MODELING AND SIMULATING INTEREST RATES VIA TIME-DEPENDENT MEAN REVERSION

by

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ABSTRACT

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The purpose of this thesis is to compare the effectiveness of several interest rate models in fitting the true value of interest rates. Up until 1990, the universally accepted models were the equilibrium models, namely the Rendleman–Bartter model, the Vasicek model, and the Cox–Ingersoll–Ross (CIR) model. While these models were probably considered relatively accurate around the time of their discovery, they do not provide a good fit to the initial term structure of interest rates, making them substandard for use by traders in pricing interest rate options. The fourth model we consider is the Hull–White one-factor model, which does provide this fit. After calibrating, simulating, and comparing these four models, we find that the Hull–White model gives the best fit to our data sets.

DEDICATION

For Jenn.

MODELING AND SIMULATING INTEREST RATES VIA TIME-DEPENDENT MEAN REVERSION

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CHAPTER 1

INTRODUCTION

1.1 BACKGROUND ON INTEREST RATES

What are interest rates? Interest rates are used to define how much money a particular borrower promises to pay to a lender. Interest rates can be classified as mortgage rates, deposit rates, prime borrowing rates, et cetera. Given any particular situation, the applicable interest rate depends on the credit risk, that is, the risk that there will be a default by the borrower in paying the promised interest. In general, the higher the credit risk, the more interest the borrower promises to pay the lender. There are two main types of interest rates; these are the US Treasury rates and the LIBOR rates. Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds. Investors may choose to trade bonds themselves, or they may choose to trade options on those bonds[1]. LIBOR is short for London Interbank Offered Rate. A LIBOR quote by a particular bank is the rate of interest at which the bank is prepared to make a large wholesale deposit with other banks (Hull, 2009). Interest rates can be measured with annual, semiannual, quarterly, monthly, weekly, or daily compounding. A 10% interest rate compounded annually means that at the end of one year, $100 will have

[1] Although option pricing is not the focus of this study, it bears mentioning here what an option is. A European option maturing at time T on an asset gives its holder the right, but not the obligation, to either buy or sell the asset at time T for a certain price, while an American option maturing at time T on an asset gives its holder the right, but not the obligation, to buy or sell the asset at any time up to and including time T for a certain price. The option that gives its holder the right to buy the asset is called a call on the asset, while the option that gives its holder the right to sell the asset is called a put.

grown to

$$\$100 \times 1.1 = \$110;$$

a 10% interest rate compounded semiannually means that 5% interest is earned every six months, so that at the end of one year, $100 will have grown to

$$\$100 \times 1.05 \times 1.05 = \$110.25;$$

a 10% interest rate compounded quarterly means that 2.5% interest is earned every three months, so that at the end of one year, $100 will have grown to

$$\$100 \times 1.025^4 = \$110.38;$$

and so on. There is also the notion of continuous compounding. With continuous compounding, an amount A invested for n years at the rate R grows to

$$Ae^{Rn}.$$

This can be thought of as the limit the principal grows to as the compounding frequency approaches infinity. It turns out there are some simple relationships between rates compounded m times per annum and rates compounded continuously. We have the following two equations:

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right)$$

and

$$R_m = m(e^{R_c/m} - 1),$$

where R_c denotes the continuously compounded interest rate and R_m denotes the interest rate that is compounded m times per annum (Hull, 2009).

The n-year zero-coupon interest rate is defined as the rate of interest earned on an investment that starts today and lasts for n years (Hull, 2009). By zero-coupon, we mean that there are no payments prior to the end of the n-year period, when all of the principal and interest are realized. Treasury "zero rates" can be determined by the market prices of coupon-bearing bonds, but this requires some calculation. To price bonds correctly, we have to add up all the payments that the owner of the bond will receive up to, and including, the end of its life. These payments are called coupons, or cash flows. When valuing a bond, the cash flows are discounted at the appropriate zero rates. Thus, for a bond with a principal of \$100 that provides coupons at the rate of 6% semiannually, if the zero rates for maturities of six months, one year, one year and six months, and two years are 5%, 5.8%, 6.4%, and 6.8%, respectively, then the current value of the bond would be

$$3e^{-0.05\times 0.5} + 3e^{-0.058\times 1.0} + 3e^{-0.064\times 1.5} + 103e^{-0.068\times 2.0} = 98.39$$

or \$98.39. This is called the theoretical price of the bond. The yield of a bond is defined as the discounting rate that would need to be applied at the maturity of each coupon to make the bond equal to its theoretical price. Note that, in our example, the theoretical price came out to be \$98.39. The bond yield can therefore be determined by solving the equation

$$3e^{-y\times 0.5} + 3e^{-y\times 1.0} + 3e^{-y\times 1.5} + 103e^{-y\times 2.0} = 98.39$$

for y. In this case, y=6.76%. Similarly, the par yield for a certain maturity is defined as the coupon rate needed to make the bond equal to its principal value. Note that, in our example, the principal value was \$100, while the compounding was semiannual. Therefore, to obtain the par yield, we would solve the equation

$$\frac{c}{2}e^{-0.05\times 0.5} + \frac{c}{2}e^{-0.058\times 1.0} + \frac{c}{2}e^{-0.064\times 1.5} + \left(100 + \frac{c}{2}\right) e^{-0.068\times 2.0} = 100$$

for c. This equation can be easily solved as c=6.87% per annum (Hull, 2009).

We had mentioned earlier that Treasury zero rates can be determined from the market prices of coupon-bearing bonds. We will now illustrate how this is done. Suppose the prices of five bonds are as given in Table 1.1. Since the first three bonds provide no coupons, the continuously compounded zero rate can be computed as follows: We know that the first bond will provide a return of 2.5 in three months on an investment of 97.5. Therefore, the quarterly compounded zero rate is $(4 \times 2.5)/97.5 =$ 10.256% per annum. Using the equation that gives R_c in terms of R_m mentioned earlier, we get

$$R_c = 4 \ln \left(1 + \frac{0.10256}{4}\right) = 0.10127.$$

Thus, the three-month zero rate is 10.127% per annum. The second bond provides a return of 5.1 in six months on an investment of 94.9. Therefore, the semianually compounded zero rate is $(2 \times 5.1)/94.9 = 10.748\%$ per annum. Therefore, we must have

$$2 \ln \left(1 + \frac{0.10748}{2}\right) = 0.10469.$$

Thus, the six-month zero rate must be 10.469% per annum. Looking at the third bond, we can see that the one-year rate is $(1 \times 10)/90 = 0.10$ compounded annually. From this we can determine the continuously compounded one-year zero rate to be

$$\ln(1 + 0.10) = 0.10536,$$

or 10.536% per annum. The fourth bond lasts for one year and six months and provides three payments: \$4 after six months, \$4 after one year, and \$104 after one year and six months. In order to compute the 1.5-year zero rate, we must use the zero rates obtained in the previous calculations to discount the annual coupons and find the value of R that solves the equation

$$4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96.$$

This gives $R = 0.10681$, or 10.681% per annum. Finally, we can use the information on the last bond to set up the equation

$$6e^{-0.10469\times0.5} + 6e^{-0.10536\times1.0} + 6e^{-0.10681\times1.5} + 106e^{-R\times2.0} = 101.6.$$

Solving this gives R=0.10808, or 10.808% per annum. Hull defines the zero curve as the chart that relates the zero rate to its maturity. The relationship between interest rates and their maturities is known as the term structure of interest rates.

