{\sigma_t} = \frac{\mu_t}{\sigma_t} + \upsilon^i_t,$$

and with $\mathbf{E}(\upsilon^i_t) = 0$, from Theorem 1.2, then $\mathbf{E}(\theta^i_t) = \frac{\mu_t}{\sigma_t}$. Therefore

$$\mathbf{E}(\theta^i_t)^2 = \left(\frac{\mu_t}{\sigma_t}\right)^2 + 2\frac{\mu_t}{\sigma_t}\mathbf{E}(\upsilon^i_t) + \mathbf{E}(\upsilon^i_t)^2 = \left(\frac{\mu_t}{\sigma_t}\right)^2 + \mathbf{E}(\upsilon^i_t)^2. \tag{3.9.17}$$

So by (3.9.16), we have

$$u^i_{T,\,d}(x) \approx \int_0^T \left(A_t \left[\left(\frac{\mu_t}{\sigma_t} \right)^2 + \mathbf{E}(\upsilon^i_t)^2 \right] + B_t \left(\frac{\mu_t}{\sigma_t} \right) + C_t \right) dt$$

$$= \int_0^T \left(A_t \left(\frac{\mu_t}{\sigma_t} \right)^2 + B_t \left(\frac{\mu_t}{\sigma_t} \right) + C_t \right) dt + \int_0^T A_t \mathbf{E} \left(\frac{\mu_t}{\sigma_t} \right)^2$$

$$= \int_0^T Q \left(\frac{\mu_t}{\sigma_t} : \sigma_t,\, M_1,\, M_2 \right) dt + \int_0^T A_t\, \mathbf{E} \left(\frac{\mu_t}{\sigma_t} \right)^2 dt.$$

□

With Corollary 3.3 in hand, we now have a major result for asymptotic optimal utilities due to jumps.

Theorem 3.14. *Assume that* $\int_{\mathbf{R}} (e^{\pm x} - 1)^k v(dx) < \infty, \quad k \geq 3$, *and* G *is restricted to* $[0,\, 1]$. *Let* $x > 0$ *be the initial wealth of the investors and* $i \in \{0, 1\}$.

(1) *As* $T \to \infty$, *the asymptotic optimal utility for the* i*–th investor is*

$$u^i_{\infty,\,d}(x) \sim \int_0^T Q \left(\frac{\mu_t}{\sigma_t} : \sigma_t, M_1, M_2 \right) dt + \frac{\lambda}{2} A_\infty (1-p)(1 + (-1)^{i+1} p) T, \tag{3.9.18}$$

where $Q(\theta)$ *is given by (3.9.12) and*

$$A_\infty = - \frac{M_2}{2(\sigma^2 + M_2)}, \quad \sigma = \lim_{t \to T} \sigma_t. \tag{3.9.19}$$

(2) *The excess asymptotic optimal utility of the informed investor over the uninformed investor, due to jumps is:*

$$u^1_{\infty,\,d}(x) - u^0_{\infty,\,d}(x) \sim \lambda\, A_\infty\, p\, (1-p)\, T. \tag{3.9.20}$$

Proof. Assume that $\int_{\mathbf{R}} (e^{\pm x} - 1)^2 v(dx) < \infty$, and G is restricted to $[0, 1]$. By Lemma

2.4, as $t \to \infty$,

$$\mathbf{E}(v_t^i)^2 \longrightarrow \frac{\lambda}{2}\,(1-p)\,(1+(-1)^{i+1}p),\;\; i \in \{0,1\}.$$

By Corollary 3.3, as $T \to \infty$, we have $u^i_{T,\,d}(x) \sim$

$$\mathbf{E}\int_0^T Q\left(\frac{\mu_t}{\sigma_t} : \sigma_t,\, M_1,\, M_2\right) dt + \int_0^T A_t \mathbf{E}(v_t^i)^2 dt = \mathbf{E}\int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + T \lim_{t\to\infty} A_t \mathbf{E}(v_t^i)^2$$

$$= \mathbf{E}\int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + T\, A_\infty \lim_{t\to\infty} \mathbf{E}(v_t^i)^2 = \mathbf{E}\int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\lambda}{2} A_\infty (1-p)(1+(-1)^{i+1}p)T,$$

where

$$A_\infty = \lim_{t\to T} A_t = \lim_{t\to\infty} -\frac{M_2}{2(\sigma_t^2 + M_2)} = -\frac{M_2}{2(\sigma_\infty^2 + M_2)} = -\frac{M_2}{2(\sigma^2 + M_2)}.$$

From part(1), it follows that the excess optimal utility of the informed investor due to the jumps is

$$u^1_{\infty,\,d}(x) - u^0_{\infty,\,d}(x) = \frac{\lambda}{2} A_\infty (1-p)(1+p)T - \frac{\lambda}{2} A_\infty (1-p)(1-p)T = \lambda\, A_\infty\, p\,(1-p)\,T.$$

□

We are now positioned to give the main theorem.

Theorem 3.15 (Main). *Let $p \in [0,\,1]$. Under quadratic approximation, the total asymptotic excess optimal utility of the informed investor is*

$$u^1_\infty(x) - u^0_\infty(x) \sim \frac{\tilde{\lambda}}{2}\, p\,(1-p)\,T, \tag{3.9.21}$$

where $x > 0$ is the initial wealth of investors, $\sigma = \lim_{t\to\infty} \sigma_t$ and

$$\tilde{\lambda} \triangleq \lambda \, \frac{\sigma^2}{\sigma^2 + M_2}, \tag{3.9.22}$$

is the adjusted mean–reversion rate.

Proof. Let $p \in [0, 1]$ and $i \in \{0, 1\}$. From Theorem 3.11, the total optimal asymptotic utility of the i–th investor is :

$$u^i_\infty(x) = u^i_{\infty,\,c}(x) + u^i_{\infty,\,d}(x).$$

By Theorems 3.10 and 3.14, respectively

$$u^i_{\infty,\,c}(x) = \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda\,T}{4}(1-p)(1+(-1)^{i+1}p) \tag{3.9.23}$$

$$u^i_{\infty,\,d}(x) = \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\lambda}{2}A_\infty(1-p)(1+(-1)^{i+1}p)T \tag{3.9.24}$$

The excess asymptotic optimal logarithmic utility of the informed investor is

$$u^1_\infty(x) - u^0_\infty(x) = (u^1_{\infty,\,c}(x) - u^0_{\infty,\,c}(x)) + (u^1_{\infty,\,d}(x) - u^0_{\infty,\,d}(x))$$

$$= \frac{\lambda}{2}\,p\,(1-p)\,T + \lambda\,A_\infty\,p\,(1-p)\,T = \frac{\lambda}{2}\,p\,(1-p)\,T(1+2\,A_\infty)$$

$$= \frac{\lambda}{2}\,p\,(1-p)\,T\left(1 - \frac{M_2}{\sigma^2_\infty + M_2}\right) = \frac{\lambda}{2}\left(\frac{M_2}{\sigma^2_\infty + M_2}\right)p\,(1-p)\,T$$

$$= \frac{\lambda}{2}\left(\frac{M_2}{\sigma^2 + M_2}\right)p\,(1-p)\,T = \frac{\tilde{\lambda}}{2}\,p\,(1-p)\,T,$$

where $\tilde{\lambda} = \lambda\,\frac{M_2}{(\sigma^2+M_2)}$ is the adjusted mean–reversion rate and $\sigma = \lim_{t\to\infty}\sigma_t$. □

Remark 3.9. *Note that Theorem 3.15 is analogous to Guasoni's major result for excess asymptotic utility (cf Thm 1.4, part 4), given by $\frac{\lambda}{2}\,p\,(1-p)\,T$, where λ is the mean–reversion rate for the Ornstein–Uhlenbeck process. However in this case, the* *mean–reversion rate for the O–U process is **reduced** to* $\tilde{\lambda} \triangleq \lambda \frac{\sigma_\infty^2}{\sigma_\infty^2 + M_2}$*, instead of* λ *as in the purely continuous GBM market. Here* $\sigma = \sigma_\infty$ *is the long–run volatility. For constant volatility models,* $\sigma_\infty = \sigma$*, which is usually the case. Note also, that we have maximum excess utility when* $p = \frac{1}{2}$.

We now give the asymptotic optimal utilities of each investor, under quadratic approximation of G.

Theorem 3.16. *Let* $p \in [0,\, 1]$ *and* $i \in \{0, 1\}$*. As* $T \longrightarrow \infty$*, the maximum expected logarithmic utility from terminal wealth for the* i*–th investor with* $x > 0$ *in initial wealth, is*

$$u_\infty^i(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\tilde{\lambda}}{4}(1-p)(1+(-1)^{i+1}p)\,T, \quad (3.9.25)$$

where $\tilde{\lambda}$ *is the adjusted mean–reversion rate of the driving O–U process.*

Proof. This follows easily by adding (3.9.23) and (3.9.24), with $A_\infty = -\frac{M_2}{2(\sigma_\infty^2 + M_2)}$. □

We now give an explicit formula for $\phi_{\pi^*}(i)$ of Theorem 3.11.

Corollary 3.4. *For each* $i \in \{0, 1\}$*, we have*

$$\phi_{\pi^*}(i) \triangleq \lim_{t \to \infty} \mathbf{E} f^i(\pi_t^*) = \frac{1}{T}\int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\lambda}{2} A_\infty (1-p)(1+(-1)^{i+1}p). \quad (3.9.26)$$

Proof. From corollary 3.1, as $T \to \infty$, we have

$$u_\infty^i(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \frac{\lambda T}{4}(1-p)(1+(-1)^{i+1}p) + T\phi_{\pi^*(i)}.$$

By Theorem 3.16,

$$u_\infty^i(x) \sim \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\tilde{\lambda}}{4}(1-p)(1+(-1)^{i+1}p)\,T.$$

Thus by (3.9.23) and (3.9.24),

$$\begin{aligned}
T\phi_{\pi^*}(i) &\sim \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\tilde{\lambda}}{4}(1-p)(1+(-1)^{i+1}p)T - \frac{\lambda}{4}(1-p)(1+(-1)^{i+1}p)T \\
&= \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{(\tilde{\lambda}-\lambda)}{4}(1-p)(1+(-1)^{i+1}p)T \\
&= \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{-M_2}{2(\sigma^2+M_2)}(1-p)(1+(-1)^{i+1}p)T \\
&= \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\lambda}{2}A_\infty\,(1-p)(1+(-1)^{i+1}p)T,
\end{aligned}$$

from which the result follows. □

3.10 The Link Between the Asymmetric and Symmetric Optimal Portfolios

Consider the situation of a symmetric Lévy market, where both investors have equal information, and the stock S has percentage returns dynamic:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t\, dB_t + \int_{\mathbf{R}} (e^x - 1)N(dt,\, dx). \tag{3.10.1}$$

By Theorem 3.8, it has unique optimal portfolio

$$\pi_t^* = \frac{\mu_t - r_t + G'(\pi_t^*)}{\sigma_t^2}, \tag{3.10.2}$$

where N is the Poisson random measure that counts the jumps of $X_t = \int_{\mathbf{R}} x\, N(t,\, dx)$, in $(0,\, t)$ and as usual, $v(dx) = \mathbf{E}N(1,\, dx)$ is the Levy measure with $G(\pi) = \int_{\mathbf{R}} \log(1+\pi(e^x - 1))v(dx)$. Let $i \in \{0, 1\}$. Under asymmetric information, the percentage returns dynamic for the i–th investor is

$$\frac{dS_t}{S_t} = \mu_t^i dt + \sigma_t \, dB_t^i + \int_{\mathbf{R}} (e^x - 1) N(dt,\, dx), \tag{3.10.3}$$

where B, B^i are $\mathcal{H}^i$–adapted standard Brownian motions, and

$$\begin{aligned}
\mu_t^i &= \mu_t + \upsilon_t^i \sigma_t, . \\
\upsilon_t^0 &= -\lambda \int_0^t e^{-\lambda(t-u)} (1 + \gamma(u)) dB_u^0. \\
\upsilon_t^1 &= -\lambda \, q U_t = -\lambda \, q \int_0^t e^{-\lambda(t-u)} dB_u.
\end{aligned}$$

Theorem 3.8 give the optimal portfolio $\pi_t^{*,\,i}$ for each investor as

$$\pi_t^{*,\,i} = \frac{\mu_t^i - r_t + G'(\pi_t^{*,\,i})}{\sigma_t^2}. \tag{3.10.4}$$

We now link π^*, the deterministic optimal portfolio in the symmetric market, to $\pi^{*,\,i}$, the random optimal portfolio in the asymmetric market.

Theorem 3.17. *Let $i \in \{0, 1\}$ and $T > 0$, be the the investment horizon. Assume that G is restricted to $[0,\, 1]$ and $\int_{\mathbf{R}} (e^{\pm x} - 1)^k v(dx) < \infty$ for some integer $k \geq 3$.*
(1) *There exists η^i between π^* and $\pi^{*,\,i}$, such that for all $t \in [0, T]$*

$$\pi_t^{*,\,i} = \pi_t^* + \frac{\upsilon_t^i \, \sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}. \tag{3.10.5}$$

That is,

$$\pi_t^{*,\,i} = \pi_t^* + noise_t^i.$$

(2) *Under quadratic approximation of G*

$$\pi_t^{*,\,i} = \pi_t^* + \frac{\upsilon_t^i \, \sigma_t}{\sigma_t^2 + M_2}. \tag{3.10.6}$$

Proof. If $\int_{\mathbf{R}}(e^{\pm x}-1)^2 v(dx) < \infty$, then $G''(\pi)$ exists for all $\pi \in [0,1]$, and by Theorem 3.8, optimal portfolios exist for both asymmetric and symmetric markets, given respectively by

$$\pi_t^* = \frac{\mu_t - r_t + G'(\pi_t^*)}{\sigma_t^2} = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{G'(\pi_t^*)}{\sigma_t^2},$$

and

$$\pi_t^{*,i} = \frac{\mu_t^i - r_t + G'(\pi_t^{*,i})}{\sigma_t^2} = \frac{\mu_t + \upsilon_t^i\,\sigma_t - r_t + G'(\pi_t^{*,i})}{\sigma_t^2} = \frac{\upsilon_t^i}{\sigma_t} + \frac{\mu_t - r_t}{\sigma_t^2} + \frac{G'(\pi_t^{*,i})}{\sigma_t^2}.$$

Define the portfolios $\beta^{*,i}$, by the prescription:

$$\begin{aligned}\beta_t^{*,i} &\triangleq \pi_t^* + \frac{\upsilon_t^i}{\sigma_t} \\ &= \frac{\mu_t - r_t}{\sigma_t^2} + \frac{G'(\pi_t^*)}{\sigma_t^2} + \frac{\upsilon_t^i}{\sigma_t}.\end{aligned} \tag{3.10.7}$$

By the Mean Value Theorem, there exists η^i between π^* and $\pi^{*,i}$ such that for all $t \in [0, T]$,

$$\pi_t^{*,i} - \beta_t^{*,i} = \frac{G'(\pi_t^{*,i}) - G'(\pi_t^*)}{\sigma_t^2} = \frac{(\pi_t^{*,i} - \pi_t^*)G''(\eta_t^i)}{\sigma_t^2}.$$

Since $G''(\pi) < 0$ for all π, then

$$\pi_t^{*,i} - (\pi_t^* + \frac{\upsilon_t^i}{\sigma_t}) = -\frac{(\pi_t^{*,i} - \pi_t^*)|G''(\eta_t^i)|}{\sigma_t^2}.$$

Rearranging, we get

$$\pi_t^{*,i}\left(\sigma_t^2 + |G''(\eta_t^i)|\right) = \upsilon_t^i\sigma_t + \pi_t^*\left(\sigma_t^2 + |G''(\eta_t^i)|\right),$$

whence,

$$\pi_t^{*,i} = \pi_t^* + \frac{\upsilon_t^i \, \sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}.$$

Part(2) follows from part(1), using the fact that under quadratic approximation of G, $G''(\pi) = -M_2$. □

We restate the last theorem as follows.

Theorem 3.18. *Let $i \in \{0, 1\}$ and $T > 0$, be the the investment horizon. Assume that G is restricted to $[0,\, 1]$ and $\int_{\mathbf{R}} (e^{\pm x} - 1)^k v(dx) < \infty$ for some integer $k \geq 3$. There exists an adjusted diffusive coefficient $0 < \tilde{\sigma_t^i} < \frac{1}{\sigma_t}$, such that, for all $t \in [0,\, T]$,*

$$\pi_t^{*,i} = \pi_t^* + \upsilon_t^i \, \tilde{\sigma_t^i}, \tag{3.10.8}$$

where

$$\tilde{\sigma_t^i} \triangleq \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}, \tag{3.10.9}$$

and η^i is a process between π^* and $\pi^{*,i}$.

Proof. This result follows from Theorem 3.17, with

$$\tilde{\sigma_t^i} = \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|} < \frac{1}{\sigma_t}.$$

□

We now compute an upper bound on the expected squared deviation of the asymmetric optimal portfolio and the symmetric optimal portfolio for each investor.

Theorem 3.19. *Let $i \in \{0, 1\}$, $p \in [0,\, 1]$ and $T > 0$ be the the investment horizon. Assume that G is restricted to $[0,\, 1]$ and $\int_{\mathbf{R}} (e^{\pm x} - 1)^k v(dx) v(dx) < \infty$ for some integer $k \geq 3$. Then for each $t \in [0,\, T]$,*

$$\mathbf{E}[\pi_t^{*,i} - \pi_t^*]^2 \leq \frac{\lambda}{2\sigma_t^2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}), \tag{3.10.10}$$

where λ is the mean–reversion rate of the fads process.

Proof. By Theorem 3.17,

$$\pi_t^{*,i} = \pi_t^* + \frac{\upsilon_t^i \,\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}.$$

The expected squared deviation of $\pi_t^{*,i}$ from the symmetric optimal portfolio is

$$\mathbf{E}[\pi_t^{*,i} - \pi_t^*]^2 = \mathbf{E}\left[\frac{\upsilon_t^i \,\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}\right]^2 \leq \frac{\mathbf{E}[\upsilon_t^i]^2}{\sigma_t^2}.$$

When $i = 1$, it was proven in Chapter 2 that

$$\mathbf{E}[\upsilon_t^1]^2 = \frac{\lambda}{2}(1-p^2)(1-e^{-2\lambda t}) = \frac{\lambda}{2}(1+p)(1-p)(1-e^{-2\lambda t}).$$

When $i = 0$, $\upsilon_t^0 = -\lambda \int_0^t e^{-\lambda(t-u)}(1+\gamma(u))dB_u^0$, and by Itô–Isometry,

$$\mathbf{E}[\upsilon_t^0]^2 = \lambda^2 \int_0^t e^{-2\lambda(t-u)}(1+\gamma(u))^2 du. \tag{3.10.11}$$

But

$$1 + \gamma(u) = \frac{1-p^2}{1+p\tanh{(p\lambda u)}},$$

where $\tanh(u) \in (-1,1)$, for each $p \in [0,1]$. Therefore $0 \leq 1+\gamma(u) \leq \frac{1-p^2}{1-p} = 1+p$, whence $(1+\gamma(u))^2 \leq (1+p)^2$, and by (3.10.11)

$$\mathbf{E}[\upsilon_t^0]^2 \leq \lambda^2 \int_0^t e^{-2\lambda(t-u)}(1+p)^2 du = \lambda^2(1+p)^2 e^{-2\lambda t}\int_0^t e^{2\lambda u} du$$

$$= \lambda^2(1+p)^2 e^{-2\lambda t}\frac{(e^{2\lambda t}-1)}{2\lambda} = \frac{\lambda}{2}(1+p)^2(1-e^{-2\lambda t}) = \frac{\lambda}{2}(1+p)(1+(-1)^0 p)(1-e^{-2\lambda t}).$$

Therefore, for $i \in \{0, 1\}$

$$\mathbf{E}[\upsilon_t^i]^2 \leq \frac{\lambda}{2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}), \tag{3.10.12}$$

and so

$$\mathbf{E}[\pi_t^{*,i} - \pi_t^*]^2 \leq \frac{\lambda}{2\sigma_t^2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}). \tag{3.10.13}$$

□

It will prove convenient to isolate equation 3.10.12 as a separate result.

Corollary 3.5. *For each $i \in \{0, 1\}$ and $p \in [0,\ 1]$,*

$$\mathbf{E}[\upsilon_t^i]^2 \leq \frac{\lambda}{2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}). \tag{3.10.14}$$

We now compute the expectation and give an upper bound on the variance of the asymmetric portfolio for each investor under quadratic approximation. It turns out that we **expect** the random asymmetric optimal portfolios to be equal to the pure deterministic symmetric optimal portfolio.

Theorem 3.20. *Let $i \in \{0, 1\}$, $p \in [0,\ 1]$ and $T > 0$, be the the investment horizon. Assume that G is restricted to $[0,\ 1]$ and $\int_{\mathbf{R}}(e^{\pm x} - 1)^k v(dx) v(dx) < \infty$ for some integer $k \geq 3$. Under quadratic approximation, for each $t \in [0,\ T]$,*

$$\mathbf{E}[\pi_t^{*,i}] = \pi_t^*, \tag{3.10.15}$$

and

$$\mathbf{Var}[\pi_t^{*,i}] \leq \frac{\lambda}{2\sigma_t^2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}), \tag{3.10.16}$$

where λ is the mean–reversion rate of the fads process.

Proof. Under quadratic approximation $G''(\eta) = -M_2$. By Theorem 3.17,

$$\mathbf{E}[\pi_t^{*,i}] = \pi_t^* + \mathbf{E}\left(\frac{\upsilon_t^i\,\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}\right) = \pi_t^* + \sigma_t\frac{\mathbf{E}(\upsilon_t^i)}{\sigma_t^2 + M_2} = \pi_t^*.$$

By Theorem 3.19, the variance of the asymmetric optimal portfolio, is

$$\mathbf{Var}[\pi_t^{*,i}] = \mathbf{E}[\pi_t^{*,i} - \pi_t^*]^2 \leq \frac{\lambda}{2\sigma_t^2}(1+p)(1+(-1)^i p)(1-e^{-2\lambda t}).$$

□

3.11 The Pure Jump Lévy Market

We study markets that have no diffusive component in his section. They are called *Pure Jump Lévy Markets*. In this case, the market is driven **only** by a pure jump Lévy process X and no Brownian process is involved. As usual, our market has two assets. There is a bond $\mathbf{B}$ earning risk–free compounded interest rate r with price given by (3.1.1). There is also a single stock S with log returns dynamic for the i–th investor, $i \in \{0, 1\}$, being:

$$d(\log S_t) = \mu_t^i\,dt + \int_{\mathbf{R}} xN(dt, dx), \tag{3.11.1}$$

where μ_t^i is the continuous returns on the stock for the i–th investor. The stock has percentage returns:

$$\frac{dS_t}{S_t} = \mu_t^i\,dt + \int_R (e^x - 1)N(dt, dx), \tag{3.11.2}$$

where X has Lévy triple $(\gamma, 0, v)$, with $\gamma = \int_{[-1,1]} x\,v(dx)$. μ_t^i is the continuous component of the total stock appreciation rate $b = \mu^i + M_1$. Note that the percentage returns has two components: one continuous and locally deterministic (μ_t^i); one is discontinuous and driven by the Poisson random measure N on $[0, T] \times (\mathbf{R} - \{0\})$.

All processes live on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$.

What are Pure Jump Models?

Consider the asymmetric diffusion market:

$$d(\log S_t) = (\mu_t^i - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t + \int_{\mathbf{R}} xN(dt, dx), \tag{3.11.3}$$

with percentage returns dynamic

$$\frac{dS_t}{S_t} = \mu_t^i \, dt + \sigma_t \, dB_t + \int_{\mathbf{R}} (e^x - 1)N(dt, dx). \tag{3.11.4}$$

Pure jump models result from diffusion models when the contribution from the diffusive component σ_t is negligible relative to the total votatility $\sigma_{Tot} = \sqrt{\sigma_t^2 + M_2}$; that is, when $\sigma_t \approx 0$. Thus a Pure Jump Market (PJM) is a diffusive market with $\sigma_t = 0$ for all t. Explicitly,

$$d(\log S_t) = \mu_t^i \, dt + \int_{\mathbf{R}} x \, N(dt, dx) = \lim_{\sigma_t \to 0} \left((\mu_t^i - \frac{1}{2} \, \sigma_t^2)dt + \sigma_t \, dB_t + \int_{\mathbf{R}} x \, N(dt, dx) \right), \tag{3.11.5}$$

and

$$\mu_t^i = \mu_t + v_t^i \sigma_t = \mu_t + v_t^i \, \sigma_t \longrightarrow \mu_t. \tag{3.11.6}$$

The optimal portfolio for the stock with dynamic of (3.11.1) is therefore the limiting optimal portfolio of the stock with dynamic (3.11.3) as $\sigma_t \to 0$. Since $\mu_t^i \longrightarrow \mu_t$ as $\sigma_t \longrightarrow 0$, then the dynamic for **both** investors is

$$d(\log S_t) = \mu_t \, dt + \int_{\mathbf{R}} x \, N(dt, dx) = \lim_{\sigma_t \to 0} \left((\mu_t \, dt + \sigma_t \, dB_t + \int_{\mathbf{R}} x \, N(dt, dx)) \right). \tag{3.11.7}$$

We have the following result.

Theorem 3.21. *For the Pure Lévy Market (3.11.1), the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth is the same for both informed and uninformed investors. It is given by the deterministic function* π_t *where*

$$G'(\pi_t) = r_t - \mu_t, \;\; t \in [0, T]. \tag{3.11.8}$$

The maximum expected utility for each investor is equal, and given by:

$$u(x) = \log(x) + \int_0^T \left(\pi_t(\mu_t - r_t) + \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))v(dx) \right) dt, \tag{3.11.9}$$

and the informed investor has no excess utility.

Proof. By Theorem 3.8, (3.11.7) yields optimal portfolio π given by

$$\sigma_t^2 \pi = \mu_t - r_t + G'(\pi). \tag{3.11.10}$$

Letting $\sigma_t \to 0$ we get

$$0 = \mu_t - r_t + G'(\pi).$$

Thus the optimal portfolio for **both** investors is the deterministic portfolio

$$\pi_t^i = \pi_t, \tag{3.11.11}$$

where

$$G'(\pi_t) = r_t - \mu_t. \tag{3.11.12}$$

Since both optimal portfolios are equal, there is no excess utility involved. □

We also have the following:

Theorem 3.22. *There exists η_t between 0 and π_t such that*

$$\pi_t = \frac{\mu_t - r_t + M_1}{|G''(\eta_t)|} = \frac{b_t - r_t}{|G''(\eta_t)|}, \tag{3.11.13}$$

where the total returns on the stock is $b_t = \mu_t + M_1$, and $M_1 = \int_{\mathbf{R}}(e^x - 1)v(dx)$ is the returns due to the jumps.

Moreover, under quadratic approximation of $G(\cdot)$

$$\pi_t = \frac{\mu_t - r_t + M_1}{M_2} = \frac{b_t - r_t}{M_2}. \tag{3.11.14}$$

Proof. Let $G(\pi) = \int_{\mathbf{R}} \log(1 + \pi\,(e^x - 1))\,v(dx)$. Then $M_1 = \int_{\mathbf{R}}(e^x - 1)\,v(dx)$. Since π is optimal, it obeys the equation

$$G'(\pi_t) = r_t - \mu_t.$$

By the Mean Value theorem, there exists $\eta_t \in (0,\, \pi_t)$ such that

$$\pi_t\, G''(\eta_t) = G'(\pi_t) - G'(0) = r_t - \mu_t - M_1.$$

Since $G''(\pi) < 0$ for all π, then

$$\pi_t = \frac{\mu_t - r_t + M_1}{-G''(\eta_t)} = \frac{\mu_t - r_t + M_1}{|G''(\eta_t)|} = \frac{b_t - r_t}{|G''(\eta_t)|}.$$

Under quadratic approximation of G, $G''(\pi) = G''(0) = -M_2$, and the result follows.

□

3.11.1 An Example of a Pure Jump Market

We now give an example of a Pure Jump Market driven by two independently linked Poisson processes.

The Double Poisson Market

Consider a stock that has price dynamic:

$$d(\log S_t) = \mu_t \, dt + dX_t, \ \ t \in [0, T], \tag{3.11.15}$$

where

$$X_t = \alpha_u \, N^u(t) + \alpha_d \, N^d(t), \tag{3.11.16}$$

with

$$\alpha_u \in (0, \log 2), \quad \alpha_d = \log(2 - e^{\alpha_u}).$$

X is called the *Double Poisson* Lévy process with parameters $\alpha_u, \alpha_d, \lambda_u, \lambda_d$ where N^u and N^d are independent Poisson processes with intensities λ_u and λ_d, respectively. In this model, N^u controls the upward jumps which have log amplitude α_u, while N^d controls the downward jumps, which have log amplitude α_d. As in the other models, μ is the continuous component of total returns. We denote a Double Poisson process by $\Pi(1, 2)$. The Lévy measure for the Double Poisson process is:

$$v(dx) \equiv v_{\Pi(1,2)}(dx) = \lambda_u \, \delta_{\alpha_u}(dx) + \lambda_d \, \delta_{\alpha_d}(dx), \tag{3.11.17}$$

where

$$0 < \lambda_d \leq \lambda_u < 1.$$

$\delta_a(\cdot)$ is the Dirac measure on $B(\mathbf{R} - \{0\})$ where $\delta_a(A) = 1$, if $a \in A$, and 0 otherwise. Let $N(t, A)$ be the Poisson random measure that counts the jumps of X in the set $A \in \mathbf{B}(\mathbf{R} - \{0\})$ in the time interval $(0, t)$.

3.11.2 Maximization of Expected Logarithmic Utility From Terminal Wealth

Let $V^\pi = (V_t^\pi) \equiv V$ be the wealth process corresponding to the portfolio process $\pi = (\pi_t)_{t\in[0,T]}$, which is $\mathcal{H}_t$–adapted, S–integrable, with $V_t^\pi > 0$, where $V_0^0 = x > 0$, is the initial capital investment or wealth. The **Admissible Set** is:

$\mathcal{A}_x = \{\pi\colon \pi \text{ is } S\text{–integrable, predictable, } \mathcal{H}_t\text{–adapted, } V_t^\pi > 0\ \}$.

The wealth process $V \equiv V^\pi$ satisfies the dynamic:

$$dV_t = (1-\pi_t)\, r_t\, V_t\, dt + \frac{\pi_t\, V_t}{S_t}\, dt. \tag{3.11.18}$$

Setting $J_t = \int_{\mathbf{R}} (e^x - 1)\, N(dt, dx)$ we get

$$\begin{aligned}
\frac{dV_t}{V_t} &= (1-\pi_t)\, r_t\, dt + \frac{\pi_t\, dS_t}{S_t} \\
&= (1-\pi_t)\, r_t\, dt + \pi_t\, (\mu_t\, dt + dJ_t) \\
&= (\pi_t\, (\mu_t - r_t) + r_t)\, dt + \pi_t\, dJ_t \\
&= (r_t + \pi_t\, (\mu_t - r_t))\, dt + \int_{\mathbf{R}} \pi_t\, (e^x - 1)\, N(dt, dx),
\end{aligned}$$

which has solution (cf. Protter [41], p. 84)

$$V_t = V_0\, \exp\left(\int_0^t r_s\, ds + \int_0^t \pi_s\, (\mu_s - r_s)\, ds\right) \Pi_{0\le s\le t}(1 + \pi_s\, (e^{\Delta X_s} - 1)). \tag{3.11.19}$$

Let the discounted wealth process be $\widetilde{V}_t \triangleq V_t\, \exp(-\int_0^t r_s\, ds)$. Then

$$\widetilde{V}_t = V_0\, \exp\left(\int_0^t \pi_s\, (\mu_s - r_s)\, ds\right) \Pi_{0\le s\le t}(1 + \pi_s (e^{\Delta X_s} - 1)). \tag{3.11.20}$$

The logarithmic utility of terminal wealth is

$$
\begin{aligned}
u(\widetilde{V}_T) &= \log \widetilde{V}_T \\
&= \log x + \int_0^T \pi_s\,(\mu_s - r_s)\,ds + \sum_{0\le s\le T} \log(1 + \pi_s(e^{\Delta X_s} - 1))
\end{aligned}
$$

$$
= \log x + \int_0^T \pi_t\,(\mu_t - r_t)\,dt + \int_0^T \int_{\mathbf{R}} \log(1 + \pi_t\,(e^x - 1))\,N(dt, dx).
$$

The expected logarithmic utility from terminal wealth is

$$
\begin{aligned}
\mathbf{E}(\log \widetilde{V}_T) &= \log x + \mathbf{E}\int_0^T \pi_t\,(\mu_t - r_t)\,dt + \mathbf{E}\int_0^T \int_{\mathbf{R}} \log(1 + \pi_t\,(e^x - 1))v(dx)dt \\
&= \log x + \int_0^T \left[\pi_t\,(\mu_t - r_t) + \int_{\mathbf{R}} \log(1 + \pi_t\,(e^x - 1))\,v(dx)\right]\,dt, \qquad (3.11.21)
\end{aligned}
$$

since all market coefficients are deterministic.

Define a function $f : \mathcal{A}_x \to \mathbf{R}$ by the pescription

$$
\begin{aligned}
f(\pi) &= \pi\,(\mu - r) + \int_{\mathbf{R}} \log(1 + \pi\,(e^x - 1))\,v(dx) \\
&= \pi(\mu - r) + G(\pi). \qquad (3.11.22)
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{E}(\log \widetilde{V}_T) &= \log x + \int_0^T \int_{\mathbf{R}} f(\pi_s)\,ds, \qquad (3.11.23) \\
u(x) &= \max_{\pi_s \in \mathcal{A}_x} \mathbf{E}(\log \widetilde{V}_T) \\
&= \log x + \max_{\pi \in \mathcal{A}_x} \int_0^T \int_{\mathbf{R}} f(\pi_s)\,ds \\
&= \log x + \int_0^T \int_{\mathbf{R}} \max_{\pi_s \in \mathcal{A}_x} f(\pi_s)\,ds, \qquad (3.11.24)
\end{aligned}
$$

where π maximizes $f(\pi)$, the objective function. Thus

$$\begin{aligned}
f(\pi) &= \pi\,(\mu - r) + G(\pi) \\
&= \pi\,(\mu - r) + \int_{\mathbf{R}} \log(1 + \pi\,(e^x - 1))\,v(dx) \\
&= \pi(\mu - r) + \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))(\lambda_u\,\delta_{\alpha_u}(dx) + \lambda_d\,\delta_{\alpha_d}(dx))
\end{aligned}$$

$$\begin{aligned}
&= \pi(\mu - r) + \lambda_u \log(1 + \pi(e^{\alpha_u} - 1)) + \lambda_d \log(1 + \pi(e^{\alpha_d} - 1)) \\
&= \pi(\mu - r) + \lambda_u \log(1 + \pi(e^{\alpha_u} - 1)) + \lambda_d \log(1 + \pi(1 - e^{\alpha_u})).
\end{aligned}$$

π falls in the admissible set $\mathcal{A}_x$, if $G(\pi)$ is well–defined. We therefore insist that

$$1 + \pi\,(e^{\alpha_u} - 1) > 0, \quad 1 + \pi\,(1 - e^{\alpha_u}) > 0.$$

Thus, setting $0 < A_u = e^{\alpha_u} - 1 < 1$ and $a = \frac{1}{A_u}$, we have $-a < \pi < a$. That is,

$$\pi \in (-a\,,\,a), \;\; a > 1$$

We can take the admissible set as $(-a\,,\,a)$, and optimize $f(\pi)$ over this set, which clearly contains [-1 , 1]. Since f is strictly concave , it admits a unique maximum π, where $f'(\pi) = 0$. Now

$$f'(\pi) = \mu - r + \frac{\lambda_u\,(e^{\alpha_u} - 1)}{1 + \pi\,(e^{\alpha_u} - 1)} + \frac{\lambda_d\,(1 - e^{\alpha_u})}{1 + \pi(1 - e^{\alpha_u})} = 0,$$

is equivalent to the quadratic equation

$$f'(\pi) = \mu - r + \frac{\lambda_u}{a + \pi} - \frac{\lambda_d}{a - \pi} = 0. \tag{3.11.25}$$

This has solution:

$$\pi = \pi_{\pm} = -\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right) \pm \sqrt{\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right)^2 + \frac{(\lambda_u - \lambda_d)a}{\mu - r} + a^2}, \tag{3.11.26}$$

provided $\pi \in \mathcal{A}_x$; $a = (e^{\alpha_u} - 1)^{-1}$. Specifically, we have

$$\pi = \begin{cases} \pi_+ & \text{if } \mu > r \\ \frac{\lambda_u - \lambda_d}{\lambda_u + \lambda_d} a & \text{if } \mu = r \\ \pi_- & \text{if } \mu < r. \end{cases}$$

The maximum expected logarithmic utility is

$$\begin{aligned} u(x) &= \log x + \int_0^T f(\pi_t)\, dt = \log(x) + \int_0^T \left(\pi_t\left(\mu_t - r_t\right) + G(\pi_t)\right)\, dt \\ &= \log(x) + \int_0^T \left[\pi_t(\mu_t - r_t) + \lambda_d \log\left(1 + \frac{\pi}{a}\right) + \lambda_u \log\left(1 - \frac{\pi}{a}\right)\right] dt. \end{aligned}$$

Remark 3.10. *If $\lambda_d = 0$, we have no downward jumps, and $G(\pi)$ is well defined if $\pi > -a = (e^{\lambda_u} - 1)^{-1}$. The Admissible set $\mathcal{A}_x$, may be as large as $(-a, \infty)$. The objective function reduces to*

$$f(\pi) = \pi(\mu - r) + \lambda_u \log(1 + \pi\,(e^{\alpha_u} - 1)),$$

with

$$f'(\pi) = \mu - r + \frac{\lambda_u}{a + \pi} = 0 \;\; \textit{iff} \;\; \frac{\lambda_u}{a + \pi} = r - \mu.$$

This has a solution only if $r - \mu > 0$, since $a + \pi$ is positive. In this case, if $r \leq \mu$, we have no optimal. If $r - \mu > 0$, we have an optimal

$$\pi = -a + \frac{\lambda_u}{r - \mu},$$

with maximum expected utility

$$u(x) = \log x + \left[a(r-\mu) - \lambda_u + \lambda_u \, \log\left(\frac{\lambda_u}{a(r-\mu)}\right)\right] T,$$

if the market coefficients μ_t, r_t *are constants.*

3.12 Conclusion

We successfully extended the theory of fads models under asymmetric information to the jump case, where analogous results were obtained. Our more general model reduces to the Guasoni [21] model, if no jumps are present. We obtained explicit formulas for the optimal portfolios and expected logarithmic utilities for both investors, under reasonable assumptions. We showed that the asymptotic excess maximal expected utility from terminal wealth of the informed investor, is $\frac{\tilde{\lambda}}{2}\, p\,(1-p)\, T$, which is similar to Guasoni's. In the jump case, the mean–reversion rate λ, is replaced by a smaller adjusted mean–reversion rate $\tilde{\lambda}$.

We also link the random optimal portfolio under asymmetric information to the entirely deterministic optimal portfolio of the symmetric market. It turns out that **under asymmetric information, the optimal portfolio is equal to the deterministic optimal portfolio plus noise.** Our model depends on parameters– the instantaneous centralized moments of returns M_j–calculated or approximated from exponential functions having Lévy measure integrators. We obtained tractable results under quadratic approximation of the portfolio estimating function
$G(\pi) = \int_{\mathbf{R}} \log(1 + \pi(e^x - 1))v(dx)$.

We also study the pure jump Lévy process and show that the informed investor has no excess utility in this market. We give explicit formulas for optimal portfolio and utility for the Double Poisson market.

Chapter 4

Asymmetric Information in Discontinuous Fads Models in Lévy Markets

In this chapter we generalize the theory developed in Chapter 3 to include the situation where the fads/mispricing jump, in addition to the jumps in the stock price. We model the jumps in the mispricing by a pure jump Lévy process $Z = (Z_t)_{t\geq 0}$ with $\mathbf{E}(Z_t) = 0$ and $\mathbf{E}(Z_t^2) < \infty$ for all $t \in [0, T]$ where $T > 0$, is the investment horizon.

The market consist of a bond $\mathbf{B}$ with price given by (3.1.1). There is also a risky asset S called stock. The stock is viewed by investors in disjoint classes populated by uninformed and informed investors, indexed by $i = 0$ and $i = 1$, respectively. Investors have filtrations $\mathcal{K}_t^i$ with

$$\mathcal{K}_t^0 \subset \mathcal{K}_t^1 \subset \mathcal{F},\ t \in [0, T].$$

All random objects are defined on a filtered probability space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{K}^i, \mathbf{P})$.

4.1 The Discontinuous Fads Model

The stock S has log returns dynamic

$$\begin{aligned} d(\log S_t) &= (\mu_t - \frac{1}{2}\sigma_t^2)\,dt + \sigma_t\,dY_t + dX_t,\ t \in [0,\,T] && (4.1.1)\\ Y_t &= p\,W_t + q\,U_t,\ p^2 + q^2 = 1, p \geq 0,\ q \geq 0, && (4.1.2)\\ dU_t &= -\lambda\,U_t\,dt + dL_t,\ \lambda > 0,\ U_0 = 0, && (4.1.3)\\ L_t &= B_t + Z_t, && (4.1.4)\\ X_t &= \int_0^t \int_{\mathbf{R}} x\,N_S(dt, dx), && (4.1.5)\\ Z_t &= \int_0^t \int_{\mathbf{R}} x\,\widetilde{N}_U(dt, dx),\ \ \mathbf{E}(Z_t) = 0,\ \mathbf{E}(Z_t^2) < \infty. && (4.1.6) \end{aligned}$$

W and B are independent standard Brownian motions. $U = (U_t)$ is a mean–reverting **Ornstein–Uhlenbeck** process with rate λ. N_S and N_U are Poisson random measures on $\mathbf{R}_+ \times \mathcal{B}(\mathbf{R} - \{0\})$ that are linked to the stock and fads/mispricing, respectively. They count the jumps of X and Z, respectively, in the time interval $(0,\,t)$. The respective Lévy measures are

$$v_S(dx) = \mathbf{E}N_S(1, dx), \tag{4.1.7}$$

and

$$v_U(dx) = \mathbf{E}N_U(1, dx). \tag{4.1.8}$$

Standing Assumption

We assume that N_S and N_U are independent random measures if $N_S \neq N_U$. That is, X and Z are independent pure jump Lévy processes with $\mathbf{E}(Z_t) = 0$ for all $t \in [0,\,T]$. Thus

$$\int_{\mathbf{R}} x v_U(dx) < \infty.$$

From (4.1.5),

$$
\begin{aligned}
Z_t &= \int_0^t \int_{\mathbf{R}} x \widetilde{N}_U(ds, dx) \\
&= \int_0^t \int_{\mathbf{R}} x (N_U(ds, dx) - v_U(dx) ds) \\
&= \int_0^t \int_{\mathbf{R}} x N_U(ds, dx) - t \int_{\mathbf{R}} x v_U(dx),
\end{aligned} \tag{4.1.9}
$$

where

$$
\widetilde{N}_U(dt, dx) \triangleq N_U(dt, dx) - v_U(dx) dt
$$

is the compensated Poisson measure. Clearly $Z = (Z_t)$ is a martingale, along with W. Equation (4.1.9) proves that we may take Z to be a martingale provided $\int_{\mathbf{R}} x v_U(dx) < \infty$. Consequently, we set

$$
Z_t \triangleq \int_0^t \int_{\mathbf{R}} x \widetilde{N}_U(dt, dx) = \int_0^t \int_{\mathbf{R}} x N_U(ds, dx) - t \int_{\mathbf{R}} x v_U(dx). \tag{4.1.10}
$$

From equations (4.1.3) and (4.1.4), we get the dynamic

$$
dU_t = -\lambda\, U_t dt + dB_t + dZ_t, \tag{4.1.11}
$$

which admits the unique solution:

$$
\begin{aligned}
U_t &= \int_0^t e^{-\lambda(t-s)} dB_s + \int_0^t e^{-\lambda(t-s)} dZ_s \\
&\triangleq U_t^B + U_t^Z,
\end{aligned} \tag{4.1.12}
$$

where

$$
dU_t^B = -\lambda\, U_t^B dt + dB_t, \quad U_0^B = 0, \ \ \lambda > 0, \tag{4.1.13}
$$

and

$$dU_t^Z = -\lambda\, U_t^Z dt + dZ_t, \quad U_0^Z = 0, \quad \lambda > 0. \tag{4.1.14}$$

Explicitly,

$$U_t^B = \int_0^t e^{-\lambda(t-s)} dB_s \tag{4.1.15}$$

and

$$U_t^Z = \int_0^t e^{-\lambda(t-s)} dZ_s. \tag{4.1.16}$$

Note that in this case the mispricing is the O–U process U, consisting of a linear combination of two independent O–U processes: a continuous component U^B driven by Brownian motion B, which is identical to the fads process used by Guasoni [21] in Chapter 3, and a jump component U^Z driven by the martingale Z. We showed in Chapter 2 that for all $t \in [0,\, T]$

$$\mathbf{E}(U_t^B) = 0, \quad \mathbf{E}(U_t^B)^2 = \frac{1 - e^{-2\lambda\, t}}{2\lambda}.$$

We have a similar result for the process U^Z.