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

Table 1.1: Example of data used in determining Treasury zero rates

Continuous compounding plays an important role in pricing certain instruments, such as bond options, swap options, and interest rate caps and floors, whose values depend on the level of interest rates. Such instruments are called interest rate derivatives (Hull, 2009). If we assume interest rates are constant, calculating expected payoffs for these instruments is easy; in reality interest rates are stochastic. Many researchers have already developed formulas for pricing options on assets whose values are resistant to changes in interest rates (such as stock options), but it turns out that many European option-pricing models (as an example, Black's model) can be extended to include the case where interest rates are stochastic. Originally this discovery led researchers to develop models for pricing all kinds of interest rate derivatives, but these early models, which were known as the standard market models, all rested on the assumption that the value of the underlying asset variable came from a lognormal distribution. As one might imagine, this was not good for the interest rate derivatives trader, and it led researchers to develop what are known as term structure models, or models of the short rate r. These examined the behavior of r over very small intervals of time, that is, as $\Delta t \to 0$. Some of the inventors of these models also developed corresponding bond-pricing models to go along with the interest rate models. We shall return to the short-rate models in the next chapter. Before we can get into a thorough discussion of these models, we must first take up the discussion of stochastic processes and Brownian motion.

1.2 BACKGROUND ON STOCHASTIC PROCESSES AND ITÔ'S FORMULA

Stochastic processes and Brownian motion are central to understanding the theory of mathematical finance. According to Øksendal (2007), a stochastic process is a parameterized collection of random variables $\{X_t\}, t \in T$, defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbf{R}^n$. Brownian motion is a special type of stochastic process $\{B_t\}, t \geq 0$, such that

$$P^x(B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k) = \int\limits_{F_1 \times \cdots \times F_k} p(t_1, x, x_1) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

where

$$p(t, x, y) = (2\pi t)^{-n/2} \cdot \exp\left(-\frac{|x-y|^2}{2t}\right)$$

for $y \in \mathbf{R}^n, t > 0$ and

$$|x-y|^2 = \sum_{i=1}^{n} (x_i - y_i)^2.$$

Brownian motion can be used to deal with many problems in the area of mathematical finance. For example, one problem that often arises in this area is the problem of solving the equation that models a stock price X_t as a function of time t. Suppose we know that the price X_t of a certain stock satisfies the differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mu X_t + \sigma X_t W_t,$$

where W_t is white noise and is equal to $\mathrm{d}B_t/\mathrm{d}t$. What is the solution X_t of this equation? What is the mean $E[X_t]$? What is the variance $Var[X_t]$? To answer these and other questions, we need a few basic results from stochastic calculus. First and foremost, we must define the Itô integral. Let $\{\mathcal{N}_t\}, t \geq 0$, be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega) : [0,\infty) \times \Omega \to \mathbf{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \to g(t,\omega)$ is $\mathcal{N}_t$-measurable (Øksendal, 2007). Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbf{R}$ such that $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$; $f(t,\omega)$ is $\mathcal{F}_t$-adapted; and

$$E\left[\int_S^T f(t,\omega)^2 \,\mathrm{d}t\right] < \infty.$$

Let $f \in \mathcal{V}(S,T)$. Then the Itô integral of f from S to T is defined by

$$\int_S^T f(t,\omega)\,\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\,\mathrm{d}B_t(\omega),$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \,\mathrm{d}t\right] \to 0$$

as $n \to \infty$ (Øksendal, 2007). The next result, which Øksendal calls the Itô isometry, is often useful:

$$E\left[\left(\int_S^T f(t,\omega)\,\mathrm{d}B_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)\,\mathrm{d}t\right]$$

for all $f \in \mathcal{V}(S,T)$.

Next, we must take up the topics of one-dimensional Itô processes and the one-dimensional Itô formula. Let B_t be one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A one-dimensional Itô process or stochastic integral is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\,\mathrm{d}s + \int_0^t v(s,\omega)\,\mathrm{d}B_s,$$

where $v \in \mathcal{V}$, so that

$$P\left[\int_0^t v(s,\omega)^2\,\mathrm{d}s < \infty \quad \text{for all}\, t \geq 0\right] = 1.$$

We also assume that u is $\mathcal{F}_t$-adapted and

$$P\left[\int_0^t |u(s,\omega)|\,\mathrm{d}s < \infty \quad \text{for all}\, t \geq 0\right] = 1.$$

The equation for the process X_t given above can be condensed to the shorter differential form

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t$$

(Øksendal, 2007). Finally, we need one of the most important results in stochastic calculus: the one-dimensional Itô formula. Let X_t be an Itô process given by

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbf{R})$ (i.e., g is twice continuously differentiable on $[0,\infty) \times \mathbf{R}$). Then $Y_t = g(t, X_t)$ is again an Itô process, and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t,X_t)\,\mathrm{d}t + \frac{\partial g}{\partial x}(t,X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(\mathrm{d}X_t)^2,$$

where

$$(\mathrm{d}X_t)^2 = (\mathrm{d}X_t)\cdot(\mathrm{d}X_t)$$

is computed according to the rules

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_t = \mathrm{d}B_t \cdot \mathrm{d}t = 0, \quad \mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t$$

(Øksendal, 2007).

Now that we have stated these critically important results, let us return our attention to the stock price model (also known as the Black–Scholes model)

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mu X_t + \sigma X_t W_t$$

mentioned earlier. $W_t = \mathrm{d}B_t/\mathrm{d}t$ and usually we assume $B_0 = 0$. Multiplying both sides of this equation by $\mathrm{d}t$ and dividing both sides by X_t gives us the transformed equation

$$\frac{\mathrm{d}X_t}{X_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

By using Itô's formula, we find that

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t},$$

which is called geometric Brownian motion. This is the general form of the solution curve followed by the differential equation for stock prices. It is also the general solution curve of the differential equation for interest rates under the Rendleman–Bartter model, as will be discussed later on. If X_0 is known and independent of B_t, then it can be shown that

$$E[X_t] = X_0 e^{\mu t}$$

and

$$Var[X_t] = X_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

Before we proceed, we first try to make sense of what is termed the risk-neutral world. In such a world, investors require no compensation for risk, and the expected return on all securities is the risk-free interest rate. That being said, let us return to our discussion of the short-rate models.

1.3 OVERVIEW

In this study, we introduce, calibrate, and simulate four popular interest rate models. These are:

1. Rendleman–Bartter model

2. Vasicek model

3. Cox–Ingersoll–Ross model

4. Hull–White one-factor model

In Chapter 2, we discuss what each of the models say about the way interest rates behave, give the differential form of each model, and, where applicable, give their closed-form solutions, expected (mean) values, and variances. We then explain how each model can be discretized so that they may be calibrated to observable market data, how the parameters in the models can be estimated (calibrated) from market data using the familiar least squares procedure, how we can use the estimates for these parameters to create a simulation equation for each of the models, and how these simulation equations can be used to simulate interest rates for each day of data used to calibrate them. In Chapter 3, we calibrate these models to three different sets of two years of interest rate data, collected by the US Department of the Treasury from January 2, 2008 to December 31, 2013, simulate them, and show how the models can be compared in terms of accuracy by comparing the measures of fit obtained for the prediction of each model. We break the data into three subsets: January 2, 2008 to December 31, 2009; January 4, 2010 to December 30, 2011; and January 3, 2012 to December 31, 2013; and perform the calibrations and simulations separately on each set. We also make use of two user-friendly software packages: Minitab and Excel.

We use Minitab primarily for calibration, Excel primarily for simulation. Finally, in Chapter 4, we conclude the study by summarizing our findings, their importance, and their impact on financial analysis.

CHAPTER 2

METHODOLOGY

In this chapter, we introduce four popular interest rate models. These are:

1. Rendleman–Bartter model

2. Vasicek model

3. Cox–Ingersoll–Ross model

4. Hull–White one-factor model

We first discuss what each of the models say about the way interest rates behave, then give the differential form of each model, and, where applicable, give their closed-form solutions, expected (mean) values, and variances. We then explain how each can be discretized so that they may be calibrated to observable market data, how the parameters in the models can be estimated (calibrated) from market data using the familiar least squares procedure, how we can use the estimates for these parameters to create a simulation equation for each of the models, and how these simulation equations can be used to simulate interest rates for each day of data used to calibrate them.

The first three models we are going to look at are called one-factor equilibrium models. These assume that the short rate r follows some evolutionary path, but that path depends only on the present value of r. This can be expressed by the relation

$$\mathrm{d}r_t = m(r_t)\,\mathrm{d}t + s(r_t)\,\mathrm{d}z_t,$$

where m is the drift, s is the standard deviation, and z_t is a standard Brownian motion. By one factor, we mean that r depends on one source of uncertainty. There exist models with multiple factors, but we do not consider them here.

2.1 RENDLEMAN–BARTTER MODEL

Rendleman and Bartter (1980) proposed a one-factor equilibrium model; under this model's assumption, the risk-neutral process for r is given by

$$\mathrm{d}r_t = \mu r_t \,\mathrm{d}t + \sigma r_t \,\mathrm{d}z_t,$$

that is, r follows a geometric Brownian motion (like the stock price X_t we considered earlier). Hence, for this model, we can infer that

$$r_t = r_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z_t},$$

with mean

$$E[r_t] = r_0 e^{\mu t}$$

and variance

$$Var[r_t] = r_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

The advantage of the Rendleman and Bartter model is that it assumes interest rates behave like stock prices, which makes them easy to calculate; the disadvantage is that the model is not very accurate.

Cescato and Lemgruber (2011) give the discretization of the Rendleman–Bartter model as

$$r_i = r_{i-1} e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} z_i},$$

where r_i is the short rate value at time $t_i = i\Delta t$, r_{i-1} is the short rate value at time $t_{i-1} = (i-1)\Delta t$, Δt is the time interval, μ is the drift coefficient, σ is the diffusion coefficient, and z_i is the standard normal random variable (Brownian motion) at time t_i. To calibrate this model, we can use the maximum likelihood method. Let

$$u_i = \ln \frac{r_i}{r_{i-1}}$$

and

$$\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i.$$

Then, we have

$$\hat{\sigma}^2 \Delta t = \frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u})^2.$$

It follows that

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{n}(u_i - \bar{u})^2$$

and

$$\hat{\mu} = \frac{1}{T}\sum_{i=1}^{n}(u_i) + \frac{\hat{\sigma}^2}{2},$$

where T is the total time in years. It turns out the first term in that last equation will always be equal to

$$\frac{1}{T}\ln \frac{r_n}{r_0},$$

where r_0 denotes the first value in the data set and r_n denotes the last value, which can greatly simplify the calculation. Once we have these values for $\hat{\mu}$ and $\hat{\sigma}^2$, we can create the simulation equation

$$\tilde{r}_i = \tilde{r}_{i-1} e^{\left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right)\Delta t + \hat{\sigma}\sqrt{\Delta t} z_i},$$

where

$$\tilde{r}_0 = r_0.$$

2.2 VASICEK MODEL

Vasicek (1977) proposed a model that assumes that r follows the risk-neutral process

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t$$

and has the advantage that it incorporates mean reversion, which is always observed in interest rates, making it a much more accurate model than Rendleman and Bartter's model. It can be shown that the Vasicek model has the closed-form solution

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{a(s-t)}\,\mathrm{d}z_s,$$

with mean

$$E[r_t] = r_0 e^{-at} + b(1 - e^{-at})$$

and variance

$$Var[r_t] = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Using the fact that bond prices depend only on the process followed by r in a risk-neutral world, Vasicek shows that the price $P(t,T)$ at time t of a zero-coupon bond that pays off \$1 at time T is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a},$$

and

$$A(t,T) = e^{\frac{(B(t,T)-T+t)(a^2b-\sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}}.$$

When $a = 0$, $B(t,T) = T - t$ and

$$A(t,T) = e^{\frac{\sigma^2(T-t)^3}{6}}$$

(Vasicek, 1977).

The discretization of the Vasicek model is given as

$$r_i = (1 - a\Delta t)r_{i-1} + ab\Delta t + \sigma\sqrt{\Delta t}z_i,$$

where r_i is the short rate value at time $t_i = i\Delta t$, r_{i-1} is the short rate value at time $t_{i-1} = (i-1)\Delta t$, Δt is the time interval, b is the long-term rate, a is the reversion speed of short rate to long-term rate, σ is the diffusion coefficient, and z_i is the standard normal random variable at time t_i (Cescato & Lemgruber, 2011). This is called an AR(1) time series model. To calibrate the Vasicek model, we first use the least squares method to find the values $\hat{a}$ and $\hat{b}$ that minimize the function

$$S_V(a,b) = \sum_{i=1}^{n} (r_i - r_{i-1} - ab\Delta t + a\Delta t r_{i-1})^2$$

(Amin, 2012). We can let $\hat{a}\hat{b}\Delta t = b_0$, $-\hat{a}\Delta t = b_1$, and run a least squares regression to find values for b_0 and b_1. Once we have values for b_0 and b_1, we can divide b_1 by $-\Delta t$ to find our estimate for a. Then we can use this estimate for a, together with b_0, to solve for our estimate for b. We then have the one-step prediction equation

$$\hat{r}_i = (1 - \hat{a}\Delta t)r_{i-1} + \hat{a}\hat{b}\Delta t.$$

We then calculate the standard deviation s of the prediction errors

$$\hat{r}_i - r_i$$

and let

$$s = \hat{\sigma}\sqrt{\Delta t}.$$

We can solve this equation to obtain

$$\hat{\sigma} = s/\sqrt{\Delta t}.$$

Once this estimate for σ is obtained, we can simulate r_i by using $\tilde{r}_i$, which are generated by the following recursive formula:

$$\tilde{r}_i = (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{a}\hat{b}\Delta t + \hat{\sigma}\sqrt{\Delta t} z_i,$$

where

$$\tilde{r}_0 = r_0.$$

2.3 COX–INGERSOLL–ROSS MODEL

Cox, Ingersoll, and Ross (1985) proposed that r follows the process

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}z_t,$$

the advantage of which might not seem obvious at first glance. The fact is, this model is not very different from Vasicek's model. Both incorporate mean reversion, but in this model the stochastic coefficient is always nonnegative, and hence the entire process is always nonnegative. Like Vasicek's model, the Cox, Ingersoll, and Ross model has formulas that allow for the valuation at time t of zero-coupon bonds that pay off \$1 at time T. Just like in the Vasicek model,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

but in the Cox, Ingersoll, and Ross model

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma},$$

and

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2ab/\sigma^2},$$

where

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

(Cox, Ingersoll, & Ross, 1985).

The discretization of the Cox–Ingersoll–Ross (CIR) model is given as

$$r_i = (1 - a\Delta t)r_{i-1} + ab\Delta t + \sigma\sqrt{r_{i-1}}\sqrt{\Delta t}z_i,$$

where r_i is the short rate value at time $t_i = i\Delta t$, r_{i-1} is the short rate value at time $t_{i-1} = (i-1)\Delta t$, Δt is the time interval, b is the long-term rate, a is the reversion speed of short rate to long-term rate, σ is the diffusion coefficient, and z_i is the standard normal random variable at time t_i (Cescato & Lemgruber, 2011). The CIR model can be calibrated in a near-identical fashion to the Vasicek model. We first find the values $\hat{a}$ and $\hat{b}$ that minimize the function

$$S_{CIR}(a,b) = \sum_{i=1}^{n}\left(\frac{r_i - r_{i-1}}{\sqrt{r_{i-1}}} - \frac{ab\Delta t}{\sqrt{r_{i-1}}} + a\Delta t\sqrt{r_{i-1}}\right)^2.$$