Proposition 4.1. *For each* $t \in [0,\, T]$

$$\begin{aligned} \mathbf{E}(U_t^Z) &= 0, \\ \mathbf{Var}(U_t^Z) &= \frac{1 - e^{-2\lambda\, t}}{2\lambda} \int_{\mathbf{R}} x^2\, v_U(dx), \\ \lim_{t\to\infty} \mathbf{E}(U_t^Z)^2 &= \frac{1}{2\lambda} \int_{\mathbf{R}} x^2\, v_U(dx). \end{aligned}$$

Proof. By the martingale property of Z,

$$\mathbf{E}(U_t^Z) = \mathbf{E} \int_0^t e^{-\lambda(t-s)} dZ_s = e^{-\lambda\, t}\mathbf{E} \int_0^t e^{\lambda\, s} dZ_s = 0.$$

By Itô-Isometry and the fact that $t \int_{\mathbf{R}} x^2 \, v_U(dx) = \mathbf{E}(Z_t^2) < \infty$, we have

$$\mathbf{Var}(U_t^Z) = \mathbf{E}(U_t^Z)^2 = \mathbf{E} \left(\int_0^t e^{-\lambda(t-s)} dZ_s \right)^2 = \mathbf{E} \int_0^t e^{-2\lambda(t-s)} d[Z,\, Z]_s$$

$$= \int_0^t e^{-2\lambda(t-s)} \int_{\mathbf{R}} x^2 \, v_U(dx) ds = \int_0^t e^{-2\lambda(t-s)} ds \int_{\mathbf{R}} x^2 \, v_U(dx) = \frac{1 - e^{-2\lambda\, t}}{2\lambda} \int_{\mathbf{R}} x^2 \, v_U(dx).$$

The last result follows directly by letting $t \to \infty$. □

We have a similar result for the mean–reverting mispricing process U.

Proposition 4.2. *For each* $t \in [0,\, T]$

$$\begin{aligned} \mathbf{E}(U_t) &= 0, \\ \mathbf{Var}(U_t) &= \left(\frac{1 - e^{-2\lambda\, t}}{2\lambda} \right) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right), \\ \lim_{t \to \infty} \mathbf{E}(U_t)^2 &= \frac{1}{2\lambda} \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right). \end{aligned}$$

Proof. By equation (4.1.12),

$$U_t \triangleq U_t^B + U_t^Z,$$

whence

$$\mathbf{E}(U_t) = \mathbf{E}(U_t^B) + \mathbf{E}(U_t^Z) = 0.$$

Since U^B and U^Z are independent processes, then it follows from Proposition 4.1 that

$$\mathbf{E}(U_t^2) = \mathbf{Var}(U_t) = \mathbf{Var}(U_t^B) + \mathbf{Var}(U_t^Z) = \frac{1 - e^{-2\lambda\, t}}{2\lambda} + \left(\frac{1 - e^{-2\lambda\, t}}{2\lambda} \right) \int_{\mathbf{R}} x^2 \, v_U(dx)$$

$$= \left(\frac{1 - e^{-2\lambda\, t}}{2\lambda} \right) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right).$$

The last result follows directly by letting $t \to \infty$. □

It now follows from (4.1.2) and equation (4.1.12) that

$$
\begin{aligned}
Y_t &= p\,W_t + q\,U_t \\
&= p\,W_t + q\int_0^t e^{-\lambda(t-s)}dB_s + q\int_0^t e^{-\lambda(t-s)}dZ_s,
\end{aligned}
\tag{4.1.17}
$$

is a martingale consisting of three independent components:–two are continuous, while one is a pure jump Lévy process. Taking differentials, and importing equation (4.1.11) yield

$$
\begin{aligned}
dY_t &= p\,dW_t + q\,dU_t \\
&= p\,dW_t + q(-\lambda U_t dt + dB_t + dZ_t) \\
&= p\,dW_t + q\,dB_t - \lambda q U_t dt + q\,dZ_t \\
&= p\,dW_t + q\,dB_t - \lambda q U_t^B dt - \lambda U_t^Z dt + q\,dZ_t \\
&= dB_t^1 + \upsilon_t^{1,B} dt + \upsilon_t^Z dt + q\,dZ_t, \\
&= dB_t^1 + \upsilon_t^1 dt + q\,dZ_t,
\end{aligned}
\tag{4.1.18}
$$

where

$$
B_t^1 = p\,W_t + q\,B_t, \tag{4.1.19}
$$

$$
\upsilon_t^1 = -\lambda q\,U_t^B - \lambda q\,U_t^Z, \tag{4.1.20}
$$

$$
\triangleq \upsilon_t^{1,B} + \upsilon_t^Z, \tag{4.1.21}
$$

and

$$
\upsilon_t^{1,B} \triangleq -\lambda\,q\,U_t^B, \tag{4.1.22}
$$

$$
\upsilon_t^Z \triangleq -\lambda\,q\,U_t^Z. \tag{4.1.23}
$$

We have a useful result for the process υ^1, which is a generalization of the process used in Chapter 3.

Proposition 4.3. *Let $t \in [0, T]$ and $i = 1$. Then for p, $q \in [0, 1]$ with $p^2 + q^2 = 1$,*

$$\begin{aligned}
\mathbf{E}[\upsilon_t^i] &= 0, \\
\mathbf{E}[\upsilon_t^i]^2 &= \mathbf{Var}[\upsilon_t^i] = \frac{\lambda}{2}(1 - e^{-2\lambda t})(1-p)(1+(-1)^{i+1}p) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right) \\
&= \frac{\lambda}{2}(1 - e^{-2\lambda t})(1+p)(1+(-1)^{i}p) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right).
\end{aligned} \tag{4.1.24}$$

Moreover

$$\lim_{t\to\infty} \mathbf{E}[\upsilon_t^i]^2 = \frac{\lambda}{2}(1-p)(1+(-1)^{i+1}p) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right).$$

Proof. We prove the case for $i = 1$. From equation (4.2.1),

$$\mathbf{E}(\upsilon_t^1) = \mathbf{E}(\upsilon_t^{1,B}) + \mathbf{E}(\upsilon_t^Z) = -\lambda \, q\mathbf{E}(U_t^B) - \lambda \, q\mathbf{E}(U_t^Z) = 0.$$

Since U^B and U^Z are independent O–U processes, then from Lemma 2.2 with $q^2 = 1 - p^2$, we have

$$\mathbf{Var}(\upsilon_t^i) = \mathbf{Var}(\upsilon_t^{i,B}) + \mathbf{Var}(\upsilon_t^Z) = \lambda^2 q^2 \, \mathbf{Var}(U_t^B) + \lambda^2 \, q^2 \, \mathbf{Var}(U_t^Z)$$

$$\begin{aligned}
&= \lambda^2 q^2 \left(\frac{1 - e^{-2\lambda t}}{2\lambda}\right) + \lambda^2 \, q^2 \left(\frac{1 - e^{-2\lambda t}}{2\lambda}\right) \int_{\mathbf{R}} x^2 \, v_U(dx) \\
&= \frac{\lambda}{2}(1 - e^{-2\lambda t}) \, q^2 \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right) \\
&= \frac{\lambda}{2}(1 - e^{-2\lambda t}) \, (1-p)(1+(-1)^{i+1}p) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right) \\
&= \frac{\lambda}{2}(1 - e^{-2\lambda t}) \, (1+p)(1+(-1)^{i}p) \left(1 + \int_{\mathbf{R}} x^2 \, v_U(dx)\right).
\end{aligned}$$

The last result follows immediately when $t \to \infty$. $\square$

4.2 Asset Price Dynamic for the Informed Investor

Define

$$\mu_t^* \triangleq \mu_t - \frac{1}{2}\sigma_t^2. \tag{4.2.1}$$

The dynamic for the stock is given by (4.1.1) as

$$d(\log S_t) = \mu_t^*\, dt + \sigma_t\, dY_t + dX_t,\ t \in [0,\, T].$$

Imposing (4.1.2), (4.1.3), and (4.1.18), yield

$$\begin{aligned} d(\log S_t) &= \mu_t^* dt + \sigma_t(dB_t^1 + \upsilon_t^1 dt + q\, dZ_t) + dX_t \\ &= \mu_t^* dt + \upsilon_t^1 \sigma_t dt + \sigma_t dB_t^1 + q\, \sigma_t dZ_t + dX_t \\ &= (\mu_t^* - q\sigma_t \int_{\mathbf{R}} x\upsilon_U(dx) + \upsilon_t^1\sigma_t)dt + \sigma_t dB_t^1 \\ &\quad + \int_{\mathbf{R}} q\, \sigma_t x N_U(dt, dx) + \int_{\mathbf{R}} x N_S(dt, dx) \\ &= \mu_t^{*,1} dt + \sigma_t dB_t^1 + \int_{\mathbf{R}} x(q\, \sigma_t N_U(dt, dx) + N_S(dt, dx)), \end{aligned} \tag{4.2.2}$$

where

$$\mu_t^{*,1} = \mu_t^* - q\sigma_t \int_{\mathbf{R}} x\upsilon_U(dx) + \upsilon_t^1\sigma_t = \mu_t + \upsilon_t^1\sigma_t - q\sigma_t \int_{\mathbf{R}} x\upsilon_U(dx) - \frac{1}{2}\sigma_t^2. \tag{4.2.3}$$

Remark 4.1. *If $N_S = N_U = N$ then $\upsilon = \upsilon_U = \upsilon_S$, and (4.2.2) becomes*

$$\mu_t^{*,1} dt + \sigma_t\, dB_t^1 + \int_{\mathbf{R}} x\,(q\sigma_t + 1)N(dt, dx). \tag{4.2.4}$$

We now quote a very useful result in Applebaum page 47.

Lemma 4.1 (Proposition 1.3.12, Applebaum [3]).
If $(N_1(t), t \geq 0)$ and $(N_2(t), t \geq 0)$ are two independent Poisson processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with arrival times $(T_n^{(j)},\ n \in \mathbf{N})$, for $j = 1, 2$, respectively, then $\mathbf{P}(T_m^{(1)} = T_n^{(2)}$ for some $m, n \in \mathbf{N}$)=0.

This result means that **independent Poisson processes do not jump at the same time**. Thus for random measures N_U and N_S and Borel set $A \in \mathcal{B}(\mathbf{R} - \{0\})$, $N_U(t, A)$ and $N_S(t, A)$ are independent Poisson processes, and so, by the last result, do not jump together.

We have the following important result.

Theorem 4.1. *For the informed investor the log returns dynamic (4.1.1) is*

$$d(\log S_t) = \mu_t^{*,1} dt + \sigma_t dB_t^1 + \int_{\mathbf{R}} q\sigma_t \, x \, N_U(dt, dx) + \int_{\mathbf{R}} x \, N_S(dt, dx), \tag{4.2.5}$$

and its percentage returns dynamic is

$$\frac{dS_t}{S_t} = \mu_t^1 dt + \sigma_t dB_t^1 + \int_{\mathbf{R}} (e^{q\sigma_t x} - 1) N_U(dt, dx) + \int_{\mathbf{R}} (e^x - 1) N_S(dt, dx), \tag{4.2.6}$$

where N_U and N_S are defined by (4.1.5)and (4.1.6) and

$$\begin{aligned} \mu_t^{*,1} &\triangleq \mu_t^1 - \frac{1}{2}\sigma_t^2, \\ \mu_t^1 &\triangleq \mu_t + \upsilon_t^1 \sigma_t - q\sigma_t \int_{\mathbf{R}} x \upsilon_U(dx). \end{aligned}$$

Proof. By equation (4.2.2), the dynamic for the informed investor is equivalently given by

$$d(I(t)) = (\mu_t^{*,1} dt + \sigma_t \, dB_t^1 + \int_{\mathbf{R}} q\sigma_t x \, N_U(dt, dx) + \int_{\mathbf{R}} x \, N_S(dt, dx), \tag{4.2.7}$$

where

$$I(t) = \log S_t \Longleftrightarrow S_t = e^{\log S_t} = e^{I(t)}.$$

Let $f(x) = e^x$. Then $f \in C^2(\mathbf{R})$, and by Itô's formula (cf Thm 4.4.7, Applebaum [3]), we have from (4.2.7), that

$$\begin{aligned}
d(f(X_t)) &= \frac{df}{dx}(X_{t-}) + \frac{1}{2}\frac{d^2 f}{dx^2}(X_{t-})d[X^c, X^c]_t \\
&\quad + \int_{\mathbf{R}} (f(X_{t-} + q\,x\,\sigma_t) - f(X_{t-}))\, N_U(dt, dx) \\
&\quad + \int_{\mathbf{R}} (f(X_{t-} + x) - f(X_{t-}))\, N_S(dt, dx) \\
&= e^{X_{t-}}\left[\mu_t^{*,1}\, dt + \sigma_t\, dB_t^1 + \frac{1}{2}\,\sigma_t^2 dt\right] + \int_{\mathbf{R}} e^{X_{t-}}(e^{q\sigma_t x} - 1)N_U(dt, dx) \\
&\quad + \int_{\mathbf{R}} e^{X_{t-}}(e^x - 1)N_S(dt, dx) \\
&= e^{X_{t-}}\left[\mu_t^1 dt + \sigma_t dB_t^1 + \int_{\mathbf{R}}(e^{q\sigma_t x} - 1)N_U(dt, dx) + \int_{\mathbf{R}}(e^x - 1)N_S(dt, dx)\right].
\end{aligned}$$

Thus, with $X_t = I(t) = \log S_t$

$$d(e^{\log S_t}) = S_{t-}\left[\mu_t^1 dt + \sigma_t dB_t^1 + \int_{\mathbf{R}}(e^{q\sigma_t\, x} - 1)N_U(dt, dx) + \int_{\mathbf{R}}(e^x - 1)N_S(dt, dx)\right]$$

which is equivalent to the percentage returns equation:

$$\frac{dS_t}{S_{t-}} = \mu_t^1 dt + \sigma_t\, dB_t^1 + \int_{\mathbf{R}}(e^{q\sigma_t\, x} - 1)N_U(dt, dx) + \int_{\mathbf{R}}(e^x - 1)N_S(dt, dx). \qquad \square$$

With Theorem 4.1 in hand, we are now in the framework of the models developed in Chapter 3.

4.2.1 Filtration of the Informed Investor

We expand the filtration developed in Chapter 3, to include the information generated by the martingale $Z = (Z_t)$. We assume that all filtrations obey the usual hypothesis;

that is, they are right continuous and complete. The filtration of the informed investor ($i = 1$) is

$$\mathcal{K}_t^1 = \mathcal{H}_t^1 \vee \sigma(Z_u : u \leq t),\; t \in [0,\, T], \tag{4.2.8}$$

where $\mathcal{H}_t^1$ is the filter for the informed investor in the models where there are no jumps in fads (cf Lemma 3.2, of Chapter 3). We denote by $\mathcal{K}^1$, the filtration for the informed investor, where

$$\mathcal{K}^1 = (\mathcal{K}_t^1)_{t \geq 0}.$$

4.3 Maximization of Logarithmic Utility from Terminal Wealth for the Informed Investor

Let $\pi = \pi^1 = (\pi_t^1),\; t \in [0,\, T]$, be the portfolio process of the informed investor. The corresponding wealth process with initial capital $x > 0$ is :

$$V_t^1 \equiv V_t^{1,\pi} \equiv V_t^{1,\pi,x}. \tag{4.3.1}$$

We now give the dymamic for V^1.

Theorem 4.2. *The percentage returns of the wealth V^1 of the informed investor with stock price dynamic (4.1.1) is:*

$$\frac{dV_t^1}{V_t^1} = (r_t + \pi_t\, \sigma_t\, \theta_t^1)dt + \pi_t\, \sigma_t\, dB_t^1 + \int_{\mathbf{R}} \pi_t (e^{q\sigma_t\, x} - 1) N_U(dt, dx) + \int_{\mathbf{R}} \pi_t (e^{x} - 1) N_S(dt, dx). \tag{4.3.2}$$

The discounted wealth process is

$$
\begin{aligned}
\widetilde{V}_t^1 &= V_t^1 \exp\left(-\int_0^t r_s ds\right) \\
&= V_0^1 \exp\left(\int_0^t (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^1\right) \\
&\quad \times \Pi_{0\le u\le t}(1+\pi_u(e^{q\sigma_u\Delta Z_u}-1)) \times \Pi_{0\le s\le t}(1+\pi_s(e^{\Delta X_s}-1)). \quad (4.3.3)
\end{aligned}
$$

Proof. The crucial component of this proof is Lemma 4.1, which states that $N_U(t,A)$ and $N_S(t,A)$ do not jump at the same time, because they are independent Poisson processes at each bounded Borel set $A \in \mathcal{B}(\mathbf{R}-\{0\})$. From Theorem 4.1, the percentage returns of V^1 is

$$
\begin{aligned}
\frac{dV_t^1}{V_t^1} &= (1-\pi_t)r_t dt + \pi_t \frac{dS_t}{S_t} \\
&= (1-\pi_t)\, r_t\, dt + \pi_t\, \mu_t^1\, dt + \pi_t\, \sigma_t\, dB_t^1 + \int_{\mathbf{R}} \pi_t(e^{q\sigma_t x}-1)N_U(dt,dx) \\
&\quad + \int_{\mathbf{R}} \pi_t(e^x-1)N_S(dt,dx). \\
&= (r_t + \pi_t(\mu_t^1 - r_t))dt + \pi_t\, \sigma_t\, dB_t^1 + \int_{\mathbf{R}} \pi_t(e^{q\sigma_t x}-1)N_U(dt,dx) \\
&\quad + \int_{\mathbf{R}} \pi_t(e^x-1)N_S(dt,dx). \\
&= (r_t + \pi_t\, \sigma_t\, \theta_t^1)dt + \pi_t\, \sigma_t\, dB_t^1 + \int_{\mathbf{R}} \pi_t(e^{q\sigma_t x}-1)N_U(dt,dx) \\
&\quad + \int_{\mathbf{R}} \pi_t(e^x-1)N_S(dt,dx).
\end{aligned}
$$

This proves equation (4.3.2). Applying Theorem 2.18 on Doleans–Dade/Stochastic exponentials, and using the fact of the independence of jumps of N_U and N_S, equation (4.3.2) yields the solution

$$
\begin{aligned}
V_t^1 &= V_0^1 \exp\left(\int_0^t r_s ds + \int_0^t (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^1\right) \\
&\quad \times \Pi_{0\le u\le t}(1+\pi_u(e^{q\sigma_u\Delta Z_u}-1)) \times \Pi_{0\le s\le t}(1+\pi_s(e^{\Delta X_s}-1)), \quad (4.3.4)
\end{aligned}
$$

with discounted wealth given by (4.3.3). □

Remark 4.2. *We use the subscripts "u" and "s" in equation* (4.3.4) *to distinguish the jumps due to N_U and N_S, respectively.*

4.3.1 The Objective Function G

For any Lévy measure $v_a(\cdot)$, $a \in \{U,\, S\}$, define the (partial) objective function G_a by the prescription

$$G_a : [0,\, 1] \longrightarrow \mathbf{R},$$

$$G_a(\alpha\,;\, s) \triangleq \int_{\mathbf{R}} \log(1 + \alpha(e^{s\,x} - 1))v_a(dx),\ a \in \{U, S\}, \tag{4.3.5}$$

where $s \in (0, \sigma_{max}] \cup \{1\}$ is a non-negative parameter or function representing volatility and $\sigma_{max} > 0$. Note that α and hence π, is restrictd to the domain $[0, 1]$.

Standing Assumption

We insist that there exists an integer $k \geq 2$ such that

$$\int_{\mathbf{R}} (e^{\pm s\,x} - 1)^k v_a(dx) < \infty\ a \in \{U, S\},\ \ s \in (0,\, \sigma_{max}] \cup \{1\}. \tag{4.3.6}$$

Define the objective function G by

$$G : [0,\, 1] \longrightarrow \mathbf{R},$$

$$G(\alpha\,;\, s) \triangleq G_U(\alpha\,;\, s) + G_S(\alpha\,;\, 1). \tag{4.3.7}$$

Lemma 4.2. *Let $\alpha \in [0,\, 1]$ and $\int_{\mathbf{R}}(e^{\pm\, sx} - 1)^2 v_a(dx) < \infty$, where $a \in \{U,\, S\}$ and $s \in (0,\, \sigma_{max}] \cup \{1\}$. Then*

$$G''(\alpha\,; s) < 0$$

Proof. Let $a \in \{U, S\}$. Since $\int_{\mathbf{R}}(e^{\pm\, sx} - 1)^2 v_a(dx) < \infty$, with $s \in (0,\, \sigma_{max}] \cup \{1\}$then

$G_U, G_S \in C^2[0,\,1]$, and for all $\alpha \in [0,\,1]$ and $s \in [0, \sigma_{max}] \cup \{1\}$,

$$G''_a(\alpha\,;\,s) = -\int_{\mathbf{R}} \frac{(e^{s\,x}-1)^2 v_a(dx)}{(1+\alpha(e^{s\,x}-1))^2} < 0.$$

Thus $G''(\alpha\,;\,s) = G''_U(\alpha\,;\,s) + G''_S(\alpha\,;\,1) < 0.$ $\square$

We now give a major result about optimal portfolios.

Theorem 4.3. *For the informed investor, the optimal portfolio π^1 that maximizes the expected logarithmic utility from terminal wealth over the investment period $[0,\,T]$ is given by*

$$\pi^1_t = \frac{\mu^1_t - r_t + G'(\pi^1_t\,;\,q\sigma_t)}{\sigma^2_t} = \frac{\mu^1_t - r_t + G'_U(\pi^1_t\,;\,q\sigma_t) + G'_S(\pi^1_t\,;\,1)}{\sigma^2_t} \tag{4.3.8}$$

In terms of the stock's total returns b^1_t, the optimal portfolio is

$$\pi^1_t = \frac{b^1_t - r_t - K_U(q\sigma_t) - K_S(1) + G'(\pi^1_t; q\sigma_t)}{\sigma^2_t}, \tag{4.3.9}$$

$$K_a(s) = \int_{\mathbf{R}} (e^{s\,x}-1) v_a(dx),\; a \in \{U, S\},\; s \geq 0. \tag{4.3.10}$$

The maximum expected logarithmic utility from terminal wealth, with $x > 0$ in initial capital, is

$$u^1(x) = \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta^1_t)^2 dt + \mathbf{E}\int_0^T f^{(1)}_{U,S}(\pi^1_t)dt, \tag{4.3.11}$$

where

$$
\begin{aligned}
f^{(1)}_{U,S}(\pi^1) &= G(\pi^1\,;\,q\sigma) - \frac{1}{2}(\pi^1\sigma - \theta^1)^2, \qquad (4.3.12)\\
&\equiv G_U(\pi^1\,;\,q\sigma) + G_S(\pi^1\,;\,1) - \frac{1}{2}(\pi^1\sigma - \theta^1)^2,\\
\mu^1_t &= \mu_t + \upsilon^1_t\sigma_t - q\sigma_t\int_{\mathbf{R}} x\upsilon_U(dx),\\
\upsilon^1_t &= -\lambda q U^B_t - \lambda q U^Z_t,\\
b^1_t &= \mu^1_t + K_U(q\sigma_t) + K_S(1). \qquad (4.3.13)
\end{aligned}
$$

Proof. Since utility is logarithmic, then using Poisson integration, the utility from terminal (discounted) wealth for the informed investor is:

$$
\begin{aligned}
\log \widetilde{V}^1_T &= \log x + \int_0^T (\pi_s\sigma_s\theta^1_s - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB^1_s\\
&\quad + \sum_{0\le u\le T}\log(1+\pi_u(e^{q\sigma_u\Delta Z_u}-1)) + \sum_{0\le s\le T}\log(1+\pi_s(e^{\Delta X_s}-1))\\
&= \log x + \int_0^T (\pi_s\sigma_s\theta^1_s - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^T \pi_s\sigma_s dB^1_s\\
&\quad + \int_0^T\int_{\mathbf{R}}\log(1+\pi_u(e^{q\sigma_u x}-1))N_U(du,dx)\\
&\quad + \int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^{x}-1))N_S(ds,dx)
\end{aligned}
$$

Taking expectation yields,

$$
\begin{aligned}
\mathbf{E}\log \widetilde{V}^1_T &= \log x + \frac{1}{2}\mathbf{E}\int_0^T(\theta^1_s)^2 ds - \frac{1}{2}\mathbf{E}\int_0^T(\pi_s\sigma_s - \theta^1_s)^2 ds\\
&\quad + \int_0^T\int_{\mathbf{R}}\log(1+\pi_u(e^{q\sigma_u x}-1))\upsilon_U(dx)ds\\
&\quad + \int_0^T\int_{\mathbf{R}}\log(1+\pi_s(e^{x}-1))\upsilon_S(dx)ds
\end{aligned}
$$

Thus

$$
\mathbf{E}\log \widetilde{V}^1_T = \log x + \frac{1}{2}\mathbf{E}\int_0^T(\theta^1_t)^2 dt + \mathbf{E}\int_0^T f^{(1)}_{U,S}(\pi_t)dt, \qquad (4.3.14)
$$

where

$$\begin{aligned}
f_{U,S}^{(1)}(\pi_t) &= -\frac{1}{2}(\pi_t\sigma_t - \theta_t^1)^2 + \int_{\mathbf{R}} \log(1+\pi_t(e^{q\sigma_t x}-1))v_U(dx) \\
&\quad + \int_{\mathbf{R}} \log(1+\pi_t(e^{x}-1))v_S(dx) \\
&= G_U(\pi_t; q\sigma_t) + G_S(\pi_t; 1) - \frac{1}{2}(\pi_t\sigma_t - \theta_t^1)^2 \\
&= G(\pi_t; q\sigma_t) - \frac{1}{2}(\pi_t\sigma_t - \theta_t^1)^2.
\end{aligned}$$

The value function is

$$\begin{aligned}
u^1(x) &= \max_{\pi\in\mathcal{A}_1(x)} \mathbf{E}\log \widetilde{V}_T^{1,\pi} \\
&= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^1)^2 dt + \max_{\pi}\mathbf{E}\int_0^T f_{U,S}^{(1)}(\pi_s)ds \\
&= \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^1)^2 dt + \mathbf{E}\int_0^T \max_{\pi} f_{U,S}^{(1)}(\pi_s)ds. \qquad (4.3.15)
\end{aligned}$$

Thus the objective function $f_{U,S}^{(1)}(\pi)$ given by (4.3.12) is strictly concave by Lemma 4.2 if $\pi \in [0,\, 1]$, $\int_{\mathbf{R}}(e^{\pm\sigma x}-1)^2 v_a(dx) < \infty$, and $\int_{\mathbf{R}}(e^{\pm x}-1)^2 v_a(dx) < \infty$, where $a \in \{U, S\}$. Thus $f_{U,S}^{(1)}(\pi)$ has a maximum at π^1 where

$$\frac{d}{d\pi}\left(f_{U,S}^{(1)}(\pi)\right)|_{\pi=\pi^1}. \qquad (4.3.16)$$

Dropping the subscripts "U" and "S", we have

$$f'(\pi) = G'(\pi\,;\, q\sigma) - \sigma\,(\pi\sigma - \theta^1) = G'_U(\pi\,;\, q\sigma) + G'_S(\pi\,;\, 1) - \sigma(\pi\sigma - \theta^1) = 0,$$

whence equation (4.3.8) holds. The maximum expected utility $u^1(x)$ follows from the value function (4.3.15), with $\max_\pi f_{U,S}^{(1)}(\pi) = f_{U,S}^{(1)}(\pi^1)$. □

Remark 4.3. *Because of the intractability of the integrals* $\int_{\mathbf{R}}\log(1+\pi(e^{\sigma x}-1))v_a(dx)$

where $a \in \{U, S\}$, we compute the optimal portfolio π^1 using approximation techniques. This is done via the instantaneous centralized moments of returns for both Lévy processes X and Z, with Lévy measures v_S and v_U, respectively.

For the stock S, we have the usual centralized moments

$$M_S(k) = \int_{\mathbf{R}} (e^x - 1)^k v_S(dx) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} K_S(j),$$

where

$$K_S(s) = \int_{\mathbf{R}} (e^{s\,x} - 1) v_S(dx), \;\; s \geq 0.$$

For the fads process linked to U, the instantaneous centralized moments of returns are dependent on the volatility σ_t and are given by:

$$M_U(k\,;\,\sigma) = \int_{\mathbf{R}} (e^{\sigma\,x} - 1)^k v_U(dx) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} K_U(j\,;\,\sigma) \tag{4.3.17}$$

where

$$K_U(j\,;\,\sigma) = \int_{\mathbf{R}} (e^{j\,\sigma\,x} - 1) v_U(dx). \tag{4.3.18}$$

In this case, if $\int_{\mathbf{R}} (e^{\pm\,\sigma x} - 1)^{k+1} v_a(dx) < \infty$ and $\int_{\mathbf{R}} (e^{\pm\,x} - 1)^{k+1} v_a(dx) < \infty$ for some $k \in \mathbf{N}$, we approximate $G_U(\alpha\,;\,\sigma)$ by $G_{U,\,k}(\alpha\,;\,\sigma)$ and $G(\alpha\,;\,\sigma)$ by $G_k(\alpha\,;\,\sigma)$, where $\alpha \in [0,\,1]$.

$$G_U(\alpha\,;\,\sigma) \approx G_{U,\,k}(\alpha\,;\,\sigma) = \sum_{j=1}^{k} (-1)^{j-1} M_U(j\,;\,\sigma) \frac{\alpha^j}{j!} \tag{4.3.19}$$

and

$$G(\alpha\,;\,\sigma) \approx G_k(\alpha\,;\,\sigma) = \sum_{j=1}^{k} (-1)^{j-1} (M_S(j) + M_U(j\,;\,\sigma)) \frac{\alpha^j}{j!}. \tag{4.3.20}$$

4.3.2 The Case $N_U = N_S = N$

We now study the case where X and Z, given by (4.1.5) and (4.1.6) respectively, have the same Poisson random measure N, and hence the same Lévy measure $v = v_U = v_S$. As already demonstrated by (4.2.4), the stock has log returns dynamic

$$d(\log S_t) = \mu_t^{*,1}\, dt + \sigma_t\, dB_t^1 + \int_{\mathbf{R}} x\,(q\sigma_t + 1)N(dt, dx). \tag{4.3.21}$$

Application of Itô's formula yields,

$$\frac{dS_t}{S_t} = \mu_t^1\, dt + \sigma_t\, dB_t^1 + \int_{\mathbf{R}} (e^{(q\sigma_t+1)\,x} - 1)N(dt, dx). \tag{4.3.22}$$

It follows from Theorem 3.1, that the dynamic of the wealth process V^1 is

$$\frac{dV_t^1}{V_t^1} = (r_t + \pi_t\, \sigma_t\, \theta_t^1)\, dt + \pi_t\, \sigma_t\, dB_t^1 + \int_{\mathbf{R}} \pi_t(e^{(q\sigma_t+1)\,x} - 1)N(dt, dx), \tag{4.3.23}$$

with value

$$\begin{aligned} V_t^1 &= V_0^1 \exp\left(\int_0^t r_s\, ds + \int_0^t (\pi_s\sigma_s\theta_s^1 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^1\right) \\ &\quad \times \Pi_{0\le s\le t}(1 + \pi_s(e^{(q\sigma_s+1)\Delta X_s} - 1)). \end{aligned} \tag{4.3.24}$$

The optimal portfolio π^1 is given by the equation

$$\begin{aligned} \pi^1 &= \frac{\sigma\theta^1 + G'(\pi^1; q\sigma)}{\sigma^2} \\ &= \frac{\mu^1 - r + G'(\pi^1; q\sigma)}{\sigma^2}, \end{aligned} \tag{4.3.25}$$

where

$$G(\alpha\,;\,\sigma) = \int_{\mathbf{R}} \log(1 + \alpha(e^{(\sigma+1)\,x} - 1))v(dx).$$

4.4 Asset Price Dynamic for Uninformed Investor

The uninformed investor observes the stock price only and does not know what the fads are, although it is known that they may exist. Uninformed investors have filtration $\mathcal{K}_t^0 \subset \mathcal{F}$, $t \in [0, T]$, which is contained in $\mathcal{K}_t^1$, the filtration of the informed investor. We shall develop the dynamics for these investors, indexed by $i = 0$, starting with the log returns dynamic for the stock given by equations (4.1.1)–(4.1.6).

Recall that

$$U_t = U_t^B + U_t^Z.$$

Definition 4.1. *Define the continuous part of the process Y by*

$$Y_t^c \triangleq Y_t - q\,U_t^Z = p\,W_t + q\,U_t^B. \tag{4.4.1}$$

We have the following major result.

Theorem 4.4. *There exists an $\mathcal{K}^0$–Brownian motion B^0 and a process $\upsilon_t^{0,\,B} \equiv -\phi_t^{0,\,B}$ adapted to $\mathcal{K}_t^0$, such that for each $t \in [0, T]$ and $q \in [0,\,1]$,*

$$B_t^0 = Y_t^c + \int_0^t \phi_s^{0,\,B} ds \equiv Y_t - q\,U_t^Z - \int_0^t \upsilon_s^{0,\,B} ds, \tag{4.4.2}$$

where

$$\begin{aligned} \upsilon_t^{0,\,B} &= -\lambda \int_0^t e^{-\lambda(t-s)}(1+\gamma(s))dB_s^0, \\ \gamma(s) &= \frac{1-p^2}{1+p\tanh(\lambda p s)} - 1,\ p \in [0,\,1], \end{aligned} \tag{4.4.3}$$

and $\lambda > 0$ is the mean–reversion rate of U.

Proof. Since

$$Y_t^c \triangleq Y_t - q\,U_t^Z = p\,W_t + q\,U_t^B$$

is a continuous Gaussian process (being the sum of two such independent processes), then by Lemma 3.4 there exists an $\mathcal{H}^0$–Brownian motion B^0 and a process $\phi_t^{0,\,B}$ adapted to $\mathcal{H}_t^0$, such that

$$B_t^0 = Y_t^c + \int_0^t \phi_s^{0,\,B} ds,$$

where

$$\phi_s^{0,\,B} = -\upsilon_s^{0,\,B} = \lambda \int_0^t e^{-\lambda(t-s)}(1+\gamma(s))dB_s^0.$$

Define the filtration $\mathcal{K}^0$ by the prescription

$$\mathcal{K}_t^0 = \mathcal{H}_t^0 \vee \sigma(Z_u : u \leq t),\; t \in [0,\, T], \tag{4.4.4}$$

where we assume that all filtrations are complete and right continuous. Clearly,

$$\mathcal{H}_t^0 \subset \mathcal{K}_t^0,$$

and since B_t^0 and $\phi_t^{0,\,B}$ are $\mathcal{H}_t^0$–adapted, they are $\mathcal{K}_t^0$–adapted. Thus,

$$B_t^0 = Y_t - q\,U_t^Z + \int_0^t \phi_s^{0,\,B} ds = Y_t - q\,U_t^Z - \int_0^t \upsilon_s^{0,\,B} ds.$$

$\square$

From equation (4.4.2), we have

$$Y_t = B_t^0 + \int_0^t \upsilon_s^{0,\,B}\, ds + q\,U_t^Z. \tag{4.4.5}$$

Therefore,

$$\begin{aligned}
dY_t &= dB_t^0 + \upsilon_t^{0,\,B}\,dt + q\,dU_t^Z \\
&= dB_t^0 + \upsilon_t^{0,\,B}\,dt - \lambda\,q\,U_t^Z\,dt + dZ_t \\
&= dB_t^0 + \upsilon_t^{0,\,B}\,dt + \upsilon_t^Z\,dt + dZ_t \\
&= dB_t^i + \upsilon_t^i\,dt + dZ_t,\ i = 0, \qquad (4.4.6)
\end{aligned}$$

where

$$\upsilon_t^0 \triangleq \upsilon_t^{0,B} + \upsilon_t^Z, \qquad (4.4.7)$$

$$\upsilon_t^Z \triangleq -\lambda\,q\,U_t^Z. \qquad (4.4.8)$$

Note that (4.4.6) has the same form as (4.1.18), with $i = 0$. Also equation (4.4.7) is similar to equation (4.2.1). We have a useful result for the process υ^0 which is similar to the result of Propostion 4.3.

Proposition 4.4. *Let $t \in [0,\,T]$, $i \in \{0,\,1\}$ and $p,\,q \in [0,1]$, where $p^2+q^2 = 1$. Then*

$$\begin{aligned}
\mathbf{E}[\upsilon_t^i] &= 0, \\
\mathbf{E}[\upsilon_t^i]^2 &= \mathbf{Var}[\upsilon_t^i] \le \frac{\lambda}{2}(1-e^{-2\lambda\,t})(1+p)(1+(-1)^i p)\left(1+\int_{\mathbf{R}} x^2\,v_U(dx)\right).
\end{aligned}$$

Moreover

$$\lim_{t\to\infty} \mathbf{E}[\upsilon_t^i]^2 = \frac{\lambda}{2}\,(1-p)(1+(-1)^{i+1}p)\left(1+\int_{\mathbf{R}} x^2\,v_U(dx)\right).$$

Proof. We proved equality for the case $i = 1$ in Proposition 4.3. The case $i = 0$ follows from

$$\mathbf{E}[\upsilon_t^i]^2 = \mathbf{E}[\upsilon_t^{i,B}]^2 + \mathbf{E}[\upsilon_t^Z]^2,$$

by importing the inequality for $\mathbf{E}(\upsilon_t^{0,B})^2$ from Corollary 3.5. $\square$

We now have a result analogous to Theorem 4.1.

Theorem 4.5. *There exists an $\mathcal{K}^0$–adapted Brownian motion B^0 such that the log returns dynamic (4.1.1) of the uninformed investor is*

$$d(\log S_t) = \mu_t^{*,0}\, dt + \sigma_t\, dB_t^0 + \int_{\mathbf{R}} q\sigma_t x\, N_U(dt, dx) + \int_{\mathbf{R}} x\, N_S(dt, dx), \tag{4.4.9}$$

and its percentage return dynamic is

$$\frac{dS_t}{S_t} = \mu_t^0\, dt + \sigma_t\, dB_t^0 + \int_{\mathbf{R}} (e^{q\sigma_t\, x} - 1) N_U(dt, dx) + \int_{\mathbf{R}} (e^x - 1) N_S(dt, dx). \tag{4.4.10}$$

N_U and N_S are defined by (4.1.5)and (4.1.6).

Proof. This follows similarly as in the proof of Theorem 4.1. $\square$

4.4.1 Maximization of Logarithmic Utility for the Uninformed Investor.

Let $V^0 \equiv V^{0,\pi} \equiv V^{0,\pi,x}$ be the wealth of the uninformed investor, where $\pi \equiv \pi^0 = (\pi_t^0)_{t\geq 0}$ is the proportion of the wealth invested in the stock, with $x > 0$ in initial capital.

Theorem 4.6. *The percentage returns dynamic for the uninformed investor is*

$$\frac{dV_t^0}{V_t^0} = (r_t + \pi_t\, \sigma_t\, \theta_t^0)\, dt + \pi_t\, \sigma_t\, dB_t^0 + \int_{\mathbf{R}} \pi_t (e^{q\sigma_t\, x} - 1) N_U(dt, dx) + \int_{\mathbf{R}} \pi_t (e^x - 1) N_S(dt, dx). \tag{4.4.11}$$

The discounted wealth process is

$$
\begin{aligned}
\widetilde{V}_t^0 &= V_t^0 \exp\left(-\int_0^t r_s ds\right) \\
&= V_0^0 \exp\left(\int_0^t (\pi_s\sigma_s\theta_s^0 - \frac{1}{2}\pi_s^2\sigma_s^2)ds + \int_0^t \pi_s\sigma_s dB_s^0\right) \\
&\quad \times \Pi_{0\le u\le t}(1+\pi_s(e^{q\sigma_u\Delta Z_u}-1)) \times \Pi_{0\le s\le t}(1+\pi_s(e^{\Delta X_s}-1)), \quad (4.4.12)
\end{aligned}
$$

Proof. This is the same as the proof for Theorem 4.2 with $\theta = \theta^0$ and $\upsilon = \upsilon^0$. $\square$

4.4.2 The Partial Objective Function $G(\cdot)$

The G function is the same for both investors, with

$$G(\alpha\,;\,s) = G_U(\alpha\,;\,s) + G_S(\alpha\,;\,1),$$

$q \in [0,1]$, $\alpha \in [0,\,1]$, and $s \in (0, \sigma_{max}] \cup \{1\}$, where

$$G_a(\alpha\,;\,s) \triangleq \int_{\mathbf{R}} \log(1+\alpha(e^{s\,x}-1))\,v_a(dx),\ a \in \{U, S\}.$$

We assume, as before, that $\int_{\mathbf{R}}(e^{\pm s\,x}-1)^2 v_a(dx) < \infty$, where $s \in (0, \sigma_{max}] \cup \{1\}$ to ensure that $G''(\alpha;\,s) < 0$. We now give the optimal portfolios and maximum expected utilities for the investors.

Theorem 4.7. *Let $i \in \{0,1\}$ and $q \in [0,\,1]$. For the i–th investor, the optimal portfolio π^i that maximizes the expected logarithmic utility from terminal wealth V_T^i over the investment period $[0,\,T]$, is given by*

$$\pi_t^i = \frac{\mu_t^i - r_t + G'(\pi_t^i\,;\,q\sigma_t)}{\sigma_t^2} = \frac{\mu_t^i - r_t + G'_U(\pi_t^i\,;\,q\sigma_t) + G'_S(\pi_t^i\,;\,1)}{\sigma_t^2}. \quad (4.4.13)$$

In terms of the stock's total returns $b_t^i = \mu_t^i + K_U(q\sigma_t) + K_S(1)$, the optimal portfolio *is*

$$\pi_t^i \;=\; \frac{b_t^i - r_t - K_U(q\sigma_t) - K_S(1) + G'(\pi_t^i\,;\,q\sigma_t)}{\sigma_t^2}, \tag{4.4.14}$$

$$K_a(s) \;=\; \int_{\mathbf{R}} (e^{s\,x}-1)\,\upsilon_a(dx),\; a \in \{U,S\}, s \geq 0. \tag{4.4.15}$$

The maximum expected logarithmic utility from terminal wealth with $x > 0$ in initial wealth, is

$$u^i(x) = \log x + \frac{1}{2}\mathbf{E}\int_0^T (\theta_t^i)^2 dt + \mathbf{E}\int_0^T f_{U,S}^{(i)}(\pi_t^i)dt, \tag{4.4.16}$$

where

$$\begin{aligned} f_{U,S}^{(i)}(\pi_t^i) &= G(\pi_t^i\,;\,q\sigma_t) - \frac{1}{2}(\pi_t^i\sigma_t - \theta_t^i)^2 \\ \mu_t^i &= \mu_t + \upsilon_t^i\,\sigma_t - q\sigma_t\int_{\mathbf{R}} x\upsilon_U(dx), \\ \upsilon_t^i &\triangleq \upsilon_t^{i,B} + \upsilon_t^Z, \\ b_t^i &= \mu_t^i + K_U(q\sigma_t) + K_S(1). \end{aligned}$$

Proof. The case $i = 1$ was proven in Theorem 4.3. For the case $i = 0$, simply replace the superscript "1" by"0" in the said theorem. $\square$

4.4.3 Approximation of Optimal Portfolios using Instantaneous Centralized Moments

As in the cases examined in Chapter 3, we need to use approximation methods to compute the optimal portfolios, since the partial objective function $G_a(\alpha\,;\,s) \triangleq \int_{\mathbf{R}} \log(1+\alpha(e^{s\,x}-1)\,\upsilon_a(dx)$, $a \in \{U,S\}$, $s \in (0,\sigma_{max}] \cup \{1\}$, is in general, not tractable. We assume as before, that $\int_{\mathbf{R}}(e^{\pm s\,x}-1)^2\upsilon_a(dx) < \infty$, to ensure that the optimal portfolios exist(i.e; $G''(\alpha) < 0$). Since

$$G(\alpha\,;\,s) = G_U(\alpha\,;\,s) + G_S(\alpha\,;\,1), \alpha \in [0,\,1], \tag{4.4.17}$$

then, if $\int_{\mathbf{R}}(e^{\pm\,sx} - 1)^{k+1} v_a(dx) < \infty$, where $s \in (0, \sigma_{max}] \cup \{1\}$, we can approximate $G(\alpha\,;\,s)$ by a k–th degree polynomial $G_k(\alpha\,;\,s)$, where

$$\begin{aligned}
G_k(\alpha\,;\,s) &\triangleq \sum_{j=1}^{k}(-1)^{j-1} M(j\,;\,s)\,\frac{\alpha^j}{j!}, && (4.4.18)\\
G_k'(\alpha\,;\,s) &\triangleq \sum_{j=1}^{k}(-1)^{j-1} M(j\,;\,s)\,\alpha^{j-1}, && (4.4.19)\\
M(j\,;\,s) &= M_U(j\,;\,s) + M_S(j\,;\,1), j = 1, 2, \cdots, k. && (4.4.20)\\
M_a(j\,;\,s) &= \int_{\mathbf{R}}(e^{s\,x} - 1)^j v_a(dx),\ a \in \{U,\,S\}, s \in (0, \sigma_{max}] \cup \{1\} && (4.4.21)\\
&= \sum_{r=1}^{j}(-1)^{r-1}\binom{j}{r} K_a(r\,;\,s) && (4.4.22)\\
K_a(j\,;\,s) &= \int_{\mathbf{R}}(e^{j\,s\,x} - 1)\, v_a(dx). && (4.4.23)
\end{aligned}$$

We have the following lemma.

Lemma 4.3. *Let $k \in \mathbf{N}$. If $\int_{\mathbf{R}}(e^{\pm s\,x} - 1)^{k+1} v_a(dx) < \infty$, where $s \in (0, \sigma_{max}] \cup \{1\}$, and $a \in \{U, S\}$, then the k–th centralized moment exists, and is given by*

$$M(k\,;\,s) = \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j} K(j\,;\,s) = M_U(k\,;\,s) + M_S(k\,;\,1), \tag{4.4.24}$$

where

$$\begin{aligned}
K(j\,;\,s) &= K_U(j\,s) + K_S(j), && (4.4.25)\\
K_a(\gamma) &= \int_{\mathbf{R}}(e^{\gamma\,x} - 1)\, v_a(dx),\ a \in \{U,\,S\}. && (4.4.26)
\end{aligned}$$

Proof. Obviously,

$$K_a(j\,;\,s) = \int_{\mathbf{R}} (e^{j\,s\,x} - 1)\,v_a(dx) = K_a(j\,s).$$

Thus

$$K(j\,;\,s) = K_U(j\,;\,s) + K_S(j\,;1) = K_U(j\,s) + K_S(j),$$

whence

$$M(k\,;\,s) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} K(j\,;\,s) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (K_U(j\,;s) + K_S(j\,;\,1))$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} K_U(j\,;\,s) + \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} K_S(j\,;\,1) = M_U(k\,;\,s) + M_S(k\,;\,1).$$

□

4.5 Asymptotic Utilities

We assume that the partial objective function $G(\alpha) \equiv G(\alpha\,;\,s) = G_U(\alpha\,;\,s) + G_S(\alpha\,;\,1)$, is restricted to the domain $[0,1]$, where $s \in (0, \sigma_{max}] \cup \{1\}$ and $\int_{\mathbf{R}} (e^{\pm s\,x} - 1)^3 v_a(dx) < \infty$, where $a \in \{U, S\}$. That is, we have at least a quadratic approximation of $G(\alpha)$. Let $Q(\theta)$ be given as in Chapter 3. That is,

$$Q(\theta) \stackrel{\triangle}{=} A\theta^2 + B\theta + C \tag{4.5.1}$$

where θ is the Sharpe ratio and

$$M_a(j\,;s) = \int_{\mathbf{R}} (e^{s\,x} - 1)^j v_a(dx),\; a \in \{U,\, S\}, s \in [0, \sigma_{max}] \cup \{1\}.$$

Set

$$
\begin{aligned}
M(j\,;\sigma) &\triangleq M_U(j\,;\sigma)+M_S(j\,;1), j=1,2,\cdots,k. && (4.5.2)\\
A &= \frac{-M(2\,;q\sigma_t)}{2(\sigma_t^2+M(2\,;\,q\sigma_t))}, && (4.5.3)\\
B &= \frac{2M(1\,;q\sigma_t)}{2(\sigma_t^2+M(2\,;\,q\sigma_t))}, && (4.5.4)\\
C &= \frac{M(1\,;q\sigma_t)}{2(\sigma_t^2+M(2\,;\,q\sigma_t))}. && (4.5.5)
\end{aligned}
$$

Note that A, B, C are functions of σ_t. Let

$$\gamma_t^2=\frac{\sigma_t^2}{\sigma_t^2+M(2\,;\,q\sigma_t)}. \tag{4.5.6}$$

Then γ_t is the proportion of total volatility due to the diffusive component of the stock with dynamic (4.1.1) We assume that

$$lim_{t\to\infty}\sigma_t=\sigma_\infty=\sigma>0,$$

and set

$$\gamma^2=lim_{t\to\infty}\gamma_t^2=\frac{\sigma_\infty^2}{\sigma_\infty^2+M(2\,;\,q\sigma_\infty)}=\frac{\sigma^2}{\sigma^2+M(2\,;\,q\sigma)}. \tag{4.5.7}$$

Note that in (4.5.3)–(4.5.5), σ is really σ_t, while in (4.5.7), σ is a positive constant.

Theorem 4.8. *Assume that* $\int_{\mathbf{R}}(e^{\pm s\,x}-1)^k v(dx)<\infty$*,* $s\in(0,\sigma_{max}]\cup\{1\},\ k\geq 3$ *and* G *is restricted to* $[0,\,1]$*. Let* $x>0$ *be the initial wealth of investors and* $i\in\{0,1\}$*.*
(1) *As* $T\to\infty$*, the asymptotic optimal utility for the* i*–th investor due to jumps is*

$$
\begin{aligned}
u^i_{\infty,\,d}(x) \ \sim\ & \int_0^T Q\left(\frac{\mu_t}{\sigma_t}:\sigma_t, M(1,q\sigma_t)\,M(2,q\sigma_t)\right)dt\\
&+\frac{\lambda}{2}A_\infty(1-p)(1+(-1)^{i+1}p)\left[1+\int_{\mathbf{R}}x^2\,v_U(dx)\right]T, \qquad (4.5.8)
\end{aligned}
$$

where $Q(\theta)$ is given by (4.5.1) and

$$A_\infty = -\frac{M(2, q\sigma)}{2(\sigma^2 + M(2, q\sigma))}, \quad \sigma = \lim_{t \to T} \sigma_t. \tag{4.5.9}$$

(2) *The excess asymptotic optimal utility of the informed investor over the uninformed investor, due to jumps is:*

$$u^1_{\infty, d}(x) - u^0_{\infty, d}(x) \sim \lambda \, A_\infty \, p \, (1-p) \left[1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right] T. \tag{4.5.10}$$

Proof. Assume that $\int_{\mathbf{R}} (e^{\pm s\, x} - 1)^2 v(dx) < \infty$, where $s \in (0, \sigma_{max}] \cup \{1\}$ and G is restricted to $[0, 1]$.

By Proposition 4.4, as $t \to \infty$

$$\mathbf{E}(\upsilon^i_t)^2 \longrightarrow \frac{\lambda}{2}(1-p)(1 + (-1)^{i+1} p) \left[1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right], \quad i \in \{0, 1\}.$$

By Corollary 3.3, as $T \to \infty$ we have

$$\begin{aligned}
u^i_{T, d}(x) &\sim \mathbf{E} \int_0^T Q\left(\frac{\mu_t}{\sigma_t} : \sigma_t, \, M(1, q\sigma_t) \, M(2, q\sigma_t) \right) dt + \int_0^T A_t \mathbf{E}(\upsilon^i_t)^2 dt \\
&= \mathbf{E} \int_0^T Q\left(\frac{\mu_t}{\sigma_t} \right) dt + T \lim_{t \to \infty} A_t \mathbf{E}(\upsilon^i_t)^2 \\
&= \mathbf{E} \int_0^T Q\left(\frac{\mu_t}{\sigma_t} \right) dt + T \, A_\infty \lim_{t \to \infty} \mathbf{E}(\upsilon^i_t)^2 \\
&= \mathbf{E} \int_0^T Q\left(\frac{\mu_t}{\sigma_t} \right) dt + \frac{\lambda}{2} A_\infty (1-p)(1 + (-1)^{i+1} p) \left[1 + \int_{\mathbf{R}} x^2 \, v_U(dx) \right] T,
\end{aligned}$$

where

$$A_\infty = \lim_{t \to T} A_t = \lim_{t \to \infty} -\frac{M(2, q\sigma_t)}{2(\sigma_t^2 + M(2, q\sigma_t))} = -\frac{M(2, q\sigma)}{2(\sigma_\infty^2 + M(2, q\sigma))} = -\frac{M(2, q\sigma)}{2(\sigma^2 + M(2, q\sigma))}.$$

From part(1), since $\frac{\lambda}{2} A_\infty \int_{\mathbf{R}} x^2 \, v_U(dx) T$, is common for both investors, it follows that the excess optimal utility of the informed investor due to the jumps is

$$u^1_{\infty,\,d}(x) - u^0_{\infty,\,d}(x) = \frac{\lambda}{2} A_\infty [(1-p)(1+p) - (1-p)(1-p)] \left[1 + \int_{\mathbf{R}} x^2\, v_U(dx)\right] T$$

$$= \lambda\, A_\infty\, p\,(1-p)\, \left[1 + \int_{\mathbf{R}} x^2\, v_U(dx)\right] T.$$

□

We have the following major result as a consequence of the quadratic approximation of $G(\alpha\,;\,\sigma)$. This is analogous to Theorems 3.18 and 3.19, with $r = 0$.

Theorem 4.9. *Assume that the conditions of Theorem 4.8 hold. Let the investment horizon $T \longrightarrow \infty$. Under quadratic approximation of $G(\alpha\,;\,s)$, we have:*

(1) *The maximum expected asymptotic logarithmic utility from terminal wealth for the i–th investor with initial capital $x > 0$ is $u^i_\infty(x)$*

$$\approx \log x + \frac{1}{2}\int_0^T \left(\frac{\mu_t}{\sigma_t}\right)^2 dt + \int_0^T Q\left(\frac{\mu_t}{\sigma_t}\right) dt + \frac{\tilde{\lambda}}{4}(1-p)(1+(-1)^{i+1}p)\left[1 + \int_{\mathbf{R}} x^2\, v_U(dx)\right] T. \tag{4.5.11}$$

(2) *The excess asymptotic logarithmic utility of the informed investor is*

$$u^1_\infty(x) - u^0_\infty(x) \approx \frac{\tilde{\lambda}}{2}\, p\,(1-p)\left[1 + \int_{\mathbf{R}} x^2\, v_U(dx)\right] T, \tag{4.5.12}$$

where $\tilde{\lambda} = \lambda\gamma^2$ is the long run adjusted mean-reversion rate and γ^2 is given by (4.5.7).

Proof. (1) follows from Theorem 3.16, with $(1-p)(1+(-1)^{i+1}p)(1+\int_{\mathbf{R}} x^2\, v_U(dx))$, replacing $(1-p)(1+(-1)^{i+1}p)$ and M_2 replaced by $M(2\,;\,q\sigma) = M_U(2\,;\,q\sigma) + M_S(2\,;\,1)$. (2) follows by taking the difference of the optimal utilities for both investors. □

Chapter 5

Numerical Approximation of Optimal Portfolios Using Centralized Moments

In this chapter, we apply some of the theories developed in Chapter 3 to Lévy markets having a non-zero diffusive coefficient. Let $i \in \{0, 1\}$. Recall from Theorem 3.17, that the random optimal portfolio π^i for the i-th investors is linked to the deterministic optimal portfolio π of the symmetric market, via the relationship:
where

$$\pi^i = \pi + noise^i, \quad noise^i = v^i\,\tilde{\sigma}, \tag{5.0.1}$$

and

$$\tilde{\sigma}_t = \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|},$$

for some η_t^i between π_t and π_t^i, $t \in [0,\, T]$. In particular, under quadratic approximation $noise^i = \frac{v^i\,\sigma_t}{\sigma_t^2 + M_2}$, where the v^is are defined in Theorem 1.2, Guasoni [21]. It is clear from the foregoing that we only need π, the deterministic optimal portfolio in the symmetric market, to obtain the approximate optimal portfolios in the asymmetric markets. Explicitly, by Theorem 3.8 give the optimal portfolio $\pi_t^{*,i}$ for each investor as

$$\pi_t^{*,i} = \frac{\mu_t^i - r_t + G'(\pi_t^{*,i})}{\sigma_t^2}. \tag{5.0.2}$$

This chapter is focused on obtaining and estimating π and $\pi_t^{*,i}$ using Newton's method and other iteration procedures, with the help of polynomial approimations of $G(\cdot)$ built from the centralized moments of returns. We study various markets including Kou, Variance Gamma, Double Poisson, and m-Double Poisson. We obtain analyic formulas for the optimal portfolios for the Kou and Double Poisson markets, while the VG and m–Double Poisson markets are estimated by numerical procedures.

5.1 The Symmetric Lévy Market

The symmetric market is driven by standard Brownian motion B and a pure jump Lévy process X. Both investors have equal knowledge. The stock price S has log returns dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t + dX_t, \tag{5.1.1}$$

with equivalent percentage returns dynamic

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dB_t + \int_R (e^x - 1)N(dt, dx), \tag{5.1.2}$$

where B is a standard Brownian motion independent of X, which has Lévy triple $(\gamma\,,\,0,\,v)$, with $\gamma = \int_{[-1,\,1]} x\, v(dx)$. σ_t is the continuous component of the stock's volatility and μ_t is the continuous component of the total stock appreciation rate $b = \mu + M_1$, where $M_1 = \int_R (e^x - 1)v(dx)$. Note that the percentage returns has three components: one continuous and locally deterministic (μ_t); one continuous and stochastic ($\sigma_t\, dB_t$), and the third is discontinuous and driven by the Poisson Random measure N on $[0, T] \times R$.

All processes live on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. The structure of (5.1.2) is analogous to that of a stock dynamic with jumps first considered by Merton [36] in 1976, and more recently in 2003 by Liu, Longstaff and Pan [34]. In these papers, stocks follow a jump-diffusion process with Poisson arrival rates. In the sequel, we examine Merton's and other models, where jumps are not constrained only to slow Poisson arrival rates, but may arrive at extremely fast, even infinite rates. We study Lévy markets that are driven by both finite and infinite activity processes. We assume that the jumps in stock returns are smooth in the sense that the jump paths are of finite variation. That is (cf Propostion 2.2)

$$\int_R \min(|x|, 1) v(dx) < \infty. \tag{5.1.3}$$

Finite activity processes drive the Merton and Kou jump-diffusion models, while the Variance Gamma (VG) and CGMY processes are (driven by) infinite activity processes. We also consider markets driven by a Double Poisson process $(\Pi_{(1,2)}(\lambda))$, and by m-Double Poisson processes $(\Pi_{(m,2)}(\lambda))$, where m is an positive integer and λ is the arrival/intensity rate. All markets consist of a single stock and a bond $\mathbf{B}$ with price $\mathbf{B}_t = \exp\left(\int_0^t r_s \, ds\right)$, where r_t is the continuously compounded risk–free interest rate, T is the investment horizon, and $t \in [0, T]$.

5.2 Instantaneous Centralized Moments of Returns

In this section we present a general result for instantaneous centralized moments of returns. Let $v(\cdot)$ be the Lévy measure of an arbitrary pure jump Lévy process X.

Definition 5.1 (Instantaneous Centralized Moments of Returns).

Define the objects M_j and K_s by the prescriptions:

$$\begin{aligned} M_j &= \int_R (e^x - 1)^j v(dx), & (5.2.1) \\ K_s &= \int_R (e^{sx} - 1) v(dx). & (5.2.2) \end{aligned}$$

M_j is called the j-th instantaneous centralized moments of returns of the Lévy process X, with measure $v(\cdot)$. K_s is a kernel used to calculate M_j.

We have the following result, which will be quite useful in the sequel.

Lemma 5.1. *If there exists $k \in N$ such that $\int_R (e^{jx} - 1) v(dx) < \infty$ for each $0 \leq j \leq k$, then M_j and K_j exist, and*

$$M_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i. \tag{5.2.3}$$

Proof. If there exists $k \in N$ such that $\int_R (e^{jx} - 1) v(dx) < \infty$ for each $0 \leq j \leq k$, then

$$K_j = \int_R (e^{jx} - 1) v(dx) < \infty.$$

Now

$$M_j = \int_R (e^x - 1)^j \, v(dx).$$

From the Binomial Theorem

$$\begin{aligned}
(e^x - 1)^j &= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} e^{ix} \\
&= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1) + \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1) + (1-1)^j \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1).
\end{aligned}$$

Therefore

$$M_j = \int_R (e^x - 1)^j \, v(dx) = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \int_R (e^{ix} - 1) \, v(dx) = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i,$$

which is clearly finite for each integer $0 \le j \le k$. $\square$

The following results are particular consequences of Lemma 5.1, and are used repeatedly in the sequel. We encapsulate them in the following corollary.

Corollary 5.1. *Let k be the largest integer such that $\int_R (e^{jx} - 1) v(dx) < \infty$ for each integer $0 \le j \le k$. Then K_j and M_j exist for each $j \le k$. In particular:*

$$M_1 = K_1, \text{ if } k=1; \tag{5.2.4}$$

$$M_2 = K_2 - 2K_1, \text{ if } k=2; \tag{5.2.5}$$

$$M_3 = K_3 - 3K_2 + 3K_1, \text{ if } k=3; \tag{5.2.6}$$

$$M_4 = K_4 - 4K_3 + 6K_2 - 4K_1, \text{ if } k=4; \text{ and} \tag{5.2.7}$$

$$M_5 = K_5 - 5K_4 + 10K_3 - 10K_2 + 5K_1, \text{ if } k=5. \tag{5.2.8}$$

Remark 5.1. *The kernels* K_j*,* $j = 1, \ldots, k$*, are easy to compute when the Lévy measure* $v(dx)$ *has the form*

$$v(dx) = const\, e^{-Mx} \frac{f(x)dx}{|x|},$$

where f *is a real bounded function on* $\mathcal{R}$*. This will be the case for most of the models considered in this dissertation, and indeed, represents a large class of Lévy processes in finance.*

5.3 Polymonial Approximation of $G(\alpha)$

The Newtons' method algorithm requires explicit values for the first and second derivatives of

$$G(\alpha) = \int_R \log(1 + \alpha(e^x - 1))v(dx),$$

which must be estimated if an analytic formula is not available. In the sequel, we estimate $G(\alpha)$ by the k–th degree polynomial $G_k(\alpha)$. We shall use the M_js to approximate the function G and hence G' and G'', by a truncated Taylor series. The larger the value of k, the better the approximation of G. We have the following result which is general for all models where M_k exists.

Theorem 5.1. *If there exists* $k \in N$ *such that* $M_j = \int_{\mathbf{R}} (e^x - 1))^j\, v(dx) < \infty$ *for each* $1 \leq j \leq k$*, then* $G(\cdot)$ *has a k-th degree polynomial approximation* $G_k(\cdot)$ *given by:*

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} \frac{M_j}{j} \alpha^j, \; \alpha \in [0, 1], \tag{5.3.1}$$

where

$$G'_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} M_j \alpha^{j-1} \;\; and \;\; G''_k(\alpha) = \sum_{j=2}^{k} (-1)^{j-1} (j-1)\, M_j \alpha^{j-2}. \tag{5.3.2}$$

Proof. Let k be the largerst integer such that M_j exists for each $j \leq k$. Expanding $G(\alpha)$ about $\alpha = 0$ as a truncated Taylor series of degree k, yields:

$$\begin{aligned} G(\alpha) &\approx \sum_{j=1}^{k} G^{(j)}(0) \frac{\alpha^j}{j!} \\ &= \sum_{j=1}^{k} (-1)^{j-1} (j-1)! \int_{\mathbf{R}} (e^x - 1)^j v(dx) \frac{\alpha^j}{j!} \\ &= \sum_{j=1}^{k} (-1)^{j-1} \int_{\mathbf{R}} (e^x - 1)^j v(dx) \frac{\alpha^j}{j}. \end{aligned}$$

The result follows with

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} \frac{M_j}{j} \alpha^j,$$

and

$$G_k'(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} M_j \alpha^{j-1}.$$

$G_k''(\alpha)$ follows directly by differentiating $G_k'(\alpha)$. □

5.4 The Error in the Approximation of the Optimal Portfolio

Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth. Let $\pi^{(k)}$ be the estimate of π based on the k-th degree truncated Taylor polynomial $G_k(\cdot)$ of $G(\cdot)$, with remainder $R_k(\cdot)$, where for $\pi \in [0, 1]$

$$G(\pi) = G_k(\pi) + R_k(\pi), \tag{5.4.1}$$

where

$$G_k(\pi) = \sum_{j=1}^{k} G^{(j)}(0)\frac{\pi^j}{j!} = \sum_{j=1}^{k}(-1)^{j-1}M_j\frac{\pi^j}{j!}.$$

We assume that $\int_{\mathbf{R}}(e^{\pm x}-1)^{k+1}v(dx) < \infty$, to ensure that all integrals, and hence all derivatives, exist. Explicitly, For $j = 1, \ldots, k+1$, we have

$$G^{(j)}(\pi) = (-1)^{j-1}(j-1)!\int_R \frac{(e^x-1)^j\,v(x)dx}{(1+\pi\,(e^x-1))^j}, \tag{5.4.2}$$

$$G^{(j)}(0) = (-1)^{j-1}(j-1)!M_j. \tag{5.4.3}$$

In particular,

$$G^{(2)}(\pi) = -\int_R \frac{(e^x-1)^2v(x)dx}{(1+\pi(e^x-1))^2} < 0. \tag{5.4.4}$$

Thus $G(\cdot)$ is strictly concave on $[0,\,\infty)$. Clearly $G_k(\pi) \to G(\pi)$ as $k \to \infty$, iff $R_k(\pi) \to 0$ as $k \to \infty$. By Taylor's theorem, there exists $\theta_\alpha \in (0,\,\alpha)$, where $\alpha \in (0,\,1]$, such that

$$R_k(\alpha) = G^{(k+1)}(\theta_\alpha)\frac{\alpha^{k+1}}{(k+1)!} = (-1)^k\frac{\alpha^{k+1}}{k+1}\int_{\mathbf{R}} \frac{(e^x-1)^{k+1}v(x)dx}{(1+\theta_\alpha(e^x-1))^{k+1}}. \tag{5.4.5}$$

Definition 5.2 (**k-th Degree Error**).

Let $t \in [0,\,T]$. The k-th degree error in the approximation of π_t is denoted by $\varepsilon_t^{(k)}$ and defined by

$$\varepsilon_t^{(k)} \triangleq \pi_t - \pi_t^{(k)}, \tag{5.4.6}$$

where $\pi_t^{(k)}$ is the k-th degree approximation of π_t.

We have the following result.

Theorem 5.2. *If $G \in C^{k+2}[0,\,1]$ for some $k \in \mathbf{N}$, then there exists $\theta_\alpha^{(k)} \in (0,\,1)$, such that*

$$\varepsilon_t^{(k)} = \frac{R_k'(\pi_t^{(k)})}{\sigma_t^2 + |G''(\theta_\alpha^{(k)})|}, \tag{5.4.7}$$

where $R_k(\cdot)$ is the remainder.

Proof. The optimal portfolio π_t is given by Theorem 5.7 as

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{1}{\sigma_t^2}\, G'(\pi_t)$$

Its k-th degree approximation is given by Theorem 5.6, as

$$\pi_t^{(k)} = \frac{\theta_t}{\sigma_t} + \frac{1}{\sigma_t^2}\, G'_k(\pi_t^{(k)}).$$

Thus

$$\varepsilon_t^{(k)} = \pi_t - \pi_t^{(k)} = \frac{1}{\sigma_t^2}\,(G'(\pi_t) - G'_k(\pi_t^{(k)})).$$

Since $G^{(k+2)}(\alpha)$ exists for $\alpha \in [0,\, 1]$ and $G(\alpha) = G_k(\alpha) + R_k(\alpha)$ then

$$G'_k(\pi_t^{(k)}) = G'(\pi_t^{(k)}) - R'_k(\pi_t^{(k)}).$$

By the Mean Value theorem, there exists $\theta_\alpha^{(k)}$ between π_t and $\pi_t^{(k)}$, such that

$$\begin{aligned}
\varepsilon_t^{(k)} &= \frac{1}{\sigma_t^2}\,(G'(\pi_t) - G'(\pi_t^{(k)}) + R'_k(\pi_t^{(k)})) \\
&= \frac{1}{\sigma_t^2}\,\Big[(\pi_t - \pi_t^{(k)})G''(\theta_\alpha^{(k)}) + R'_k(\pi_t^{(k)})\Big] \\
&= \frac{1}{\sigma_t^2}\,(\varepsilon_t^{(k)} G''(\theta_\alpha^{(k)}) + R'_k(\pi_t^{(k)})).
\end{aligned}$$

Thus $\varepsilon_t^{(k)}(\sigma_t^2 - G''(\theta_\alpha^{(k)})) = R'_k(\pi_t^{(k)})$ and so with $G''(\alpha) < 0$ for all α, we have

$$\varepsilon_t^{(k)} = \frac{R'_k(\pi_t^{(k)})}{\sigma_t^2 - G''(\theta_\alpha^{(k)})} = \frac{R'_k(\pi_t^{(k)})}{\sigma_t^2 + |G''(\theta_\alpha^{(k)})|}.$$

Moreover $\theta_\alpha^{(k)} \in (0,\, 1)$, since π_t and $\pi_t^{(k)} \in [0,\, 1]$. $\square$

An immediate consequence of the last result is:

Corollary 5.2.

$$|\varepsilon_t^{(k)}| \le \frac{|R'_k(\pi_t^{(k)})|}{\sigma_t^2}. \tag{5.4.8}$$

Theorem 5.3. *Let $G \in C^{k+2}[0,\,1]$ for some $k \in \mathbf{N}$ and $\alpha \in [0,\,1)$. Then there exists $\theta_\alpha^{(k)}, \theta_\alpha \in (0,\,1)$, such that*

$$\varepsilon_t^{(k)} = \frac{1}{\sigma_t^2 + |G''(\theta_\alpha^{(k)})|}\left(\Delta_k\left(\pi_t^{(k)}, \theta_\alpha\right) + \theta'_\alpha \Delta_{k+1}(\pi_t^{(k)}, \theta_\alpha)\right), \tag{5.4.9}$$

where

$$\Delta_k(\alpha, \theta_\alpha) = \frac{\alpha^k}{k!} G^{(k+1)}(\theta_\alpha) \tag{5.4.10}$$

and $\alpha = \pi_t^{(k)}$ is the k-th degree approximation of π_t.

Proof. Since $G^{(k+2)}(\alpha)$ exists, then from (5.4.5) there exists $\theta_\alpha \in (0, \alpha)$, such that

$$R_k(\alpha) = \frac{\alpha^{k+1}}{(k+1)!} G^{(k+1)}(\theta_\alpha).$$

Thus

$$\begin{aligned} R'_k(\alpha) &= \frac{\alpha^k}{k!} G^{(k+1)}(\theta_\alpha) + \frac{\alpha^{k+1}}{(k+1)!} G^{(k+2)}(\theta_\alpha)\theta'_\alpha, \\ &= \Delta_k(\alpha, \theta_\alpha) + \theta'_\alpha \Delta_{k+1}(\alpha, \theta_\alpha.) \end{aligned} \tag{5.4.11}$$

Setting $\alpha = \pi_t^{(k)}$ and applying Theorem 5.2, yield the result. $\square$

Corollary 5.3. *Suppose there is $k \in \mathbf{N}$ such that $G^{(k+2)}(\alpha)$ exists for all $\alpha \in [0,\,1)$. Then there exists $\theta_\alpha \in (0,\,1)$ such that*

$$|\varepsilon_t^{(k)}| \le \frac{1}{\sigma_t^2}\left[\frac{(\pi_t^{(k)})^k}{k!}|G^{k+1}(\theta_\alpha)| + \frac{(\pi_t^{(k)})^{k+1}}{(k+1)!}|G^{k+2}(\theta_\alpha)|\right], \tag{5.4.12}$$

where $\alpha = \pi_t^{(k)}$.

Proof. Without loss of generality, we take $0 < |\theta'_\alpha| < 1$ since $\theta_\alpha \in (0,\, \alpha) \subset (0,\, 1)$. Applying the triangle inequality to (5.4.11), yields

$$\begin{aligned} |R'_k(\alpha)| &\leq \frac{\alpha^k}{k!}|G^{(k+1)}(\theta_\alpha)| + \frac{\alpha^{k+1}}{(k+1)!}|G^{(k+2)}(\theta_\alpha)||\theta'_\alpha| \\ &\leq \frac{\alpha^k}{k!}|G^{(k+1)}(\theta_\alpha)| + \frac{\alpha^{k+1}}{(k+1)!}|G^{(k+2)}(\theta_\alpha)|. \end{aligned}$$

Set $\alpha = \pi_t^{(k)}$, and apply Corollary 5.2 to the above inequality to get the result. □

Corollary 5.4. *If there exists $C > 0$ such that for $j = k+1$ and $j = k+2$, $\left|\frac{G^{(j)}(\alpha)}{(j-1)!}\right| \leq C$, for all $\alpha \in [0,\, 1]$, then*

$$|\varepsilon_t^{(k)}| \leq 2\,C\,\frac{(\pi_t^{(k)})^k}{\sigma_t^2}. \tag{5.4.13}$$

Proof. Set $\alpha = \pi_t^{(k)} \in [0,\, 1]$. Imposing $|\frac{G^{(j)}(\theta_\alpha)}{(j-1)!}| \leq C$ for $j = k+1$ and $j = k+2$, onto equation (5.4.12) yields $|\varepsilon_t^{(k)}| \leq \frac{1}{\sigma_t^2}\left[(\pi_t^{(k)})^k C + (\pi_t^{(k)})^{k+1} C\right] \leq 2\,C\,\frac{(\pi_t^{(k)})^k}{\sigma_t^2}$. □

It is obvious from the last result that if $|G^{(j)}(\alpha)| \leq C(j-1)!$, $C > 0$, and $j \in \{k+1, k+2\}$, then $\varepsilon_t^{(k)} \to 0$ as $k \to \infty$. In other words, if the derivatives of $G(\cdot)$ are bounded, we get convergence of the approximation $\pi_t^{(k)}$ to the optimal portfolio π.

Theorem 5.4. $\pi_t^{(k)} \to 0$ *iff* $\;R_k(\alpha) \to 0,\;\; \forall\, \alpha \in [0,\, 1]$.

Proof. Asssume that $R_k(\alpha) \to 0\;\forall\,\alpha \in [0,\, 1]$ and take $\alpha = \pi_t^{(k)}$, $t \in [0,\, T]$. Since $G(\alpha) = G_k(\alpha) + R_k(\alpha)$ then as $k \to \infty$, $G_k(\alpha) \to G(\alpha)$. Thus $G'_k(\alpha) \to G'(\alpha)$, and so

$$\varepsilon_t^{(k)} = \pi_t - \pi_t^{(k)} = \frac{1}{\sigma_t^2}(G'_k(\alpha) - G'(\alpha)) \to 0.$$

Now assume that $\pi_t^{(k)} \to 0$. Then

$$G(\alpha) = G_k(\alpha) + R_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} \frac{M_j}{j} \alpha^j + R_k(\alpha), \; : \alpha \in [0, 1].$$

Thus $0 = G(0) = G_k(0) + R_k(0) \Rightarrow R_k(0) = 0$. By the Mean Value theorem, there exists $\theta_\alpha \equiv \theta(\alpha) \in (0, \alpha)$ such that

$$R_k(\alpha) = R_k(\alpha) - R_k(0) = \frac{R_k'(\theta(\alpha))}{\alpha}.$$

For each $x \in (0, 1)$, define

$$\theta^{-1}(x) = \alpha \, if \; \theta(\alpha) = x$$

where $\alpha \in (0, 1)$. Then

$$R_k'(x) = \frac{R_k(\theta^{-1}(x))}{\theta^{-1}(x)}.$$

By Theorem 5.2, with $x = \pi_t^{(k)}$, $\varepsilon_t^{(k)} \to 0$ implies that

$$R_k'(\pi_t^{(k)}) = \frac{R_k(\theta^{-1}(\pi_t^{(k)}))}{\theta^{-1}(\pi_t^{(k)})} \to 0.$$

Thus with $\alpha = \theta^{-1}(\pi_t^{(k)})$, we get $R_k(\theta^{-1}(\pi_t^{(k)})) = R_k(\alpha) \to 0$. $\square$

Remark 5.2. *This result proves that the error vanishes iff the remainder of the approximating series vanishes at all points in* $[0, 1]$*. So if there is divergence of the series, the error diverges.*

5.5 Estimation of Optimal Portfolio Using Newton's Method

By Theorem 3.8, the exact unique optimal portfolio in a symmetric market is

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2}.$$

This is a non-linear equation which must be solved numerically. In this section, we employ Newton's method to achieve this objective. For each $t \in [0, T]$, define $g : [0, 1] \longrightarrow \mathbf{R}$ by the prescription $g(\pi_t) = \pi_t - \frac{\theta_t}{\sigma_t} - \frac{G'(\pi_t)}{\sigma_t^2}$. We generate a sequence $\{\pi_t(n)\}$ which converges to π_t, via the algorithm:

$$\begin{aligned} \pi_t(0) &= \frac{\theta_t}{\sigma_t} \\ \pi_t(n+1) &= \pi_t(n) - \frac{g(\pi_t(n))}{g'(\pi_t(n))}, \quad n \geq 0. \end{aligned} \tag{5.5.1}$$

In term of the derivatives $G'(\alpha)$ and $G''(\alpha)$, we have the equivalent algorithm:
Set $\epsilon = 0.5 \times 10^{-d}$, where $d \in \{5, 6, 7, 8, 9, 10\}$.

$$\pi_t(0) = \frac{\theta_t}{\sigma_t} \tag{5.5.2}$$

$$\pi_t(n+1) = \frac{-\pi_t(n)\, G''(\pi_t(n)) + \theta_t\, \sigma_t + G'(\pi_t(n))}{\sigma_t^2 - G''(\pi_t(n))}, \; n \geq 0. \tag{5.5.3}$$

$$\epsilon_t(n) = |\pi_t(n+1) - \pi_t(n)| \tag{5.5.4}$$

Stop if $\epsilon_t(n) < \epsilon$, and take $\pi_t = \pi_t(n+1)$. Else, set $n = n+1$ and repeat search. In the event that $G'(\alpha)$ is estimated by a k–th polynomial $G'_k(\alpha)$, then π_t is estimated by $\pi_t^{(k)}$, where

$$\pi_t^{(k)}(0) = \frac{\theta_t}{\sigma_t} \tag{5.5.5}$$

$$\pi_t^{(k)}(n+1) = \frac{-\pi_t^{(k)}(n)\, G_k''(\pi_t^{(k)}(n)) + \theta_t\,\sigma_t + G_k'(\pi_t^{(k)}(n))}{\sigma_t^2 - G_k''(\pi_t^{(k)}(n))}, \; n \geq 0. \tag{5.5.6}$$

$$\epsilon_t^{(k)}(n) = |\pi_t^{(k)}(n+1) - \pi_t^{(k)}(n)| \tag{5.5.7}$$

Stop if $\epsilon_t^{(k)}(n) < \epsilon$, and take $\pi_t^{(k)} = \pi_t^{(k)}(n+1)$. Else, set $n = n+1$ and repeat search.

5.5.1 Estimation of Optimal Portfolios: Asymmetric Market

Let $i \in \{0, 1\}$. We use Newtons' method to estimate the optimal portfolios π_t^i at each time $t \in [0, T]$. We restate Theorem 3.8 as follows.

Theorem 5.5. *Let $i \in \{0, 1\}$, $t \in [0, T]$, and $p, q \in [0, 1]$, $p^2 + q^2 = 1$. Assume $\int_{\mathbf{R}} (e^{\pm x} - 1)^2\, v(dx) < \infty$. Then the optimal portfolio for the i–th investor is*

$$\pi_t^i = \frac{\theta_t}{\sigma_t} + \frac{\upsilon_t^i}{\sigma_t} + \frac{G'(\pi_t^i)}{\sigma_t^2}, \tag{5.5.8}$$

which is estimated by $\pi_t^i(n)$ using Newtons' method given by (5.5.2)–(5.5.4) or (5.5.5)–(5.5.7), where

$$\upsilon_t^1 = -\lambda\, q\, B_t^1, \quad \textit{and} \quad \upsilon_t^0 = -\lambda\, \tilde{B}_t^0, \tag{5.5.9}$$

and

$$\begin{aligned} B_t^1 &\sim \mathcal{N}(0,\, t), \\ \tilde{B}_t^0 &= \int_0^t e^{-\lambda\,(t-s)}(1+\gamma(s))\, dB_s^0 \sim \mathcal{N}\left(0,\, \int_0^t e^{-2\lambda\,(t-s)}(1+\gamma(s))^2 ds\right), \\ \gamma(s) &= \frac{1-p^2}{1+p\,\tanh(\lambda\, p\, s)} - 1. \end{aligned}$$

$\mathcal{N}(0, \delta^2)$ is a Gaussian distribution with mean 0 and variance δ^2; B_t^0 and B_t^1 are standard Brownian motion previously defined by Lemmas 3.4 and 3.2.

Remark 5.3. *Since B_t^1 and $\tilde{B}_t^0$ are Gaussian random variables for each $t \in [0, T]$, we randomly generate these values for use in the Newtons' method algorithm at each time point t.*

Consider the normal random variable $\tau \sim \mathcal{N}(0, \delta^2)$. Then

$$\xi = \frac{\tau}{\delta} \sim \mathcal{N}(0,\, 1).$$

We further refine Theorem 5.5 as follows:

The Algorithm

Let $i \in \{0,\, 1\}$, $t \in [0, T]$, and $p,\ q \in [0,\, 1]$, $p^2 + q^2 = 1$.

Step 1. Randomly generate u_i, independent uniform variables on $[0, 1]$.

Step 2. Generate independent standard normal random variables ξ^0 and ξ^1 by the **Box–Muller** method as follows:

$$\xi^0 = \sqrt{-2 \ln u_1}\, \cos(2\pi u_0) \qquad (5.5.10)$$

$$\xi_1 = \sqrt{-2 \ln u_1}\, \sin(2\pi u_0). \qquad (5.5.11)$$

Step 3 The optimal portfolios for the investors are generated numerically from

$$\pi_t^1 = \frac{\theta_t}{\sigma_t} - \frac{\lambda\, q\, \xi^1 \sqrt{t}}{\sigma_t} + \frac{G'(\pi_t^1)}{\sigma_t^2}, \qquad (5.5.12)$$

and

$$\pi_t^0 = \frac{\theta_t}{\sigma_t} - \frac{\lambda\, q\, \xi^0 \sqrt{\int_0^t e^{-2\lambda\,(t-s)}(1+p\,\tanh(\lambda\, p\, s))^{-2} ds}}{\sigma_t} + \frac{G'(\pi_t^1)}{\sigma_t^2}, \tag{5.5.13}$$

and are estimated by $\pi_t^i(n)$ using Newtons' method given by

$$\begin{aligned} \pi_t^i(0) &= \frac{\theta_t}{\sigma_t} \\ \pi_t^i(n+1) &= \frac{-\pi_t^i(n)\, G''(\pi_t^i(n)) + \theta_t^i\, \sigma_t + G'(\pi_t^i(n))}{\sigma_t^2 - G''(\pi_t^i(n))},\ \ n \geq 0, \end{aligned}$$

with Sharpe ratios

$$\theta_t^1 = \frac{\theta_t - \lambda\, q\, \xi^1 \sqrt{t}}{\sigma_t}, \quad \theta_t = \frac{\mu_t - r_t}{\sigma_t}, \tag{5.5.14}$$

$$\theta_t^0 = \frac{\theta_t - \lambda\, q\, \xi^0 \sqrt{\int_0^t e^{-2\lambda\,(t-s)}(1+p\,\tanh(\lambda\, p\, s))^{-2} ds}}{\sigma_t}. \tag{5.5.15}$$

Remark 5.4. *This result follows directly from Theorem 5.5 with B_t^1 generated by $\xi^1 \sqrt{t}$ and $\tilde{B}_t^0$ generated by $q\, \xi^0 \sqrt{\int_0^t e^{-2\lambda\,(t-s)}(1+p\,\tanh(\lambda\, p\, s))^{-2} ds}$; $\xi^i \sim \mathcal{N}(0,\, 1)$.*

5.5.2 Linear Iteration

We can also generate estimates of the optimal portfolios using the following algorithm.

For each $t \in [0,\, T]$, define $g : [0,\, 1] \longrightarrow \mathbf{R}$ by the prescription

$$g(\pi_t) = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2}. \tag{5.5.16}$$

We generate a sequence $\{\pi_t(n)\}$ which converges to π_t, via the algorithm:

$$\begin{aligned} \pi_t(0) &= \frac{\theta_t}{\sigma_t} \\ \pi_t(n+1) &= g(\pi_t(n)), \ \ n \geq 0. \end{aligned} \tag{5.5.17}$$

In term of the derivative $G'(\alpha)$, we have the equivalent algorithm:

Set $\epsilon = 0.5 \times 10^{-d}$, where $d \in \{5, 6, 7, 8, 9, 10\}$.

$$\begin{aligned} \pi_t(0) &= \frac{\theta_t}{\sigma_t} \\ \pi_t(n+1) &= \frac{\mu_t - r_t + G'(\pi_t(n))}{\sigma_t^2}, \ n \geq 0. \qquad (5.5.18) \\ \epsilon_t(n) &= |\pi_t(n+1) - \pi_t(n)| \qquad (5.5.19) \end{aligned}$$

Stop if $\epsilon_t(n) < \epsilon$, and take $\pi_t = \pi_t(n+1)$. Else, set $n = n+1$ and repeat search. In the event that $G'(\alpha)$ is estimated by a k–th polynomial $G'_k(\alpha)$ then π_t is estimated by $\pi_t^{(k)}$ where

$$\begin{aligned} \pi_t^{(k)}(0) &= \frac{\theta_t}{\sigma_t} \\ \pi_t^{(k)}(n+1) &= \frac{\mu_t - r_t + G'_k(\pi_t^{(k)}(n))}{\sigma_t^2}, \ n \geq 0. \qquad (5.5.20) \\ \epsilon_t^{(k)}(n) &= |\pi_t^{(k)}(n+1) - \pi_t^{(k)}(n)| \qquad (5.5.21) \end{aligned}$$

Stop if $\epsilon_t^{(k)}(n) < \epsilon$, and take $\pi_t^{(k)} = \pi_t^{(k)}(n+1)$. Else, set $n = n+1$ and repeat search.

Remark 5.5. *Equation (5.5.17) has a unique root provided $g'(\alpha) \neq 1$ on the admissible set. For illustration, assume that the admisible set is $[0,\,1]$ and let $\alpha \in [0,\,1]$. This condition is equivalent to $G''(\alpha) \neq \sigma^2$ in (5.5.18) and $G''_k(\alpha) \neq \sigma^2$ in (5.5.20) where G_k is any k–th degree polynomial approximation of G. A* ***sufficient*** *condition of convergence of this algorithm to a unique root is $|G''_k(\alpha)| < \sigma^2$. Thus from (5.3.2), if*

$$\sum_{j=2}^{k} (j-1)\,|M_j| < \sigma^2, \qquad (5.5.22)$$

we are assured of a unique portfolio in $[0,\,1]$. While (5.5.22) gives a very ***crude*** *bound on $G''(\alpha)$, it works in most models that are simulated in this dissertation. We* *may further relax this condition and use the* ***necessary*** *requirement that the equation*

$$G_k''(\alpha) = \sigma^2 \tag{5.5.23}$$

must not have any roots in $[0,\, 1]$*, the admissible set. In partilcular,* $G_k''(0) = -M_2 \neq \sigma^2$ *is automatically satisfied. We also require that*

$$G_k''(1) = \sum_{j=2}^{k} (-1)^{j-1} (j-1)\, M_j \neq \sigma^2,$$

and that because of the continuity of the polynomial G_k''*,*

$$\min G_k''(\alpha) \neq \sigma^2 \quad \textit{and} \quad \max G_k''(\alpha) \neq \sigma^2.$$

By solving the equation $G_k'''(\alpha) = 0$ *we obtain* $l \leq k-3$ *turning points* $r_1, r_2, \cdots, r_l$ *in the admissile set, and provided* $G_k''(r_j) \neq \sigma^2$*, we are assured that the iterative procedure returns a unique optimal portfolio.*
Example: Take the case of $k = 4$*.*

$$G_4''(\alpha) = -M_2 + 3M_3\, \alpha - 3M_4\, \alpha^2,$$

which has a maximum at $r = \frac{M_3}{3\, M_4}$*. If* $G_4''(1)$ *and* $G_4''(r)$ *are different from* σ^2*, we are asssured of a unique optimal portfolio.*

We are now in a position to give an approximation of the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth.

Theorem 5.6. *Let* π *be the portfolio that maximizes the expected logarithmic utility from terminal wealth in the Lévy market. Then* π *is approximated by* $\pi^{(k)}$ *given by Newton's method in (5.5.5)–(5.5.7). We may also compute* $\pi^{(k)}$ *by linear iteration,* *provided* $G_k''(\alpha_t) \neq \sigma_t^2,\ \ \alpha_t \in [0,\, 1]$, *using the equation*

$$\pi_t^{(k)} = \frac{\mu_t - r_t + G_k'(\pi_t^{(k)})}{\sigma_t^2} = \frac{\theta_t}{\sigma_t} + \frac{1}{\sigma_t^2}\sum_{j=1}^{k}(-1)^{j-1}\, M_j\, (\pi_t^{(k)})^{j-1}, \tag{5.5.24}$$

where $G_k(\cdot)$ *is given in the last theorem,* μ_t *is the continuous appreciation rate,* r_t *is the risk-free interest rate, and* θ_t *is the stock's Sharpe ratio.*

The approximate maximum expected logarithmic utility with $x > 0$ *in initial wealth, is:*

$$u^{(k)}(x) = \log x + \frac{1}{2}\int_0^T \theta_t^2 dt + \int_0^T \left(G_k(\pi_t^{(k)}) - \frac{1}{2}(\pi_t^{(k)}\,\sigma_t - \theta_t)^2\right) dt. \tag{5.5.25}$$

Proof. By Theorem 5.7, π is given exactly by the equation

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2},$$

Since k be the largest integer such that M_j exists for each $j \leq k$. By Theorem 5.1 we can approximate $G(\alpha)$ by the kth–degree polynomial $G_k(\alpha)$ which implies that $G_k(\alpha) \approx G(\alpha)$. Thus if $G_k''(\alpha) \neq \sigma^2$, we have a unique optimal portfolio $\pi_t^{(k)}$ approximating π_t given by (5.9.1). The utility formula follows similarly. □

5.6 Jump–Diffusion Markets

A jump–diffusion market has stock price dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)\, dt + \sigma_t\, dB_t + d(\Sigma_{i=1}^{N(t)}(V_i - 1)). \tag{5.6.1}$$

N is the driving Poisson process ($\Pi(\lambda)$) with intensity λ. The jump amplitude V_i are independent and identically distributed variables drawn from a random variable V.