If we consider a prediction equation of the form

$$\frac{\hat{r}_i - r_{i-1}}{\sqrt{r_{i-1}}} = b_0 + b_1 \cdot \frac{1}{\sqrt{r_{i-1}}} + b_2\sqrt{r_{i-1}},$$

we can let $0 = b_0$, $\hat{a}\hat{b}\Delta t = b_1$, $-\hat{a}\Delta t = b_2$, and run a least squares regression to find values for b_1 and b_2. Note that unlike what we did for the Vasicek model, we are not fitting an intercept term here. Once we have values for b_1 and b_2, we can divide b_2 by $-\Delta t$ to find our estimate for a. Then we can use this estimate for a, together with b_1, to get our estimate for b. We then have the one-step prediction equation

$$\frac{\hat{r}_i - r_{i-1}}{\sqrt{r_{i-1}}} = \frac{\hat{a}\hat{b}\Delta t}{\sqrt{r_{i-1}}} - \hat{a}\Delta t\sqrt{r_{i-1}},$$

or

$$\hat{r}_i = (1 - \hat{a}\Delta t)r_{i-1} + \hat{a}\hat{b}\Delta t.$$

We then calculate the standard deviation s of the weighted prediction errors

$$\frac{\hat{r}_i - r_i}{\sqrt{r_{i-1}}}$$

and let

$$s = \hat{\sigma}\sqrt{\Delta t}.$$

From this we can obtain $\hat{\sigma}$ as we did for the Vasicek model, and the desired simulation model

$$\tilde{r}_i = (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{a}\hat{b}\Delta t + \hat{\sigma}\sqrt{\tilde{r}_{i-1}}\sqrt{\Delta t} z_i,$$

where

$$\tilde{r}_0 = r_0.$$

The main purpose of the equilibrium models is to describe how interest rates can evolve over time. In the real world, however, not only are interest rates functions of time—the parameters in the equations used to model those interest rates are also functions of time! As such, the initial term structure of interest rates should be considered before attempting to model them. Next we will consider a type of interest rate model—called a no-arbitrage model—that takes this into account.

2.4 HULL–WHITE ONE-FACTOR MODEL

Ho and Lee proposed the first no-arbitrage model in 1986, which allowed for a time-dependent mean. Hull and White (1990) extended this model to incorporate mean-reversion. They proposed that r follows the stochastic differential equation

$$\begin{aligned} \mathrm{d}r_t &= [\theta(t) - ar_t]\,\mathrm{d}t + \sigma\,\mathrm{d}z_t \\ &= a\left[\frac{\theta(t)}{a} - r_t\right]\mathrm{d}t + \sigma\,\mathrm{d}z_t \end{aligned}$$

which we can deduce many features from. We know, for instance, that the short rate r is pulled to the level $\theta(t)/a$ at the rate of a. Thus, this model is similar to Vasicek's equilibrium model, except that b is replaced by the parameter $\theta(t)/a$, which depends on time. For the Hull–White one-factor model, it can be shown that

$$r_t = r_0 e^{-at} + \int_0^t e^{a(s-t)} \theta(s)\, \mathrm{d}s + \sigma \int_0^t e^{a(s-t)}\, \mathrm{d}z_s,$$

with mean

$$E[r_t] = r_0 e^{-at} + \int_0^t e^{a(s-t)} \theta(s)\, \mathrm{d}s$$

and variance

$$Var[r_t] = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Hull and White show that

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}),$$

where the $F(0,t)$ is the instantaneous forward rate for a maturity t as seen at time zero and the subscript t denotes a partial derivative with respect to t. Park (2004) gives the formula for $F(0,t)$ as

$$F(0,t) = t\frac{\partial R(0,t)}{\partial t} + R(0,t),$$

where

$$R(t,T) = -\frac{\log\, P(t,T)}{T-t}.$$

We also have

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a},$$

and

$$\ln\, A(t,T) = \ln\, \frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1)$$

(Hull & White, 1990).

The Hull–White one-factor model can be discretized as

$$r_i = \theta_{i-1}\Delta t + (1 - a\Delta t)r_{i-1} + \sigma\sqrt{\Delta t}z_i,$$

where r_i is the short rate value at time $t_i = i\Delta t$, r_{i-1} is the short rate value at time $t_{i-1} = (i-1)\Delta t$, Δt is the time interval, θ_{i-1} is the value of the parameter $\theta(t)$ at time t_{i-1}, σ is the diffusion coefficient, and z_i is the standard normal random variable at time t_i. Thus, the Hull–White one-factor model can be regarded as a dynamic Vasicek model that changes depending on the day. Up until now, calibrating our models has been fairly easy; the Hull–White model is an exception. As mentioned earlier, the no-arbitrage models require knowledge of the initial term structure before they can be calibrated. Luckily, Park (2004) gives a method for calibrating the Hull–White one-factor model using such initial term structure data. We wish to use this data to approximate the forward curve $F(0,t)$. Suppose we are willing to assume that this curve is polynomial in nature. Then we could easily implement a least squares package such as Minitab to approximate the curve. Once this curve has been estimated, we can use a simple formula to calculate $\theta(t)$:

$$\theta(t) = \frac{\partial F(0,t)}{\partial t} + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

(Park, 2004). However, Hull (2009) says that the term

$$\frac{\sigma^2}{2a}(1 - e^{-2at}),$$

as seen in the expression for $\theta(t)$, is usually fairly small and can be ignored when calculating $\theta(t)$. Once we have an expression for $\theta(t)$, we can estimate a and σ using procedures similar to those used to estimate these parameters for the previous three models. We begin by noting that

$$\begin{aligned} r_i - r_{i-1} &= \left(F_t\left(0, t_{i-1}\right) + aF\left(0, t_{i-1}\right)\right)\Delta t - a\Delta t r_{i-1} + \sigma\sqrt{\Delta t} z_i \\ &= F_t\left(0, t_{i-1}\right)\Delta t + a\left(F\left(0, t_{i-1}\right) - r_{i-1}\right)\Delta t + \sigma\sqrt{\Delta t} z_i. \end{aligned}$$

We then proceed by finding the estimate of a that minimizes the function

$$S_{HW}(a) = \sum_{i=1}^{n} \left(r_i - r_{i-1} - F_t\left(0, t_{i-1}\right)\Delta t - a\left(F\left(0, t_{i-1}\right) - r_{i-1}\right)\Delta t\right)^2,$$

which looks complicated but will be relatively straightforward since we will have already known all of the other quantities in this expression. If we consider a prediction equation of the form

$$\hat{r}_i - r_{i-1} - F_t\left(0, t_{i-1}\right)\Delta t = b_1\left(F\left(0, t_{i-1}\right) - r_{i-1}\right),$$

we can let $0 = b_0$, $\hat{a}\Delta t = b_1$, and run a least squares regression to obtain a value for b_1. Once we have a value for b_1, we can divide this value by Δt to obtain an estimate for a. We then have the one-step prediction of r_i:

$$\hat{r}_i = \left(F_t\left(0, t_{i-1}\right) + \hat{a}F\left(0, t_{i-1}\right)\right)\Delta t + (1 - \hat{a}\Delta t)r_{i-1}.$$

We then calculate the standard deviation s of the prediction errors

$$\hat{r}_i - r_i$$

and let

$$s = \hat{\sigma}\sqrt{\Delta t}.$$

We can then obtain $\hat{\sigma}$ in the usual way. Once the estimate for σ is obtained, we can simulate r_i by using $\tilde{r}_i$, which is generated from the following recursive formula:

$$\tilde{r}_i = \left(F_t\left(0, t_{i-1}\right) + \hat{a}F\left(0, t_{i-1}\right)\right)\Delta t + (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{\sigma}\sqrt{\Delta t} z_i,$$

where

$$\tilde{r}_0 = r_0.$$

There exist other no-arbitrage models, such as the Black–Karasinski, Heath–Jarrow–Morton, and Libor Market models, but we do not consider them further in this study.