The random variable V_i is the i^{th} jump of the stock returns in the interval $(0,\ t)$. The log jump amplitude $X = \log V$ has distribution $F_X(\cdot)$, and $X_i = logV_i$, $i =$ 1,2,..., $N(t)$ are iid random variables.

Our model (5.6.1), has percentage returns:

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dB_t + d(\Sigma_{i=1}^{N(t)}(e^{X_i} - 1)), \tag{5.6.2}$$

while the **total stock returns** and Lévy measure for the model are:

$$b_t = \mu_t + \lambda \int_R (e^x - 1) F_X(dx) \ \text{ and } \ v(dx) = \lambda F_X(dx). \tag{5.6.3}$$

5.7 The Kou Jump–Diffusion Model

The stock price dynamic for this model is:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\,\sigma_t^2)\, dt + \sigma_t\, dB_t + d(\Sigma_{i=1}^{N(t)}(V_i - 1)), \tag{5.7.1}$$

where the log jump amplitude $X = \log(V)$ has *double exponential* distribution with density $f = f_{kou}$, dependent on 3 parameters p, η_1 and η_2, and given by

$$f_{kou}(x) = p\,\eta_1\, \exp\left(-\eta_1\, x\right) I_{\{x>0\}} + q\,\eta_2\, \exp\left(-\eta_2\, |x|\right) I_{\{x<0\}}, \tag{5.7.2}$$

$$\eta_1 > 1,\ \eta_2 > 0,\ p + q = 1, p \geq 0,\ q \geq 0.$$

X can be expressed as:

$$X = \begin{cases} X^u, & \text{with probability } p. \\ X^d, & \text{with probability } q. \end{cases}$$

Equivalently,

$$X = X^u \, I_{\{x>0\}} - X^d \, I_{\{x<0\}}, \tag{5.7.3}$$

where

$$X^u \sim \exp(\eta_1), \quad X^d \sim \exp(\eta_2),$$

are exponential random variables with means $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$, respectively. X^u, the upward jump log amplitude, occurs with probability p, which is not expected to exceed 100%. This leads to the constraint

$$\mathbf{E}(X^u) = \frac{1}{\eta_1} \; < 1. \tag{5.7.4}$$

X^d is the log amplitude of the downward movements in returns, which occurs with probabilty $q = 1 - p$. For this model (cf Kou [30]), we have:

$$\begin{aligned} \mathbf{E}(X) &= \frac{p}{\eta_1} - \frac{q}{\eta_2}, & (5.7.5) \\ \mathbf{Var}(X) &= p\, q \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2 + \left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2} \right), & (5.7.6) \end{aligned}$$

and

$$\mathbf{E}(V) = \frac{q\, \eta_2}{(\eta_2 + 1)} - \frac{p\, \eta_1}{(\eta_1 - 1)}, \tag{5.7.7}$$

is the expected jump amplitude. The Kou Lévy density v_{kou} is given by:

$$v_{kou}(x) = \lambda f_{kou}(x) = \lambda\, p\, \eta_1 \, \exp\left(-\eta_1\, x\right) I_{\{x>0\}} + \lambda\, q\, \eta_2 \, \exp\left(-\eta_2\, |x|\right) I_{\{x<0\}}. \tag{5.7.8}$$

The stock dynamic (5.7.1) can be written as:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dB_t + \int_R (e^x - 1)\, N(dt, dx), \tag{5.7.9}$$

with total returns:

$$b_t = \mu_t + \lambda \int_R (e^x - 1)\, f_{kou}(x)dx = \mu_t + \lambda \left(\frac{p}{\eta_1 - 1} - \frac{q}{\eta_2 + 1} \right). \tag{5.7.10}$$

Standing Assumptions

(1) For all markets, the stock's Sharpe ratio or market price of risk, is:

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}. \tag{5.7.11}$$

(2) The Performance Function $G(\cdot)$ is restricted to the domain [0, 1], where

$$G(\pi) = \int_R \log(1 + \pi\,(e^x - 1))\, v(dx). \tag{5.7.12}$$

Theorem 5.7. *Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth. Then*

(1)

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2}. \tag{5.7.13}$$

(2 *The maximum expected logarithmic utility with $x > 0$ in initial wealth, is*

$$u(x) = \log(x) + \frac{1}{2}\,\mathbf{E} \int_0^T \theta_t^2\, dt + \mathbf{E} \int_0^T f(\pi_t)\, dt, \tag{5.7.14}$$

where

$$f(\pi_t) = -\frac{1}{2}(\pi_t\, \sigma_t - \theta_t)^2 + G(\pi_t). \tag{5.7.15}$$

Proof. This follows directly from Theorem 3.8. □

Remark 5.6. *Because $G(\pi)$, and hence $G'(\pi)$, is in general very difficult to compute, we resort to approximation methods. This leads to an approximation $\pi^{(k)}$ of π, based on a k-th degree truncated Taylor series expansion of G.*

Definition 5.3. *Define the objects $\widehat{M}_j(\eta)$ and $\widehat{K}_s(\eta)$ by the prescriptions:*

$$\widehat{M}_j(\eta) = \int_0^\infty (e^x - 1)^j e^{-\eta x} dx, \ \eta > 0, \ j < \eta. \tag{5.7.16}$$

$$\widehat{K}_s(\eta) = \int_0^\infty (e^{sx} - 1) e^{-\eta x} dx, \ s < \eta. \tag{5.7.17}$$

$$\mathbf{B}(\alpha, \beta) = \int_0^\infty x^{\alpha-1}(x+1)^{-\alpha-\beta} dx, \ \alpha > 0, \ \beta > 0. \tag{5.7.18}$$

Lemma 5.2. *Let $\mathbf{B}(\alpha, \beta)$ be the Beta function above.*

(1) *If $j < \eta$, then*

$$\widehat{M}_j(\eta) = \mathbf{B}(j+1, \eta - j) = \frac{\Gamma(j+1)\Gamma(\eta - j)}{\Gamma(\eta + 1)}. \tag{5.7.19}$$

(2) *If $s < \eta$, then*

$$\widehat{K}_s(\eta) = \frac{s}{\eta(\eta - s)}. \tag{5.7.20}$$

Proof. (1) Let $y = e^x - 1$. Then $x = \log(1+y)$ and $dy = e^x \, dx$. Thus

$$\begin{aligned} \widehat{M}_j(\eta) &= \int_0^\infty y^j e^{-\eta \log(1+y)} dx = \int_0^\infty y^j (1+y)^{-\eta-1} dy \\ &= \int_0^\infty y^{(j+1)-1}(1+y)^{-(j+1)-(\eta-j)} dy \\ &= \mathbf{B}(j+1, \eta - j) = \frac{\Gamma(j+1)\Gamma(\eta-j)}{\Gamma(\eta+1)}. \end{aligned}$$

(2) If $\eta > s$, then

$$\widehat{K}_s(\eta) = \int_0^\infty (e^{sx} - 1) \, e^{-\eta x} dx = \frac{1}{\eta - s} - \frac{1}{\eta} \ = \ \frac{s}{\eta(\eta - s)}.$$

$\square$

With Lemma 5.2 in hand, we are now able to compute the kernels $K_s(\eta)$ and instantaneous centralized moments of returns $M_j(\eta)$, defined by:

Definition 5.4 (Instantaneous Centralized Moments of Returns).

$$M_j \equiv M_j(\eta_1, \eta_2, p, \lambda) \triangleq \int_R (e^x - 1)^j v_{kou}(x)dx = \lambda \int_R (e^x - 1)^j f_{kou}(x)dx,$$

and

$$K_s \equiv K_s(\eta_1, \eta_2, p, \lambda) \triangleq \int_R (e^{sx} - 1) v_{kou}(x)dx = \lambda \int_R (e^{sx} - 1) f_{kou}(x)dx.$$

Lemma 5.3. *For the Kou jump–diffusion market given by (5.7.1), we have the following. Let $\eta_1 > 1$, $\eta_2 > 0$, $p + q = 1$, $p \geq 0$, $q \geq 0$, with Poisson intensity rate $\lambda > 0$. If $\max(s, j) < \eta_1$, then*

$$M_j = (-1)^j \, (j!) \, \frac{\lambda \, q \, \eta_2 \, \Gamma(\eta_2)}{\Gamma(\eta_2 + j + 1)} + (j!) \, \frac{\lambda \, p \, \eta_1 \, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)}, \tag{5.7.21}$$

and

$$K_s = \frac{\lambda \, p \, s}{(\eta_1 - s)} + \frac{\lambda \, q \, s}{(\eta_2 + s)} = \lambda \frac{s \, (p \, \eta_2 - q \, \eta_1 + s)}{(\eta_1 - s)(\eta_2 + s)}. \tag{5.7.22}$$

Proof. (1). Let $j < \eta_1$ and $s < \eta_2$. From (5.7.20), we get:

$$\begin{aligned}
\frac{M_j}{\lambda} &= \int_R (e^x - 1)^j f_{kou}(x)dx \\
&= \int_R (e^x - 1)^j \left(p\eta_1 \exp{(-\eta_1 x)} I_{\{x>0\}} + q\eta_2 \exp{(-\eta_2 |x|)} I_{\{x<0\}} \right) dx \\
&= q\eta_2 \int_{-\infty}^0 (e^x - 1)^j \exp{(-\eta_2 |x|)} dx + p\eta_1 \int_0^\infty (e^x - 1)^j \exp{(-\eta_1 x)} dx \\
&= q\eta_2 \int_0^\infty (1 - e^x)^j \exp{(-jx)} \exp{(-\eta_2 x)} dx + p\eta_1 \widehat{M_j}(\eta_1),
\end{aligned}$$

and from equation (5.7.16) and Lemma 5.2,

$$\begin{aligned}
\frac{M_j}{\lambda} &= (-1)^j \, q \, \eta_2 \int_0^\infty (e^x - 1))^j \exp\left(-(j+\eta_2)\, x\right) dx + p \, \eta_1 \, \widehat{M}_j(\eta_1) \\
&= (-1)^j \, q \, \eta_2 \, \widehat{M}_j \, (j+\eta_2) + p \, \eta_1 \, \widehat{M}_j \, (\eta_1) \\
&= (-1)^j q \, \eta_2 \, \frac{\Gamma(j+1)\, \Gamma(j+\eta_2-j)}{\Gamma(\eta_2+j+1)} + p \, \eta_1 \, \frac{\Gamma(j+1)\, \Gamma(\eta_1-j)}{\Gamma(\eta_1+1)} \\
&= (-1)^j \, j! \, \frac{\lambda \, q \, \eta_2 \, \Gamma(\eta_2)}{\Gamma(\eta_2+j+1)} + j! \, \frac{\lambda \, p \, \eta_1 \, \Gamma(\eta_1-j)}{\Gamma(\eta_1+1)},
\end{aligned}$$

and the result follows.

For (2):

$$\begin{aligned}
\frac{K_s}{\lambda} &= \int_R (e^{sx}-1) f_{kou}(x) dx \\
&= \int_R (e^{sx}-1) \left(p\eta_1 \exp\left(-\eta_1 x\right) I_{\{x>0\}} + q\eta_2 \exp\left(-\eta_2 |x|\right) I_{\{x<0\}}\right) dx \\
&= q\eta_2 \int_{-\infty}^0 (e^{sx}-1) \exp\left(-\eta_2 |x|\right) dx + p\eta_1 \int_0^\infty (e^{sx}-1) \exp\left(-\eta_1 x\right) dx \\
&= q\eta_2 \int_0^\infty (e^{-sx}-1) \exp\left(-\eta_2 x\right) dx + p\eta_1 \int_0^\infty (e^{sx}-1) \exp\left(-\eta_1 x\right) dx
\end{aligned}$$

$$\begin{aligned}
&= q\eta_2 \int_0^\infty (\exp(-(s+\eta_2)x) - \exp(-\eta_2 x)) dx \\
&\quad + p\eta_1 \int_0^\infty (\exp((s-\eta_1)x) - \exp\left(-\eta_1 x\right)) dx \\
&= q\eta_2 \widehat{K}_{-s}(\eta_2) + p\eta_1 \widehat{K}_s(\eta_1), \\
&= q\eta_2 \left[\frac{-s}{\eta_2(\eta_2+s)}\right] + p\eta_1 \left[\frac{s}{\eta_1(\eta_1-s)}\right], \\
&= \frac{s\,(p\,\eta_2 - q\,\eta_1 + s)}{(\eta_1-s)(\eta_2+s)},
\end{aligned}$$

which completes the proof. □

For the Kou model $\eta_1 = \frac{1}{\mathbb{E}(X^u)} > 1$, and M_j exists provided $j < \eta_1$. Thus M_1 always exists for this model, where

$$M_1 \;=\; \frac{\lambda p}{\eta_1 - 1} - \frac{\lambda q}{\eta_2 + 1} = \lambda(\mathbf{E}(e^X - 1)) = \lambda(\mathbf{E}(V) - 1)), \tag{5.7.23}$$

where $\mathbf{E}(V)$ is the mean jump amplitude. In this case M_1 is the jump component of the total stock appreciation rate $b = \mu + M_1$, where μ is the continuous component of stock returns.

Corollary 5.5.

For the double exponential Kou jump-diffusion model with $p = q = \frac{1}{2}$ *and* $\eta_1 = \eta_2 = \eta$*, we have the following:*

(1) *If* $j < \eta$*, then*

$$M_j = \frac{\lambda}{2}\, j! \left[\frac{(-1)^j}{\Pi_{r=1}^{j}(\eta + r)} + \frac{1}{\Pi_{r=1}^{j}(\eta - r)} \right]. \tag{5.7.24}$$

(2) *If* $|s| < \eta$*, then*

$$K_s = \frac{\lambda s^2}{\eta^2 - s^2}. \tag{5.7.25}$$

Proof. This result follows from Lemma 5.3 with $p = q = \frac{1}{2}$ and $\eta_1 = \eta_2 = \eta$. □

Remark 5.7. *There are many combinatorial identities resulting from Lemma 5.3 and its corollary. They are contained in Appendix A.*

In the next section we develop an analytic formula for the derivatives of $G(\pi)$ in terms of the cumulative distribution function of a Beta random variable.

5.8 Analytic Formulas for $G'(\pi)$ and $G''(\pi)$ for Kou Market

We show that $G'(\pi)$ and $G''(\pi)$ can be expressed analytically in terms of the cumulative distribution function of Beta random variables. The proofs for the results of this section involve lots of calculations and are located in appendix A. Let $\pi \in [0,\, 1]$ and $\eta > 0$. Define

$$D(\pi, \eta) \triangleq \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta\, x} dx. \tag{5.8.1}$$

Define $J(\pi, \eta)$ by the prescription:

$$J(\pi, \eta) \triangleq \int_0^\infty \frac{dy}{(1 + \pi\, y)(1 + y)^\eta} \;\; \eta > 0. \tag{5.8.2}$$

We now give a major result.

Theorem 5.8. *Let $0 < \eta < 1$, $\pi \in (0,\, 1)$ and $\beta = 1 - \pi$. Then*

$$J(\pi, \eta) = \frac{1}{\pi} \left(\frac{\pi}{\beta}\right)^\eta B(1 - \eta, \eta) \left[1 - F\left(\frac{\pi}{\beta}; 1 - \eta, \eta\right)\right], \tag{5.8.3}$$

and

$$D(\pi, \eta) = \frac{1}{\beta\, \pi} \left(\frac{\pi}{\beta}\right)^\eta B(1 - \eta, \eta) \left[1 - F\left(\frac{\pi}{\beta}; 1 - \eta, \eta\right)\right] - \frac{1}{\beta\, \pi}, \tag{5.8.4}$$

where $F(x; a, b)$ is the cumulative distribution function of a Beta random variable with parameters a and b, and $B(a, b)$ is the corresponding Beta function.

We now have the following result.

Proposition 5.1. *For the Kou model,with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$,*

$p+q=1,\ p,\ q\ \geq 0,$

$$G'(\pi) = \lambda\, p\, \eta_1\, D(\pi,\eta_1) - \lambda\, q\, \eta_2\, D(1-\pi,\eta_2). \tag{5.8.5}$$

We now give the main result for the Kou model.

Theorem 5.9. *Let $\pi \in (0,1)$ and $\beta = 1-\pi$. For the Kou model with parameters $\eta_1 \geq 1,\ \eta_2 > 0,\ \lambda > 0,\ p+q=1,\ p,\ q\ \geq 0,$ Sharpe ratio θ_t and volatility σ_t, the optimal portfolio is the unique solution of the equation*

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{(\lambda\, p\, \eta_1\, D(\pi_t,\eta_1) - \lambda\, q\, \eta_2\, D(1-\pi_t,\eta_2))}{\sigma_t^2}, \tag{5.8.6}$$

where $D(\pi_t,\eta_1)$ and $D(1-\pi_t,\eta_2)$ can be computed from equation (A.1.18).

An explicit formula for $G''(\pi)$ is given below

Theorem 5.10. *Let $\pi \in (0,1)$, $\beta = 1-\pi$ and $a = \frac{\pi}{\beta}$. For the Kou model with parameters $\eta_1 \geq 1,\ \eta_2 > 0,\ \lambda > 0,\ p+q=1,\ p,\ q\ \geq 0,$*

$$G''(\pi_t) = -\lambda\, p\, \eta_1\, A(\pi_t,\eta_1) - \lambda\, q\, \eta_2\, A(1-\pi_t,\eta_2), \tag{5.8.7}$$

where

$$\begin{aligned} A(\pi,\eta) &= \pi^{\eta-2}\left(I_1 - 2I_2 + I_3\right), \\ I_1 &= \frac{1}{\eta\,\pi^{\eta}}, \\ I_2 &= \frac{1}{\pi^{\eta-1}} J(\pi,\eta+1), \\ I_3 &= \left(\frac{a}{\pi}\right)^{\eta+2} H(\eta+1,\, a), \end{aligned}$$

where

$$J(\pi,\eta+1) = \frac{1}{\beta\eta} - a\, J(\pi,\eta),$$

and for $0 < \eta < 1$,

$$J(\pi, \eta) = \frac{1}{\pi} a^{\eta} B(1 - \eta, \eta) [1 - F(a; 1 - \eta, \eta)],$$

and

$$H(\eta+1, a) = \frac{1}{\eta a^{\eta}} - B(1-\eta, \eta)[1-F(a; 1-\eta, \eta)] - B(1-\eta, 1+\eta)[1-F(a; 1-\eta, \eta+1)], \tag{5.8.8}$$

where $F(x; u, v)$ *is the cumulative distribution function of a Beta random variable with parameters* u *and* v*, and* $B(u, v)$ *is the corresponding Beta function.*

5.9 Polymonial Approximation of $G(\alpha)$–Kou Model

We now use the M_js to approximate the function G, and hence G', by a truncated Taylor series which is controlled by η_1, the inverse of the average upward log jump amplitude. Since $\eta_1 \to \infty$, as $\mathbb{E}(X^u) \downarrow 0$, then the smaller the average upward jump size, the better the approximation of G. We have the following result.

Theorem 5.11. *Let* $\eta_1 > 1$*, be the inverse of the average upward jump amplitude. If* $k \leq \eta_1$ *then* $G(\cdot)$ *has a* k*-th degree polynomial approximation* $G_k(\cdot)$ *given by:*

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} \frac{M_j}{j} \alpha^j, \ \alpha \in [0, 1], \tag{5.9.1}$$

where M_j *is given by Lemma 5.3 with*

$$G'_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} M_j \alpha^{j-1} \ \text{and} \ G''_k(\alpha) = \sum_{j=2}^{k} (-1)^{j-1} (j-1) M_j \alpha^{j-2}. \tag{5.9.2}$$

Proof. By Lemma 5.3, M_j exists for all $j \leq k = [\eta_1]$. The result then follows from Theorem 5.1.3. □

Remark 5.8. *Since M_j is the main ingredient required for an approximation of $G(\alpha)$, it follows that when η_1 is large, we get a better approximation. In particular, if $\eta_1 \in$ (2, 3) we have quadratic approximation of G; $\eta_1 \in (3, 4)$ leads to cubic approximation, while $\eta_1 \in (k,\, k+1)$ leads to a k-th degree polynomial approximation of $G(\alpha)$.*

Theorem 5.12. *An approximation $\pi_t^{(1)}$ of the optimal portfolio π_t always exists for the Kou market. It is given by:*

$$\pi_t^{(1)} = \frac{\mu_t - r_t + M_1}{\sigma_t^2} = \frac{b_t - r_t}{\sigma_t^2}, \tag{5.9.3}$$

where $b = \mu + M_1$ is the total stock appreciation rate, and r is the risk-free interest rate.

Proof. Since $\eta_1 > 1$, then M_1 always exists, and we approximate $G(\cdot)$ by a linear function $G_1(\alpha) = M_1\alpha$. Thus $G_1'(\alpha) = M_1$, and the result follows from Theorem 5.6. □

Remark 5.9. *Note that, in this case $\frac{M_1}{\sigma_t^2}$ is the excess optimal stock holdings required over the Merton [36] (GBM) case, with dynamic $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t$.*

We give particular values of $\pi_t^{(k)}$ when η_1 falls in the interval $(k, k+1)$, where k is a positive integer.

Theorem 5.13. *Let π be the optmial portfolio that maximizes the expected logarithmic utility from terminal wealth in the Kou market. Let $\eta_1 > 1$ and $b_t = \mu_t + M_1$ be the total stock appreciation rate and r_t is the risk-free interest rate.*

(1) *If $\eta_1 \in (2, 3]$, we have quadratic approximation of π given by*

$$\pi_t^{(2)} = \frac{\mu_t - r_t + M_1}{\sigma_t^2 + M_2} = \frac{b_t - r_t}{\sigma_t^2 + M_2}. \tag{5.9.4}$$

(2) *If* $\eta_1 \in (3, 4]$, *we have cubic approximation of* π *given by*

$$\pi_t^{(3)} = \pi_\pm = \frac{(\sigma_t^2 + M_2) \pm \sqrt{(\sigma_t^2 + M_2)^2 - 4M_3(b_t - r_t)}}{2M_3}, \qquad (5.9.5)$$

where

$$\pi_t^{(3)} = \begin{cases} \pi_-, & \text{if } M_3 < 0 \text{ or } b_t - r_t > 0\ , \\ \pi_+, & \text{if } M_3 > 0 \text{ and } b_t - r_t < 0, \\ \frac{\sigma_t^2 + M_2}{M_3}, & \text{if } M_3 > 0 \text{ and } b_t - r_t = 0, \\ 0, & \text{if } M_3 < 0 \text{ and } b_t - r_t = 0, \end{cases}$$

with $\pi_t^{(3)} = \pi_t^{(2)}$ *if* $M_3 = 0$. *If* $M_3 > 0$ *and* $b_t - r_t > 0$ *then at most one of* $\pi_\pm$ *is in* $[0,\ 1]$ *and* $\pi_t^{(3)}$ *is 0, 1 or this value.*

Proof. (1) If $\eta_1 \in (2, 3]$ then $2 < \eta_1$, whence M_2 exists. By Theorem 5.1, $G(\alpha)$ can be approximated by $G_2(\alpha) = M_1\alpha - \frac{1}{2}M_2\alpha^2$, where $\alpha \in [0,\ 1]$. By Theorem 5.6, π can be aproximated by $\pi^{(2)}$, where

$$\pi_t^{(2)} \quad = \quad \frac{\mu_t - r_t + G_k'(\pi_t^{(2)})}{\sigma_t^2} = \frac{\mu_t - r_t + M_1 - M_2\pi_t^{(2)}}{\sigma_t^2},$$

which yields (5.9.4).

(2) If $\eta_1 \in (3, 4]$, then the average upward jump amplitude is at most 33.3%. Since $3 < \eta_1$, then M_3 exists and by Theorem 5.1, we may approximate $G(\alpha)$ by the 3rd degree polynomial $G_3(\alpha) = M_1\alpha - \frac{1}{2}M_2\alpha^2 + \frac{1}{3}M_3\alpha^3$. By Theorem 5.6, π has approximation

$$\pi_t^{(3)} = \frac{\mu_t - r_t + M_1 - M_2\pi_t^{(3)} + M_3(\pi_t^{(3)})^2}{\sigma_t^2},$$

whence

$$\mu_t - r_t + M_1 - (\sigma_t^2 + M_2)\pi_t^{(3)} + M_3(\pi_t^{(3)})^2 = 0,$$

and so, (5.9.5) holds. $\square$

We generalize the last result and Theorem 5.6 as follows:

Theorem 5.14. *Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the Kou market. Let η_1 be the inverse of the average upward log jump amplitude, r the risk-free interest rate, and μ the continuous component of total stock returns. Let $k = [\eta_1]$. There exists a k-th degree polynomial $G_k(\cdot)$ such that $\pi^{(k)}$, the k-th degree approximation of π, is the unique root of the the equation*

$$\pi = \frac{\mu - r}{\sigma^2} + \frac{1}{\sigma^2}\left(\sum_{j=1}^{k}(-1)^{j-1}M_j\pi^{j-1}\right) \tag{5.9.6}$$

provided $G_k''(\cdot) \neq \sigma^2$.

Proof. If $k = [\eta_1]$ then $\eta_1 \in [k,\, k+1)$, and $k \leq \eta_1$. Thus M_k exists, and by Theorem 5.6, π has k-th degree approximation:

$$\begin{aligned}\pi^{(k)} &= \frac{\mu - r + \left(\sum_{j=1}^{k}(-1)^{j-1}M_j(\pi^{(k)})^{j-1}\right)}{\sigma^2},\\ &= \frac{\mu - r}{\sigma^2} + \frac{1}{\sigma^2}\left(\sum_{j=1}^{k}(-1)^{j-1}M_j(\pi^{(k)})^{j-1}\right).\square\end{aligned}$$

We now give explicit relationships between the optimal portfolios in the asymmetric setting, to that of the symmetric setting for the Kou market.

Theorem 5.15. *Let π be the optimal portfolio that maximizes the expected logarithmic utililty from terminal wealth in the symmetric Kou market with parameters r, μ, σ, λ, p, q, η_1, η_2. Assume that $k \leq [\eta_1]$ and that $\pi_t^{(k)}$ is the approximation of π based on the k-th degree polynomial approximation G_k, of G. Let π^i be the optimal portfolio for the i-th investor, $i \in \{0, 1\}$. Then for each $t \in [0,\, T]$, there exists $\eta_t^{(i,k)}$ between π and $\pi_t^{(k)}$ such that*

$$\pi_t^i = \pi_t + v^i \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i)})|}, \tag{5.9.7}$$

where η^i *lies between* π *and* π^i*, with approximation given by*

$$\pi_t^i \approx \pi_t^{(k)} + v_t^i \, \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i,k)})|}. \tag{5.9.8}$$

Under quadratic approximation of G*,*

$$\pi_t^i \approx \pi_t^{(2)} + v_t^i \, \frac{\sigma_t}{\sigma_t^2 + 2\lambda \left[\frac{q}{(\eta_2+2)(\eta_2+1)} + \frac{p}{(\eta_1-2)(\eta_1-1)}\right]}. \tag{5.9.9}$$

Proof. The result follows from Theorem 3.20 with π_t replaced by $\pi_t^{(k)}$. Under quadratic approximation, $G''(\pi) = -M_2$, which is explicitly given in Lemma 5.3. $\square$

5.10 Diffusion Market Driven by a Variance Gamma Process

Let $X^{VG} \equiv (X_t)_{t\geq 0}$ be a Variance Gamma process with Lévy triple $(\gamma, 0, v_{VG})$, where $\gamma = \int_{-1}^{1} x v_{VG}(x) dx$. We assume that X^{VG} has parameters, C , G, M, and Lévy measure

$$v(x) \equiv v_{VG}(x) = \frac{C \, \exp\left(-G\,|x|\right)}{|x|} \, I_{\{x<0\}} + \frac{C \, \exp\left(-M\,x\right)}{x} \, I_{\{x>0\}}, \tag{5.10.1}$$

where $C > 0,\ G > 0,\ M > 0,\ G \geq M.$

5.10.1 The Symmetric VG Market

This market consists of a single bond **B** that earns risk–free interest r_t with price dynamic giveb by (3.1.1). There is also a single risky asset S called stock, with log returns dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\,\sigma_t^2)\,dt + \sigma_t\,dB_t + dX_t^{VG}, \tag{5.10.2}$$

which is equivalent to the percentage returns dynamic:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t\,dt + \sigma_t\,dB_t + \int_{\mathbf{R}} (e^x - 1)\,N(dt, dx), && (5.10.3)\\ X_t^{VG} &= \int_{\mathbf{R}} x\,N(t, dx), && (5.10.4) \end{aligned}$$

where $N(t, A)$ is the Poisson Random measure on $\mathbf{R}^+ \times (\mathbf{R} - \{0\})$, $A \in B(\mathbf{R} - \{0\})$, that counts the jumps of X^{VG} in $(0,\,t)$, $t \in [0,\,T]$, $T > 0$. The total returns on the stock is; $b_t = \mu_t + M_1 \equiv \mu_t + \int_{\mathbf{R}} (e^x - 1)\,v(x)dx$. The VG process is a pure jump Lévy process with an infinite arrival rate of small jumps, ($\int_{\mathbf{R}} v(x)dx = \infty$) and having paths of finite variation ($\int_{-1}^{1} |x|v(x)dx < \infty$).

5.10.2 The Centralized Moments of Instantaneous Returns for the VG Market

Definition 5.5. *The k-th centralized moment of instantaneous returns for the VG market is M_k, with kernel K_s, define by the prescriptions:*

$$M_k = \int_{\mathbf{R}} (e^x - 1)^k v_{VG}(x)dx \quad \textit{and} \quad K_s = \int_{\mathbf{R}} (e^{sx} - 1) v_{VG}(x)dx. \tag{5.10.5}$$

The following result helps with future computations.

Lemma 5.4. *Let u, $v > 0$. Then*

$$\int_{0+}^{\infty} (e^{-ux} - e^{-vx})\frac{dx}{x} = \ln\left(\frac{v}{u}\right). \tag{5.10.6}$$

Proof. Fix $v > u > 0$ and $t > 0$.

$$\begin{aligned}
\int_{0+}^{\infty}(e^{-ux} - e^{-vx})\frac{dx}{x} &= \int_{0+}^{\infty}\int_{u}^{v} d(e^{-t\,x})\frac{dx}{x} = \int_{0+}^{\infty}\left[\int_{u}^{v} -xe^{-t\,x}dt\right]\frac{dx}{x} \\
&= -\int_{0+}^{\infty}\left[\int_{u}^{v} e^{-t\,x}dt\right]dx = -\int_{u}^{v}\left[\int_{0+}^{\infty} e^{-t\,x}dx\right]dt \\
&= -\int_{u}^{v}\left[\frac{e^{-t\,x}}{-t}|_{0+}^{\infty}\right]dt = \int_{u}^{v}\frac{1}{t}dt \\
&= \ln\left(\frac{v}{u}\right).
\end{aligned}$$

□

We are now able to compute the VG kernel K_s.

Lemma 5.5. *Let X^{VG} be a Variance Gamma process with parameters C, G, M. Then for $0 \leq s < M$,*

$$K_s = C\log\left[\frac{GM}{(G+s)(M-s)}\right]. \tag{5.10.7}$$

Proof. Let $0 \leq s < M$. From Lemma 5.4, we get

$$\begin{aligned}
K_s &= \int_{\mathbf{R}}(e^{sx} - 1)v_{VG}(x)dx \\
&= C\int_{-\infty}^{0-}(e^{sx} - 1)\frac{e^{Gx}}{-x}dx + C\int_{0+}^{\infty}(e^{sx} - 1)\frac{e^{-Mx}}{x}dx \\
&= C\int_{0+}^{\infty}(e^{-sx} - 1)\frac{e^{-Gx}}{x}dx + C\int_{0+}^{\infty}(e^{sx} - 1)\frac{e^{-Mx}}{x}dx \\
&= C\int_{0+}^{\infty}(e^{-(G+s)x} - e^{-Gx} + e^{-(M-s)x} - e^{-Mx})\frac{dx}{x}dx
\end{aligned}$$

Thus

$$K_s = C\left[\log(\frac{G}{G+s}) + \log(\frac{M}{M-s})\right] = C\log\left[\frac{GM}{(G+s)(M-s)}\right].$$

□

With these explicit values for $K_1, K_2, \ldots, K_5$, in hand, we can now compute the first five (5) instantaneous centralized moments of returns, $M_1, M_2, \ldots, M_5$, which are required for the estimation of the optimal portfolio, π.

Theorem 5.16. *Let X be the VG process with parameters C, G, M. Then*

$$\begin{aligned}
M_1 &= K_1 = C\log\left[\frac{GM}{(G+1)(M-1)}\right], \quad \textit{if } M > 1. \\
M_2 &= C\log\left[\frac{(G+1)^2(M-1)^2}{GM(G+2)(M-2)}\right], \quad \textit{if } M > 2. \\
M_3 &= C\log\left[\frac{GM(G+2)^3(M-2)^3}{(G+3)(M-3)(G+1)^3(M-1)^3}\right], \quad \textit{if } M > 3 \\
M_4 &= C\log\left[\frac{(G+3)^4(M-3)^4(G+1)^4(M-1)^4}{GM(G+4)(M-4)(G+2)^6(M-2)^6}\right], \quad \textit{if } M > 4. \\
M_5 &= C\log\left[\frac{GM(G+4)^5(M-4)^5(G+2)^{10}(M-2)^{10}}{(G+5)(M-5)(G+3)^{10}(M-3)^{10}(G+1)^5(M-1)^5}\right], \quad \textit{if } M > 5.
\end{aligned}$$

Proof. This follows easily from Lemma 5.1. □

We now give a general formula for M_k.

Theorem 5.17. *Let X be a VG process with parameters C, G, M. Let $k \in \mathbf{N}$, with $k < M$. Then*

$$M_k = C\ \log A_k, \tag{5.10.8}$$

where

$$A_k \triangleq \Pi_{j=0}^{k}\left[(G+j)(M-j)\right]^{\left((-1)^{(k-j+1)}\binom{k}{j}\right)}. \tag{5.10.9}$$

Proof. If $M > k$ then by Lemma 5.1, K_s exist for $s = 0, 1, \ldots, k$, and

$$\begin{aligned}
M_k &= C \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \log \left[\frac{GM}{(G+j)(M-j)} \right] \\
&= C \sum_{j=1}^{k} \log \left[\frac{GM}{(G+j)(M-j)} \right] (-1)^{k-j} \binom{k}{j} \\
&= C \sum_{j=1}^{k} \log [GM] (-1)^{k-j} \binom{k}{j} + C \sum_{j=1}^{k} \log [(G+j)(M-j)] (-1)^{k-j+1} \binom{k}{j}.
\end{aligned}$$

Therefore

$$\begin{aligned}
M_k &= C \log [GM]^{\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j}} + C \log (\prod_{j=1}^{k} [(G+j)(M-j)]^{(-1)^{k-j+1} \binom{k}{j}} \\
&= C \log [GM]^{(-1)^{k+1} \binom{k}{j}} + C \log (\prod_{j=1}^{k} [(G+j)(M-j)]^{(-1)^{k-j+1} \binom{k}{j}} \\
&= C \prod_{j=0}^{k} [(G+j)(M-j)]^{\left((-1)^{(k-j+1)} \binom{k}{j} \right)} \triangleq C \log A_k. \qquad \square
\end{aligned}$$

Maximization of Expected Logarithmic Utility from Terminal Wealth in VG model

Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth at time $T > 0$, with initial wealth $x > 0$. We now give explicit formulas for the approximation of π, based on linear, quadratic and cubic approximation of $G_{VG}(\pi)$.

Theorem 5.18. *Let π be the optimal portfolio that maximizes the expected logarithmic utililty from terminal wealth in the VG market with parameters r, μ, σ, C, G, M. Assume that $k \leq [M]$, and that $\pi_t^{(k)}$ is the approximation of π based on the k–th degree polynomial approximation, G_k, of G. Let $b_t = \mu_t + M_1$ be the total stock returns, where $M_1 = K_1 = C \log \left[\frac{GM}{(G+1)(M-1)} \right]$.*

(1) *Under linear approximation of* G,

$$\pi_t^{(1)} = \frac{\mu_t - r_t + M_1}{\sigma_t^2} = \frac{\mu_t - r_t + C\,\log\left[\frac{G\,M}{(G+1)(M-1)}\right]}{\sigma_t^2} = \frac{b_t - r_t}{\sigma_t^2}. \tag{5.10.10}$$

(2) *Under quadratic approximation of* G,

$$\pi_t^{(2)} = \frac{\mu_t - r_t + M_1}{\sigma_t^2 + M_2} = \frac{\mu_t - r_t + C\,\log\left[\frac{GM}{(G+1)(M-1)}\right]}{\sigma_t^2 + C\,\log\left[\frac{(G+1)^2(M-1)^2}{GM(G+2)(M-2)}\right]} = \frac{b_t - r_t}{\sigma_t^2 + C\,\log\left[\frac{(G+1)^2(M-1)^2}{GM(G+2)(M-2)}\right]}. \tag{5.10.11}$$

(3) *Under quadratic approximation of* G,

$$\pi_t^{(3)} = \pi_{\pm} = \frac{(\sigma_t^2 + M_2) \pm \sqrt{(\sigma_t^2 + M_2)^2 - 4M_3(b_t - r_t)}}{2M_3}, \tag{5.10.12}$$

where

$$\pi_t^{(3)} = \begin{cases} \pi_-, & \text{if } M_3 < 0 \text{ or } b_t - r_t > 0\ , \\ \pi_+, & \text{if } M_3 > 0 \text{ and } b_t - r_t < 0\ , \\ \frac{\sigma_t^2 + M_2}{M_3}, & \text{if } M_3 > 0 \text{ and } b_t - r_t = 0, \\ 0, & \text{if } M_3 < 0 \text{ and } b_t - r_t = 0, \end{cases}$$

with $\pi_t^{(3)} = \pi_t^{(2)}$ *if* $M_3 = 0$, *and*

$$M_3 = C\,\log\left[\frac{G\,M(G+2)^3(M-2)^3}{(G+3)(M-3)(G+1)^3(M-1)^3}\right].$$

Proof. M_k exists if $k \leq [M]$, and are given by Theorem 5.16. The results numbered 1, 2 and 3 follow by imposing these constants onto Theorems **??** and 5.13. □

Remark 5.10. *For the VG market with parameters* C, G, , M, *the approximating polynomial* G_k *is controlled by* M. *The errors* $\varepsilon_t^{(k)}$ *are controlled by* M_{k+1} *and* M_{k+2}. *Thus we are able to get an actual bound on* $\varepsilon_t^{(k)}$, *when* $k+2 \leq [M]$. *An observation of the data in Table 1, in Carr, Geman, Madan and Yor [12], shows that the values of*

M range between 25 and 138. Thus polynomials of degree $k \in [23, 136]$ may be used to approximate π, in addition to giving a bound for the absolute error $|\varepsilon_t^{(k)}|$.

We now give explicit relationships between the optimal portfolios in the asymmetric setting to those of the symmetric case, for the VG market.

Theorem 5.19. *Let π be the optimal portfolio that maximizes the expected logarithmic utililty from terminal wealth in the symmetric VG market with parameters r, μ, σ, C, G, M. Assume that $k \leq [M]$, and that $\pi_t^{(k)}$ is the approximation of π based on the k-th degree polynomial approximation G_k, of G. Let π^i be the optimal portfolio for the i-th investor, $i \in \{0, 1\}$. Then for each $t \in [0, T]$, there exist $\eta_t^{(i)}$ between π and π^i, and $\eta_t^{(i,k)}$ between π and $\pi_t^{(k)}$, such that*

$$\pi_t^i = \pi_t + v_t^i \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i)})|}, \tag{5.10.13}$$

$$\pi_t^i \approx \pi_t^{(k)} + v_t^i \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i,k)})|}. \tag{5.10.14}$$

Under quadratic approximation of G,

$$\pi_t^i \approx \pi_t^{(2)} + v_t^i \frac{\sigma_t}{\sigma_t^2 + C \log\left[\frac{(G+1)^2(M-1)^2}{GM(G+2)(M-2)}\right]}. \tag{5.10.15}$$

Proof. The result follows from Theorem 3.17 with π_t replaced by $\pi_t^{(k)}$. Under quadratic approximation $G''(\pi) = -M_2$, which is explicitly given in Theorem 5.17. □

5.11 The Jump Diffusion Market Driven by the Double Poisson Process: $\Pi(1, 2)$

The Double Poisson jump diffusion market consist of a single stock S and a bond $\mathbf{B}$ which earns the risk–free interest r and has price (3.1.1). The risky asset (stock) has log return dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)\, dt + \sigma_t\, dB_t + dX_t, \tag{5.11.1}$$

where

$$X_t = \alpha_u\, N^u(t) + \alpha_d\, N^d(t), \tag{5.11.2}$$

with

$$\alpha_u \in (0,\, \log 2), \quad \alpha_d = \log(2 - e^{\alpha_u}). \tag{5.11.3}$$

X is called the ***Double Poisson*** Lévy process with parameters $\alpha_u,\ \alpha_d,\ \lambda_u,\ \lambda_d$, where N^u and N^d are independent Poisson processes with intensities λ_u and λ_d, respectively.

In this model, N^u controls the upward jumps, which have log amplitude α_u, while N^d controls the downward jumps, with log amplitude α_d. As in the other models, B is standard Brownian motion; σ and $\mu - \frac{1}{2}\sigma^2$, are continuous components of total volatility and log returns, respectively. We denote a Double Poisson process by $\Pi(1, 2)$. In the next section, we generalize this idea to a process driven by m independent Double Poisson processes, which we denote by $\Pi(m, 2)$.

The Lévy measure for the Double Poisson process is:

$$v(dx) \equiv v_{\Pi(1,2)}(dx) \overset{\triangle}{=} \lambda_u\, \delta_{\alpha_u}(dx) + \lambda_d\, \delta_{\alpha_d}(dx), \tag{5.11.4}$$

where

$$0 \le \lambda_d \le \lambda_u < 1. \tag{5.11.5}$$

$\delta_a(\cdot)$ is the Dirac measure on $\mathcal{B}(\mathbf{R} - \{0\})$, where $\delta_a(A) = 1$, if $a \in A$, and 0, otherwise. Let $N(t, A)$ be the Poisson Random measure that counts the jumps of X in the set $A \in \mathcal{B}(\mathbf{R} - \{0\})$, in the time interval $(0,\, t)$. Then $X_t = \int_{\mathbf{R}} x\, N(t, dx)$. Applying Itô's change of variable formula to (5.11.1), yields the percentage returns dynamic:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dB_t + \int_R (e^x - 1) N(dt, dx). \tag{5.11.6}$$

The total returns on the stock is $b_t = \mu_t + M_1$, where

$$M_1 = \int_{\mathbf{R}} (e^x - 1) v(dx) = (\lambda_u - \lambda_d)(e^{\alpha_u} - 1) = (\lambda_u - \lambda_d) A_u, \tag{5.11.7}$$

and

$$A \equiv A_u = e^{\alpha_u} - 1 \tag{5.11.8}$$

is the upward jump size. Note that since $\alpha_u \in (0, \log 2)$, then

$$0 < A_u < 1. \tag{5.11.9}$$

5.12 Maximization of Logarithmic Utility from Terminal Wealth

Because of the relatively simple nature of the Lévy measure for the Double Poisson process, we are able to explicity compute $G(\cdot)$. The other models do not allow for this, hence approximation methods are required. We now compute $G(\alpha)$.

$$\begin{aligned} G(\alpha) &= \int_{\mathbf{R}} \log(1 + \alpha \, (e^x - 1)) v_{\Pi(1,2)}(dx) \\ &= \int_{\mathbf{R}} \log(1 + \alpha \, (e^x - 1))(\lambda_u \, \delta_{\alpha_u}(dx) + \lambda_d \, \delta_{\alpha_d}(dx)) \\ &= \lambda_u \, \log(1 + \alpha \, (e^{\alpha_u} - 1)) + \lambda_d \, \log(1 + \alpha(e^{\alpha_d} - 1)) \end{aligned} \tag{5.12.1}$$

Since $\alpha_d = \log(2 - e^{\alpha_u})$ and $A_u = e^{\alpha_u} - 1$, then

$$
\begin{aligned}
G'(\alpha) &= \frac{\lambda_u(e^{\alpha_u} - 1)}{1 + \alpha(e^{\alpha_u} - 1)} + \frac{\lambda_d(e^{\alpha_d} - 1)}{1 + \alpha(e^{\alpha_d} - 1)} \\
&= \frac{\lambda_u(e^{\alpha_u} - 1)}{1 + \alpha(e^{\alpha_u} - 1)} + \frac{\lambda_d(1 - e^{\alpha_u})}{1 + \alpha(1 - e^{\alpha_u})} \\
&= \frac{\lambda_u A_u}{1 + \alpha A_u} - \frac{\lambda_d A_u}{1 - \alpha A_u} = \frac{\lambda_u}{a + \alpha} - \frac{\lambda_d}{a - \alpha},
\end{aligned}
$$

and

$$
G''(\alpha) = -\frac{\lambda_u}{(a + \alpha)^2} - \frac{\lambda_d}{(a - \alpha)^2},
$$

where a is the inverse upward jump size. Explicitly, $a \equiv a_u = \frac{1}{A_u}$ with $a > 1$ and $G(\alpha)$ is well defined for $\alpha \in (-a\,,\, a)$. The optimal portfolio π that maximizes logarithmic utility from terminal wealth for the symmetric Double Poisson market is given by

$$
\pi = g(\pi) = \frac{\mu - r}{\sigma^2} + \frac{1}{\sigma^2}\left[\frac{\lambda_u}{a + \pi} - \frac{\lambda_d}{a - \pi}\right], \tag{5.12.2}
$$

where $\pi \in (-a\,,\, a)$, $a > 1$, and $a = \frac{1}{A_u}$, $A_u = e^{\alpha_u} - 1$. We now give an analytic formula for the optimal portfolio for the symmetric Double Poisson market.

5.12.1 Analytic Solution of Optimal Portfolio

It is easy to show that equation (5.12.2) reduces to the cubic equation (5.12.3), and the optimal portfolio π, is its unique root, where

$$
\pi^3 + b\pi^2 + c\pi + d = 0 \tag{5.12.3}
$$

and

$$
b = -m = -\frac{(\mu - r)}{\sigma^2}, \quad c = -(\tilde{\lambda_u} + \tilde{\lambda_d} + a^2), \quad d = a(-\tilde{\lambda}_u - \tilde{\lambda}_d + ma), \tag{5.12.4}
$$

where

$$\tilde{\lambda_u} = \frac{\lambda_u}{\sigma^2}, \quad \tilde{\lambda_d} = \frac{\lambda_d}{\sigma^2}, \quad a > 1. \tag{5.12.5}$$

Theorem 5.20. *Let $\sigma > 0$ and $m > 0$. The optimal portfolio π that maximizes logarithmic utility from terminal wealth for the symmetric Double Poisson market is*

$$\pi = -\frac{\left(b + \sqrt[3]{\frac{L+I}{2}} + \sqrt[3]{\frac{L-I}{2}}\right)}{3}, \tag{5.12.6}$$

where a is the inverse average upward jump amplitude; b, c and d are given by (5.12.5) and

$$L = 2b^3 - 9bc + 27d. \tag{5.12.7}$$

$$K = b^2 - 3c. \tag{5.12.8}$$

$$I = \sqrt{L^2 - 4K^3}. \tag{5.12.9}$$

Proof. This is the standard unique root of a cubic equation with coefficients 1, b, c and d,and can be found on the internet. □

Remark 5.11. *We see that the optimal portfolio is the fixed point of $g(.)$, whence it may be obtained iteratively by the sequence:*

$$\pi_{(n+1)} = g(\pi_{(n)}). \tag{5.12.10}$$

We will employ this procedure to compute the optimal portfolio. Note also that equation (5.12.2) is cubic, and can also be easily solve using Newton's method or linear iteration.

Theorem 5.21. *Let $\{\pi_n\}$ be defined recursively by*

$$\pi_{(n+1)} = g(\pi_{(n)}), \quad \pi_0 \in (-a\,,\,a), \;\; a = \frac{1}{A_u}.$$

If there exists $0 \le \kappa < 1$ such that $|g'(\alpha)| \le \kappa$, then $\pi_{(n)} \to \pi$, where π is the optimal portfolio for the Double Poisson market and $g(\cdot)$ is defined by (5.12.2).

Proof. The optimal portfolio π is given exactly by the equation $\pi = g(\pi)$. Let $\pi_{(n+1)} = g(\pi_{(n)})$, with error $e_n = \pi_n - \pi$. Then there exists α_n between π and π_n such that

$$e_{n+1} = \pi_{n+1} - \pi = g(\pi_n) - g(\pi) = (\pi_n - \pi)g'(\alpha_n) = e_n g'(\alpha_n)$$

Thus, $|e_{n+1}| \le |e_n|\kappa$ and so $|e_n| \le \kappa^{n-1} \to 0$ as $n \to \infty$. Therefore $e_n = \pi_n - \pi \to 0$, whence $\pi_n \to \pi$. □

We now examine the instantaneous centralized moments for this market.

5.12.2 The Centralized Moments of Instantaneous Returns for the Double Poisson Market

The k-th instantaneous centralized moment of returns is given by:

$$M_k = \int_{\mathbf{R}} (e^x - 1)^k v_{\Pi(1,2)}(dx).$$

Lemma 5.6. *Let $k \in \mathbf{N}$ and M_k be the k-th instantaneous centralized moment of returns for a stock in the Double Poisson market, with parameters λ_u, λ_d, α_u. Let $A_u = e^{\alpha_u} - 1$ be the upward jump size. Then*

$$M_k = (\lambda_u + (-1)^k \lambda_d)\, A_u^k. \tag{5.12.11}$$

In particular,

$$M_{2k} = (\lambda_u + \lambda_d)\, A_u^{2k}, \tag{5.12.12}$$

$$M_{2k-1} = (\lambda_u - \lambda_d)\, A_u^{2k-1}. \tag{5.12.13}$$

Proof. Let $k \in \mathbf{N}$. Since $\alpha_d = \log(2 - e^{\alpha_u})$, we have

$$\begin{aligned} M_k &= \int_{\mathbf{R}} (e^x - 1)^k \upsilon_{\Pi(1,2)}(dx) = \int_{\mathbf{R}} (e^x - 1)^k (\lambda_u\, \delta_{\alpha_u}(dx) + \lambda_d\, \delta_{\alpha_d}(dx)) \\ &= \lambda_u\, (e^{\alpha_u} - 1)^k + \lambda_d\, (e^{\alpha_d} - 1)^k = \lambda_u\, (e^{\alpha_u} - 1)^k + \lambda_d\, (1 - e^{\alpha_u})^k \\ &= (\lambda_u + (-1)^k \lambda_d)(e^{\alpha_u} - 1)^k = (\lambda_u + (-1)^k\, \lambda_d)\, A_u^k. \end{aligned}$$

The particular results follow from k being even and odd, respectively. $\square$

Remark 5.12. *From equation (5.12.13), it is obvious that there is no skewness in the returns iff* $\lambda_u = \lambda_d$. *However, since* $M_4 = (\lambda_u + \lambda_d)\, A_u^4$ *is always positive, there is always excess Kurtosis, given by:*

$$KURT - 3 = \frac{M_4}{\sigma^2 + M_2} = \frac{(\lambda_u + \lambda_d)}{[(a\,\sigma)^2 + \lambda_u + \lambda_d]^2}, \tag{5.12.14}$$

where $a = \frac{1}{A_u}$. *Note also, that if* λ_u *or* λ_d *gets very large, there is no excess kurtosis and we essentially revert to GBM.*

Corollary 5.6. *Assume that the stock has Double Poisson dynamic. There exists* $0 < A_u < 1$, *and a bounded sequence* $\{a_k\}$, *such that*

$$M_k = a_k\, A^k. \tag{5.12.15}$$

Moreover, $M_k \to 0$ *as* $k \to \infty$.

Proof. For each $k \in \mathbf{N}$, set $a_k = (\lambda_u + (-1)^k \lambda_d)$. Then $\{a_k\}$ is a binary sequence, and hence, is bounded. Set $A = A_u = e^{\alpha_u} - 1$. From Lemma 5.6, $M_k = a_k\, A_u^k = a_k\, A^k$. Since $\alpha_u \in (0, \log 2)$ then $1 < e^{\alpha_u} < 2$, whence $0 < A_u = e^{\alpha_u} - 1 < 1$. Therefore

$$|M_k| = |a_k|\, A^k \leq (\lambda_u + \lambda_d)\, A^k \to 0,$$

as $k \to \infty$, whereby $M_k \to 0$. $\square$

We state without proof the following:

Corollary 5.7. *The approximating polynomial* $G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} M_j \frac{\alpha^j}{j}$ *converges absolutely to* $G(\alpha)$ *for all* $\alpha \in \mathbf{R}$.

5.12.3 Optimal Portfolios for Asymmetric Double Poisson Market

As with the other models, we now give the optimal portfolios for the investors in the asymmetric Double Poisson market.

Theorem 5.22. *Let* π *be the optimal portfolio that maximizes the expected logarithmic utililty from terminal wealth in the symmetric Double Poisson market with parameters* r, μ, σ^2, λ_u, λ_d, α_u, α_d. *Let* π^i *be the optimal portfolio for the* i*-th investor, where* $i \in \{0, 1\}$. *Then for each* $t \in [0, T]$, *there exists* η_t^i *between* π *and* π^i *, such that*

$$\pi_t^i = \pi_t + v_t^i \, \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i)})|}, \tag{5.12.16}$$

where $a = \frac{1}{A_u}$, $\; A_u = e^{\alpha_u} - 1$ *and*

$$G''(\alpha) = -\left[\frac{\lambda_u}{(a + \alpha)^2} + \frac{\lambda_d}{(a - \alpha)^2}\right]. \tag{5.12.17}$$

Under quadratic approximation of G,

$$\pi_t^i \approx \pi_t + v_t^i \frac{\sigma_t}{[\sigma_t^2 + (\lambda_u + \lambda_d)\, A_u^2]}, \tag{5.12.18}$$

where A_u is the upward jump size and the v^is are defined as in Chapter 3.

Proof. The result follows from Theorem 3.17. Under quadratic approximation, $G''(\pi) = -M_2 = -(\lambda_u + \lambda_d)A_u^2$, which is explicitly given in Lemma 5.6. □

5.13 The m-Double Poisson Jump Diffusion Market: $\Pi(m, 2)$

The m-Double Poisson jump diffusion market consist of a single stock S and a bond **B**, which earns the risk–free interest r and having price given by (ref3.0). The risky asset (stock) has log return dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t + dX_t, \tag{5.13.1}$$

where

$$X_t = \sum_{i=1}^{m} \alpha_{u_i}\, N^{u_i}(t) + \alpha_{d_i}\, N^{d_i}(t), \tag{5.13.2}$$

with

$$\alpha_{u_i} \in (0,\, \log 2), \quad \alpha_{d_i} = \log(2 - e^{\alpha_{u_i}}), \quad i = 1, 2, \cdots, m. \tag{5.13.3}$$

X is called the *m-Double Poisson* Lévy process with parameters α_{u_i}, α_{d_i}, λ_{u_i}, λ_{d_i}, where N^{u_i}, and N^{d_i} are independent Poisson processes with intensities λ_{u_i} and λ_{d_i} respectively, with

$$0 < \lambda_{d_i} \le \lambda_{u_i} < 1. \tag{5.13.4}$$

In this model, N^{u_i} controls the upward jumps, which have log amplitude $\alpha_{u_i} \in (0, \log 2)$, while N^{d_i} controls the downward jumps, with log amplitude α_{d_i}. As in the other models, B is standard Brownian motion; σ and $\mu - \frac{1}{2}\sigma^2$, are continuous components of total volatility and log returns, respectively. We denote a m-Double Poisson process by $\Pi(m,\, 2)$, with $\Pi(1,\, 2)$ denoting the Double Poisson.

The Intensities of $\Pi(m,\, 2)$

We expect smaller jumps to occur more frequently that larger jumps. In addition, we expect upward jumps to occur more frequently than downward jumps (large downward jumps represent market crashes, etc). Consequently, we have the following restrictions on this model:

$$1 \;>\; \lambda_{u_1} > \lambda_{u_2} >, \ldots, > \lambda_{u_m} > 0. \tag{5.13.5}$$

$$0 \;<\; \lambda_{d_1} < \lambda_{d_2} <, \ldots, < \lambda_{d_m} < 1. \tag{5.13.6}$$

$$\lambda_{d_1} \;\leq\; \lambda_{u_1}, \lambda_{d_2} \leq \lambda_{u_2}, \ldots, \lambda_{d_m} \leq \lambda_{u_m}. \tag{5.13.7}$$

The requirement that $0 < \lambda_{d_i} \leq \lambda_{u_i} < 1$, ensures that the Poisson processes do not interrupt the continuous Geometric Brownian Motion component of the stock's trajectory too often. Indeed, we expect $\lambda_{u_i} \leq \frac{1}{10}$ and $\lambda_{d_i} \leq \frac{1}{20}$. The greater the amplitude of the jump, the lower its frequency.

The Lévy Measure of $\Pi(m,\, 2)$

The Lévy measure for the Double Poisson process

$$X_t^i = \alpha_{u_i}\, N^{u_i}(t) + \alpha_{d_i}\, N^{d_i}(t), \tag{5.13.8}$$

is

$$v_i(dx) = \lambda_{u_i}\,\delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\,\delta_{\lambda_{d_i}}(dx), \quad i = 1, 2, \cdots, m, \tag{5.13.9}$$

where $\delta(\cdot)$ is the Dirac measure on $\mathcal{B}(\mathbf{R}-\{0\})$. It follows easily that the Lévy measure for the m-Double Poisson process: $X_t = \sum_{i=1}^{m} \alpha_{u_i}\,N^{u_i}(t) + \alpha_{d_i}\,N^{d_i}(t)$, is

$$v(dx) \equiv v_{\Pi(m,2)}(dx) \triangleq \sum_{i=1}^{m} v_i(dx) = \sum_{i=1}^{m} \lambda_{u_i}\,\delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\,\delta_{\lambda_{d_i}}(dx). \tag{5.13.10}$$

Let $N(t, A)$ be the Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}-\{0\})$ that counts the jumps of X in the time interval $(0, t)$. Then we can express X and its Lévy measure v, respectively, as $X_t = \int_{\mathbf{R}} x\,N(t, dx)$ and $v(dx) = \mathbf{E}[N(1, dx)]$. The first instantaneous centralized moment of returns is

$$M_1 = \int_{\mathbf{R}} (e^x - 1)\,v(dx) = \sum_{i=1}^{m} \int_{\mathbf{R}} (e^x - 1) v_i(dx) = \sum_{i=1}^{m} M_1(i),$$

where $M_1(i)$, the first instantaneous centralized return of the i-th double, is given by

$$\begin{aligned} M_1(i) &= \int_{\mathbf{R}} (e^x - 1) v_i(dx) = \int_{\mathbf{R}} (e^x - 1)(\lambda_{u_i}\delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\delta_{\lambda_{d_i}}(dx)) \\ &= \lambda_{u_i}(e^{\alpha_{u_i}} - 1) + \lambda_{d_i}(e^{\alpha_{d_i}} - 1) = \lambda_{u_i}A_i - \lambda_{d_i}A_i = (\lambda_{u_i} - \lambda_{d_i})A_i, \end{aligned}$$

where the upward jump size of the i-th double component is $A_i = e^{\alpha_{u_i}} - 1$. Thus

$$M_1 = \sum_{i=1}^{m} (\lambda_{u_i} - \lambda_{d_i}) A_i. \tag{5.13.11}$$

By applying Itô's change of variable formula to the log returns dynamic (5.13.1), we get the percentage returns dynamic:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t + \int_R (e^x - 1) N(dt, dx). \tag{5.13.12}$$

The total returns on the stock is

$$b_t = \mu_t + M_1 = \mu_t + \sum_{i=1}^{m} (\lambda_{u_i} - \lambda_{d_i}) A_i, \tag{5.13.13}$$

which is the sum of the continuous and expected returns from each of the jump doubles.

The $G(\cdot)$ Function for the $\Pi(m,\, 2)$ model.

We can easily compute the G function because of the relatively simple nature of the Lévy measure.

Theorem 5.23. *Let $v \equiv v_{\Pi(m,\,2)}$ be the Lévy measure for the m–Double Poisson market. There exists $a > 1$, such that for all $\alpha \in (-a\,,\, a)$,*

$$G(\alpha) = \log\left[\Pi_{i=1}^{m} (1 + \alpha A_i)^{\lambda_{u_i}} (1 - \alpha A_i)^{\lambda_{d_i}}\right], \tag{5.13.14}$$

where $0 < A_i < 1,\ \frac{1}{a_i} = A_i = e^{\alpha_{u_i}} - 1,$ *and* $a = \min_{i=1,\ldots,m}\{a_i\} > 1.$

Proof. Set $A_i = e^{\alpha_{u_i}} - 1$ and $a_i = \frac{1}{A_i}$ for each $i = 1, \ldots, m$.

$$\begin{aligned} G(\alpha) &= \int_{\mathbf{R}} \log(1 + \alpha(e^x - 1)) v(dx) \\ &= \sum_{i=1}^{m} \int_{\mathbf{R}} \log(1 + \alpha(e^x - 1)) v_i(dx) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \left[\lambda_{u_i} \log(1+\alpha(e^{\alpha_{u_i}}-1)) + \lambda_{d_i} \log(1+\alpha(e^{\alpha_{d_i}}-1))\right] \\
&= \sum_{i=1}^{m} \left[\lambda_{u_i} \log(1+\alpha(e^{\alpha_{u_i}}-1)) + \lambda_{d_i} \log(1-\alpha(e^{\alpha_{u_i}}-1))\right] \\
&= \sum_{i=1}^{m} \left[\lambda_{u_i} \log(1+\alpha A_i) + \lambda_{d_i} \log(1-\alpha A_i)\right] \\
&= \sum_{i=1}^{m} \log\left[(1+\alpha A_i)^{\lambda_{u_i}}(1-\alpha A_i)^{\lambda_{d_i}}\right],
\end{aligned}
$$

whence, provided the product exists,

$$
G(\alpha) = \log\left[\Pi_{i=1}^{m}(1+\alpha A_i)^{\lambda_{u_i}}(1-\alpha A_i)^{\lambda_{d_i}}\right]. \tag{5.13.15}
$$

Clearly $G(\alpha)$ exists, if for each $i = 1, \ldots, m$, $1+\alpha A_i > 0$, $\quad 1-\alpha A_i > 0$, which is equivalent to $a_1+\alpha > 0$, $a_i - \alpha > 0$. Thus $\alpha \in (-a_i\,,\, a_i)$ and $a_i > 1$, since $0 < A_i < 1$. Now take $a \stackrel{\triangle}{=} \min_{i=1,\ldots,m}\{a_i\}$. Then $a > 1$ and $G(\alpha)$ exists for all $\alpha \in (-a\,,\, a)$. □

Remark 5.13. *This model allows for borrowing* $(\pi > 1)$ *and short selling* $(\pi < 0)$, *since* $G(\pi)$ *exists in* $(-a\,,\, a)$, *where* $a > 1$.

5.14 Maximizing of Utility from Terminal Wealth

Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth at time $T > 0$, with $x > 0$ in initial investment. Unlike most of the other models considered in this dissertation, we will allow for the possibility of short–selling $(\pi < 0)$ by borrowing stocks, selling them, and investing the proceeds in the bank account. We also allow for the possibility of borrowing money at the risk–free interest rate, to buy stocks $(\pi > 1)$. The relaxing of the assumption $\pi \in [0, 1]$, is possible for this, and hence, the Double Poisson model, because $G(\alpha)$ exists in the interval $(-a\,,\, a) \supset (-1\,,\, 1)$. The optimal portfolio for the symmetric market follows.

Theorem 5.24. *Let the stock be driven by a symmetric m-Double Poisson market with parameters m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, where $i = 1, 2, \cdots, m$. The optimal portfolio π can be solved by Newtons method using (5.5.2)–(5.5.4) or recursively from the equation*

$$\pi = \frac{\mu - r}{\sigma^2} + \sum_{i=1}^{m} \left(\frac{\tilde{\lambda_{u_i}}}{a_i + \pi} + \frac{\tilde{\lambda_{d_i}}}{a_i - \pi} \right) \tag{5.14.1}$$

using (5.5.18)–(5.5.19), where

$$G'(\alpha) = \sigma^2 \sum_{i=1}^{m} \left[\frac{\tilde{\lambda_{u_i}}}{a_i + \alpha} + \frac{\tilde{\lambda_{d_i}}}{a_i - \alpha} \right] \tag{5.14.2}$$

and

$$G''(\alpha) = -\sigma^2 \sum_{i=1}^{m} \left[\frac{\tilde{\lambda_{u_i}}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda_{d_i}}}{(a_i - \alpha)^2} \right]. \tag{5.14.3}$$

Proof. By Theorem 3.8, the optimal portfolio is given by

$$\pi = \frac{\mu - r}{\sigma^2} + \frac{G'(\pi)}{\sigma^2},$$

provided $G'(\pi)$ exists. Since $G(\alpha)$ exists for all $\alpha \in (-a, a)$, where $a = min\{a_i\}$, then $G(\alpha)$ exists within said interval. From the proof of Theorem 5.23, we have

$$G(\alpha) = \sum_{i=1}^{m} \left[\lambda_{u_i} \log(1 + \alpha A_i) + \lambda_{d_i} \log(1 - \alpha A_i) \right],$$

whence

$$G'(\alpha) \;=\; \sum_{i=1}^{m} \left[\frac{\lambda_{u_i} A_i}{1 + \alpha A_i} + \frac{\lambda_{d_i} A_i}{1 - \alpha A_i} \right] = \sum_{i=1}^{m} \left[\frac{\lambda_{u_i}}{\frac{1}{A_i} + \alpha} + \frac{\lambda_{d_i}}{\frac{1}{A_i} - \alpha} \right],$$

which yields

$$\frac{G'(\alpha)}{\sigma^2} \;=\; \sum_{i=1}^{m} \left[\frac{\frac{\lambda_{u_i}}{\sigma^2}}{a_i + \alpha} + \frac{\frac{\lambda_{d_i}}{\sigma^2}}{a_i - \alpha} \right] = \sum_{i=1}^{m} \left[\frac{\tilde{\lambda_{u_i}}}{a_i + \alpha} + \frac{\tilde{\lambda_{d_i}}}{a_i - \alpha} \right],$$

and

$$\frac{G''(\alpha)}{\sigma^2} = -\sum_{i=1}^{m}\left[\frac{\tilde{\lambda_{u_i}}}{(a_i+\alpha)^2} + \frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2}\right].$$

Thus

$$\pi \;=\; \frac{\mu - r}{\sigma^2} + \sum_{i=1}^{m}\left(\frac{\tilde{\lambda_{u_i}}}{a_i+\pi} + \frac{\tilde{\lambda_{d_i}}}{a_i-\pi}\right) \triangleq g^{(m)}(\pi) \equiv g(\pi), \tag{5.14.4}$$

provided $\pi \in (-a\,,\, a)$. □

We give an equivalent form of Theorem 5.24 as follows. For each $i = 1, 2, \ldots, m$, define functions:

$$g,\ g_i : (-a\,,\, a) \to \mathbf{R},$$

by the prescription

$$g_i(\alpha) \;=\; \frac{\tilde{\lambda_{u_i}}}{a_i+\alpha} + \frac{\tilde{\lambda_{d_i}}}{a_i-\alpha}, \tag{5.14.5}$$

$$g(\alpha) \;=\; \pi_{Mer} + \sum_{i=1}^{m} g_i(\alpha), \tag{5.14.6}$$

$$\pi_{Mer} \;=\; \frac{\mu - r}{\sigma^2}. \tag{5.14.7}$$

As in the case of the Double Poisson market, this sequence will converge if $|g'(\alpha)| < 1$, for all $\alpha \in (-a, a)$. We have an analogous result for the m–Double Poisson market.

Theorem 5.25. *Let π be the optimal portfolio for the m–Double Poisson market. The sequence $\{\pi_n\}$, defined by*

$$\pi_{n+1} = g(\pi_n),\ \ \pi_0 = \pi_{Mer}$$

converges to π, if $g'(\alpha) > -1$ for all $\alpha \in (-a\,,\, a)$.

Proof. Convergence is assured if $-1 < g'(\alpha) < 1$. From equation (5.14.4), we have for all $\alpha \in (-a\,,\,a)$, where $a = min\{a_i\}$, that

$$\begin{aligned} g'(\alpha) &= \frac{G''(\alpha)}{\sigma^2} \\ &= \frac{d}{d\alpha}\left(\sum_{i=1}^{m}\left[\frac{\tilde{\lambda_{u_i}}}{a_i+\alpha}+\frac{\tilde{\lambda_{d_i}}}{a_i-\alpha}\right]\right) \\ &= -\sum_{i=1}^{m}\left[\frac{\tilde{\lambda_{u_i}}}{(a_i+\alpha)^2}+\frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2}\right], \end{aligned} \tag{5.14.8}$$

which is striclty negative. Thus for all $\alpha \in (-a, a)$,

$$g'(\alpha) < 0.$$

This reduces the convergence condtion to $g'(\alpha) > -1$. □

We have the following corollary which give an equivalent condition in terms of the market parameters.

Corollary 5.8. *If*

$$\sum_{i=1}^{m}\left[\frac{\tilde{\lambda_{u_i}}}{(a_i+\alpha)^2}+\frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2}\right] < 1, \tag{5.14.9}$$

for all $\alpha \in (-a, a)$*, then* $\pi_n \to \pi$ *as* $n \to \infty$*.*

Proof. This follows directly from equation (5.14.8) and the fact that $g'(\alpha) > -1$. □

If no short–selling or borrowing from the bank account is allowed, then α is restricted to the interval $[0, 1]$, and so if

$$\max_{\alpha\in[0,1]}\sum_{i=1}^{m}\left[\frac{\tilde{\lambda_{u_i}}}{(a_i+\alpha)^2}+\frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2}\right] < 1, \tag{5.14.10}$$

we get convergence. This leads to the following:

Theorem 5.26. *Let $a = min\{a_i\}$ and $\alpha \in [-1\,,\,1]$ in the m–Double Poisson market, then*

$$|g'(\alpha)| < \frac{2m\,\lambda_{u_1}}{\sigma^2(a-1)^2}. \tag{5.14.11}$$

Moreover, if $\lambda_{u_1} < \frac{\sigma^2\,(a-1)^2}{2m}$ then $\pi_n \to \pi$, as $n \to \infty$.
Equivalently, if $\lambda_{u_1} < \frac{\sigma^2\,(1-A)^2}{2mA^2}$, we have convergence, where $A = \frac{1}{a}$.

Proof. Assume that $\alpha \in [0,\,1]$. Then $a_i \geq a$, where $a = min\{a_i\}$. Let

$$g_i(\alpha) = \frac{\tilde{\lambda_{u_i}}}{a_i+\alpha} - \frac{\tilde{\lambda_{d_i}}}{a_i-\alpha}.$$

Then

$$|g_i'(\alpha)| = \frac{\tilde{\lambda_{u_i}}}{(a_i+\alpha)^2} + \frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2} \leq \frac{\tilde{\lambda_{u_i}}}{a_i^2} + \frac{\tilde{\lambda_{d_i}}}{(a_i-\alpha)^2} \leq \frac{\tilde{\lambda_{u_i}}}{a^2} + \frac{\tilde{\lambda_{d_i}}}{(a-\alpha)^2}$$

$$\leq \frac{\tilde{\lambda_{u_i}}}{(a-1)^2} + \frac{\tilde{\lambda_{d_i}}}{(a-1)^2} \leq \frac{2\tilde{\lambda_{u_i}}}{(a-1)^2} \leq \frac{2\tilde{\lambda_{u_1}}}{(a-1)^2}.$$

Thus

$$|g'(\alpha)| = |\sum_{i=1}^m g_i'(\alpha)| \leq \sum_{i=1}^m |g_i'(\alpha)| \leq \sum_{i=1}^m \frac{2\tilde{\lambda_{u_1}}}{(a-1)^2} = \frac{2m\tilde{\lambda_{u_1}}}{(a-1)^2}$$

By symmetry, if $\alpha \in [-1,0]$, then $-\alpha \in [0,1]$, and by replacing α by $-\alpha$, we get

$$|g'(\alpha)| = \sum_{i=1}^m \frac{\tilde{\lambda_{u_i}}}{(a_i-\alpha)^2} + \frac{\tilde{\lambda_{d_i}}}{(a_i+\alpha)^2} \leq \frac{2\tilde{\lambda_{u_1}}}{(a-1)^2}.$$

Therefore, if $\alpha \in [-1,1]$

$$|g'(\alpha)| < \frac{2m\lambda_{u_1}}{\sigma^2(a-1)^2}.$$

Convergence of the sequence is assured if $|g'(\alpha)| < 1$, for all $\alpha \in [-1,1]$. This obtains, if

$$\frac{2m\lambda_{u_1}}{\sigma^2(a-1)^2} < 1.$$

That is, if $\lambda_{u_1} < \frac{\sigma^2(a-1)^2}{2m}$. $\square$

Theorem 5.26 tells us that to ensure convergence in $[-1, 1]$, we must select the largest intensity to satisfy the condition

$$\lambda_{u_1} < \frac{\sigma^2(a-1)^2}{2m} = \frac{\sigma^2(1-A)^2}{2mA^2},$$

which shows that $\lambda_{u_i} \to 0$ as $m \to \infty$, and consequently $\lambda_{d_i} \to 0$. This proves that when $m \to \infty$ we revert to GBM.

Note also, that to ensure convergence, the intensities λ_{u_i}, of upward jumps must all satisfy the condition

$$\lambda_{u_i} < \frac{\sigma^2(a-1)^2}{2m}.$$

Since $\lambda_{u_1} < 1$, to calibrate our model we can choose m such that $2m > \sigma^2(a-1)^2$. Likewise, for fixed m, the volatility σ^2, must obey the condition $\sigma^2 < \frac{2m}{(a-1)^2}$.

5.15 Instantaneous Centralized Moments of Returns for the $\Pi(m, 2)$ Market

The k–th instantaneous centralized moment of returns for the m–Double Poisson market is

$$M_k^{(m)} \equiv M_k = \int_{\mathbf{R}} (e^x - 1)^k v_{\Pi(m,2)}(dx). \tag{5.15.1}$$

We have the following result.

Theorem 5.27. *Let M_k be the k–th instantaneous centralized moment of returns for any Lévy market with dynamic (5.1.2). The total instantaneous variance at time $t \in [0, T]$ is*

$$VAR = \sigma_t^2 + M_2. \tag{5.15.2}$$

Moreover, the Skewness and excess Kurtosis of the instantaneous returns at time t are respectively:

$$SKEW = \frac{M_3}{(\sigma_t^2 + M_2)^{\frac{3}{2}}}. \tag{5.15.3}$$

$$KURT - 3 = \frac{M_4}{(\sigma_t^2 + M_2)^2}. \tag{5.15.4}$$

Proof. The results follow from the variance of $\frac{dS_t}{S_t}$, which is given in Chapter 2, as $\sigma_t^2 + M_2$. □

Remark 5.14.

Note that when $\frac{dS_t}{S_t}$ is driven only by Brownian motion with drift, it has zero skewness and a Kurtosis of 3. Therefore skewness and excess kurtosis in the model comes only from the jump component. This result is general and applies to any model with diffusive and jump components.

We now give the M_ks in terms of market parameters.

Theorem 5.28. *Let $M_k^{(m)} \equiv M_k$ be the k–th centralized moment of the instantaneous returns for a stock in a m–Double Poisson market with parameters m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, where $i = 1, 2, \ldots, m$. Then*

$$M_k = \sum_{i=1}^{m} M_k(i),$$

$$M_k(i) = (\lambda_{u_i} + (-1)^k \lambda_{d_i}) A_i^k. \tag{5.15.5}$$

where A_i is the upward jump size in the i–th interval, or the i-th jump size.

Proof. Set $A_i = e^{\alpha_{u_i}} - 1$. Then

$$\begin{aligned} M_k(i) &= \int_{\mathbf{R}} (e^x - 1)^k (\lambda_{u_i} \delta_{\lambda_{u_i}}(dx) + \lambda_{d_i} \delta_{\lambda_{d_i}}(dx)) = \lambda_{u_i} (e^{\alpha_{u_i}} - 1)^k + \lambda_{d_i} (e^{\alpha_{d_i}} - 1)^k \\ &= \lambda_{u_i} (e^{\alpha_{u_i}} - 1)^k + \lambda_{d_i} (1 - e^{\alpha_{u_i}})^k = (\lambda_{u_i} + (-1)^k \lambda_{d_i}) A_i^k. \end{aligned}$$

Since $v_{\Pi(m,2)}(dx) = \sum_{i=1}^m v_i(dx)$, then

$$\begin{aligned} M_k &= \int_{\mathbf{R}} (e^x - 1)^k v_{\Pi(m,2)}(dx) = \int_{\mathbf{R}} (e^x - 1)^k (\sum_{i=1}^m v_i(dx)) \\ &= \int_{\mathbf{R}} \sum_{i=1}^m (e^x - 1)^k v_i(dx) = \sum_{i=1}^m \int_{\mathbf{R}} (e^x - 1)^k v_i(dx) = \sum_{i=1}^m M_k(i). \square \end{aligned}$$

A direct consequence of the last result is:

Corollary 5.9. *Let $M_k^{(m)} \equiv M_k$ be the k–th centralized moment of the instantaneous returns for a stock in a m–Double Poisson market with parameters m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i} where $i = 1, 2, \ldots, m$. Then*

$$M_{2k} = \sum_{i=1}^m (\lambda_{u_i} + \lambda_{d_i})\, A_i^{2k}, \tag{5.15.6}$$

$$M_{2k+1} = \sum_{i=1}^m (\lambda_{u_i} - \lambda_{d_i})\, A_i^{2k+1}, \tag{5.15.7}$$

where A_i is the upward jump size in the i–th interval, or the i-th jump size. Moreover, the Variance, Skewness, and Excess Kurtosis of the instantaneous returns $\frac{dS_t}{S_t}$, are respectively

$$VARIANCE = \sigma_t^2 + \sum_{i=1}^m \lambda_i^+ A_i^2, \tag{5.15.8}$$

$$SKEWNESS = \frac{\sum_{i=1}^m \lambda_i^- A_i^3}{(\sigma_t^2 + \sum_{i=1}^m \lambda_i^+ A_i^2)^{\frac{3}{2}}}, \tag{5.15.9}$$

$$KURT - 3 = \frac{\sum_{i=1}^m \lambda_i^+ A_i^4}{(\sigma_t^2 + \sum_{i=1}^m \lambda_i^+ A_i^2)^2}, \tag{5.15.10}$$

where

$$\lambda_i^+ \triangleq \lambda_{u_i} + \lambda_{d_i}, c \quad \textit{and} \quad \lambda_i^- \triangleq \lambda_{u_i} - \lambda_{d_i}. \tag{5.15.11}$$

Remark 5.15.

Observe from (5.15.9) that since $\lambda_{u_i} \geq \lambda_{d_i}$, then we always have zero or positive *skewness. In fact, if $\lambda_{u_i} > \lambda_{d_i}$ for some $i \leq m$, we have positive skewness, and the returns are skewed to the right. This leads to*

Theorem 5.29. *The m–Double Poisson market has stock returns that always have excess kurtosis, and are either symmetrically distributed or positively skewed. Further, as $m \to \infty$, skewness and kurtosis go to zero iff $\sum_{i=1}^{m} \lambda_{u_i} A_i^2 \to \infty$.*

Proof. Fix $m \in \mathbf{N}$. Then $\lambda_{u_i} \geq \lambda_{d_i}$ and $0 < A_i < 1$ for each $i \leq m$. It follows immediately from (5.15.9), that $Skewness > 0$, unless $\lambda_{u_i} = \lambda_{d_i}$, in which case, $Skewness = 0$. Likewise from (5.15.10) $\lambda_{u_i} + \lambda_{d_i} > 0$ implies that $KURT - 3 > 0$. With $\lambda_i^{\pm} = \lambda_{u_i} \pm \lambda_{d_i}$, assume that

$$\sum_{i=1}^{m} \lambda_{u_i} A_i^2 \longrightarrow \infty.$$

Then

$$SKEW = \frac{\sum_{i=1}^{m} \lambda_i^- A_i^3}{(\sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2)^{\frac{3}{2}}} \leq \frac{\sum_{i=1}^{m} \lambda_i^- A_i^2}{VAR^{\frac{3}{2}}} \leq \frac{\sum_{i=1}^{m} \lambda_i^+ A_i^2}{VAR^{\frac{3}{2}}} \leq \frac{(\sigma^2 + \sum_{i=1}^{m} \lambda_i^- A_i^2)}{VAR^{\frac{3}{2}}}$$

$$= \frac{VAR}{VAR^{\frac{3}{2}}} = \frac{1}{VAR^{\frac{1}{2}}} = \frac{1}{(\sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{\sum_{i=1}^{m} \lambda_i^+ A_i^2}}$$

$$= \frac{1}{\sqrt{\sum_{i=1}^{m} \lambda_{u_i} A_i^2}} \longrightarrow 0$$

By equation(5.15.10),

$$KURT - 3 = \frac{\sum_{i=1}^{m} \lambda_i^+ A_i^4}{(\sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2)^2)} < \frac{\sum_{i=1}^{m} \lambda_i^+ A_i^2}{VAR^2} < \frac{\sigma^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2}{VAR^2}$$

$$= \frac{VAR}{VAR^2} = \frac{1}{VAR} = \frac{1}{\sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2}$$

$$< \frac{1}{\sum_{i=1}^{m} \lambda_i^+ A_i^2} \leq \frac{1}{\sum_{i=1}^{m} \lambda_{u_i} A_i^2} \longrightarrow 0.$$

Conversely, suppose $SKEW \to 0$, and let $0 < \sum_{i=1}^{\infty} \lambda_{u_i} A_i^2 < \infty$. Then since

$\lambda_{u_i} \leq \lambda_{d_i}$, we have $\sum_{i=1}^{\infty} \lambda_i^+ A_i^2 < \infty$. Therefore, $VAR < \infty$. Since

$$SKEW = \frac{\sum_{i=1}^{m} \lambda_i^- A_i^3}{(\sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2)^{\frac{3}{2}}} = \frac{\sum_{i=1}^{m} \lambda_i^- A_i^3}{VAR^{\frac{3}{2}}},$$

then as $m \longrightarrow \infty$,

$$\sum_{i=1}^{m} \lambda_i^- A_i^3 = VAR^{\frac{3}{2}} SKEW \longrightarrow 0.$$

Since $\lambda_i^- A_i^3 \geq 0$, then $\lambda_i^- A_i^3 = 0$ for all i. But $A_i \neq 0$. Thus $\lambda_{u_i} = \lambda_{d_i}$, for all i. Consequently, $SKEW = 0$ for each m. But since $SKEW \neq 0$ for at lease one m, we have a contradiction. Thus

$$\sum_{i=1}^{m} \lambda_{u_i} A_i^2 \longrightarrow \infty.$$

Similarly, if $0 < \sum_{i=1}^{\infty} \lambda_{u_i} A_i^2 < \infty$, and $KURT - 3 \longrightarrow 0$, then $VAR < \infty$. Thus

$$\sum_{i=1}^{m} \lambda_i^+ A_i^4 = (KURT - 3)(VAR)^2 \longrightarrow 0.$$

Therefore, $\lambda_i^+ A_i^4 = 0$ for all i, whence $\lambda_{u_i} = \lambda_{d_i} = 0$, since $A_i \neq 0$. Therefore $\sum_{i=1}^{\infty} \lambda_{u_i} A_i^2 = 0$, which contradicts $\sum_{i=1}^{\infty} \lambda_{u_i} A_i^2 > 0$. Thus $\sum_{i=1}^{m} \lambda_{u_i} A_i^2 \longrightarrow \infty$. $\square$

Theorem 5.30. *For the m–Double Poisson market with parameters* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, *where* $i = 1, 2, \ldots, m$, *there exists* $a > 1$ *such that*

$$M_k \leq \frac{2m\, \lambda_{u_1}}{a^k} \leq \frac{2m}{a^k}. \tag{5.15.12}$$

Moreover, as $k \longrightarrow \infty$

$$M_k \longrightarrow 0. \tag{5.15.13}$$

Proof. From Theorem 5.28, $A_i \in (0,\, 1)$ and

$$M_k = \sum_{i=0}^{m} (\lambda_{u_i} + (-1)^k \lambda_{d_i}) A_i^k.$$

Thus

$$|M_k| \leq \sum_{i=0}^{m} (\lambda_{u_i} + \lambda_{d_i}) A_i^k \leq \sum_{i=0}^{m} 2\lambda_{u_i} A_i^k \leq 2\lambda_{u_1} \sum_{i=1}^{m} A_i^k.$$

But $A_i = \frac{1}{a_i}$, where $a = \min\{a_i\}$. Thus $A_i = \frac{1}{a_i} \leq \frac{1}{a}$, for all $i = 1, 2, \ldots, m.$. Hence

$$|M_k| < 2\lambda_{u_1} \sum_{i=1}^{m} \frac{1}{a^k} = \frac{2m\lambda_{u_1}}{a^k},$$

from which $M_k \longrightarrow 0$ when $k \longrightarrow \infty$. $\square$

Remark 5.16. *Since* $1 > \lambda_{u_i} \geq \lambda_{d_i}$*, we see that* $\lambda_{u_i} + (-1)^k \lambda_{d_i} \geq 0$*. Thus for each* $k \leq m$,

$$0 \leq M_k < \frac{2m}{a^k}.$$

However, if we relax the condition that $\lambda_{u_i} \geq \lambda_{d_i}$ *then* (5.15.12) *still holds.*

Optimal Portfolios for Asymmetric m–Double Poisson Market.

We now give the relation between the symmetric and asymmeric optimal portfolios. As with the other models, we give the optimal portfolios for the investors in the asymmetric Doule Poisson market.

Theorem 5.31. *Let* π *be the optimal portfolio that maximizes the expected logarithmic utililty from terminal wealth in the symmetric Double Poisson market with parameters* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}*, where* $i = 1, 2, \ldots, m$*. Let* π^i *be the optimal portfolio for the i-th investor,* $i \in \{0, 1\}$*. Then for each* $t \in [0, T]$ *there exists* η_t^i *between* π *and* π^i *such that*

$$\pi_t^i = \pi_t + v_t^i \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i)})|}, \tag{5.15.14}$$

where $a_i = \frac{1}{A_i}$, $\quad A_i = e^{\alpha_{u_i}} - 1$ *and*

$$G''(\alpha) = -\sum_{i=1}^{m}\left[\frac{\lambda_{u_i}}{(a_i+\alpha)^2} + \frac{\lambda_{d_i}}{(a_i-\alpha)^2}\right].$$

Under quadratic approximation of G,

$$\pi_t^i \approx \pi_t + v_t^i \frac{\sigma_t}{[\sigma_t^2 + \sum_{i=0}^{m}(\lambda_{u_i} + \lambda_{d_i})\, A_i^2]}, \tag{5.15.15}$$

where A_i *is the i–th upward jump size, and the* v^i*s are defined as in Chapter 3.*

Proof. The result follows from Theorem 3.17. Under quadratic approximation $G''(\pi) = -M_2 = -\sum_{i=0}^{m}(\lambda_{u_i} + \lambda_{d_i})\, A_i^2$, which is explicitly given in Corollary 5.9. □

Example 5.1 (**Optimal Portfolio of Symmetric 4–Double Poisson Market**).

Input parameters*:* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, *where* $i = 1, 2, \ldots, 4$

Set $m = 4$.

$\pi_{Mer} = \frac{\mu - r}{\sigma^2}$.

$\tilde{\lambda}_{u_i} = \frac{\lambda_{u_i}}{\sigma^2}$, $\quad \tilde{\lambda}_{d_i} = \frac{\lambda_{d_i}}{\sigma^2}$.

$A_i = e^{\lambda_{u_i}} - 1$.

$a_i = \frac{1}{A_i}$, $\quad a = min\{a_i\}$.

For $\pi \in (-a\,,\, a)$, *set*

$$g(\pi) \equiv g^{(4)}(\pi) = \sum_{i=1}^{4} \frac{\tilde{\lambda}_{u_i}}{a_i + \alpha} + \frac{\tilde{\lambda}_{d_i}}{a_i - \alpha}.$$

1. *Set error* $\epsilon = 0.5 \times 10^{-d}$, *where* $d \in \{4, 5, 6, 7\}$
2. *Set* $\pi_0 = \pi_{Mer}$
3. *Generate a sequence* $\{\pi_n\}$ *by the prescription* $\pi_{n+1} = g(\pi_n)$
4. *Stop if* $|\pi_{n+1} - \pi_n| \le \epsilon$ *and take optimal to be* $\pi \approx \pi_{n+1}$.
5. *Otherwise,set n=n+1, and go to 3.*

Chapter 6

Numerical Approximation of Optimal Portfolios Using Quasi-Centralized Moments

In this chapter, we construct two series expansions to approximate the function $G'(\pi)$, which is partitioned as the sum of two independent derivatives. Each series is convergent, and is built from quasi–centralized moments defined on the half–line which, like the instantaneous centralized moments, are dependents of the Lévy measure. With this in hand, we then approximate the optimal portfolio and maximum expected utility from terminal wealth for various models, including the Kou jump–diffusion, Variance Gamma, CGMY and Generalized Tempered Stable markets.

6.1 Series Expansion of $G'(\pi)$

Recall that

$$\begin{aligned} G'(\pi) &= \int_{\mathbf{R}} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx \qquad (6.1.1) \\ &= \int_0^{\infty} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(dx) + \int_{-\infty}^0 \frac{e^x - 1}{1 + \pi(e^x - 1)} v(dx). \end{aligned}$$

Thus

$$G'(\pi) = G'_{+}(\pi) + G'_{-}(\pi), \tag{6.1.2}$$

where

$$G'_{\pm}(\pi) \triangleq \int_{\mathbf{R}_{\pm}} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx. \tag{6.1.3}$$

6.1.1 Expansion of $G'_{-}(\pi)$

$$G'_{-}(\pi) = \int_{(-\infty,\,0)} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx$$

Since $x \in \mathbf{R}_{-} = (-\infty,\, 0)$, then $|e^x - 1| < 1$, whence $|\pi(e^x - 1)| < |\pi|$.