CHAPTER 3
DATA ANALYSIS AND RESULTS

In this chapter, we calibrate the models considered in the previous chapter to three different sets of two years of interest rate data, collected by the US Department of the Treasury from January 2, 2008 to December 31, 2013, simulate them, and show how the models can be compared in terms of accuracy by comparing the measures of fit obtained for the prediction of each model. We break the data into three subsets: January 2, 2008 to December 31, 2009; January 4, 2010 to December 30, 2011; and January 3, 2012 to December 31, 2013; and perform the calibrations and simulations separately on each set. In this chapter, we also make use of two user-friendly software packages: Minitab and Excel. We use Minitab primarily for calibration, Excel primarily for simulation.

3.1 DATA PREPARATION

For this study, we consider the US Department of the Treasury's Daily Treasury Yield Curve Rates data set, taken from http://www.treasury.gov/resource-center/data-chart-center/interest-rates/. For the sake of simplicity, we only consider one-year rates. A snapshot of the data is included in Table 3.1. We will break the data into three subsets: January 2, 2008 to December 31, 2009; January 4, 2010 to December 30, 2011; and January 3, 2012 to December 31, 2013. Using these three data sets, we will generate interest rates by first calibrating and then simulating the Rendleman–Bartter, Vasicek, CIR, and Hull–White models. For each of these three subsets, we will calibrate the models using all data points, and then simulate the models for each day of data used to calibrate them. Thus, in each subset, we are considering two complete years of interest rate data. Since the data were collected daily, we will let $\Delta t = 1/252$, as is the usual convention, during calibration. Once the models are calibrated, we will simulate each one for each of the days we have collected data (i.e. $\Delta t = 1/252$) to generate new interest rates based on the original data sets. We shall call the true value of r the observed value for each day in our data sets. Since 0.12 is the first value occurring in the 2012-2013 set, we will use this value for r_0 when simulating values for the second day in our data set, i.e., January 4, 2012. In other words,

$$\tilde{r}_0 = r_0 = 0.12.$$

We do the same thing for the 2010-2011 and 2008-2009 sets. For brevity, we only narrate the results for the 2012-2013 period. For the other two sets, we just give the simulation equations and the measures of fit. The calibrations of the Vasicek, CIR, and Hull–White one-factor models can be easily performed in Minitab, using Minitab's regression feature. The Rendleman–Bartter model can be calibrated by hand. The simulations for each of the models can be efficiently performed in Excel, which is particularly adept at iterating recursive formulas.

When comparing mathematical models, there are four universally accepted measures of fit. These are defined as follows:

1. Root mean square error

$$rmse = \sqrt{\sum_{\text{Rates}} \frac{(\text{True rate} - \text{Model rate})^2}{\text{Number of rates}}}$$

2. Average absolute error as a percentage of the mean rate

$$ape = \frac{1}{\text{Mean rate}} \sum_{\text{Rates}} \frac{|\text{True rate} - \text{Model rate}|}{\text{Number of rates}}$$

3. Average absolute error

$$aae = \sum_{\text{Rates}} \frac{|\text{True rate} - \text{Model rate}|}{\text{Number of rates}}$$

4. Average relative percentage error

$$arpe = \frac{1}{\text{Number of rates}} \sum_{\text{Rates}} \frac{|\text{True rate} - \text{Model rate}|}{\text{True rate}}$$

In general, the smaller these quantities are, the better the fit the model has.

3.2 RENDLEMAN–BARTTER MODEL

3.2.1 2012-2013

For the Rendleman–Bartter model over the 2012-2013 period, using the aforementioned methods, we get approximately

$$\hat{\mu} - \frac{\hat{\sigma}^2}{2} = 0.04002$$

and

$$\hat{\sigma}^2 = 0.83446.$$

Date	One-Year Rate
1/2/08	3.17
1/3/08	3.13
1/4/08	3.06
⋮	⋮
12/31/13	0.13

Table 3.1: Snapshot of Daily Treasury Yield Curve Rates data set

Therefore, we must have

$$\begin{aligned}\hat{\sigma} &= \sqrt{0.83446} \\ &\approx 0.91349\end{aligned}$$

and

$$\begin{aligned}\hat{\mu} &= 0.04002 + \frac{0.83446}{2} \\ &= 0.45725.\end{aligned}$$

Putting all of our previous results together, we can simulate the interest rate on day i by

$$\begin{aligned}\tilde{r}_i &= \tilde{r}_{i-1} e^{\left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right)\Delta t + \hat{\sigma}\sqrt{\Delta t} z_i} \\ &= \tilde{r}_{i-1} e^{0.04002 \cdot \frac{1}{252} + 0.91349\sqrt{\frac{1}{252}} z_i} \\ &= \tilde{r}_{i-1} e^{0.00016 + 0.05754 z_i}, \quad i = 1, 2, \ldots, 500,\end{aligned}$$

where $\tilde{r}_0 = r_0 = 0.12$. The measures of fit for the Rendleman–Bartter model are summarized in Table 3.2. The Excel output is included in the appendix. From this we can infer that the Rendleman–Bartter model

$$r_t = 0.12 e^{0.04002t + 0.91349 z_t},$$

where t is measured in years, is not a great fit for the data. The simulation depends too much on the initial value, 0.12, and this greatly distorts the results. Note that the simulated values for r_t "agree" somewhat with the theoretical mean

$$E[r_t] = 0.12 e^{0.45725t}$$

for this model.

3.2.2 2010-2011

For the Rendleman–Bartter model over the 2010-2011 period, the simulation equation used was

$$\tilde{r}_i = \tilde{r}_{i-1} e^{-0.00262+0.06783 z_i}, \quad i = 1, 2, \ldots, 502,$$

where $\tilde{r}_0 = r_0 = 0.45$.

3.2.3 2008-2009

For the Rendleman–Bartter model over the 2008-2009 period, the simulation equation used was

$$\tilde{r}_i = \tilde{r}_{i-1} e^{-0.00379+0.05340 z_i}, \quad i = 1, 2, \ldots, 501,$$

where $\tilde{r}_0 = r_0 = 3.17$.

	2008-2009	2010-2011	2012-2013
rmse	0.47668	0.04778	0.04258
ape	0.34913	0.15589	0.22254
aae	0.40202	0.03888	0.03405
arpe	0.53252	0.17152	0.20268

Table 3.2: Measures of fit for Rendleman–Bartter model

3.3 VASICEK MODEL

3.3.1 2012-2013

For the Vasicek model over the 2012-2013 period, using Minitab to simplify the calculation, we get approximately

$$\hat{a}\hat{b}\Delta t = 0.00616$$

and

$$-\hat{a}\Delta t = -0.0401.$$

Solving for $\hat{a}$ and $\hat{b}$ yields $\hat{a} = 10.1052$ and $\hat{b} = 0.15362$. Minitab gave s as being approximately 0.00839, which means that

$$\begin{aligned}\hat{\sigma} &= \frac{s}{\sqrt{\Delta t}} \\ &= \frac{0.00839}{\sqrt{1/252}} \\ &\approx 0.13316.\end{aligned}$$

Putting all of this information together, can simulate r_i by $\tilde{r}_i$, which is given by:

$$\begin{aligned}\tilde{r}_i &= (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{a}\hat{b}\Delta t + \hat{\sigma}\sqrt{\Delta t} z_i \\ &= 0.9599\tilde{r}_{i-1} + 0.00616 + 0.00839 z_i, \quad i = 1, 2, \ldots, 500,\end{aligned}$$

where $\tilde{r}_0 = r_0 = 0.12$. The measures of fit for the Vasicek model are summarized in Table 3.3. The Minitab and Excel output are included in the appendix, indicative that the Vasicek model

$$r_t = 0.12e^{-10.1052t} + 0.15362(1 - e^{-10.1052t}) + 0.13316 \int_0^t e^{10.1052(s-t)}\,\mathrm{d}z_s,$$

where t is measured in years, is a better fit for the data than the Rendleman–Bartter model was. The simulation is quickly pulled to the value 0.15362, a much more accurate portrayal of the data's true mean. Again we see that the simulated values for r_t exhibit agreement with the theoretical mean

$$E[r_t] = 0.12e^{-10.1052t} + 0.15362(1 - e^{-10.1052t}).$$

3.3.2 2010-2011

For the Vasicek model over the 2010-2011 period, the simulation equation used was

$$\tilde{r}_i = 0.9862\tilde{r}_{i-1} + 0.00278 + 0.01449 z_i, \quad i = 1, 2, \ldots, 502,$$

where $\tilde{r}_0 = r_0 = 0.45$.