If $\quad |\pi| \leq 1$, then

$$\frac{e^x - 1}{1 + \pi(e^x - 1)} = (e^x - 1) \sum_{k=0}^{\infty} (-1)^k \pi^k (e^x - 1)^k = -\sum_{k=0}^{\infty} \pi^k (1 - e^x)^{k+1},$$

whence

$$\begin{aligned} G'_{-}(\pi) &= \int_{\mathbf{R}_{-}} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx \\ &= -\sum_{k=0}^{\infty} \pi^k \int_{-\infty}^{0} (1 - e^x)^{k+1} v(dx) nonumber && (6.1.4) \\ &= -\sum_{k=0}^{\infty} \pi^k M^{-}_{k+1} && (6.1.5) \end{aligned}$$

where the negative quasi–centralized moment is

$$M^{-}_{k} \triangleq \int_{-\infty}^{0} (1 - e^x)^k v(dx). \tag{6.1.6}$$

Remark 6.1. *Note that for each $k \in \mathbf{N}$, the negative quasi–centralized moment M^{-}_{k} is always positive.*

6.1.2 Series Expansion of $G'_+(\pi)$

$$G'_+(\pi) \;=\; \int_{\mathbf{R}_+} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx = -\int_0^\infty \frac{(1 - e^{-x})v(x)}{(e^{-x} + \pi(1 - e^{-x}))} dx. \quad (6.1.7)$$

Put $x = -y$ and $\beta = 1 - \pi$. Then

$$G'_+(\pi) = -\int_0^{-\infty} \frac{(1 - e^y)v(-y)}{(e^y + \pi(1 - e^y))} dy = \int_0^{-\infty} \frac{(e^y - 1)v(-y)}{(1 + (1 - \pi)(e^y - 1))} dy$$

$$= \int_0^{-\infty} \frac{(e^y - 1)v(-y)}{(1 + \beta(e^y - 1))} dy = \int_{-\infty}^0 \frac{(1 - e^y)v(-y)}{(1 - \beta(1 - e^y))} dy.$$

Since $y \in (-\infty, 0)$, then $|e^y - 1| < 1$. If $|\beta| \le 1$, then

$$\begin{aligned} G'_+(\pi) &= \int_{-\infty}^0 \sum_{k=0}^\infty (-1)^k(-\beta)^k(1 - e^y)^{k+1} v(-y)dy \\ &= \sum_{k=0}^\infty \beta^k \int_{-\infty}^0 (1 - e^y)^{k+1} v(-y)dy \\ &= \sum_{k=0}^\infty \beta^k M_{k+1}^+, \end{aligned} \quad (6.1.8)$$

where

$$M_k^+ \triangleq \int_{-\infty}^0 (1 - e^y)^k v(-y)dy. \quad (6.1.9)$$

By putting $y = -x$ in the last equation, we get

$$M_k^+ \;=\; -\int_\infty^0 (1 - e^{-x})^k v(x)dx = \int_0^\infty (1 - e^{-x})^k v(x)dx. \quad (6.1.10)$$

Definition 6.1 (**Quasi-Centralized Moments of Returns**).

The Quasi-Centralized Moments of Returns are defined by:

$$M_k^\pm \triangleq \int_{\mathbf{R}_\pm} (1 - e^{-|x|})^k v_\pm(x)dx, \quad (6.1.11)$$

where the Levy density $v(\cdot)$ is explicitly given as:

$$v(x) = \begin{cases} v_-(x) & \textit{if } x \in (-\infty, 0)\ ; \\ v_+(x) & \textit{if } x \in (0, \infty). \end{cases}$$

Note that $M_k^{\pm}$ are always positive numbers.

Theorem 6.1. *Let $\pi \in [0,1]$. Then*

$$G'(\pi) = G'_-(\pi) + G'_+(\pi), \tag{6.1.12}$$

where

$$G'_-(\pi) = -\sum_{k=0}^{\infty} \pi^k M_{k+1}^-, \quad |\pi| \le 1, \tag{6.1.13}$$

$$G'_+(\pi) = \sum_{k=0}^{\infty} (1-\pi)^k M_{k+1}^+, \quad |1-\pi| \le 1. \tag{6.1.14}$$

Explicitly, for all $\pi \in [0,\, 1]$

$$G'(\pi) = -\sum_{k=0}^{\infty} \pi^k\, M_{k+1}^- + \sum_{k=0}^{\infty} (1-\pi)^k\, M_{k+1}^+. \tag{6.1.15}$$

6.2 Application to General Models

This section contains the main result which will be applied to general Lévy markets in the sequel.

Theorem 6.2 (**Main**).

For each $\pi \in [0,1]$,

$$G'(\pi) = \sum_{k=0}^{\infty} (1-\pi)^k\, M_{k+1}^+ - \sum_{k=0}^{\infty} \pi^k\, M_{k+1}^-, \tag{6.2.1}$$

and

$$G''(\pi) = -\sum_{k=1}^{\infty} k\,(1-\pi)^{k-1}\,M_{k+1}^{+} - \sum_{k=0}^{\infty} k\,\pi^{k-1}\,M_{k+1}^{-}, \tag{6.2.2}$$

where $M_k^{\pm}$ *is given by (6.1.11). The optimal portfolio* π *that maximizes the logarithmic utility from terminal wealth for a stock with dynamic (5.1.2): is given by the equation:*

$$\sigma^2\,\pi = \mu - r + \sum_{k=0}^{\infty}(1-\pi)^k\,M_{k+1}^{+} - \sum_{k=0}^{\infty}\pi^k\,M_{k+1}^{-}, \tag{6.2.3}$$

which is approximated by Newtons method or by recursion from the equation:

$$\sigma^2\,\pi = \mu - r + \sum_{k=0}^{m_2}(1-\pi)^k\,M_{k+1}^{+} - \sum_{k=0}^{m_1}\pi^k\,M_{k+1}^{-}, \tag{6.2.4}$$

where m_1 *and* m_2 *are chosen truncation points.*

Proof. Equations (6.2.1) and (6.2.2) follow from Theorem 6.1. A stock with dynamic (**??**) has optimal portfolio π given by Theorem 3.8 as $\sigma^2\pi = \mu - r + G'(\pi)$, which yields (6.2.3). By truncating the series $\sum_{k=0}^{\infty}(1-\pi)^k\,M_{k+1}^{+}$ and $\sum_{k=0}^{\infty}\pi^k\,M_{k+1}^{-}$ respectively, at the cut-off points m_1 and m_2 respectively, we get (6.2.4). □

We now apply Theorem 6.2 to various models.

6.3 Kou Jump Diffusion Model

In this case the Lévy density is given in terms of the intensity rate λ.

$$\begin{aligned} \frac{v(x)}{\lambda} &= v_-(x) I_{\{x<0\}} + v_+(x) I_{\{x>0\}} \\ &= f_{kou}(x) \\ &= q\,\eta_2\,e^{-\eta_2\,|x|} I_{\{x<0\}} + p\,\eta_1\,e^{-\eta_1\,x} I_{\{x>0\}}, \end{aligned} \tag{6.3.1}$$

where $\eta_1 \geq 1$, $\eta_2 > 0$, $p+q=1$, $p,\ q\ \geq 0$. Changing x to $-x$, yields:

$$\begin{aligned} M_k^- &= \int_{-\infty}^{0} (1-e^x)^k v_-(x)dx = \lambda\, q\eta_2 \int_{-\infty}^{0} (1-e^x)^k e^{-\eta_2|x|}dx \\ &= \lambda\, q\eta_2 \int_0^\infty (1-e^{-x})^k e^{-\eta_2 x}dx = \lambda\, q\eta_2 \int_0^\infty (e^x-1)^k e^{-(\eta_2+k)x}dx. \end{aligned}$$

Setting $t = e^x - 1$, we get

$$\begin{aligned} M_k^- &= \lambda\, q\eta_2 \int_0^\infty t^k (t+1)^{-(\eta_2+k+1)}dt \\ &= \lambda\, q\eta_2 \mathbf{B}(k+1, \eta_2). \end{aligned} \tag{6.3.2}$$

Since

$$\mathbf{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{6.3.3}$$

we have

$$\begin{aligned} \mathbf{B}(k+1,\eta_2) &= \frac{\Gamma(k+1)\,\Gamma(\eta_2)}{\Gamma(k+1+\eta_2)} = \frac{k!\,\Gamma(\eta_2)}{(k+\eta_2)\,\Gamma(k+\eta_2)} \\ &= \frac{k!\,\Gamma(\eta_2)}{\Pi_{j=0}^k (j+\eta_2)\,\Gamma(\eta_2)} = \frac{k!}{\Pi_{j=0}^k (j+\eta_2)}. \end{aligned}$$

Similarly, we have from equation (6.1.11) that

$$\begin{aligned} M_k^+ &= \int_0^\infty (1-e^{-x})^k v_+(x)dx = \lambda\, p\, \eta_1 \int_0^\infty (1-e^{-x})^k e^{-\eta_1 x}dx \\ &= \lambda\, p\, \eta_1\, \mathbf{B}(k+1,\eta_1) = \frac{\lambda\, p\, k!}{\Pi_{j=0}^k (j+\eta_1)}. \end{aligned} \tag{6.3.4}$$

We now have the following:

Theorem 6.3. *For the Kou model with parameters $p \geq 0$, $q \geq 0$, $\lambda > 0$, $\eta_1 > 1$ and*

$\eta_2 \geq 0$,

$$G'(\pi) = -\lambda\, q\, \eta_2 \sum_{k=0}^{\infty} \pi^k\, \mathbf{B}(k+2, \eta_2) + \lambda\, p\, \eta_1 \sum_{k=0}^{\infty} (1-\pi)^k\, \mathbf{B}(k+2, \eta_1),$$

and

$$G''(\pi) = -\lambda\, q\, \eta_2 \sum_{k=1}^{\infty} k\, \pi^k\, \mathbf{B}(k+2, \eta_2) - \lambda\, p\, \eta_1 \sum_{k=1}^{\infty} k\, (1-\pi)^k\, \mathbf{B}(k+2, \eta_1). \quad (6.3.5)$$

Explicitly,

$$G'(\pi) = -\lambda q \sum_{k=0}^{\infty} \pi^k \frac{k!}{\Pi_{j=0}^{k}(j+\eta_2)} + \lambda p \sum_{k=0}^{\infty} (1-\pi)^k \frac{k!}{\Pi_{j=0}^{k}(j+\eta_1)} \quad (6.3.6)$$

and

$$G''(\pi) = -\lambda q \sum_{k=0}^{\infty} k\, \pi^{k-1} \frac{k!}{\Pi_{j=0}^{k}(j+\eta_2)} - \lambda p \sum_{k=1}^{\infty} k\, (1-\pi)^{k-1} \frac{k!}{\Pi_{j=1}^{k}(j+\eta_1)} \quad (6.3.7)$$

Proof. This follows from Theorem 6.1 with $M_k^{\pm}$ given by (6.1.11). □

We now give the optimal portfolio for the Kou model.

Theorem 6.4. *The optimal portfolio π for the Kou jump–diffusion market with parameters r, μ, σ^2, λ, p, q, η_1 and η_2, is given exactly by:*

$$\sigma^2\, \pi = \mu - r + \lambda\, p\eta_1 \sum_{k=0}^{\infty} (1-\pi)^k\, \mathbf{B}(k+2, \eta_1) - \lambda\, q\, \eta_2 \sum_{k=0}^{\infty} \pi^k\, \mathbf{B}(k+2, \eta_2), \quad (6.3.8)$$

which can be solved by iteration using (5.5.18)–(5.5.19) or by Newton's method using (5.5.2)–(5.5.4) where the derivatives are given by Theorem 6.3.

Proof. The optimal portfolio is given by Theorem 3.8 as $\sigma^2\, \pi = \mu - r + G'(\pi)$, on which we impose Theorem 6.3 to get the result. □

6.4 Kou Approximations by Recursion

We must truncate the infinite series in Theorem 6.4 to obtain approximations of the optimal portfolio using the simpler recursion method. Choose integers m_1 and m_2 such that

$$\sum_{k=0}^{\infty}(1-\pi)^k\,\mathbf{B}(k+2,\eta_1) \approx \sum_{k=0}^{m_1}(1-\pi)^k\,\mathbf{B}(k+2,\eta_1) \stackrel{\triangle}{=} \mathbf{S}_1(\pi;\,m_1), \tag{6.4.1}$$

and

$$\sum_{k=0}^{\infty}\pi^k\,\mathbf{B}(k+2,\eta_2) \approx \sum_{k=0}^{m_2}\pi^k\,\mathbf{B}(k+2,\eta_2) \stackrel{\triangle}{=} \mathbf{S}_2(\pi;\,m_2). \tag{6.4.2}$$

Theorem 6.5. *The optimal portfolio π for the Kou jump–diffusion market is approximated recursively by the equation:*

$$\sigma^2\,\pi \;=\; \mu - r + \lambda\,p\,\eta_1\,\mathbf{S}_1(\pi;\,m_1) - \lambda\,q\,\eta_2\,\mathbf{S}_2(\pi;\,m_2). \tag{6.4.3}$$

Explicitly,

$$\sigma^2\,\pi \;=\; \mu - r + \lambda\,p\,\eta_1\sum_{k=0}^{m_1}(1-\pi)^k\,\mathbf{B}(k+2,\eta_1) - \lambda\,q\,\eta_2\sum_{k=0}^{m_2}\pi^k\,\mathbf{B}(k+2,\eta_2).$$

Proof. This follows from Theorem 6.4 and equation (6.4.1). □

Example 6.1.

Let $\eta_1 = \eta_2 = \eta = 1$ and $m_1 = m_2 = m$. Then

$$\mathbf{B}(k+2,\eta) = \mathbf{B}(k+2,1) = \frac{\Gamma(k+2)\Gamma(1)}{\Gamma(k+3)} = \frac{(k+1)!}{(k+2)!} = \frac{1}{k+2}.$$

Thus

$$G'(\pi) \;=\; \lambda\, p \sum_{k=0}^{m} \frac{(1-\pi)^k}{k+2} - \lambda\, q \sum_{k=0}^{m} \frac{\pi^k}{k+2} = \lambda \sum_{k=0}^{m} \frac{p\,(1-\pi)^k - q\,\pi^k}{k+2}. \quad (6.4.4)$$

The approximate optimal portfolio is π *where*

$$\sigma^2\,\pi = \mu - r + \lambda \sum_{k=0}^{m} \frac{p\,(1-\pi)^k - q\,\pi^k}{k+2}. \quad (6.4.5)$$

If $p = q = \frac{1}{2}$ *and* $m = 4$, *then from (6.4.5)*

$$\begin{aligned} G'(\pi) \;&\approx\; \frac{1}{2}\,\lambda \sum_{k=0}^{4} \frac{p\,(1-\pi)^k - q\,\pi^k}{k+2} \\ &=\; \frac{1}{2}\lambda \left[\frac{1-2\pi}{3} + \frac{1-2\pi}{4} + \frac{1-3\pi+3\pi^2-2\pi^3}{5} + \frac{(1-2\pi)(1-2\pi+2\pi^2)}{6}\right], \end{aligned}$$

where

$$\sigma^2\,\pi \;=\; \mu - r + \frac{1}{2}\,\lambda \left[\frac{7(1-2\pi)}{13} + \frac{1-3\pi+3\pi^2-2\pi^3}{5} + \frac{(1-2\pi)(1-2\pi+2\pi^2)}{6}\right]$$

gives the approximate optimal portfolio π *which is solved numerically using a recursion method.*

Example 6.2. *Let* $\eta_1 = \eta_2 = \eta \in \mathbf{N}$, $m_1 = m_2 = m \in \mathbf{N}$, p, q, *and* $\lambda \geq 0$ *be arbitrary. In this case*

$$\begin{aligned} \mathbf{B}(k+1,\eta) \;&=\; \frac{\Gamma(k+1)\,\Gamma(\eta)}{\Gamma(k+1+\eta)} = \frac{(k+1)!\,(\eta-1)!}{(k+1+\eta)!} \\ &=\; \frac{(k+1)!\,(\eta-1)!}{\Pi_{j=0}^{k+\eta}(k+1+\eta-j)} = \frac{(k+1)!\,(\eta-1)!}{\Pi_{j=0}^{\eta-1}(k+1+\eta-j)(k+1)!} \\ &=\; \frac{(\eta-1)!}{\Pi_{j=0}^{\eta-1}(k+1+\eta-j)}. \end{aligned} \quad (6.4.6)$$

Thus

$$G'_+(\pi) \;\approx\; \lambda\, p\, \eta \sum_{k=0}^{m} (1-\pi)^k\, \mathbf{B}(k+2,\eta) = \lambda\, p\,(\eta!) \sum_{k=0}^{m} \frac{(1-\pi)^k}{\Pi_{j=0}^{\eta-1}(k+1+\eta-j)},$$

and

$$G'_-(\pi) \;\approx\; -\lambda\, q\, \eta \sum_{k=0}^{m} \pi^k\, \mathbf{B}(k+2,\eta) = -\lambda\, q\,(\eta!) \sum_{k=0}^{m} \frac{\pi^k}{\Pi_{j=0}^{\eta-1}(k+1+\eta-j)}.$$

Thus

$$G'(\pi) = \lambda\,(\eta!) \sum_{k=0}^{m} \frac{p\,(1-\pi)^k - q\,\pi^k}{k+2},$$

and the approximate optimal portfolio is given by

$$\sigma^2\,\pi = \mu - r + \lambda\,(\eta!) \sum_{k=0}^{m} \frac{p\,(1-\pi)^k - q\,\pi^k}{k+2}. \tag{6.4.7}$$

We now consider the most general case.

Example 6.3.

Let $\eta_1 \neq \eta_2 \in \mathbf{N}$*, with truncation points* $m_1 \neq m_2 \in \mathbf{N}$*. Then*

$$G'_+(\pi) \;\approx\; \lambda\, p\, \eta_2 \sum_{k=0}^{m_2} (1-\pi)^k\, \mathbf{B}(k+2,\eta_2) = \lambda\, p\,(\eta_2!) \sum_{k=0}^{m_2} \frac{(1-\pi)^k}{\Pi_{j=0}^{\eta_2-1}(k+1+\eta_2-j)},$$

and

$$G'_-(\pi) \;\approx\; -\lambda\, q\, \eta_1 \sum_{k=0}^{m_1} \pi^k\, \mathbf{B}(k+2,\eta_1) = -\lambda\, q\,(\eta_1!) \sum_{k=0}^{m_1} \frac{\pi^k}{\Pi_{j=0}^{\eta_1-1}(k+1+\eta_1-j)}.$$

Therefore

$$G'(\pi) = \lambda\, p\,(\eta_2!) \sum_{k=0}^{m_2} \Phi(k,\eta_2)\,(1-\pi)^k - \lambda\, q\,(\eta_1!) \sum_{k=0}^{m_1} \Phi(k,\eta_1)\,\pi^k, \tag{6.4.8}$$

where

$$\Phi(k,\eta) \triangleq \frac{1}{\Pi_{j=0}^{\eta-1}(k+1+\eta-j)} = \frac{1}{\Pi_{j=1}^{\eta}(k+2+\eta-j)}. \tag{6.4.9}$$

If $m_1 = 4$, $m_2 = 2$, $\eta_1 = 1$, $\eta_2 = 2$, $p = \frac{1}{4}$, $q = \frac{3}{4}$ *and* $\lambda = \frac{1}{3}$, *then*

$$\Phi(k,\eta_1) = \Phi(k,1) = \frac{1}{k+2}$$

and

$$\Phi(k,\eta_2) = \Phi(k,2) = \frac{1}{(k+2)(k+3)}.$$

Thus

$$\begin{aligned}
\sigma^2\pi &= \mu - r + \frac{1}{2}\lambda\sum_{k=0}^{2}\frac{(1-\pi)^k}{(k+2)(k+3)} - \frac{3}{4}\lambda\sum_{k=0}^{4}\frac{\pi^k}{k+2} \\
&= \mu - r + \frac{1}{6}\sum_{k=0}^{2}\frac{(1-\pi)^k}{(k+2)(k+3)} - \frac{1}{4}\sum_{k=0}^{4}\frac{\pi^k}{k+2} \\
&= \mu - r - \left[\frac{3}{40} + \frac{41}{360}\pi + \frac{13}{240}\pi^2 + \frac{1}{20}\pi^3 5 + \frac{1}{24}\pi^4\right].
\end{aligned}$$

We solve this equation numerically for π *by recursion or by Newtons' method.*

6.5 The Variance Gamma Model

We now apply the main theorem to the Variance Gamma (VG) Model. The Lévy density for this model is given by:

$$v(x) = \begin{cases} C\frac{e^{-G|x|}}{|x|} & \text{if } x < 0; \\ C\frac{e^{-Mx}}{x} & \text{if } x > 0, \end{cases}$$

where C, G, and M are parameters of the model and are non-negative constants. Recall that the quasi-centralized moments are:

$$M_k^{\pm} \triangleq \int_{\mathbf{R}_{\pm}} (1 - e^{-|x|})^k v_{\pm}(x)dx,$$

We now define kernels $K_s^{\pm}$ that are useful in calculating the quasi-centralized moments.

Definition 6.2. *The kernels of the quasi-centralized moments are defined by the prescription:*

$$K_s^{\pm} = \int_0^{\infty} (e^{-sx} - 1)v_{\pm}(x)dx, s \geq 0.$$

We have the following result.

Theorem 6.6. *For the VG model with parameters C, G, and M, we have:*

$$M_k^{+} = C\log(A_k(M)) \tag{6.5.1}$$

and

$$M_k^{-} = C\log(A_k(G)), \tag{6.5.2}$$

where

$$A_k(\alpha) \triangleq \Pi_{j=0}^{k}(\alpha + j)^{\left[(-1)^{(j+1)}\binom{k}{j}\right]}. \tag{6.5.3}$$

Proof.

$$M_k^{+} = \int_0^{\infty} (1 - e^{-x})^k v_{+}(x)dx$$

$$= (-1)^k \sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}\int_0^{\infty} e^{-jx}v_{+}(x)dx = \sum_{j=0}^{k}(-1)^{-j}\binom{k}{j}\int_0^{\infty} (e^{-jx} - 1)v_{+}(x)dx$$

$$= (-1)^k \sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}K_j^{+}.$$

Recall that

$$\int_{0+}^{\infty} (e^{-ux} - e^{-vx}) \frac{dx}{x} = \log(\frac{v}{u}).$$

Now

$$K_s^+ = \int_0^{\infty} (e^{-sx} - 1) v_+(x) dx = C \int_0^{\infty} (e^{-sx} - 1)\, e^{-Mx} \frac{dx}{x} = C \,\log\left(\frac{M}{s+M}\right).$$

Thus with $A = \log(M)$, we get

$$K_s^+ = C \,\log(\frac{M}{M+s}) = C\,(A - \log(M+s)). \tag{6.5.4}$$

$$\begin{aligned} M_k^+ &= (-1)^k\, C \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} (A - \log(M+s)) \\ &= (-1)^k\, C\, A \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} + (-1)^k\, C \sum_{j=0}^{k} (-1)^{(k-j+1)} \binom{k}{j} \log(M+j) \end{aligned}$$

$$\begin{aligned} &= (-1)^k\, C\, A (1-1)^k + C \sum_{j=0}^{k} \log(M+j)^{(-1)^{(j+1)} \binom{k}{j}} \\ &= C\, \Pi_{j=0}^{k} (\alpha + j)^{\left[(-1)^{(j+1)} \binom{k}{j}\right]} = C \,\log(A_k(M)), \end{aligned}$$

where

$$A_k(\alpha) \triangleq \Pi_{j=0}^{k} (\alpha + j)^{\left[(-1)^{(j+1)} \binom{k}{j}\right]}.$$

Similarly, by putting $x = -y$ and using $v_-(x) = v_-(-x)$, we get

$$\begin{aligned} M_k^- &= \int_{-\infty}^{0} (1 - e^{x})^k v_-(x)dx = -\int_{\infty}^{0} (1 - e^{-x})^k v_-(-x)dx \\ &= \int_0^{\infty} (1 - e^{-x})^k v_-(x)dx = \sum_{j=0}^{k} (-1)^{(-j)} \binom{k}{j} \int_0^{\infty} (e^{-jx} - 1) v_-(x)dx \\ &= (-1)^k \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} K_j^-. \end{aligned}$$

Now

$$K_s^- = \int_0^{\infty} (e^{-sx} - 1) v_-(x)dx = C \int_0^{\infty} (e^{-sx}\, e^{-G|x|} - e^{-G|x|}) \frac{dx}{|x|}$$
$$= C \int_0^{\infty} (e^{-(s+G)x} - e^{-Gx}) \frac{dx}{x} = C \log(\frac{G}{G+s}).$$

Therefore

$$K_s^- = C\,(\log(G) - \log(G+s)),$$

which is the same as equation(6.5.4) with M replaced by G. It now follows that

$$M_k^- = C \log(A_k(G)). \qquad \square$$

We are now in a position to compute $G'(\pi)$ and $G''(\pi)$ explicitly.

Theorem 6.7. *Let $\pi \in [0, 1]$. For the VG market with parameters C, G and M*

$$G'(\pi) = C \left[\sum_{k=0}^{\infty} (1-\pi)^k \log(A_{k+1}(M)) - \sum_{k=0}^{\infty} \pi^k \log(A_{k+1}(G)) \right] \tag{6.5.5}$$

and

$$G''(\pi) = -C \left[\sum_{k=1}^{\infty} k\,(1-\pi)^{k-1} \log(A_{k+1}(M)) + \sum_{k=1}^{\infty} k\,\pi^{k-1} \log(A_{k+1}(G)) \right]. \tag{6.5.6}$$

The optimal portfolio π that maximizes the logarithmic utility from terminal wealth for a stock with dynamic (5.1.2): is given by the equation:

$$\sigma^2 \pi = \mu - r + C \left[\sum_{k=0}^{\infty} (1-\pi)^k \log(A_{k+1}(M)) - \sum_{k=0}^{\infty} \pi^k \log(A_{k+1}(G)) \right] \tag{6.5.7}$$

which can be approximated by Newton's method using Theorem 6.2 or recursively, using equation

$$\sigma^2 \pi = \mu - r + C \left[\sum_{k=0}^{m_2} (1-\pi)^k \log(A_{k+1}(M)) - \sum_{k=0}^{m_1} \pi^k \log(A_{k+1}(G)) \right], \tag{6.5.8}$$

where m_1 and m_2 are chosen truncation points and

$$A_k(\alpha) \triangleq \Pi_{j=0}^{k} (\alpha + j)^{\left[(-1)^{(j+1)} \binom{k}{j}\right]}. \tag{6.5.9}$$

Proof. The result follows from Theorems 6.1, 6.2 and 6.6. $\square$

Example 6.4. *Let $m_1 = m_2 = 3$. It is easy to show that*

$$A_1(\alpha) = \frac{1}{\alpha(\alpha+1)^2}. \tag{6.5.10}$$

$$A_2(\alpha) = \frac{1}{\alpha(\alpha+1)^4(\alpha+2)^3}. \tag{6.5.11}$$

$$A_3(\alpha) = \frac{1}{\alpha(\alpha+1)^6(\alpha+2)^9(\alpha+3)^4}. \tag{6.5.12}$$

$$A_4(\alpha) = \frac{1}{\alpha(\alpha+1)^8(\alpha+2)^{18}(\alpha+3)^{16}(\alpha+4)^5}. \tag{6.5.13}$$

Assuming that $G = M$ we get

$$\begin{aligned} G'(\pi) &= C\left[\sum_{k=0}^{3}(1-\pi)^k \log(A_{k+1}(M)) - \sum_{k=0}^{3}\pi^k \log(A_{k+1}(G))\right] \\ &= C\left[\sum_{k=0}^{3}((1-\pi)^k - \pi^k) \log(A_{k+1}(G))\right], \end{aligned}$$

and π is given by

$$\sigma^2 \pi \approx \mu - r + C\left[\sum_{k=0}^{3}((1-\pi)^k - \pi^k) \log(A_{k+1}(G))\right].$$

6.6 The Carr, Geman, Madan, and Yor Model

We now apply the main theorem to the CGMY Model. The Lévy density for this model is given by:

$$v(x) = \begin{cases} C\frac{e^{-G|x|}}{|x|^{(1+Y)}} & \text{if } x < 0; \\ C\frac{e^{-Mx}}{x^{(1+Y)}} & \text{if } x > 0, \end{cases}$$

where C, G, M and $Y < 2$ with $Y \neq 1$, are parameters of the model. Recall that

$$M_k^{\pm} = \sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}K_j^{\pm} \quad \text{and } K_s^{\pm} = \int_0^{\infty} e^{-sx}v_{\pm}(x)dx.$$

We now have the following result.

Theorem 6.8. *The quasi-centralized moments for the CGMY model with parameters C, G, M, Y, where $Y < 2$ with $Y \neq 1$, are:*

$$M_k^{-} = C\,\Gamma(-Y)\sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}(G+j)^Y. \tag{6.6.1}$$

$$M_k^{+} = C\,\Gamma(-Y)\sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}(M+j)^Y. \tag{6.6.2}$$

Proof.

$$K_s^- = \int_0^\infty e^{-sx} v_-(x)dx = C \int_0^\infty e^{-sx}\, e^{-G|x|} \frac{dx}{|x|^{Y+1}} = C \int_0^\infty x^{-Y} e^{-(s+G)x} \frac{dx}{x}.$$

Putting $t = (G+s)x$ gives

$$K_s^- \;=\; C \int_0^\infty (G+s)^Y t^{-Y} e^{-t} \frac{dt}{t} = C\,(G+s)^Y \int_0^\infty t^{-Y} e^{-t} \frac{dt}{t} = C\,(G+s)^Y \Gamma(-Y),$$

whence

$$M_k^- = C\,\Gamma(-Y) \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} (G+j)^Y. \tag{6.6.3}$$

Similarly,

$$K_s^+ = \int_0^\infty e^{-sx} v_+(x)dx = C \int_0^\infty e^{-sx}\, e^{-Mx} \frac{dx}{x^{Y+1}} = C\Gamma(-Y)(M+s)^Y.$$

Thus $M_k^+ = C\,\Gamma(-Y) \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} (M+j)^Y$. □

We now offer the main result.

Theorem 6.9. *Let $\Psi_+ = M$ and $\Psi_- = G$ in the CGMY model. For each $k \in \mathbf{N}$ define the partial sum*

$$S_k(\Psi) \triangleq \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} (\Psi + j)^Y. \tag{6.6.4}$$

Then for each $\pi \in [0,1]$

$$M_k^\pm \;=\; C\,\Gamma(-Y)\, S_k(\Psi_\pm), \tag{6.6.5}$$

$$G'(\pi) \;=\; C\,\Gamma(-Y)\, \left[\sum_{k=0}^{\infty} (1-\pi)^k\, S_{k+1}(M) - \sum_{k=0}^{\infty} \pi^k\, S_{k+1}(G) \right]. \tag{6.6.6}$$

The optimal portfolio π *is given exactly by*

$$\sigma^2\,\pi = \mu - r + C\,\Gamma(-Y)\left[\sum_{k=0}^{\infty}(1-\pi)^k\,S_{k+1}(M) - \sum_{k=0}^{\infty}\pi^k\,S_{k+1}(G)\right], \tag{6.6.7}$$

which is approximated by Newton's method or recursion using the equation

$$\sigma^2\,\pi = \mu - r + C\,\Gamma(-Y)\,\left[\sum_{k=0}^{m_2}(1-\pi)^k\,S_{k+1}(M) - \sum_{k=0}^{m_1}\pi^k\,S_{k+1}(G)\right], \tag{6.6.8}$$

where m_1 *and* m_2 *are truncation points.*

Proof. This result follows from the Main Theorem 6.2 and Theorem 6.8. □

6.7 The Generalized Tempered Stable Model

The GTS model is a generalization of the CGMY model and depends on six parameters satisfying

$$\alpha_\pm < 2,\ \text{with}\ \alpha_\pm \notin \{0,1\};\ \ C_\pm \geq 0,\ C_- + C_+ > 0;\ G_\pm \geq 0,\ G_+ + G- > 0\ G_+ \geq G_-.$$

The Lévy density is

$$\begin{aligned} v(x) &= v_-(x)I_{\{x<0\}} + v_+(x)I_{\{x>0\}}, &(6.7.1)\\ v_\pm(x) &= C_\pm e^{-G_\pm|x|}\frac{1}{|x|^{1+\alpha_\pm}}. &(6.7.2) \end{aligned}$$

The CGMY model is the GTS model with $\alpha_1 = \alpha_2 = Y,\ C_\pm = C,\ G_- = G$ and $G_+ = M$. Define

$$S_k(G,\alpha) \triangleq \sum_{j=0}^{k}(-1)^{(k-j)}\binom{k}{j}(G+j)^\alpha. \tag{6.7.3}$$

The main result is analogous to the last.

Theorem 6.10. *For the GTS model with parameters $C_\pm$, $G_\pm$, $\alpha_\pm \notin \{0,1\}$*

$$\begin{aligned} K_s^\pm &= C_\pm \,\Gamma(-\alpha_\pm)\,(G_\pm + s)^{\alpha_\pm}, && (6.7.4)\\ M_k^\pm &= C_\pm \,\Gamma(-\alpha_\pm)\, S_k(G_\pm, \alpha_\pm). && (6.7.5) \end{aligned}$$

For each $\pi \in [0,1]$, *and truncation points* m_1 *and* m_2,

$$G'(\pi) = C_+\,\Gamma(-\alpha_+)\sum_{k=0}^{\infty}(1-\pi)^k\, S_{k+1}(G_+,\alpha_+) - C_-\,\Gamma(-\alpha_-)\sum_{k=0}^{\infty}\pi^k\, S_{k+1}(G_-,\alpha_-). \tag{6.7.6}$$

The optimal portfolio π is given exactly by

$$\sigma^2\,\pi = \mu - r + C_+\,\Gamma(-\alpha_+)\sum_{k=0}^{\infty}(1-\pi)^k\, S_{k+1}(G_+,\alpha_+) - C_-\,\Gamma(-\alpha_-)\sum_{k=0}^{\infty}\pi^k\, S_{k+1}(G_-,\alpha_-), \tag{6.7.7}$$

which is approximated by Newton's method or recursion using the equation

$$\sigma^2\,\pi \approx \mu - r + C_+\,\Gamma(-\alpha_+)\sum_{k=0}^{m_2}(1-\pi)^k\, S_{k+1}(G_+,\alpha_+) - C_-\,\Gamma(-\alpha_-)\sum_{k=0}^{m_1}\pi^k\, S_{k+1}(G_-,\alpha_-). \tag{6.7.8}$$

Proof. Assume that $\alpha_\pm \neq \{0,1\}$.

$$\begin{aligned} K_s^\pm &= \int_0^\infty e^{-sx}\, v_\pm(x)dx. = C_\pm \int_0^\infty e^{-sx} e^{-G_\pm |x|}\frac{dx}{|x|^{1+\alpha_\pm}} \\ &= C_\pm \int_0^\infty x^{-\alpha_\pm} e^{-(s+G_\pm)x}\frac{dx}{x}. = C_\pm\,\Gamma(-\alpha_\pm)\,(G_\pm + s)^{\alpha_\pm}. \end{aligned}$$

The result then follows by importing the last formula in the Main Theorem 6.2. □

Remark 6.2. *The GTS model generalizes the CGMY model, which generalizes the VG model. From empirical data (cf. CGMY [12]), G and M are quite large, falling between 25 and 130. In this case $\int_{\mathbf{R}}(e^{jx}-1)\,v(x)dx$ exists for each integer $0 \le j \le k$ where $k \in [25,130]$. Consequently, we may use the more practical approximation of*

*$G'(\pi)$ using the instantaneous centralizes moments; that is, by a **single** Taylor series expansion about $\pi = 0$, given by*

$$G_k'(\pi) = \sum_{j=0}^{m} (-1)^j \, \pi^j \, M_{j+1},$$

where M_j is the the instantaneous centralized moments of returns and $m \le k-1$ is the truncation point. This approach is computationally more efficient in the sense that we rely on only one infinite series to approximate $G'(\pi)$, instead of two, as in the Main Theorem 6.2. For the Merton jump–diffusion model with $v(x) = const\, e^{(-ax^2+bx+c)}$, the kernel K_s exists for all s, and so (6.2) is valid as it converges to $G'(\pi)$. However if $\int_{\mathbf{R}} (e^{kx}-1)v(x)dx < \infty$ for only small values of k, as in the Kou model, using two series to approximate $G'(\pi)$ is more valid.

6.8 Comparison of Series Approximation of $G'(\pi)$

Since $G(\pi) = \int_{\mathbf{R}} \log(1+\pi(e^x-1))v(x)dx$, then

$$G^{k+1}(\pi) = (-1)^k \; k! \int_{\mathbf{R}} \frac{(e^x-1)^{k+1} v(x)dx}{(1+\pi(e^x-1))},$$

whence $G^{k+1}(0) = (-1)^k \; k! \int_{\mathbf{R}} (e^x-1)^{k+1} v(x)dx$ exists only if $\int_{\mathbf{R}} (e^{jx}-1)v(x)dx < \infty$ for each integer $0 \le j \le k+1$. If this condition holds, we may expand $G'(\pi)$ about $\pi = 0$ as a truncated Taylor series of degree k given by (6.2): Note that M_k exists iff $\int_{\mathbf{R}} (e^{jx}-1)v(x)dx < \infty$ for each integer $j \le k$. Since $\int_{-\infty}^{0} (e^{jx}-1)v(x)dx$ always exists then M_j exists iff $\int_{0}^{\infty} (e^{jx}-1)v(x)dx < \infty$.

Although $\int_0^\infty (e^x-1)^j v(x)dx$ may not always exist–that is, $\int_0^\infty (e^{jx}-1)v(x)dx$ may not exist–we are certain that $\int_0^\infty (1-e^{-x})^j v(x)dx$ always exists $\forall j$, because $\int_0^\infty (e^{-jx} - 1)v(x)dx$ exists. Consequently $G^k(1)$, the k-th derivative of G at $\pi = 1$ exists $\forall k$, and so we may expand $G'(\pi)$ about $\pi = 1$. Moreover, this expansion is always convergent although it consists of two infinite series given by:

$$G'(\pi) = \sum_{k=0}^{\infty}(1-\pi)^k M_{k+1}^{+} - \sum_{k=0}^{\infty} \pi^k M_{k+1}^{-},$$

where $M_k^{\pm} \triangleq \int_{\mathbf{R}_\pm}(1-e^{-|x|})^k v_\pm(x)dx$. Consider the Kou model with say, $\eta_1 = 2$ and $v_+(x) =$constant e^{-2x}. Then $\int_0^\infty (e^{kx}-1)v(x)dx$ exists only for $k \leq 2$. Thus we are only assured of the existence of M_1 and M_2, and a first degree polynomial $M_1 - M_2\pi$ to approximate $G'(\pi)$. However, with the quasi-centralized moments, there is no such restriction since $M_k^{\pm}$ always exist. In conclusion, if $\int_0^\infty (e^{kx}-1)v(x)dx$ exists for only small values of k, say $k \leq 4$, we use quasi-centralized momemts. Otherwise we use the centralized moments to build the approximation polynomial when $k > 4$.

Chapter 7

Future Research and Numerical Results

7.1 Future Research

We may extend the research done in this dissertation in some promising directions, as follows:

A: Non–Logarithmic Utility Functions and One Stock

We successfully extended the theory of asymmetric information in fads models from the purely continuous case to the general setting, where both the fads and underlying stock price are allowed to jump. These jumps are modelled by pure jump Lévy processes. In our study, the utility function was always logarithmic. One may consider other utility functions, such as the power utility ($\frac{x^\theta}{\theta}$, $\theta < 1$) and the negative exponential utility function ($-e^{-x}$). We still keep one stock and one bond in this development.

B: Multiple Stocks and Under Logarithmic Utility

A further promising direction is the consideration of a portfolio consisting of a single bond and multiple stocks. In this case, n stocks $S_1, S_2, \ldots, S_n$ are considered, which are driven by n independent pure jump Lévy processes $X_1, X_2, \ldots, X_n$ with n independent Ornstein–Uhlenbeck processes $U_1, U_2, \ldots, U_n$, representing the fads for each stock. In the background there are also n independent standard Brownian motions $B_1, B_2, \ldots, B_n$ and Poisson random measures $N_1, N_2, \ldots, N_n$. One may also relax the independence condition by allowing some kind of correlation between the processes. This will clearly be more difficult, however one may "test" the model by first considering a model with two dependent stocks. Utility is taken to be logarithmic.

C: Multiple Stocks and Non–Logarithmic Utility

One may further extend this study, by considering other utility functions, such as the power and negative exponential utilities.

7.2 Numerical Results

We present numerical outputs of approximate optimal portfolios for various symmetric Lévy markets. These include the Kou, Merton, VG, CGMY, Double Poisson and m–Double Poisson.

Let $i \in \{0, 1\}$. The optimal portfolios for the investors under asymmetric information are generated numerically from the equations:

$$\pi_t^i = \frac{\theta_t}{\sigma_t} - \frac{\lambda\, q\, \xi^i \sqrt{Var_t^i}}{\sigma_t} + \frac{G_k'(\pi_t^i)}{\sigma_t^2}, \tag{7.2.1}$$

where

$$Var_t^1 = t \quad \text{and} \quad Var_t^0 = \sqrt{\int_0^t e^{-2\lambda\,(t-s)}(1 + p\ \tanh(\lambda\, p\, s))^{-2} ds}. \tag{7.2.2}$$

π_t^i are estimated by $\pi_t^{(k)}(\omega)$, where $\omega \in \Omega$ is the state of the world, and G_k is the k–th degree polynomial estimate of G built from **quasi–centralized moments**, with $k \in \{2, 3, 4, \cdots, 8\}$. ξ^1 and ξ^0 are drawn from independent standard normal variables. We present data for the 0% and 100% fads market for the Kou model.

The optimal portfolios for investors in the asymmetric market ($q \neq 0$) can also be approximated from the symmetric ($q = 0$) optimal portfolio by the formula:

$$\pi_t^i = \pi_t + \upsilon_t^i\, \tilde{\sigma}_t = \pi_t + noise_t^i \sim \pi_t - \frac{\lambda\, \sigma_t\, q\, \xi^i \sqrt{Var_t^i}}{M_2 + \sigma_t^2} \;\; i \in \{0,\, 1\}, \tag{7.2.3}$$

where υ^i and $\tilde{\sigma}$ are known, and π_t is given by

$$\pi_t = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2}, \;\; t \in [0,\, T]. \tag{7.2.4}$$

G is approximated by the k–th degree polynomial $G_k(\alpha)$ built from the instantaneous **centralized moments** of returns M_j where

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1}\, M_j\, \frac{\alpha^j}{j}.$$

In each simulation, we assume that the market coefficients r_t, μ_t, σ_t^2 are constants. We compare the computed optimal portfolios to the benchmark Merton [36] optimal

$$\pi_{mer} = \frac{\mu_t - r_t}{\sigma_t^2}.$$

7.2.1 Kou Jump–Diffusion Model: Asymmetric Information

Table: Kou1A–100% FADS

Lévy Density: $v_{kou}(x) = \lambda\,(p\,\eta_1\, \exp{(-\eta_1\, x)} I_{\{x>0\}} + q\,\eta_2\, \exp{(-\eta_2\,|x|)} I_{\{x<0\}})$

Parameters: $p = p^{up} = .8,\ q = q^{up} = .2,\ intensity = \frac{1}{5},\ \eta_1 = 8,\ \eta_2 = 2.$

$Fads = 100\%, \lambda = .2,\ \mu = .140685,\ r = .0876,\ \ \sigma^2 = .2428^2,\ \pi_{mer} = .900481.$

t	i	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$	$\pi^{(6)}$	$\pi^{(7)}$	$\pi^{(8)}$
0.0	1	.904788	.904993	.905004	.905005	.905005	.905005	.905005
	0	.904788	.904993	.905004	.905005	.905005	.905005	.905005
0.1	1	.888139	.888422	.888441	.888441	.888441	.888441	.888441
	0	.636569	.639556	.640177	.640177	.640352	.640367	.640367
0.2	1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	0	.551013	.555572	.556742	.557070	.557169	.557213	.557226
0.3	1	.808049	.809715	.809807	.809818	.809819	.809819	.809819
	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.4	1	.876590	.876935	.876959	.876961	.876961	.876961	.876961
	0	.932555	.932658	.932662	.932662	.932662	.932662	.932662
0.5	1	.716454	.718272	.718567	.718619	.718629	.718632	.718632
	0	.392367	.400718	.403617	.404718	.405164	.405435	.405519
0.6	1	.632366	.635422	.636065	.636212	.636248	.636262	.636265
	0	.886861	.887151	.887171	.887171	.887171	.887171	.887171
0.7	1	.887408	.887694	.887713	.887714	.887714	.887714	.887714
	0	.855079	.855554	.855593	.855597	.855597	.855597	.855597

t	i	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$	$\pi^{(6)}$	$\pi^{(7)}$	$\pi^{(8)}$
0.8	1	.881503	.881820	.881843	.881843	.881844	.881844	.881844
	0	.084560	.103514	.113429	.119102	.122564	.127215	.128215
0.9	1	.688762	.690953	.691343	.691419	.691439	.691440	.691440
	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1.0	1	.779562	.780661	.780800	.780819	.780822	.780822	.780822
	0	.821113	.821837	.821911	.821919	.821920	.821920	.821920

7.2.2 Kou Model-2: Asymmetric Information

Table: Kou2A–100% FADS

Lévy Density: $v_{kou}(x) = \lambda\,(p\,\eta_1\,\exp{(-\eta_1\,x)}I_{\{x>0\}} + q\,\eta_2\,\exp{(-\eta_2\,|x|)}I_{\{x<0\}})$

Parameters: $p = p^{up} = .8,\; q = q^{up} = .2,\; intensity = \frac{1}{5},\; \eta_1 = 8,\; \eta_2 = 2.$

$Fads = 100\%, \lambda = .1,\;\; \mu = .114913, r = .0740,\;\; \sigma^2 = .32^2,\; \pi_{mer} = .399541.$

t	i	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$	$\pi^{(6)}$	$\pi^{(7)}$	$\pi^{(8)}$
0.0	1	.707250	.709188	.709375	.709409	.709415	.709418	.709418
	0	.707250	.709188	.709375	.709409	.709415	.709418	.709418
0.1	1	.698925	.700975	.701178	.701217	.701224	.701227	.701227
	0	.572472	.576606	.577187	.577387	.577406	.577411	.577411
0.2	1	.786056	.787091	.787164	.787175	.787176	.787176	.787176
	0	.528602	.533628	.534408	.534637	.534743	.534755	.534755

t	i	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$	$\pi^{(6)}$	$\pi^{(7)}$	$\pi^{(8)}$
0.3	1	.658880	.664144	.664439	.664502	.664516	.664522	.664522
	0	.789141	.790146	.790216	.790227	.790227	.790227	.790227
0.4	1	.693151	.695281	.695495	.695537	.695545	.695548	.695548
	0	.721408	.723164	.723325	.723353	.723358	.723359	.723359
0.5	1	.613083	.616469	.616900	.617004	.617043	.617044	.617044
	0	.444717	.451691	.452965	.453407	.453662	.453698	.453698
0.6	1	.571039	.575200	.575788	.575990	.576009	.576015	.576015
	0	.698022	.700084	.700289	.700328	.700328	.700328	.700328
0.7	1	.698560	.700615	.700819	.700867	.700867	.700867	.700867
	0	.681541	.683835	.684076	.684123	.684137	.684137	.684137
0.8	1	.695607	.697703	.697913	.697953	.697961	.697964	.697964
	0	.281080	.292770	.295535	.296777	.297372	.297800	.297800
0.9	1	.599237	.602869	.603348	.603500	.603513	.603517	.603517
	0	.868419	.868810	.868827	.868829	.868829	.868829	.868829
1.0	1	.644637	.647493	.647827	.647919	.647925	.647926	.647926
	0	.663577	.665940	.666224	.666224	.666283	.666297	.666302

7.2.3 Kou Model-3: Symmetric Information

Table: Kou3–0% Fads

Model: Kou Jump–Diffusion

Lévy Density: $v_{kou}(x) = \lambda\,(p\,\eta_1\,\exp\,(-\eta_1\,x) I_{\{x>0\}} + q\,\eta_2\,\exp\,(-\eta_2\,|x|) I_{\{x<0\}})$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, v_{kou}(x)\, dx$

$$M_1 = -0.011,\; M_2 = 0.012,\; M_3 = -0.0055,\; M_4 = 0.01,\; M_5 = 0.015,$$
$$\sigma^2 = .2428^2 = .0589$$

Input Parameters: $p = .4,\; q = .6,\; \lambda = \frac{1}{20},\; \eta_1 = 6,\; \eta_2 = 1$

Error: $\epsilon = 0.5 \times 10^{-6}$

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0230	.070127	.24	.80	.706704	.641421	.623298	.604557	.623824
2	.0876	.140685	.24	.90	.806703	.732183	.708751	.681802	.715138
3	.0931	.098990	.24	.10	.006703	.006084	.006083	.006083	.006083
4	.0094	.059490	.24	.85	.756704	.686802	.666105	.643504	.668986
5	.8087	.131188	.24	.75	.656704	.596040	.580329	.565030	.579364
6	.0297	.062125	.24	.55	.456704	.414515	.406796	.401314	.404727
7	.0037	.027260	.24	.40	.306704	.278371	.274848	.273120	.273835
8	.0024	.002400	.24	.00	.000000	.000000	.000000	.000000	.000000
9	.0732	.108598	.24	.60	.506704	.459896	.450431	.443055	.448189
10	.0545	.110514	.24	.95	.856704	.777565	.751238	.719630	.762675

Table: Kou4–0% Fads

Model: Kou Jump–Diffusion

Lévy Density: $v_{kou}(x) = \lambda\,(p\,\eta_1\,\exp\,(-\eta_1\,x)I_{\{x>0\}} + q\,\eta_2\,\exp\,(-\eta_2\,|x|)I_{\{x<0\}})$

ICMR: $M_k = \int_{\mathbf{R}}(e^x - 1)^k\,v_{kou}(x)\,dx$

$M_1 = .009524,\ M_2 = .014286,\ M_3 = .000571,\ M_4 = .007238,\ M_5 = .005714,$

$\sigma^2 = .1024,\quad IMOP \approx .60$

Input Parameters: $p = .8,\ q = .2,\ \lambda = \frac{1}{5},\ \eta_1 = 8,\ \eta_2 = 2$

Error: $\epsilon = 0.5 \times 10^{-6}$

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0244	.116551	.32	.90	.946503	.884785	.886980	.864171	.880979
2	.0015	.011748	.32	.10	.146503	.136950	.137002	.136912	.136921
3	.0595	.146553	.32	.85	.896503	.838046	.840014	.820493	.833991
4	.0788	.155568	.32	.75	.796503	.744566	.746119	.732244	.740639
5	.0007	.056986	.32	.55	.596503	.557607	.558478	.552516	.555167
6	.0740	.114913	.32	.40	.446503	.417388	.417876	.415343	.416180
7	.0257	.025679	.32	.00	.046503	.043471	.043476	.043473	.043473
8	.0338	.095275	.32	.60	.640503	.604347	.605370	.597819	.601470
9	.0428	.140055	.32	.95	.996503	.931524	.933959	.907541	.928257
10.	.0219	.103870	.32	.80	.846503	.791306	.793061	.776513	.787229

7.2.4 Merton Model: Symmetric Information

Table: Merton1–0% Fads

Model: Merton Jump–Diffusion

Lévy Density: $v_{mer}(x) = \lambda \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}\left(\frac{x-m}{\delta}\right)^2}$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k \, v_{Mer}(x)\, dx$

$M_1 = .003625,\ M_2 = .006312,\ M_3 = .003453,\ M_4 = .004360,\ M_5 = .005294,$

$\sigma^2 = .2428^2 = .05895, \quad IMOP \approx .60$

Input Parameters: $mean = m = .02,\ variance = \delta^2 = .1,\ \ \lambda = \frac{1}{20}$

Error: $\epsilon = 0.5 \times 10^{-6}$

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0810	.134093	.24	.90	.930749	.883456	.907582	.882194	.913458
2	.0883	.094210	.24	.10	.130749	.124105	.124560	.124488	.124499
3	.0201	.070161	.24	.85	.880749	.835996	.857535	.835935	.860562
4	.0545	.098757	.24	.75	.780749	.741077	.757902	.742750	.757538
5	.0695	.101874	.24	.55	.580749	.551240	.560440	.554147	.558517
6	.0476	.071143	.24	.40	.430749	.408862	.413879	.411306	.412607
7	.0421	.042100	.24	.00	.030749	.029186	.029211	.029211	.029211
8	.0848	.120220	.24	.60	.630749	.598700	.609583	.601534	.607657
9	.0298	.085846	.24	.95	.980749	.930915	.957784	.928212	.928212
10	.0348	.081966	.24	.80	.830749	.788537	.807642	.789449	.808650

Table: Merton2–0% Fads

Model: Merton Jump–Diffusion

Lévy Density: $v_{mer}(x) = \lambda \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}\left(\frac{x-m}{\delta}\right)^2}$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k \, v_{Mer}(x)\, dx$

$$M_1 = .014502,\ M_2 = .025425,\ M_3 = .013814,\ M_4 = .017439,\ M_5 = .021178,$$
$$\sigma^2 = .32^2 = .1024, \quad IMOP \approx .60$$

Input Parameters: $mean = m = .02,\ variance = \delta^2 = .10,\ \ \lambda = \frac{1}{5}$

Error: $\epsilon = 0.5 \times 10^{-6}$

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0099	.102022	.32	.90	.970809	.864267	.921549	.866205	.950668
2	.0237	.033905	.32	.10	.170809	.152063	.153656	.153389	.153406
3	.0886	.175617	.32	.85	.920809	.819754	.870914	.823380	.887548
4	.0884	.165200	.32	.75	.820809	.730729	.770803	.738750	.773828
5	.0483	.104644	.32	.55	.620809	.552679	.574977	.560217	.571014
6	.0687	.109640	.32	.40	.470809	.419140	.431710	.425166	.428657
7	.0497	.049724	.32	.00	.070809	.063038	.063308	.063267	.063288
8	.0358	.097227	.32	.60	.670809	.597191	.623404	.604586	.619851
9	.0023	.099625	.32	.96	1.020809	.908780	.972581	.908689	1.00000
10	.0543	.136255	.32	.80	.870809	.775241	.820669	.780224	.829083

7.2.5 Variance Gamma Model: Symmetric Information

Table: VG1–0% Fads

Lévy Density: $\upsilon_{VG}(x) = \frac{C}{|x|}\, e^{-\lambda_-\,|x|} I_{\{x<0\}}(x) + \frac{C}{x}\, e^{-\lambda_+\,x} I_{\{x>0\}}(x)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, \upsilon_{VG}(x)\, dx$

$$M_1 = .041304,\ M_2 = .059114,\ M_3 = .000272,\ M_4 = .000163,\ M_5 = 3.96E-06,$$
$$\sigma^2 = .2428^2 = .05895, \quad IMOP \approx .70$$

Input Parameters: $C = 65.65,\ G = 47.38, \ \ M = 46.98$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: Bank of America

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0267	.079730	.24	.90	1.250319	.832783	.834388	.833588	.833604
2	.0013	.051439	.24	.85	1.200319	.799480	.800959	.800252	.800265
3	.0800	.085850	.24	.10	.450319	.299938	.300146	.300108	.300106
4	.0062	.050373	.24	.75	1.100319	.732874	.734117	.733572	.733582
5	.0508	.083179	.24	.55	.900319	.599663	.600494	.600196	.600200
6	.0323	.055838	.24	.40	.750319	.499755	.500332	.500159	.500161
7	.0728	.072775	.24	.00	.350319	.233332	.233458	.233440	.233440
8	.0933	.126631	.24	.60	.950319	.632966	.633892	.633541	.633547
9	.0183	.074295	.24	.95	1.300319	.866086	.867822	.866922	.866941
10	0.888	.135973	.24	.80	1.150319	.760177	.767535	.766912	.766924
11	.0041	.045330	.24	.70	1.050319	.699572	.700703	.700230	.700238

Table: VG2–0% Fads

Model: Variance Gamma

Lévy Density: $\upsilon_{VG}(x) = \frac{C}{|x|}\, e^{-\lambda_- \,|x|} I_{\{x<0\}}(x) + \frac{C}{x}\, e^{-\lambda_+ \,x} I_{\{x>0\}}(x)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k \, \upsilon_{VG}(x)\, dx$

$$M_1 = .021090,\; M_2 = .017778,\; M_3 = 9.73162E-6,\; M_4 = 4.5996E-5,$$
$$M_5 = 1.20696E-6,\; \sigma^2 = .1331^2 = .0177, \quad IMOP \approx 1.00$$

Input Parameters: $C = 21.34$, $G = 49.78$, $M = 48.40$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: General Electric

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0072	.023230	.13	.90	1.49525	.995672	.998410	.997136	.997170
2	.0033	.005042	.13	.10	.69525	.462960	.463550	.463422	.463424
3	.0049	.019968	.13	.85	1.44525	.962377	.964935	.963785	.963814
4	.0750	.087430	.13	.70	1.29525	.862494	.864547	.863719	.863738
5	.0038	.013567	.13	.55	1.14525	.762610	.764214	.763643	.763654
6	.0810	.088110	.13	.40	.99525	.662727	.663938	.663562	.663569
7	.0222	.022162	.13	.00	.59525	.396371	.396803	.396723	.396724
8	.0667	.077378	.13	.60	1.19525	.795905	.797652	.797002	.797016
9	.0721	.088882	.13	.95	1.54525	1.00000	1.00000	1.000000	1.00000
10	.0731	.087248	.13	.80	1.39525	.929083	.931466	.930431	.930457

7.2.6 Carr, Geman, Madan & Yor (CGMY) Model: Symmetric Information

Table: CGMY1–0% Fads

Model: CGMY

Lévy Density: $\upsilon_{CGMY}(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x),$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, \upsilon_{CGMY}(x)\, dx$

$M_1 = .032327,\ M_2 = .046272,\ M_3 = .000226,\ M_4 = .000135,\ M_5 = 3.96E - 06,$

$\sigma^2 = .2428^2 = .05895,\quad IMOP \approx .70$

Input Parameters: $C = 65.65,\ G = 47.38,\quad M = 46.98,\ Y = -0.0719$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: BANK of AMERICA

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0239	.077000	.24	.90	1.174186	.843248	.844613	.843924	.843938
2	.0301	.035972	.24	.10	.374186	.268723	.268862	.268839	.268840
3	.0836	.133737	.24	.85	1.124186	.807340	.808591	.807986	.807999
4	.0432	.084460	.24	.70	.974186	.699617	.700556	.700162	.700169
5	.0093	.041746	.24	.55	.824186	.591893	.592565	.592327	.592331
6	.0980	.121628	.24	.40	.674186	.484170	.484619	.484489	.484491
7	.2020	.202043	.24	.00	.274186	.196908	.196982	.196973	.196973
8	.9072	.132583	.24	.60	.874186	.627801	.628557	.628273	.628277
9	.0218	.077795	.24	.95	1.224186	.879155	.880640	.879859	.879876
10	.0010	.048155	.24	.80	1.074186	.771432	.772574	.772047	.772057

Table: CGMY2–0% Fads

Model: CGMY

Lévy Density: $\upsilon_{CGMY}(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x),$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, \upsilon_{CGMY}(x)\, dx$

$M_1 = .025377,\ M_2 = .026979,\ M_3 = .000262,\ M_4 = .0001402,\ M_5 = 6.63E-06,$

$\sigma^2 = .1905^2 = .03629, \quad IMOP \approx .9454$

Input Parameters: $C = 23.70,\ G = 36.59,\ \ M = 35.70,\ Y = -0.0963$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: WALMART

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0091	.041762	.19	.90	1.24964	.911004	.914024	.912557	.912620
2	.0451	.048758	.19	.10	.449640	.327793	.328183	.328114	.328115
3	.0410	.071882	.19	.85	1.19964	.874553	.877336	.876038	.876092
4	.0150	.040364	.19	.70	1.04964	.765201	.767330	.766460	.766492
5	.0816	.101551	.19	.55	.89964	.6558497	.657412	.656864	.656881
6	.0389	.053391	.19	.40	.74964	.546497	.547581	.547265	.547273
7	.0002	.000203	.19	.00	.34964	.254892	.255127	.255095	.255095
8	.0272	.048968	.19	.60	.94964	.692300	.694041	.693398	.693419
9	.0552	.089665	.19	.95	1.29964	.947455	.950722	.949072	.949146
10	.0204	.049414	.19	.80	1.14964	.838103	.840658	.839515	.839560

Table: CGMY3–0% Fads

Model: CGMY

Lévy Density: $\upsilon_{CGMY}(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x),$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, \upsilon_{CGMY}(x)\, dx$

$M_1 = .030125,\ M_2 = .060285,\ M_3 = .000107,\ M_4 = 7.302E-05,$

$M_5 = 2.589E-06,\ \sigma^2 = .1888^2 = .03565,\quad IMOP \approx .50025$

Input Parameters: $C = 0.0823,\ G = 25.04,\ \ M = 25.04,\ Y = 1.5063$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: MCDONALDS

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.079	.111923	.19	.90	1.322564	.71558	.716355	.715976	.715985
2	.0479	.051416	.19	.10	.522564	.284149	.284269	.284245	.284245
3	.0276	.057925	.19	.85	1.272564	.688615	.689333	.688995	.689003
4	.0976	.122580	.19	.70	1.225640	.607722	.608280	.608048	.608053
5	.0158	.035452	.19	.55	.972564	.520829	.527247	.527096	.527099
6	.0204	.034608	.19	.40	.822564	.445935	.446234	.446143	.446144
7	.0835	.083531	.19	.00	.422564	.230220	.230298	.230286	.230286
8	.0718	.093139	.19	.60	1.022564	.553793	.554255	.554081	.554084
9	.0809	.114784	.19	.95	1.372564	.742544	.743379	.742955	.742966
10	.0513	.079775	.19	.80	1.222564	.661651	.662313	.662013	.662020
11	.0683	.086170	.19	.50	.922564	.499864	.500240	.500112	.500114

Table: CGMY4–0% Fads

Model: CGMY

Lévy Density: $\upsilon_{CGMY}(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x),$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, \upsilon_{CGMY}(x)\, dx$

$M_1 = .017097,\ M_2 = .034202,\ M_3 = .0000211,\ M_4 = .0000141,$

$M_5 = 8.18932E - 08,\ \sigma^2 = .1905^2 = .03629,\quad IMOP \approx 0.50025$

Input Parameters: $C = 1.7454,\ G = 73.39,\quad M = 73.39,\ Y = 0.9315$

Error: $\epsilon = 0.5 \times 10^{-6}$

Stock: DRG

n	r	μ	σ	π_{mer}	$\pi^{(1)}$	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$
1	.0065	.039132	.19	.90	1.13556	.77185	.772023	.771933	.771934
2	.0802	.083807	.19	.10	.33556	.228085	.228100	.228097	.228097
3	.0916	.122461	.19	.85	1.08556	.737864	.738023	.737944	.737945
4	.0517	.077072	.19	.70	.93556	.635908	.636026	.635976	.635976
5	.0477	.067672	.19	.55	.78556	.533952	.534035	.534006	.534006
6	.0267	.041208	.19	.40	.63556	.431996	.432051	.432035	.432005
7	.0089	.008896	.19	.00	.23556	.160114	.160121	.160121	.160121
8	.0124	.034201	.19	.60	.83556	.567938	.568031	.567996	.567996
9	.0082	.042626	.19	.95	1.18556	.805835	.806024	.805922	.805922
10	.0709	.099909	.19	.80	1.03556	.703879	.704023	.703955	.703955
11	.0763	.094489	.19	.50	.73556	.499967	.500040	.500015	.500016

7.2.7 Double Poisson Model: Symmetric Information

Table: DP1–0% Fads

Model: Double Poisson

Lévy Density: $\upsilon_{\Pi(1,2)}(dx) = \lambda_u\,\delta_{\alpha_u}(dx) + \lambda_d\,\delta_{\alpha_d}(dx)$

ICMR: $M_k = \int_{\mathbf{R}}(e^x - 1)^k\,\upsilon_{\Pi(1,2)}(dx)$

$$M_1 = .005,\ M_2 = .0015,\ M_3 = .00005,\ M_4 = .000015,\ M_5 = 5E-07,$$

$$\sigma^2 = .2428^2 = .1024, \quad IMOP \approx 0.667$$

Input Parameters: $A_u = .5,\ \alpha_u = \ln(1.5),\ \lambda_u = \frac{1}{5},\ \lambda_d = \frac{1}{10}$

Error: $\epsilon = 0.5 \times 10^{-6}$

Admissible Set : $(-2\,,\,2)$

n	r	μ	σ	π_{mer}	π^*	λ_u	λ_d	A_u	a
1	.0281	.081198	.2428	.90	.799480	.2	.1	.5	2
2	.0141	.019972	.2428	.10	.323907	.2	.1	.5	2
3	.0830	.133135	.2428	.85	.771592	.2	.1	.5	2
4	.0570	.098263	.2428	.70	.686036	.2	.1	.5	2
5	.0616	.094030	.2428	.55	.597981	.2	.1	.5	2
6	.0882	.111732	.2428	.40	.507934	.2	.1	.5	2
7	.0366	.036643	.2428	.00	.261943	.2	.1	.5	2
8	.0111	.046501	.2428	.60	.627579	.2	.1	.5	2
9	.0814	.137424	.2428	.95	.827057	.2	.1	.5	2
10	.0455	.092612	.2428	.80	.743381	.2	.1	.5	2

Table: DP2–0% Fads

Lévy Density: $v_{\Pi(1,2)}(dx) = \lambda_u\,\delta_{\alpha_u}(dx) + \lambda_d\,\delta_{\alpha_d}(dx)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, v_{\Pi(1,2)}(dx)$

$M_1 = .005,\ M_2 = .0015,\ M_3 = .00005,\ M_4 = .000015,\ M_5 = 5E-07,$

$\sigma^2 = .32^2 = .1024, \quad IMOP \approx 3.333$

Input Parameters: $A_u = .10,\ \alpha_u = \ln(1.1),\ \lambda_u = \frac{1}{10},\ \lambda_d = \frac{1}{20}$

Error: $\epsilon = 0.5 \times 10^{-6}$

Admissible Set : $(-10\,,\,10)$

n	r	μ	σ	π_{mer}	π^*	λ_u	λ_d	A_u	a
1	.0566	.148718	.32	.90	.917842	.1	.05	.10	10
2	.0126	.022877	.32	.10	.123513	.1	.05	.10	10
3	.0988	.185858	.32	.85	.868192	.1	.05	.10	10
4	.0996	.171269	.32	.70	.719246	.1	.05	.10	10
5	.0944	.150727	.32	.55	.570303	.1	.05	.10	10
6	.0040	.044953	.32	.40	.421366	.1	.05	.10	10
7	.0676	.067622	.32	.00	.024237	.1	.05	.10	10
8	.0851	.146493	.32	.60	.619950	.1	.05	.10	10
9	.0164	.113697	.32	.95	.967492	.1	.05	.10	10
10	.0403	.122218	.32	.80	.818543	.1	.05	.10	10
11	.0488	.390103	.32	3.333	3.333	.1	.05	.10	10

7.2.8 4–Double Poisson Model: Symmetric Information

Table: 4–DP1–0% Fads

Lévy Density: $v_{\Pi(4,2)}(dx) = \sum_{i=1}^{4} \lambda_{u_i}\, \delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\, \delta_{\lambda_{d_i}}(dx)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, v_{\Pi(4,2)}(dx)$

$$M_1 = .072548,\ M_2 = .07496,\ M_3 = .015297,\ M_4 = .025064,\ M_5 = .005556,$$

$$\sigma^2 = .2428^2 = .058952,\quad IMOP \approx 0.863842,\quad \epsilon = .5 \times 10^{-6}$$

Input Parameters:

$$\lambda_{u_1} = \tfrac{1}{5},\, \lambda_{u_2} = \tfrac{1}{10}\ \ \lambda_{u_3} = \tfrac{1}{15},\, \lambda_{u_4} = \tfrac{1}{20} \qquad \lambda_{d_1} = \tfrac{1}{25},\, \lambda_{u_2} = \tfrac{1}{30}\ \ \lambda_{u_3} = \tfrac{1}{35},\, \lambda_{u_4} = \tfrac{1}{40}$$

$$\alpha_{u_1} = \ln(1.1),\ \alpha_{u_2} = \ln(1.3),\ \alpha_{u_3} = \ln(1.5),\ \alpha_{u_4} = \ln(1.7),$$

Admissible Set : $(-1.4282\,,\, 1.4286)$

n	r	μ	σ	π_{mer}	π^*	A_{u_1}	A_{u_2}	A_{u_3}	A_{u_4}
1	.0036	.056616	.2428	.90	.899982	.1	.3	.5	.7
2	.0243	.074392	.2428	.10	.101731	.1	.3	.5	.7
3	.0243	.074392	.2428	.85	.850044	.1	.3	.5	.7
4	.0546	.095889	.2428	.70	.700101	.1	.3	.5	.7
5	.0082	.040606	.2428	.55	.550260	.1	.3	.5	.7
6	.0339	.057438	.2428	.40	.401467	.1	.3	.5	.7
7	.0084	.008396	.2428	.00	.000609	.1	.3	.5	.7
8	.0241	.059429	.2428	.60	.600661	.1	.3	.5	.7
9	.0807	.136730	.2428	.95	.948994	.1	.3	.5	.7
10	.0739	.121015	.2428	.80	.800576	.1	.3	.5	.7

Table: 4–DP2–0% Fads

Model: 4–Double Poisson

Lévy Density: $v_{\Pi(4,2)}(dx) = \sum_{i=1}^{4} \lambda_{u_i}\, \delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\, \delta_{\lambda_{d_i}}(dx)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k \, v_{\Pi(4,2)}(dx)$

$$M_1 = .083524,\ M_2 = .100819,\ M_3 = .017855,\ M_4 = .040604,\ M_5 = .007315,$$

$$\sigma^2 = .32^2 = .1024, \quad IMOP \approx 0.730888, \quad \epsilon = .5 \times 10^{-6}$$

Input Parameters:

$$\lambda_{u_1} = \tfrac{1}{5},\ \lambda_{u_2} = \tfrac{1}{10}\ \ \lambda_{u_3} = \tfrac{1}{20},\ \lambda_{u_4} = \tfrac{1}{25} \qquad \lambda_{d_1} = \tfrac{1}{25},\ \lambda_{u_2} = \tfrac{1}{30}\ \ \lambda_{u_3} = \tfrac{1}{55},\ \lambda_{u_4} = \tfrac{1}{40}$$

$$\alpha_{u_1} = \ln(1.1),\ \alpha_{u_2} = \ln(1.3),\ \alpha_{u_3} = \ln(1.5),\ \alpha_{u_4} = \ln(1.7),$$

Admissible Set : $(-1.25\,,\, 1.25)$

n	r	μ	σ	π_{mer}	π^*	A_{u_1}	A_{u_2}	A_{u_3}	A_{u_4}
1	.0042	.096317	.32	.90	.899843	.2	.4	.6	.8
2	.0426	.052882	.32	.10	.103125	.2	.4	.6	.8
3	.0820	.169077	.32	.85	.848040	.2	.4	.6	.8
4	.0833	.154969	.32	.70	.700423	.2	.4	.6	.8
5	.0760	.132294	.32	.55	.551973	.2	.4	.6	.8
6	.0480	.088973	.32	.40	.402074	.2	.4	.6	.8
7	.0389	.038860	.32	.00	.003233	.2	.4	.6	.8
8	.0072	.068646	.32	.60	.600142	.2	.4	.6	.8
9	.0689	.166148	.32	.95	.946328	.2	.4	.6	.8
10	.0375	.119383	.32	.80	.799514	.2	.4	.6	.8

Table: 4–DP3–0% Fads

Model: 4–Double Poisson

Lévy Density: $v_{\Pi(4,2)}(dx) = \sum_{i=1}^{4} \lambda_{u_i}\, \delta_{\lambda_{u_i}}(dx) + \lambda_{d_i}\, \delta_{\lambda_{d_i}}(dx)$

ICMR: $M_k = \int_{\mathbf{R}} (e^x - 1)^k\, v_{\Pi(4,2)}(dx)$

$$M_1 = .083524,\ M_2 = .100819,\ M_3 = .017855,\ M_4 = .040604,\ M_5 = .007315,$$

$$\sigma^2 = .32^2 = .1024, \quad IMOP \approx 0.730888, \quad \epsilon = .5 \times 10^{-6}$$

Input Parameters:

$\lambda_{u_1} = \frac{1}{5},\ \lambda_{u_2} = \frac{1}{10}\ \ \lambda_{u_3} = \frac{1}{20},\ \lambda_{u_4} = \frac{1}{25} \quad \lambda_{d_1} = \frac{1}{25},\ \lambda_{u_2} = \frac{1}{30}\ \ \lambda_{u_3} = \frac{1}{55},\ \lambda_{u_4} = \frac{1}{40}$

$$\alpha_{u_1} = \ln(1.1),\ \alpha_{u_2} = \ln(1.3),\ \alpha_{u_3} = \ln(1.5),\ \alpha_{u_4} = \ln(1.7),$$

Admissible Set : $(-1.25\,,\,1.25)$

n	r	μ	σ	π_{mer}	π^*	A_{u_1}	A_{u_2}	A_{u_3}	A_{u_4}
1	.0395	.092515	.2428	.90	.898522	.2	.4	.6	.8
2	.0718	.077736	.2428	.10	.105249	.2	.4	.6	.8
3	.0924	.142553	.2428	.85	.847797	.2	.4	.6	.8
4	.0051	.046317	.2428	.70	.700026	.2	.4	.6	.8
5	.0484	.080873	.2428	.55	.551263	.2	.4	.6	.8
6	.0256	.049179	.2428	.40	.401108	.2	.4	.6	.8
7	.0247	.024686	.2428	.00	.002057	.2	.4	.6	.8
8	.0066	.041929	.2428	.60	.600130	.2	.4	.6	.8
9	.0261	.082904	.2428	.95	.948587	.2	.4	.6	.8
10	.0793	.126422	.2428	.80	.798980	.2	.4	.6	.8

Appendix A

Combinatorial Identities Derived from the Kou Model

Define the objects M_j and K_s by the prescriptions:

$$M_j = \int_R (e^x - 1)^j v(dx), \tag{A.0.1}$$

$$K_s = \int_R (e^{sx} - 1) v(dx), \quad s \geq 0. \tag{A.0.2}$$

M_j is called the j–th instantaneous centralized moments of returns of the Lévy process X, with measure $v(\cdot)$. K_s is a kernel used to calculate M_j. We have the following result, which will be quite useful in the sequel.

Lemma A.1. *If there exists $k \in N$, such that $\int_R (e^{jx} - 1) v(dx) < \infty$ for each $0 \leq j \leq k$, then M_j and K_j exist, and*

$$M_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i. \tag{A.0.3}$$

Proof. If there exists $k \in N$, such that $\int_R (e^{jx} - 1) v(dx) < \infty$ for each $0 \leq j \leq k$, then

$$K_j = \int_R (e^{jx} - 1) v(dx) < \infty.$$

Now

$$M_j = \int_R (e^x - 1)^j \, v(dx).$$

From the Binomial Theorem

$$\begin{aligned}
(e^x - 1)^j &= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} e^{ix} \\
&= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1) + \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1) + (1 - 1)^j \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix} - 1).
\end{aligned}$$

Therefore

$$M_j = \int_R (e^x - 1)^j \, v(dx) = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \int_R (e^{ix} - 1) \, v(dx) = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i,$$

which is clearly finite for each integer $0 \leq j \leq k$. □

We obtain new combinatorial identities by applying Lemma A.1 to the Kou jump-diffusion model. The central result follows.

Theorem A.1. *Let $k \geq 1$ be an integer, $\eta_1 > k$, $\eta_2 > 0$, $p + q = 1$, $p \geq 0$, $q \geq 0$. Then*

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{j \, (j + p \, \eta_2 - q \, \eta_1)}{(\eta_1 - j)(\eta_2 + j)} = k! \left[(-1)^k \, q \, \eta_2 \, \frac{\Gamma(\eta_2)}{\Gamma(\eta_2 + k + 1)} + p \, \eta_1 \, \frac{\Gamma(\eta_1 - k)}{\Gamma(\eta_1 + 1)} \right]. \tag{A.0.4}$$

Proof. From Lemma A.1, if $\eta_1 > k$ then K_j and M_j exist for each $0 \leq j \leq k$, and

$$M_k = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} K_j.$$

For the Kou model, K_j and M_j are given explicitly by Lemma 5.3, with

$$M_j = (-1)^j \, (j!) \, \frac{\lambda \, q \, \eta_2 \, \Gamma(\eta_2)}{\Gamma(\eta_2 + j + 1)} + (j!) \, \frac{\lambda \, p \, \eta_1 \, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)}, \tag{A.0.5}$$

and

$$K_s = \frac{\lambda \, p \, s}{(\eta_1 - s)} + \frac{\lambda \, q \, s}{(\eta_2 + s)} = \lambda \frac{s \, (p \, \eta_2 - q \, \eta_1 + s)}{(\eta_1 - s)(\eta_2 + s)}. \tag{A.0.6}$$

The result then follows by dividing both sides by λ, the intensity rate of the driving Poisson process. □

In the sequel, we assume without loss of generality that $\lambda = 1$.

Corollary A.1. *Let $k, \alpha \in N$ and $\eta > k$. Then*

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{j^2}{(\eta^2 - j^2)} = \frac{\eta}{2} \, k! \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right]. \tag{A.0.7}$$

$$\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{(\eta^2 - j^2)} = \frac{\eta}{2} \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right]. \tag{A.0.8}$$

$$\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{((k+\alpha)^2 - j^2)} = \frac{1}{2} \left[(-1)^{k+\alpha} \frac{(k+\alpha)!}{(2k+\alpha)!} + \frac{(\alpha - 1)!}{(k + \alpha - 1)!} \right]. \tag{A.0.9}$$

Proof. Put $p = q = \frac{1}{2}$, $\eta_1 = \eta_2 = \eta > k$ in Theorem A.1 to get (A.0.7) and (A.0.8). Set $\eta = k + \alpha$, $\alpha \in N$. Then by said theorem, we get the last identity as follows:

$$\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{((k+\alpha)^2 - j^2)} = \frac{1}{2} \eta \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + p\eta \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right]$$

$$= \frac{1}{2} \eta \left[(-1)^k \frac{(\eta - 1)!}{(\eta + k)!} + \frac{(\eta - k - 1)!}{(\eta)!} \right] = \frac{1}{2} \left[(-1)^k \frac{(\eta)!}{(\eta + k)!} + \frac{(\eta - k - 1)!}{(\eta - 1)!} \right]$$

$$= \frac{1}{2}\left[(-1)^{k+\alpha}\frac{(k+\alpha)!}{(2k+\alpha)!} + \frac{(\alpha-1)!}{(k+\alpha-1)!}\right].$$

□

Theorem A.1 yields other special results based on the choice of the parameters p, η_1 and η_2.

Theorem A.2. *Let $k \in N$ and $\eta > k$. Then*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta-j} = k!\frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{k!}{\Pi_{j=1}^{k}(\eta-j)}. \tag{A.0.10}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{j!(k-j)!}\frac{j}{\eta-j} = \frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{1}{\Pi_{j=1}^{k}(\eta-j)}. \tag{A.0.11}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k+1-j} = 1. \tag{A.0.12}$$

Proof. Put $p = 1$, $q = 0$, and $\eta_1 = \eta_2 = \eta$ in Theorem A.1. Then

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\frac{j(j+\eta)}{(\eta-j)(\eta+j)} = k!\,p\,\eta\frac{\Gamma(\eta-k)}{\Gamma(\eta+1)}.$$

Thus starting with $j = 0$, we get

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta-j} = k!\,\frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{k!}{\Pi_{j=1}^{k}(\eta-j)}.$$

Dividing by $k!$ yields the second identity. Putting $\eta = k+1$ in equation(A.0.10), yields

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k+1-j} = \frac{k!}{\Pi_{j=1}^{k}(k+1-j)} = \frac{k!}{k!} = 1.$$

This completes the proof. □

Theorem A.3. *Let $\eta > 0$ and k, m be positive integers. Then*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta+j} = (-1)^k k!\frac{\Gamma(\eta+1)}{\Gamma(\eta+k+1)} = (-1)^k\frac{k!}{\Pi_{j=1}^{k}(\eta+j)}. \tag{A.0.13}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{j!(k-j)!}\frac{j}{\eta+j} = (-1)^k\frac{\Gamma(\eta+1)}{\Gamma(\eta+k+1)} = (-1)^k\frac{1}{\Pi_{j=1}^{k}(\eta+j)}. \tag{A.0.14}$$

$$\sum_{j=0}^{2m}(-1)^{j-1}\binom{2m}{j}\frac{j}{\eta+j} = \frac{\Gamma(2m+1)\Gamma(\eta+1)}{\Gamma(\eta+2m+1)} = \frac{1}{\Pi_{j=1}^{2m}(\eta+j)}. \tag{A.0.15}$$

$$\sum_{j=0}^{2m-1}(-1)^{j}\binom{2m-1}{j}\frac{j}{\eta+j} = \frac{\Gamma(2m)\Gamma(\eta+1)}{\Gamma(\eta+2m)} = \frac{(2m-1)!}{\Pi_{j=1}^{2m-1}(\eta+j)}. \tag{A.0.16}$$

Proof. Put $p = 0$, $q = 1$ and $\eta_1 = \eta_2 = \eta$ in Theorem A.1. Then

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j(j-\eta)}{(\eta-j)(\eta+j)} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta+j}$$

$$= (-1)^k k!\eta\frac{\Gamma(\eta)}{\Gamma(\eta+k+1)} = (-1)^k\frac{k!}{\Pi_{j=1}^{k}(\eta+j)}.$$

Equation (A.0.14) follows by dividing both sides by $k!$.

Put $k = 2m$, where m is an integer. Then

$$\sum_{j=0}^{k}(-1)^{2m-j}\binom{2m}{j}\frac{j}{\eta+j} = (-1)^{2m}(2m)!\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+2m+1)}$$

$$= (2m)!\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+2m+1)} = \frac{2m!}{\Pi_{j=1}^{2m}(\eta+j)}.$$

Similarly, setting $k = 2m - 1$ and multiplying by (-1), we have

$$\sum_{j=0}^{2m-1} (-1)^{2m-1-j} \binom{2m-1}{j} \frac{j}{\eta + j} = \sum_{j=0}^{2m-1} (-1)^{j} \binom{2m-1}{j} \frac{j}{\eta + j}$$
$$= (-1)^{2m-1}(2m-1)! \frac{\Gamma(\eta+1)}{\Gamma(\eta+2m)} = (2m-1)! \frac{\Gamma(\eta+1)}{\Gamma(\eta+2m)} = \frac{(2m-1)!}{\Pi_{j=1}^{2m-1}(\eta+j)}.$$

□

We have more interesting identities:

Theorem A.4. *For any positive integers k and m, we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = (-1)^k \frac{1}{k+1}. \tag{A.0.17}$$

$$\sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} \frac{j}{j+1} = \frac{1}{k+1}. \tag{A.0.18}$$

$$\sum_{j=0}^{2m} (-1)^{j+1} \binom{2m}{j} \frac{j}{j+1} = \frac{1}{2m+1}. \tag{A.0.19}$$

$$\sum_{j=0}^{2m-1} (-1)^{j} \binom{2m-1}{j} \frac{j}{j+1} = -\frac{1}{2m}. \tag{A.0.20}$$

Proof. Putting $\eta = 1$ in Theorem A.3 yields,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = (-1)^k k! \frac{\Gamma(2)}{\Gamma(k+2)} = (-1)^k \frac{k!}{\Pi_{j=1}^{k}(j+1)} = (-1)^k \frac{1}{k+1}.$$

Multiplying (A.0.17) by $(-1)^k$, yields (A.0.18).

Let m be an integer. Put $k = 2m$ in (A.0.17). Then

$$\sum_{j=0}^{2m} (-1)^{2m-j} \binom{2m}{j} \frac{j}{j+1} = \sum_{j=0}^{2m} (-1)^{j} \binom{2m}{j} \frac{j}{j+1} = (-1)^{2m} \frac{1}{2m+1} = \frac{1}{2m+1}.$$

Similarly, with $k = 2m - 1$, we get

$$-\sum_{j=0}^{2m-1} (-1)^{2m-1-j} \binom{2m-1}{j} \frac{j}{j+1} = \sum_{j=0}^{2m-1} (-1)^{j} \binom{2m-1}{j} \frac{j}{j+1} = (-1)^{2m} \frac{1}{2m} = \frac{1}{2m}.$$

□

We have some results involving double sums.

Corollary A.2.

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} (-1)^{j} \binom{2^n-1}{j} \frac{j}{j+1} = 1. \tag{A.0.21}$$

$$\sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = \log(2) - 1. \tag{A.0.22}$$

Proof. Put $m = 2^{n-1}$ in the third identity of Theorem A.4.

Then $2m - 1 = 2^n - 1$, and so

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} (-1)^{j} \binom{2^n-1}{j} \frac{j}{j+1} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1.$$

Recall that

$$\log(1+x) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{j}, \text{ whence } \log 2 = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} = \sum_{j=1}^{\infty} \frac{(-1)^k}{k+1}.$$

By Theorem (A.3) ,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = (-1)^k \frac{1}{k+1}.$$

Therefore

$$\sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} - 1 = \log 2 - 1.$$

$\square$

We state without proof some identities that follow directly from the third identity of Theorem A.2, which states that for all positive integer k

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k-j+1} = 1.$$

Example A.1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{(k-j+1)k^2} = \frac{\pi^2}{3}. \tag{A.0.23}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{(k-j+1)(k(k+1))} = 1. \tag{A.0.24}$$

$$\sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{jk}{k-j+1} = \frac{1}{2}n(n+1). \tag{A.0.25}$$

$$\sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{jk^2}{k-j+1} = \frac{1}{6}n(n+1)(2n+1). \tag{A.0.26}$$

A.1 Analytic Formula for $G'(\pi)$ and $G''(\pi)$ for Kou Market

We will show that $G'(\pi)$ can be expressed analytically in terms of the cumulative distribution function of Beta random variables. Let $\pi \in [0, 1]$ and

$$G(\pi) = \int_R \log(1+\pi(e^x-1))v(dx).$$

For the Kou model with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$, $p + q = 1$, p, $q \geq 0$, the Lévy density is

$$\begin{aligned} v(x) &= \lambda\, v_-(x) I_{\{x<0\}} + \lambda\, v_+(x) I_{\{x>0\}} \qquad (A.1.1) \\ &= \lambda\, f_{kou}(x). \\ f_{kou}(x) &= q\, \eta_2\, e^{-\eta_2\,|x|} I_{\{x<0\}} + p\, \eta_1\, e^{-\eta_1\, x} I_{\{x>0\}}, \qquad (A.1.2) \end{aligned}$$

and

$$v_\pm(x) = \lambda f_{kou}(x) I_{\mathbf{R}_\pm}. \qquad (A.1.3)$$

Recall that

$$\begin{aligned} G'(\pi) &= \int_{\mathbf{R}} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx, \qquad (A.1.4) \\ &= \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} v(dx) + \int_{-\infty}^0 \frac{e^x - 1}{1 + \pi(e^x - 1)} v(dx). \end{aligned}$$

Thus

$$G'(\pi) = G'_+(\pi) + G'_-(\pi), \qquad (A.1.5)$$

where

$$G'_\pm(\pi) \triangleq \int_{\mathbf{R}_\pm} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx. \qquad (A.1.6)$$

Let $\eta > 0$ and $\pi \in [0, 1]$. Define

$$D_\pm(\pi, \eta) \triangleq \int_{\mathbf{R}_\pm} \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta|x|} dx. \qquad (A.1.7)$$

We have the following result.

Proposition A.1. *Let $\eta > 0$ and $\pi \in [0,\, 1]$. Then*

$$D_-(\pi, \eta) = -D_+(1-\pi, \eta). \tag{A.1.8}$$

Proof.

$$D_-(\pi, \eta) \triangleq \int_{\mathbf{R}_-} \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta\,|x|} dx$$

$$= \int_{\mathbf{R}_-} \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{\eta\, x} dx = \int_0^\infty \frac{(1 - e^y)}{(e^y + \pi(1 - e^y))}\, e^{-\eta\, y} dy.$$

Put $\beta = 1 - \pi$ and set $v(y) = e^{\eta\, y}$. Then

$$D_-(\pi, \eta) = \int_0^\infty \frac{(e^y - 1) v(-y)}{(-\pi + (\pi - 1)e^y)} dy = -\int_0^\infty \frac{(e^y - 1)e^{-\eta\, y}}{(1 + \beta(e^y - 1))} dy = -D_+(1 - \pi, \eta).$$

□

By the last result, we dispose of the subscripts "±", and simply re-define $D_+(\pi, \eta)$ as

$$D(\pi, \eta) \triangleq \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta\, x} dx. \tag{A.1.9}$$

We now have the following result.