3.3.3 2008-2009

For the Vasicek model over the 2008-2009 period, the simulation equation used was

$$\tilde{r}_i = 0.99096\tilde{r}_{i-1} + 0.00502 + 0.06735z_i, \quad i = 1, 2, \ldots, 501,$$

where $\tilde{r}_0 = r_0 = 3.17$.

	2008-2009	2010-2011	2012-2013
rmse	0.48295	0.07357	0.02975
ape	0.32543	0.25128	0.16917
aae	0.37473	0.06267	0.02588
arpe	0.39364	0.31817	0.17734

Table 3.3: Measures of fit for Vasicek model

3.4 COX–INGERSOLL–ROSS MODEL

3.4.1 2012-2013

For the Cox–Ingersoll–Ross model over the 2012-2013 period, we get approximately

$$\hat{a}\hat{b}\Delta t = 0.00634$$

and

$$-\hat{a}\Delta t = -0.0413.$$

Solving for $\hat{a}$ and $\hat{b}$ yields $\hat{a} = 10.4076$ and $\hat{b} = 0.15351$. Minitab gave s as being approximately 0.02185. From this we know that $\hat{\sigma}$=0.34686. Putting all of this information together, we can simulate r_i by $\tilde{r}_i$, which is given by:

$$\begin{aligned}\tilde{r}_i &= (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{a}\hat{b}\Delta t + \hat{\sigma}\sqrt{\tilde{r}_{i-1}}\sqrt{\Delta t} z_i \\ &= 0.9587\tilde{r}_{i-1} + 0.00634 + 0.02185\sqrt{\tilde{r}_{i-1}} z_i, \quad i = 1, 2, \ldots, 500,\end{aligned}$$

where $\tilde{r}_0 = r_0 = 0.12$. The measures of fit for the CIR model are summarized in Table 3.4. The Minitab and Excel output are included in the appendix. Based on the measures of fit, the CIR model appears to provide a slightly better fit for the data than the Rendleman–Bartter and Vasicek models did.

3.4.2 2010-2011

For the Cox–Ingersoll–Ross model over the 2010-2011 period, the simulation equation used was

$$\tilde{r}_i = 0.9864\tilde{r}_{i-1} + 0.00273 + 0.02962\sqrt{\tilde{r}_{i-1}} z_i, \quad i = 1, 2, \ldots, 502,$$

where $\tilde{r}_0 = r_0 = 0.45$.

3.4.3 2008-2009

For the Cox–Ingersoll–Ross model over the 2008-2009 period, the simulation equation used was

$$\tilde{r}_i = 0.99186\tilde{r}_{i-1} + 0.00399 + 0.05606\sqrt{\tilde{r}_{i-1}} z_i, \quad i = 1, 2, \ldots, 501,$$

where $\tilde{r}_0 = r_0 = 3.17$.

	2008-2009	2010-2011	2012-2013
rmse	0.46536	0.07327	0.02972
ape	0.30899	0.25025	0.16898
aae	0.35580	0.06241	0.02585
arpe	0.36204	0.31650	0.17710

Table 3.4: Measures of fit for CIR model

3.5 HULL–WHITE ONE-FACTOR MODEL

3.5.1 2012-2013

Finally, we have the Hull–White one-factor model. As mentioned earlier, before we can calibrate this model, we need to approximate the function $\theta(t)$, which requires knowledge of the initial term structure. To accomplish this, we used Minitab's Fitted Line Plot feature, which gave for the 2012-2013 period

$$F(0,t) = 0.1289 + 0.2631t - 0.3344t^2 + 0.1034t^3,$$

from which we know that

$$\frac{\partial F(0,t)}{\partial t} = 0.2631 - 0.6688t + 0.3102t^2.$$

We therefore must have that

$$\begin{aligned}\theta(t) &\approx \frac{\partial F(0,t)}{\partial t} + aF(0,t) \\ &= 0.2631 - 0.6688t + 0.3102t^2 \\ &+ a(0.1289 + 0.2631t - 0.3344t^2 \\ &+ 0.1034t^3).\end{aligned}$$

Using the procedures mentioned earlier, we get approximately

$$\hat{a}\Delta t = 0.154.$$

Solving for the estimate of a yields $\hat{a} = 38.808$. Minitab gave s as being approximately 0.00813. From this we know that $\hat{\sigma}$=0.12911. Putting all of this information together, we can simulate r_i by $\tilde{r}_i$, which is given by:

$$\begin{aligned}\tilde{r}_i &= \left(F_t\left(0, \frac{i-1}{252}\right) + \hat{a}F\left(0, \frac{i-1}{252}\right)\right)\Delta t + (1 - \hat{a}\Delta t)\tilde{r}_{i-1} + \hat{\sigma}\sqrt{\Delta t}z_i \\ &= 0.00397F_t\left(0, \frac{i-1}{252}\right) + 0.154F\left(0, \frac{i-1}{252}\right) + 0.846\tilde{r}_{i-1} \\ &+ 0.00813z_i, \quad i = 1, 2, \ldots, 500,\end{aligned}$$

where $\tilde{r}_0 = r_0 = 0.12$. The measures of fit for the Hull–White one-factor model are summarized in Table 3.5. The Minitab and Excel output are included in the appendix.

3.5.2 2010-2011

For the Hull–White one-factor model over the 2010-2011 period, the simulation equation used was

$$\tilde{r}_i = 0.9395\tilde{r}_{i-1} + \hat{\theta}_{i-1}\Delta t + 0.01433z_i, \quad i = 1, 2, \ldots, 502,$$

where

$$\begin{aligned}\hat{\theta}_{i-1}\Delta t = 0.00397(&-0.1378 + 0.02846t_{i-1} - 0.03483t_{i-1}^2) \\ &+ 0.0605(0.3906 - 0.1378t_{i-1} + 0.01423t_{i-1}^2 \\ &- 0.01161t_{i-1}^3),\end{aligned}$$

and $\tilde{r}_0 = r_0 = 0.45$.

3.5.3 2008-2009

For the Hull–White one-factor model over the 2008-2009 period, the simulation equation used was

$$\tilde{r}_i = 0.9735\tilde{r}_{i-1} + \hat{\theta}_{i-1}\Delta t + 0.06721z_i, \quad i = 1, 2, \ldots, 501,$$

where

$$\begin{aligned}\hat{\theta}_{i-1}\Delta t = {} & 0.00397(0.3650 - 5.2660t_{i-1} + 3.072t_{i-1}^2) \\ & + 0.0265(2.245 + 0.3650t_{i-1} - 2.633t_{i-1}^2 \\ & + 1.024t_{i-1}^3),\end{aligned}$$

and $\tilde{r}_0 = r_0 = 3.17$.

	2008-2009	2010-2011	2012-2013
rmse	0.37558	0.04399	0.01508
ape	0.24526	0.14375	0.07839
aae	0.28242	0.03585	0.01199
arpe	0.31740	0.15724	0.08276

Table 3.5: Measures of fit for Hull–White one-factor model

Comparisons of the results for each of the four models considered in this study over each of the periods are given in Tables 3.6-3.8. As can obviously be inferred from the tables, the Hull–White one-factor model provides the best fit for our data over each of the periods. A graphical comparison of the four models is given in the appendix, where the x-axis represents the time in days, and the y-axis represents the level of one-year rates. These graphs demonstrate agreement with the statistics given in the tables; all indicate that the Hull–White one-factor model provides the best fit for the given data.