Proposition A.2. *For the Kou model,with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$, $p + q = 1$, p, $q\ \geq 0$,*

$$G'(\pi) = \lambda\, p\, \eta_1\, D(\pi, \eta_1) - \lambda\, q\, \eta_2\, D(1 - \pi, \eta_2). \tag{A.1.10}$$

Proof.

$$G'(\pi) = G'_+(\pi) + G'_-(\pi),$$

with

$$G'_+(\pi) = \int_{\mathbf{R}_+} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx = \lambda\, p\, \eta_1 \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta_1 x} dx$$

$$= \lambda\, p\, \eta_1\, D(\pi, \eta_1), \text{ and}$$

$$G'_-(\pi) = \int_{\mathbf{R}_-} \frac{e^x - 1}{1 + \pi(e^x - 1)} v(x)dx = \lambda\, q\, \eta_2 \int_{-\infty}^0 \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta_2 |x|} dx$$

$$= \lambda\, q\, \eta_2\, D_-(\pi, \eta) = -\lambda\, q\, \eta_2\, D_+(1 - \pi, \eta) = -\lambda\, q\, \eta_2\, D(1 - \pi, \eta). \tag{A.1.11}$$

Thus (A.1.10) holds. □

We now compute an explicit formula for $D(\pi, \eta)$. Define $J(\pi, \eta)$ by the prescription:

$$J(\pi, \eta) \triangleq \int_0^\infty \frac{dy}{(1 + \pi\, y)(1 + y)^\eta}\ \ \eta > 0. \tag{A.1.12}$$

Proposition A.3. *Let $\eta > 0$, $\pi \in (0,\, 1)$ and $\beta = 1 - \pi$. Then*

$$D(\pi, \eta) = J(\pi, \eta) - J(\pi, \eta + 1), \tag{A.1.13}$$

and

$$J(\pi, \eta + 1) = \frac{1}{\beta\eta} - \frac{\pi}{\beta} J(\pi, \eta). \tag{A.1.14}$$

Proof. Set $y = e^x - 1$. Then $x = \ln(1 + y)$ and $dx = \frac{dy}{1+y}$.

$$\begin{aligned} D(\pi, \eta) &= \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta\, x} dx = \int_0^\infty \frac{y dy}{(1 + \pi\, y)(1 + y)^{\eta + 1}} \\ &= \int_0^\infty \left(1 - \frac{1}{1 + y}\right) \frac{dy}{(1 + \pi\, y)(1 + y)^\eta} \\ &= J(\pi, \eta) - J(\pi, \eta + 1). \end{aligned}$$

Let $\eta > 1$. Then

$$\begin{aligned}
J(\pi, \eta - 1) &= \int_0^\infty \frac{dy}{(1 + \pi\, y)(1 + y)^{\eta - 1}} = \int_0^\infty \frac{(1 + y)dy}{(1 + \pi\, y)(1 + y)^\eta} \\
&= \int_0^\infty \frac{dy}{(1 + \pi\, y)(1 + y)^\eta} + \int_0^\infty \frac{ydy}{(1 + \pi\, y)(1 + y)^\eta} \\
&= J(\pi, \eta) + \frac{1}{\pi} \int_0^\infty \frac{\pi\, ydy}{(1 + \pi\, y)(1 + y)^\eta} \\
&= J(\pi, \eta) + \frac{1}{\pi} \int_0^\infty \frac{(1 + \pi\, y - 1)dy}{(1 + \pi\, y)(1 + y)^\eta} \\
&= J(\pi, \eta) + \frac{1}{\pi(\eta - 1)} - \frac{1}{\pi}\, J(\pi, \eta) \\
&= -\frac{\beta}{\pi}\, J(\pi, \eta) + \frac{1}{\pi(\eta - 1)} = -\frac{\beta}{\pi}\, J(\pi, \eta) + \frac{1}{\pi(\eta - 1)}.
\end{aligned}$$

Change $\eta - 1$ to η to obtain

$$J(\pi, \eta) = -\frac{\beta}{\pi}\, J(\pi, \eta + 1) + \frac{1}{\pi\, \eta},$$

whence

$$J(\pi, \eta + 1) = \frac{1}{\beta\eta} - \frac{\pi}{\beta} J(\pi, \eta).$$

□

We now give $D(\pi, \eta)$ in terms of $J(\pi, \eta)$.

Proposition A.4. *Let $\eta > 0$, $\pi \in (0, 1)$ and $\beta = 1 - \pi$. Then*

$$D(\pi, \eta) = \frac{1}{\beta}\, J(\pi, \eta) - \frac{1}{\beta\eta}. \tag{A.1.15}$$

Proof. By Proposition A.3,

$$D(\pi, \eta) = J(\pi, \eta) - J(\pi, \eta + 1) = J(\pi, \eta) - \frac{1}{\beta\eta} + \frac{\pi}{\beta} J(\pi, \eta) = \frac{1}{\beta} J(\pi, \eta) - \frac{1}{\beta\eta}.$$

□

We also have the following recursive formula for $D(\pi,\eta)$.

Proposition A.5. *Let $\eta > 0$, $\pi \in (0,1)$ and $\beta = 1-\pi$. Then*

$$D(\pi,\eta+1) = \frac{1}{\beta\eta(\eta+1)} - \frac{\pi}{\beta}\, D(\pi,\eta). \tag{A.1.16}$$

Proof. By Proposition A.4,

$$D(\pi,\eta) = \frac{1}{\beta}\, J(\pi,\eta) - \frac{1}{\beta\eta}$$

and so

$$D(\pi,\eta+1) = \frac{1}{\beta}\, J(\pi,\eta+1) - \frac{1}{\beta(\eta+1)}.$$

By Proposition A.3,

$$D(\pi,\eta) = J(\pi,\eta) - J(\pi,\eta+1) = \beta\, D(\pi,\eta) + \frac{1}{\eta} - \beta\, D(\pi,\eta+1) + \frac{1}{\eta+1}.$$

Thus

$$(1-\beta)D(\pi,\eta) = \frac{1}{\eta} - \frac{1}{\eta+1} - \beta\, D(\pi,\eta+1),$$

whence

$$\beta\, D(\pi,\eta+1) = \frac{1}{\eta(\eta+1)} - (1-\beta)\, D(\pi,\eta),$$

which implies that (A.1.16) holds. $\square$

Remark A.1. *It follows from the recursive formulals contained in Propositions A.3–A.5, that we only need to examine $J(\pi,\eta)$ and $D(\pi,\eta)$ for $0 < \eta < 1$, where $\pi \in$ (0, 1).*

We now give a major result.

Theorem A.5. *Let* $0 < \eta < 1$, $\pi \in (0,\, 1)$ *and* $\beta = 1 - \pi$. *Then*

$$J(\pi, \eta) = \frac{1}{\pi} \left(\frac{\pi}{\beta}\right)^{\eta} B(1-\eta, \eta) \left[1 - F\left(\frac{\pi}{\beta}; 1-\eta, \eta\right)\right], \tag{A.1.17}$$

and

$$D(\pi, \eta) = \frac{1}{\beta\,\pi} \left(\frac{\pi}{\beta}\right)^{\eta} B(1-\eta, \eta) \left[1 - F\left(\frac{\pi}{\beta}; 1-\eta, \eta\right)\right] - \frac{1}{\beta\,\pi}, \tag{A.1.18}$$

where $F(x; a, b)$ *is the cumulative distribution function of a Beta random variable with parameters* a *and* b*, and* $B(a, b)$ *is the corresponding Beta function.*

Proof. Let $\pi \in (0,\, 1)$, $\beta = 1 - \pi$ and $z = 1 + \pi\, y$. Then $y = \frac{z-1}{\pi}$, and $1 + y = \frac{z-\beta}{\pi}$, with $z = 1$ when $y = 0$. Now

$$J(\pi, \eta) = \int_0^\infty \frac{dy}{(1+\pi\, y)(1+y)^\eta} = \frac{1}{\pi} \int_1^\infty \frac{dz}{z(\frac{z-\beta}{\pi})^\eta} = \pi^{\eta-1} \int_1^\infty \frac{dz}{z(z-\beta)^\eta}.$$

Put $t = z - \beta$. Thus

$$J(\pi, \eta) = \pi^{\eta-1} \int_{1-\beta}^\infty \frac{dt}{(t+\beta)t^\eta} = \frac{\pi^{\eta-1}}{\beta} \int_\pi^\infty \frac{dt}{(1+\frac{t}{\beta})t^\eta}.$$

Set $t = \beta\, y$. Therefore

$$\begin{aligned}
J(\pi,\eta) &= \frac{\pi^{\eta-1}}{\beta}\int_{\frac{\pi}{\beta}}^{\infty}\frac{\beta\, dy}{(1+y)(\beta\, y)^{\eta}} \\
&= \frac{1}{\pi}\left(\frac{\pi}{\beta}\right)^{\eta}\left[\int_0^{\infty}\frac{dy}{(1+y)y^{\eta}} - \int_0^{\frac{\pi}{\beta}}\frac{dy}{(1+y)y^{\eta}}\right] \\
&= \frac{1}{\pi}\left(\frac{\pi}{\beta}\right)^{\eta}\left[B(1-\eta,\eta) - \int_0^{\frac{\pi}{\beta}} y^{1-\eta-1}(1+y)^{-(1-\eta)-\eta}dy\right] \\
&= \frac{1}{\pi}\left(\frac{\pi}{\beta}\right)^{\eta} B(1-\eta,\eta)\left[1 - \frac{1}{B(1-\eta,\eta)}\int_0^{\frac{\pi}{\beta}} y^{1-\eta-1}(1+y)^{-(1-\eta)-\eta}dy\right] \\
&= \frac{1}{\pi}\left(\frac{\pi}{\beta}\right)^{\eta} B(1-\eta,\eta)\left[1 - P\left(X_{(1-\eta,\eta)} \le \frac{\pi}{\beta}\right)\right] \\
&= \frac{1}{\pi}\left(\frac{\pi}{\beta}\right)^{\eta} B(1-\eta,\eta)\left[1 - F\left(\frac{\pi}{\beta}\,; 1-\eta,\eta\right)\right],
\end{aligned}$$

where $F(x; a, b) = P(X_{(a,b)} \le x)$ is the cumulative distribution function of the Beta random variable $X_{(a,b)}$ with parameters a, b, and $B(a, b)$ is the corresponding Beta function. It is clear that (A.1.18) follows directly from Proposition A.4 and (A.1.17).

□

Remark A.2. *The Beta cumulative distribution function (BCDF) $F(x; a, b)$, which is also called the Incomplete Beta function, is readily available in many statistical packages. It can also be directly implemented in MATHLAB. Consequently, $J(\pi, \eta)$ and equivalently $D(\pi, \eta)$, can be computed directly from the BCDF.*

Example A.2. *We compute $D(\pi, 1.5)$. By Proposition A.4,*

$$D(\pi, 1.5) = \frac{1}{\beta(.5)(1.5)} - \frac{\pi}{\beta}\, D(\pi, .5) = \frac{4}{3\beta} - \frac{\pi}{\beta}\, D(\pi, .5),$$

and from Theorem A.5,

$$D(\pi, .5) = \frac{1}{\beta\,\pi}\sqrt{\frac{\pi}{\beta}}\, B(.5, .5)\left[1 - F\left(\frac{\pi}{\beta}; .5, .5\right)\right] - \frac{1}{\beta\,\pi}.$$

Putting $y = t^2$, we have

$$F(x; .5, .5) = \int_0^x \frac{dy}{\sqrt{y}(1+y)} = 2\int_0^{\sqrt{x}} \frac{dt}{1+t^2} = 2\,\tan^{-1}(\sqrt{x}).$$

Thus

$$D(\pi, .5) = \frac{1}{\beta\,\pi}\sqrt{\frac{\pi}{\beta}}\,B(.5, .5)\left[1 - 2\,\tan^{-1}\left(\sqrt{\frac{\pi}{\beta}}\right)\right] - \frac{1}{\beta\,\pi},$$

whence

$$D(\pi, 1.5) = \frac{4}{3\beta} - \frac{1}{\beta^2}\sqrt{\frac{\pi}{\beta}}\,B(.5, .5)\left[1 - 2\,\tan^{-1}\left(\sqrt{\frac{\pi}{\beta}}\right)\right] + \frac{1}{\beta^2}.$$

We now give the main result for the Kou model.

Theorem A.6. *Let $\pi \in (0,1)$ and $\beta = 1 - \pi$. For the Kou model with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$, $p + q = 1$, p, $q \geq 0$, Sharpe ratio θ_t and volatility σ_t, the optimal portfolio is the unique solution of the equation*

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{(\lambda\,p\,\eta_1\,D(\pi_t, \eta_1) - \lambda\,q\,\eta_2\,D(1-\pi_t, \eta_2))}{\sigma_t^2}, \tag{A.1.19}$$

where $D(\pi_t, \eta_1)$ and $D(1-\pi_t, \eta_2)$ can be computed from equation (A.1.18).

Proof. Theorem 3.8 gives the unique optimal portfolio as

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2}.$$

The result follows by importing Propositions A.2, A.5 and Theorem 5.7. □

Analytic Formula for $G''(\pi)$

Consider the Kou model with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$, $p + q = 1$, $p,\ q\ \geq 0$. By Propostion A.2,

$$G'(\pi) = \lambda\, p\, \eta_1\, D(\pi, \eta_1) - \lambda\, q\, \eta_2\, D(1-\pi, \eta_2), \tag{A.1.20}$$

where

$$D(\pi, \eta) \triangleq \int_0^\infty \frac{e^x - 1}{1 + \pi(e^x - 1)} e^{-\eta\, x} dx.$$

Thus

$$G''(\pi) = \lambda\, p\, \eta_1\, D'(\pi, \eta_1) + \lambda\, q\, \eta_2\, D'(1-\pi, \eta_2), \tag{A.1.21}$$

where

$$D'(\pi, \eta) = -\int_0^\infty \left(\frac{e^x - 1}{1 + \pi(e^x - 1)} \right)^2 e^{-\eta\, x} dx.$$

For $\eta > 0$ and $\pi \in (0,\ 1)$, define

$$A(\pi, \eta) \triangleq \int_0^\infty \left(\frac{e^x - 1}{1 + \pi(e^x - 1)} \right)^2 e^{-\eta\, x} dx. \tag{A.1.22}$$

Clearly $A(\pi, \eta) > 0$ and from equation (A.1.20),

$$G''(\pi) = -\lambda\, p\, \eta_1\, A(\pi, \eta_1) - \lambda\, q\, \eta_2\, A(1-\pi, \eta_2), \tag{A.1.23}$$

which confirms that $G''(\pi) < 0$ for all $\pi \in (0,\ 1)$.

Proposition A.6. *For $\eta > 0$ and $\pi \in (0,\ 1)$*

$$A(\pi, \eta) = \pi^{\eta - 2} \left[I_1 - 2I_2 + I_3 \right], \tag{A.1.24}$$

where

$$I_1 = \frac{1}{\eta\,\pi^{\eta}}, \tag{A.1.25}$$

$$I_2 = \frac{1}{\pi^{\eta-1}} J(\pi, \eta+1), \tag{A.1.26}$$

and

$$I_3 = \int_1^{\infty} \frac{dz}{z^2 (z-\beta)^{\eta+1}}. \tag{A.1.27}$$

Proof. Set $y = e^x - 1$. Then

$$A(\pi, \eta) = \int_0^{\infty} \frac{y^2\, dy}{(1+\pi\, y)^2\, (1+y)^{\eta+1}},\ \ \eta > 0. \tag{A.1.28}$$

Set $z = 1 + \pi\, y$ and $\beta = 1 - \pi$. Then $y = \frac{z-1}{\pi}$ and $1 + y = \frac{z-\beta}{\pi}$ and

$$\begin{aligned}
A(\pi, \eta) &= \pi^{\eta-2} \int_1^{\infty} \frac{(z-1)^2\, dz}{z^2\, (z-\beta)^{\eta+1}}, \\
&= \pi^{\eta-2} \left[\int_1^{\infty} \frac{dz}{(z-\beta)^{\eta+1}} - 2 \int_1^{\infty} \frac{dz}{z\, (z-\beta)^{\eta+1}} + \int_1^{\infty} \frac{dz}{z^2\, (z-\beta)^{\eta+1}} \right], \\
&= \pi^{\eta-2} \left[I_1 - 2 I_2 + I_3 \right],
\end{aligned}$$

where

$$I_1 = \int_1^{\infty} \frac{dz}{(z-\beta)^{\eta+1}} = \frac{1}{\eta (1-\beta)^{\eta}} = \frac{1}{\eta\, \pi^{\eta}}.$$

By equation (A.1.12),

$$I_2 = \int_1^{\infty} \frac{dz}{z\, (z-\beta)^{\eta+1}} = \frac{1}{\pi^{\eta-1}} J(\pi, \eta+1),$$

which can be computed in detail using Theorem A.5, and

$$I_3 = \int_1^{\infty} \frac{dz}{z^2\, (z-\beta)^{\eta+1}}. \tag{A.1.29}$$

$\square$

Proposition A.7. *Let $0 < \eta < 1$, $\pi \in (0,1)$, $\beta = 1-\pi$ and $a = \frac{\pi}{\beta}$. Then*

$$I_3 = \left(\frac{a}{\pi}\right)^{\eta+2}\left[\frac{1}{\eta\, a^{\eta}} - B(1-\eta,\,\eta)P[X_{(1-\eta,\eta)} > a] - B(1-\eta,\,1+\eta)P[X_{(1-\eta,\eta+1)} > a]\right], \tag{A.1.30}$$

where $X_{(u,v)}$ is a Beta random variable with parameters u and v, and $B(u,v)$ is the corresponding Beta function.

Proof. Let $t = z - \beta$. Then

$$I_3 = \int_{\pi}^{\infty} \frac{dt}{(t+\beta)^2\, t^{\eta+1}} = \frac{1}{\beta^2}\int_{\pi}^{\infty} \frac{dt}{(1+\frac{t}{\beta})^2\, t^{\eta+1}}.$$

Set $y = \frac{t}{\beta}$, and $a = \frac{\pi}{\beta}$. Thus

$$I_3 = \frac{1}{\beta^{\eta+2}}\int_{a}^{\infty} \frac{dy}{(1+y)^2\, y^{\eta+1}} = \frac{1}{\beta^{\eta+2}}\, H(\eta+1,\,a) = \left(\frac{a}{\pi}\right)^{\eta+2} H(\eta+1,\,a),$$

where

$$H(\eta,\,a) = \int_{a}^{\infty} \frac{dy}{(1+y)^2\, y^{\eta}},\ \ a > 0. \tag{A.1.31}$$

Now

$$\begin{aligned}
H(\eta+1,\,a) &= \int_{a}^{\infty} \frac{dy}{y(1+y)(1+y)\, y^{\eta}} = \int_{a}^{\infty}\left(\frac{1}{y} - \frac{1}{1+y}\right)\frac{dy}{(1+y)\, y^{\eta}} \\
&= \int_{a}^{\infty} \frac{dy}{(1+y)\, y^{\eta+1}} - \int_{a}^{\infty} \frac{dy}{(1+y)^2\, y^{\eta}} \\
&= \int_{a}^{\infty} \frac{dy}{y^{\eta+1}} - \int_{a}^{\infty} \frac{dy}{(1+y)\, y^{\eta}} - \int_{a}^{\infty} \frac{dy}{(1+y)^2\, y^{\eta}} \\
&= \frac{1}{\eta\, a^{\eta}} - B(1-\eta,\eta)P[X_{(1-\eta,\eta)} > a] - B(1-\eta,1+\eta)P[X_{(1-\eta,\eta+1)} > a],
\end{aligned}$$

where $X_{(u,v)}$ is a Beta random variable with parameters, u and v and $B(u,v)$ is the corresponding Beta function. $\square$

We now give an explicit formula for $G''(\pi)$.

Theorem A.7. *Let $\pi \in (0,1)$, $\beta = 1-\pi$ and $a = \frac{\pi}{\beta}$. For the Kou model with parameters $\eta_1 \geq 1$, $\eta_2 > 0$, $\lambda > 0$, $p+q=1$, p, $q \geq 0$,*

$$G''(\pi_t) = -\lambda\, p\, \eta_1\, A(\pi_t, \eta_1) - \lambda\, q\, \eta_2\, A(1-\pi_t, \eta_2), \tag{A.1.32}$$

where

$$\begin{aligned} A(\pi,\eta) &= \pi^{\eta-2}\left(I_1 - 2I_2 + I_3\right), \\ I_1 &= \frac{1}{\eta\,\pi^{\eta}}, \\ I_2 &= \frac{1}{\pi^{\eta-1}} J(\pi,\eta+1), \\ I_3 &= \left(\frac{a}{\pi}\right)^{\eta+2} H(\eta+1,\,a), \end{aligned}$$

where $J(\pi,\eta+1) = \frac{1}{\beta\eta} - a\,J(\pi,\eta)$, and for $0 < \eta < 1$,

$$J(\pi,\eta) = \frac{1}{\pi}\, a^{\eta}\, B(1-\eta,\eta)\left[1 - F(a; 1-\eta,\eta)\right],$$

and

$$H(\eta+1,a) = \frac{1}{\eta\, a^{\eta}} - B(1-\eta,\eta)[1-F(a;1-\eta,\eta)] - B(1-\eta,1+\eta)[1-F(a;1-\eta,\eta+1)], \tag{A.1.33}$$

where $F(x;u,v)$ is given in Proposition A.7.

Proof. The result follows from Propositions A.7, A.6, A.3 and A.2, and equation A.1.23. □

Remark A.3. *We can easily set up a recursive formula for $H(\eta+1,a)$ in terms of*

$H(\eta, a)$ when $\eta > 1$, as follows. Define

$$J(\eta, a) = \int_a^\infty \frac{dy}{(1+y)\, y^\eta}, \;\; a > 0, \eta > 0. \tag{A.1.34}$$

Since

$$H(\eta,\, a) = \int_a^\infty \frac{dy}{(1+y)^2\, y^\eta}, \;\; a > 0, \eta > 0, \tag{A.1.35}$$

it is easy to show that

$$H(\eta, a) = J(\eta + 1, a) - H(\eta + 1, a), \tag{A.1.36}$$

where

$$J(\eta + 1, a) = \frac{1}{\eta\, a^\eta} - J(\eta, a). \tag{A.1.37}$$

Specifically, if $0 < \eta < 1$ and $a = \frac{\pi}{1-\pi}$, then

$$J(\eta, a) = B(1-\eta, \eta)\, [1 - F(a; 1-\eta, \eta)], \tag{A.1.38}$$

where $F(x; u, v)$ is the cumulative distribution function of a Beta random variable with parameters u and v, and $B(u, v)$ is the corresponding Beta function.

Appendix B

The CGMY Diffusion Market

The C, $G\,M$, Y Lévy process $X^{(CGMY)}$ was developed by Carr, Geman, Madan and Yor [12], and is named after its creators. It is a generalization of the Variance Gamma process VG (C, G, M) by the addition of a stability parameter Y. It reverts to the the VG (C, G, M) process when $Y = 0$. Its Lévy density and characteristic function are, respectively:

$$v(x) \equiv v_{CGMY}(x) = \frac{C}{|x|^{1+Y}}\, e^{-G\,|x|} I_{\{x<0\}}(x) + \frac{C}{x^{1+Y}}\, e^{-M\,x} I_{\{x>0\}}(x), \tag{B.0.1}$$

and

$$\begin{aligned} \phi_t(u) &= e^{t\,\eta(u)}, \;\; u \in \mathcal{R}, && \text{(B.0.2)} \\ \eta(u) &= C\,\Gamma(-Y)\, \left\{(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right\}, && \text{(B.0.3)} \end{aligned}$$

where C, $M \geq 0\,, G > 0$, $Y < 2$, where $Y \neq 1$. It is a pure jump process with Lévy triple (γ, 0, v_{CGMY}), where

$$\gamma = C\, \left(\int_0^1 e^{-M\,x}\, x^{-Y}\, dx - \int_{-1}^0 e^{G\,x}\, |x|^{-Y}\, dx \right). \tag{B.0.4}$$

The behaviour of this process is controlled by the stability parameter Y. If $Y < 0$, the paths have finite jumps in any finite interval; that is, it has finite activity. If $Y \geq 0$, the process exhibits infinitely many jumps in any finite time interval; that is, it has infinite activity.

Its Lévy measure $v_{CGMY}(x)$, is a Completely Monotone (CM) Lévy measure, in that, it relates arrival rates of large jump sizes to smaller jump sizes by requiring that large jumps arrive less frequently than smaller jumps. The CGMY process may be of finite activity (FA), infinite activity (IA), finite variation (FV), quadratic variation (QV), or infinite variation (IV), depending on the value of Y the stability parameter, as shown below (cf, CGMY [12]). We summarize these facts in the following theorem:

Range of Y values	Properties of $CGMY$ Process
$-\infty < Y < -1$	NCM, FA
$-1 < Y < 0$	CM, FA
$0 < Y < 1$	CM, IA, FV
$1 < Y < 2$	CM, IA, IV, QV

Theorem B.1 (Carr et al [12]). *The $CGMY$ Lévy process satisfies the following properties.*

(1) *It has completely monotone Lévy density if $Y > -1$.*

(2) *It has infinite activity ($\int_{\mathbf{R}} v(x)dx = \infty$) if $Y > 0$.*

(3) *It has infinite variation ($\int_{|x|\leq 1} |x|\, v(x)dx = \infty$) if $Y > 1$.*

B.1 The CGMY Market

The $CGMY$ diffusion market consist of a bond $\mathbf{B}$ that earns the risk–free interest rate r_t with price $\mathbf{B}_t = \exp\left(\int_0^t r_s\, ds\right)$, and a single stock S which has log returns price dynamic:

$$d(\log S_t) = (\mu_t - \frac{1}{2}\sigma_t^2)\, dt + \sigma_t\, dB_t + dX_t^{CGMY}, \tag{B.1.1}$$

which is equivalent (by Itô's formula) to the percentage returns dynamic:

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dB_t + \int_R (e^x - 1)\, N(dt, dx), \tag{B.1.2}$$

where

$$X_t^{CGMY} = \int_R x\, N(t, dx), \tag{B.1.3}$$

and

$$b_t = \mu_t + \int_R (e^x - 1)\, v_{CGMY}(x)\, dx, \tag{B.1.4}$$

is the stock's total expected returns, and μ_t is the continuous component of the total returns, with volatility $\sigma_t > 0$. $N(dt, dx)$ is the Poisson random measure independent of the standard Brownian motion B, that counts the jumps of X^{CGMY}. The Lévy measure is given by: $v_{CGMY}(dx) = \mathbf{E}N(1, dx)$.

B.2 CGMY Instantaneous Centralized Moments

These are the usual candidates, defined by the prescriptions:

$$M_k \;=\; \int_R (e^x - 1)^k\, v_{CGMY}(x)\, dx \quad \text{and} \quad K(s) = \int_R (e^{s\,x} - 1)\, v_{CGMY}(x)\, dx,\; s \geq 0.$$

We now give a brief review of the negative Gamma function.

B.2.1 The Negative Gamma Function

The positive Gamma function is defined for positive argument $a > 0$, as follows:

$$\Gamma(a) = \int_{0+}^{\infty} x^a \, e^{-x} \frac{dx}{x}. \tag{B.2.1}$$

which yields the formulas:

$$\Gamma(a+1) = a\,\Gamma(a) \quad \text{and} \quad \Gamma(a+2) = \ a\,(a+1)\,\Gamma(a). \tag{B.2.2}$$

We define the negative Gamma function by the prescriptions:

$$\Gamma(a) = \frac{1}{a}\,\Gamma(a+1), \quad a \in (-1,\, 0), \tag{B.2.3}$$

$$= \frac{1}{a\,(a+1)}\,\Gamma(a+2), \quad a \in (-2,\, -1). \tag{B.2.4}$$

For example,

$$\Gamma(-\frac{3}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{3}{2}\,(-\frac{1}{2})} = \frac{4}{3}\Gamma(\frac{1}{2}) = \frac{4}{3}\sqrt{\pi}.$$

The following result will be useful in the sequel.

Lemma B.1. *For each $a > 0$ and $y < 2$ with $y \neq 0, 1$,*

$$\int_{0+}^{\infty} e^{-a\,x} \frac{dx}{x^{1+y}} = a^y \, \Gamma(-y). \tag{B.2.5}$$

Proof. $\int_{0+}^{\infty} e^{-a\,x} \frac{dx}{x^{1+y}} = \int_{0+}^{\infty} x^{-y}\, e^{-a\,x} \frac{dx}{x} = a^y \int_{0+}^{\infty} (a\,x)^{-y}\, e^{-a\,x} \frac{dx}{x} = a^y \, \Gamma(-y)$, by (B.2.1). □

Lemma B.2. *Let $X = (X_t^{CGMY})_{t \geq 0}$ be the CGMY process. Then for each $s < M$*

$$K(s) = C\,\Gamma(-Y)\,[(M-s)^Y - M^Y + (G+s)^Y - G^Y]. \tag{B.2.6}$$

Proof. Applying Lemma B.2, we get

$$\begin{aligned}
\frac{K(s)}{C} &= \int_{-\infty}^{0-}(e^{sx}-1)e^{G\,x}\frac{dx}{(-x)^{1+y}} + \int_{0+}^{\infty}(e^{sx}-1)e^{-M\,x}\frac{dx}{x^{1+y}} \\
&= \int_{0+}^{\infty}(e^{-sx}-1)e^{-G\,x}\frac{dx}{x^{1+y}} + \int_{0+}^{\infty}(e^{sx}-1)e^{-M\,x}\frac{dx}{x^{1+y}} \\
&= \int_{0+}^{\infty}(e^{-(G+s)\,x}-e^{-G\,x})\frac{dx}{x^{1+y}} + \int_{0+}^{\infty}(e^{-(M-s)\,x}-e^{-M\,x})\frac{dx}{x^{1+y}} \\
&= \Gamma(-Y)((G+s)^Y - G^Y) + \Gamma(-Y)((M-s)^Y - M^Y),
\end{aligned}$$

whence $K(s) = C\,\Gamma(-Y)[(M-s)^Y - M^Y + (G+s)^Y - G^Y]$. □

We express the last result in a more useful form.

Corollary B.1. *Let* $\phi(t) = t^Y,\ t \geq 0$. *Then for each* $s \leq M$

$$K(s) = C\,\Gamma(-Y)[\phi(G+s) - \phi(G) + \phi(M-s) - \phi(M)]. \qquad \text{(B.2.7)}$$

With these explicit values of $K(s)$ in hand, we are now able to compute M_1, M_2, M_3, M_4, using the formula $M_k = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}K(j)$.

Theorem B.2. *Let* X *be the CGMY process with parameters* C, G, M, Y *and* $\Delta\phi(j) = \phi(j+1) - \phi(j)$. *Then*

(1) *If* $M > 1$, *then* $K_1 = M_1 = C\,\Gamma(-Y)[\Delta\phi(G) - \Delta\phi(M-1)]$.

(2) *If* $M > 2$, *then* $M_2 = C\,\Gamma(-Y)[\Delta^2\phi(G) - \Delta^2\phi(M-2)]$.

(3) *If* $M > 3$, *then* $M_3 = C\,\Gamma(-Y)[\Delta^3\phi(G) - \Delta^3\phi(M-3)]$.

(4) *If* $M > 4$, *then* $M_4 = C\,\Gamma(-Y)[\Delta^4\phi(G) - \Delta^4\phi(M-4)]$.

Proof. In each case, we assume that $M > k$, for $k = 1, 2, 3, 4$. By Corollary B.1, $M_1 = C\,\Gamma(-Y)[\phi(M-1) - \phi(M) + \phi(G+1) - \phi(G)] = C\,\Gamma(-Y)[\Delta\phi(G) - \Delta\phi(M-1)]$.

(2) If $M > 2$, then K_2 exists and $M_2 = K_2 - 2K_1$. Thus

$$\begin{aligned}
M_2 &= C\,\Gamma(-Y)[\phi(M-2) - \phi(M) + \phi(G+2) - \phi(G) \\
&\quad - 2(\phi(M-1) - \phi(M) + \phi(G+1) - \phi(G))] \\
&= C\,\Gamma(-Y)[\phi(G+2) - 2\phi(G+1) - \phi(G) + (\phi(M) - 2\phi(M-1) - \phi(M-2)] \\
&= C\,\Gamma(-Y)[\Delta\phi(G+1) - \Delta\phi(G) + \Delta\phi(M-1) - \Delta\phi(M-2)] \\
&= C\,\Gamma(-Y)[\Delta^2\phi(G) + \Delta^2\phi(M-2)] \\
&= C\,\Gamma(-Y)[\Delta^k\phi(G) + (-1)^k\,\Delta^2\phi(M-k)],\ \ k = 2.
\end{aligned}$$

(3) If $M > 3$, then K_3 exits and $M_3 = K_3 - 3K_2 + 3K_1$. Thus

$$\begin{aligned}
M_3 &= C\,\Gamma(-Y)[\phi(G+3) - 3\phi(G+2) + 3\phi(G+1) - \phi(G) \\
&\quad - (\phi(M) - 3\phi(M-1) + 3\phi(M-2) - \phi(M-3)] \\
&= C\,\Gamma(-Y)[\Delta^3\phi(G) - \Delta^3\phi(M-3)] \\
&= C\,\Gamma(-Y)[\Delta^3\phi(G) + (-1)^3\Delta^3\phi(M-3)] \\
&= C\,\Gamma(-Y)[\Delta^k\phi(G) + (-1)^k\Delta^k\phi(M-k)],\ \ k = 3.
\end{aligned}$$

(4) If $M > 4$, then K_4 exists and $M_4 K_4 + 6K_2 - 4K_3 - 4K_1$. Thus

$$\begin{aligned}
M_4 &= C\,\Gamma(-Y)[(\phi(G+4) - 4\phi(G+3) + 6\phi(G+2) - 4\phi(G+1) + \phi(G) \\
&\quad + (\phi(M) - 4\phi(M-1) + 6\phi(M-2) - 4\phi(M-3) + \phi(M-4))] \\
&= C\,\Gamma(-Y)[\Delta^4\phi(G) + \Delta^4\phi(M-4)] \\
&= C\,\Gamma(-Y)[\Delta^k\phi(G) + \Delta^k\phi(M-k)],\ \ k = 4. \qquad \square
\end{aligned}$$

We may extend the last result inductively to obtain:

Theorem B.3. *Let X be the CGMY process with parameters C, G, M, Y and*

$\phi(t) = t^Y$, $t \geq 0$. *Then for each positive integer* $k < M$,

$$M_k = C\,\Gamma(-Y)[\Delta^k \phi(G) + \Delta^k \phi(M-k)]. \tag{B.2.8}$$

We are now in a position to give the optimal portfolio π and its estimate $\pi^{(k)}$, based on the k–th degree polynomial approximation of $G(\alpha) = \int_R \log(1+\alpha(e^x-1))v_{CGMY}(dx)$.

B.3 Maximization of Logarithmic Utility from Terminal Wealth

Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth at time T in the CGMY market. Applying Theorem 3.8 yields

Theorem B.4. *Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the CGMY market. Then, for each $t \in [0, T]$*

(1)

$$\pi_t = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi_t)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi_t)}{\sigma_t^2}, \tag{B.3.1}$$

(2) *The maximum expected utility starting with $x > 0$ in wealth, is*

$$u(x) = \log(x) + \frac{1}{2}\mathbf{E}\int_0^T \theta_t^2\,dt + \mathbf{E}\int_0^T f(\pi_t)\,dt, \tag{B.3.2}$$

where $f(\pi_t) = -\frac{1}{2}(\pi_t\,\sigma_t - \theta_t)^2 + G(\pi_t)$.

Proof. This follows directly from Theorem 3.8. □

Remark B.1. *Because $G(\pi)$, and hence $G'(\pi)$, is in general very difficult to compute, we resort to approximation methods. This leads to an approximation π_k of π, based on a k–degree truncated Taylor series expansion of G.*

Theorem B.5. *Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the CGMY market with parameters C, G, M, Y.*
(1) *For each integer $k < M$ there is an approximation $\pi^{(k)}$ such that for each $t \in [0, T]$*

$$\pi_t^{(k)} = \pi_{mer}(t) + \frac{G_k'(\pi_t^{(k)})}{\sigma_t^2} = \frac{\mu_t - r_t + G_k'(\pi_t^{(k)})}{\sigma_t^2}, \tag{B.3.3}$$

where $M_k = C\,\Gamma(-Y)[\Delta^k\phi(G) + \Delta^k\phi(M-k)]$ and $G_k(\alpha) = \sum_{j=1}^{k}(-1)^{j-1} M_j \frac{\alpha^j}{j}$.
(2) *The maximum expected logarithmic utility $u(x)$ with $x > 0$ in initial wealth, is approximated by $u^{(k)}(x)$ given by*

$$u^{(k)}(x) = \log(x) + \frac{1}{2}\mathbf{E}\int_0^T \theta_t^2\,dt + \mathbf{E}\int_0^T \left[G_k(\pi_t^{(k)}) - \frac{1}{2}(\pi_t^{(k)}\,\sigma_t - \theta_t)^2\right] dt. \tag{B.3.4}$$

(3) *If the market parameters μ_t and r_t are constants, then*

$$u^{(k)}(x) = \log(x) + T\left[G_k(\pi_t^{(k)}) + \pi_t^{(k)}(\mu_t - r_t) - \frac{1}{2}(\pi_t^{(k)}\,\sigma_t)^2\right]. \tag{B.3.5}$$

Proof. Replace G by G_k in Theorem B.4, and the result follows. □

Theorem B.6. *Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the CGMY market given by (B.1.2) with parameters C, G, M, Y. Suppose $G(\alpha)$ is approximated by $G_k(\alpha)$for each integer $k < M$. Let $0 < \rho < 1$, $t \in [0, T]$ and $b_t = \mu_t + M_1$ is the stock's total returns. Then π is approximated by $\pi^{(k)}$ and*
(1) *Under linear approximation of G,*

$$\begin{aligned}\pi_t^{(1)} &= \frac{\mu_t - r_t + C\,\Gamma(-Y)(\phi(G+1) - \phi(G) + \phi(M-1) - \phi(M))}{\sigma_t^2} \qquad \text{(B.3.6)}\\ &= \frac{\mu_t - r_t + M_1}{\sigma_t^2} = \frac{b_t - r_t}{\sigma_t^2},\end{aligned}$$

(2) *Under quadratic approximation of G,*

$$\begin{aligned}\pi_t^{(2)} &= \frac{\mu_t - r_t + C\,\Gamma(-Y)(\phi(G+1) - \phi(G) + \phi(M-1) - \phi(M))}{\sigma_t^2 + C\,\Gamma(-Y)[\Delta^2\phi(G) + \Delta^2\phi(M-2)]} \\ &= \frac{\mu_t - r_t + M_1}{\sigma_t^2 + M_2} = \frac{b_t - r_t}{\sigma_t^2 + M_2}.\end{aligned} \tag{B.3.7}$$

(3) *Under cubic approximation of G,*

$$\begin{aligned}\pi_t^{(3)} &= \pi_\pm = \frac{(\sigma_t^2 + M_2) \pm \sqrt{(\sigma_t^2 + M_2)^2 - 4\,M_3\,(\mu_t - r_t + M_1)}}{2\,M_3} \\ &= \frac{\mathbf{Var_t} \pm \sqrt{(\mathbf{Var_t})^2 - 4\,M_3\,(b_t - r_t)}}{2\,M_3}\end{aligned} \tag{B.3.8}$$

where $\mathbf{Var_t} = \sigma_t^2 + M_2$. (4) *The maximum expected logarithmic utility from terminal wealth with* $x > 0$ *in initial wealth is approximated by* $u^{(k)}(x)$ *and given by*

$$u^{(k)}(x) = \log(x) + \frac{1}{2}\mathbf{E}\int_0^T \theta_t^2\,dt + \mathbf{E}\int_0^T \left[G_k(\pi_t^{(k)}) - \frac{1}{2}(\pi_t^{(k)}\,\sigma_t - \theta_t)^2\right]\,dt. \tag{B.3.9}$$

Proof. This is the same as the proof of Theorem 5.18. □

Bibliography

[1] Ait-Sahalia, Y., *Disentangling Diffusion from Jumps*, Journal of Financial Economics, **74**, 487–528, (2004).

[2] Akgiray, G., and Booth, G., *The Stable-Law Model of Stock Returns*, Journal of Business and Economic Statistics, **6**, 51–57, (1988).

[3] Applebaum, D., *Lévy Processes and Stochastic Calculus*, Cambridge University Press, (2004).

[4] Barndorff-Nielson, O., *Normal Inverse Gaussian Distribution and the Modelling of Stock Returns*, Reseach Report **no. 300**, Department of Theorectical Statistics, Aarhus University, (1995).

[5] Barndorff-Nielson, O., *Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling*, Scandinavian Journal of Statistics, **24**, 1–13, (1997).

[6] Barndorff-Nielson, O., *Processes of Normal Inverse Gaussian Type*, Finance and Stochastics, Vol **2**, No 1, 41–68, (1998).

[7] Bertoin, J., *Lévy Processes*, Cambridge University Press, (1996).

[8] Billingsley, P., *Probability and Measure*, John Wiley and Sons, 3rd Ed, (1995).

[9] Bingham, N., and Kiesel, R., *Risk-Neutral Valuation, Pricing and Hedging of Financial Derivatives*, Springer-Verlag, (1998).

[10] Black, F., and Scholes, M., *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, **81**, 637–659, (1973).

[11] Brunnermeier, M., *Asset Pricing Under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding*, Oxford University Press, (2001).

[12] Carr, P., Geman, H.,Madan, D., and Yor, M., *The Fine Structure of Asset Returns: An Empirical Investigation*, Journal of Business, **76**, 305–332, (2002).

[13] Chan, T., *Pricing Contingent Claims on Stocks Driven by Lévy Processes*, Annals of Probability, **Vol. 9, No. 2**, 504–528, (1999).

[14] Cheridito, P., *Representation of Gaussian Measures that are Equivalent to Wiener Measure*, Springer, **vol. 1832**, 81–89, (2003).

[15] Cont, R., da Fonseca, J., *Dynamics of Implied Volatility Surfaces*, Quantitative Finance, **2**, 45–60, (2002).

[16] Cont, R., Tankov, P., *Financial Modelling with Jump Processes*, Chapman and Hall/CRC, (2004).

[17] Delbean, F., and Schachermayer, W., *A General Version of the Fundamental Theorem of Asset Pricing*, Math. Annals, **300**, 463–520, (1994).

[18] Eberlein, E., and Keller, U., *Hyperbolic Distributions in Finance*, Bernoulli **1**, 281–299, (1995).

[19] Eberlein, E., and Prause, K., *The Generalized Hyperbolic Model: Financial Derivatives and Risk Measures*, FDM Preprint **56**, University of Freiberg, (1998).

[20] Fama, E., F., *Efficient Capital Markets: A Review of Theory and Empirical Work*, Journal of Finance, **25**, 383–417, (1970).

[21] Guasoni, P., *Asymmetric Information in Fads Models*, Finance and Stochastic, **10**, 159–177, (2006).

[22] Grimmett, G., and Stirzaker, D., *Probability and Random Processes*, Oxford University Press, 3rd Ed, (2001).

[23] Harrison, J., and Pliska, S., *Martingales and Stochastic Integrals in the Theory of Continuous Trading*, Stochastic Processes Applications, **11**, 215–260, (1981).

[24] Hitsuda, M., *Representation of Gaussian Processes Equivalent to Wiener Process*, Osaka Journal of Math, **5**, 299–312, (1968).

[25] Imkeller, P., and Amendinger, J., *Additional Logarithmic Utility of an Insider*, Stochastic Process Application, **75**, 263-286, (1998).

[26] Imkeller, P., and Ankirchner, S., *Financial Markets with Asymmetric Information, Information Drift, Additional Utility and Entrophy*, Humboldt University, August 25, 2006.

[27] Karatzas, I., and Pikovsky, *Anticipative Portfolio Optimization*, Annals of Probability,**28**, 1095–1122, (1996).

[28] Karatzas, I., and Sheves, S., *Martingale and Duality Methods for Utility Maximization in an Incomplete Market*, SIAM Journal on Control and Optimization, Volume **29**, Issue **3**, 702–730, (1991).

[29] Karatzas, I., and Shreve, S., *Brownian Motion and Stochastic Calculus*, 2nd ed, Springer-Verlag, (1999).

[30] Kou, S., *A Jump-Diffusion Model for Option Pricing*, Management Science, **48**, 1086–1101, (2002).

[31] Lamberton, D., and Lapeyre, B., *Intoduction to Stochastic Calculus Applied to Finance*, Chapman & Hall/CRC, (2000).

[32] LeRoy, S., and Porter, R., *The Present Value Relationship*, Econmetrica, **49(3)**, 555–574, (1981).

[33] Lewis, A., *Fear of Jumps*, Wilmott Magazine, (2002).

[34] Liu, J., Longstaff, F., Pan, J., *Dynamic Asset Allocation and Event Risk*, Journal of Finance, **58**, 231–259, (2003).

[35] Madan, D.,Seneta, E., *The VG Model for Share Market Returns*, Journal of Business, **63**, 511–524, (1990).

[36] Merton, R., C., *Optimum Consumption and Portfolio Rules in a Continuous-time Model*, Journal of Economic Theory, **3**, 373–413, (1971).

[37] Merton, R., C., *The Pricing of Options and Corporate Liabilities*, Bell Journal of Economics and Management Science, **4**, 141–183, (1973).

[38] Merton, R., C., *Option Pricing When Underlying Stock Returns are Discontinuous*, J. Financial Economics, **3**, 125–144, (1976).

[39] Merton, R., C., *Continuous–Time Finance*, Blackwell Publishers Ltd., (1992).

[40] Oksendal, B., *Stochastic Integration Equations*, Springer–Verlag, (2005).

[41] Protter, P., *Stochastic Integration and Differential Equations*, Springer Verlag, (2004).

[42] Revuz, D., and Yor, M., *Continuous Martingales and Brownian Motion*, 3rd ed, Springer Verlag, (1999).

[43] Rohatgi, V., and Ehsanes Saleh, A,; *An Introduction to Probability and Statistics*, John Wiley and Sons, (2001)

[44] Rydberg, T., A., *The Normal Inverse Gaussian Lévy Process: Simulation and Approximation*, Comm. Statistics, Stochastic Models, **13**, 887–910, (1997).

[45] Sato, K., *Lévy Processes and Infinite Divisibility*, Cambridge University Press, (1999).

[46] Schoutens, W., *Lévy Processes in Finance*, John Wiley and Sons, (2005).

[47] Shiller, R., *Do Stock Prices Move to Much to be Justified by Subsequent Changes in Dividends*, American Economic Review, **71**, 421–436, (1981).

[48] Shreve, S., *Stochastic Calculus for Finance II- Continuous Time Model*, Springer Finance, (2004).

[49] Summers, L., *Does the Stock Market Rationally Reflect Fundamental Value*, Journal of Finance, **41**, 591–601, (1986).

[50] Summers, L., and Poterba, J., *Mean Reversion in Stock Prices: Evidence and Implications*, Journal of Financial Economics, Vol **21(1)**, 27–29, (1988).

[51] Tankov, P., *Lévy Processes in Finance: Inverse Problems and Dependence Modelling*, ECOLE POLYTECHNIQUE, France, **PhD Thesis**, (2004).

[52] Varadhan, S., R., S., *Probability Theory*, Courant Lecture Notes, Volume **7**, American Mathematical Society, (2001).

[53] Wang, J., *A Model of Intertemporal Asset Prices Under Asymmetric Information*, Review of Economic Studies, **60**, 249–282, (1993).