	RB	V	CIR	HW
rmse	0.47668	0.48295	0.46536	0.37558
ape	0.34913	0.32543	0.30899	0.24526
aae	0.40202	0.37473	0.35580	0.28242
arpe	0.53252	0.39364	0.36204	0.31740

Table 3.6: Comparison of models for 2008-2009 period

	RB	V	CIR	HW
rmse	0.04778	0.07357	0.07327	0.04399
ape	0.15589	0.25128	0.25025	0.14375
aae	0.03888	0.06267	0.06241	0.03585
arpe	0.17152	0.31817	0.31650	0.15724

Table 3.7: Comparison of models for 2010-2011 period

	RB	V	CIR	HW
rmse	0.04258	0.02975	0.02972	0.01508
ape	0.22254	0.16917	0.16898	0.07839
aae	0.03405	0.02588	0.02585	0.01199
arpe	0.20268	0.17734	0.17710	0.08276

Table 3.8: Comparison of models for 2012-2013 period

CHAPTER 4
CONCLUSION

In this chapter, we conclude the study by summarizing our findings, their importance, and their impact on financial analysis.

We simulated interest rate values over three two-year periods using four different models: the Rendleman–Bartter model, the Vasicek model, the Cox–Ingersoll–Ross model, and the Hull–White one-factor model. Overall, the Hull–White model gave the lowest amount of error between the true interest rate value and the predicted rate for each day. It should be noted, however, that there are many different ways to calibrate each of these models, and different calibration techniques will yield different results. For instance, we could have estimated the parameters using the maximum likelihood (MLE) or method of moments techniques. However, for the purposes of this study, the least squares method (LSM) was the safest and easiest to implement. In addition, our choice to use a cubic polynomial function to estimate the $F(0,t)$ in the Hull–White model was not necessarily "ideal" since in actuality interest rates do not increase (or decrease) without bound. Our estimate does, however, convey the general idea that interest rates do behave like time-dependent mean reversion processes; and, at least on our restricted two-year domain, seem to follow a path given by a cubic polynomial. Now, had our data sets been somewhat larger, say, five to ten years of interest rate values, a higher-degree polynomial or even a trigonometric path might have been more preferable.

As investors trade more bond and swap options, their reliance on more accurate interest rate models is indisputable. This may be why so many investors got "fed up" with the equilibrium models of interest rates. Besides being unreliable, the equilibrium models do not provide an exact fit to today's term structure of interest rates, making them unrealistic as well, not that they cannot sometimes be useful. If an analyst chooses the parameters of an equilibrium model wisely enough, he or she can make very accurate predictions given the historical data. In fact, before option trading became popular, the equilibrium models of interest rates actually priced the underlying bonds quite reasonably well. But now that investors are actively trading options on those bonds, the pricing processes have become quite complicated. It is no longer possible to predict with near-certainty what the value of an interest rate derivative will be years from now, given that the variability in value of an option on a bond must be at least as high as the variability in value of the bond itself. As Hull (2009) remarks, "A 1% error in the price of the underlying bond may lead to a 25% error in an option price" (p. 678).

APPENDIX A

GRAPHS, MINITAB OUTPUT, AND EXCEL SIMULATION SNAPSHOTS

A.1 RENDLEMAN–BARTTER MODEL

A.1.1 2008-2009

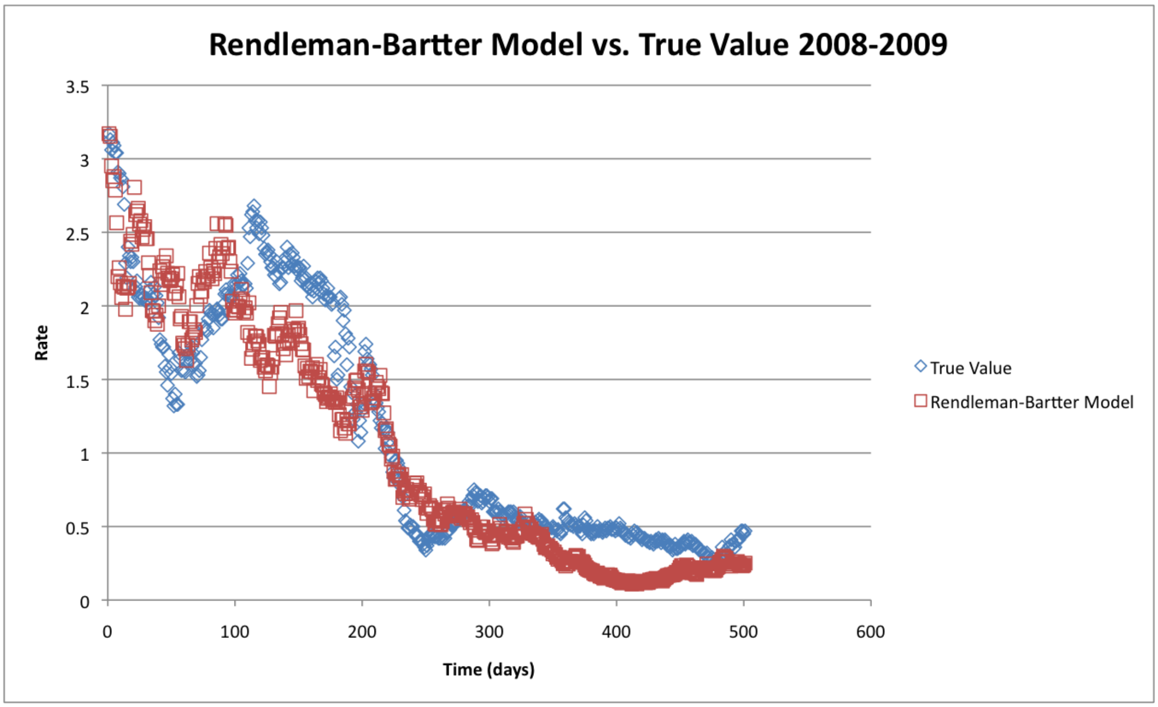

Figure A.1: Rendleman-Bartter model vs. true value 2008-2009

Date	Simulated Value
1/2/08	3.17000
1/3/08	3.15254
1/4/08	2.95101
⋮	⋮
12/31/09	0.25108

Table A.1: Snapshot of Excel simulation for Rendleman-Bartter model 2008-2009

A.1.2 2010-2011

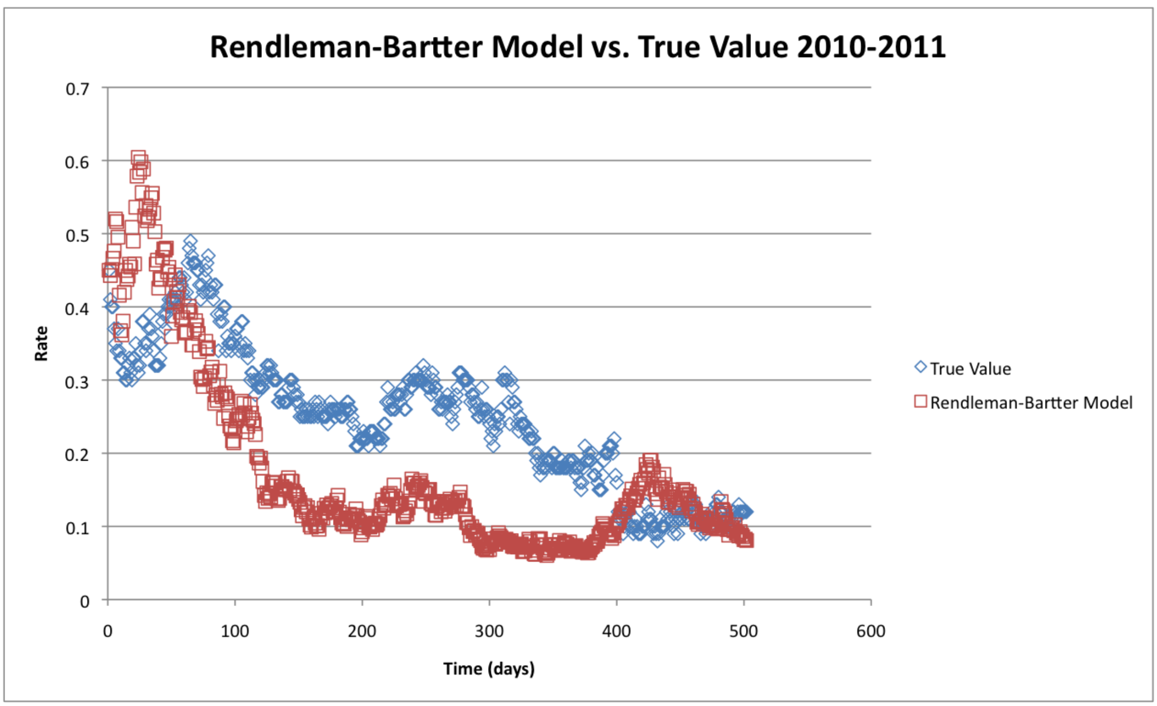

Figure A.2: Rendleman-Bartter model vs. true value 2010-2011

Date	Simulated Value
1/4/10	0.45000
1/5/10	0.44282
1/6/10	0.45102
⋮	⋮
12/30/11	0.08079

Table A.2: Snapshot of Excel simulation for Rendleman-Bartter model 2010-2011

A.1.3 2012-2013

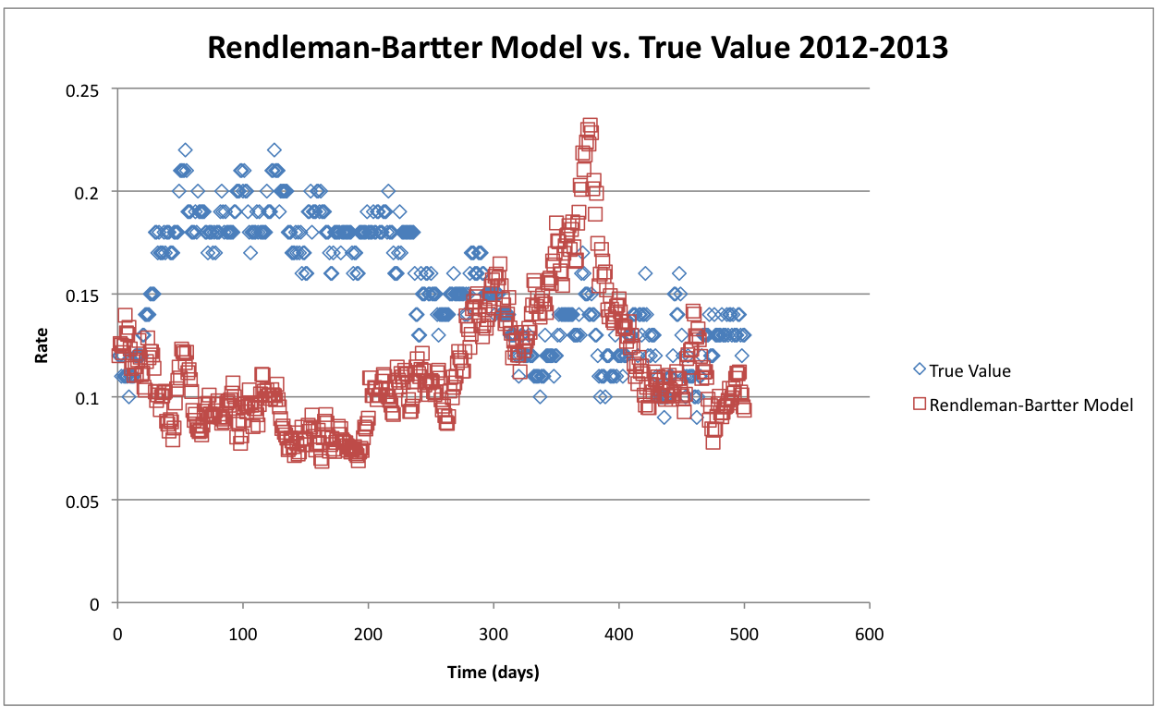

Figure A.3: Rendleman-Bartter model vs. true value 2012-2013

Date	Simulated Value
1/3/12	0.12000
1/4/12	0.12223
1/5/12	0.12196
⋮	⋮
12/31/13	0.11463

Table A.3: Snapshot of Excel simulation for Rendleman-Bartter model 2012-2013

A.2 VASICEK MODEL

A.2.1 2008-2009

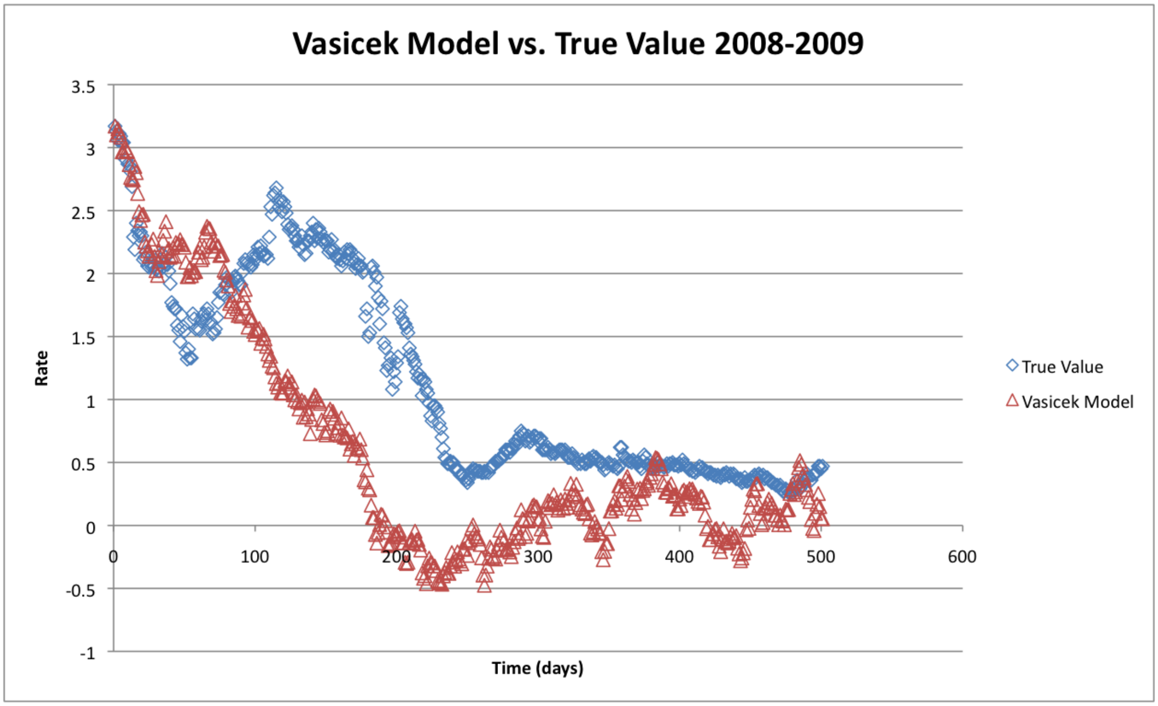

Figure A.4: Vasicek model vs. true value 2008-2009

Predictor	Coef	SE Coef	T	P
Constant	0.005017	0.005247	0.96	0.339
Lag 1	-0.009036	0.003727	-2.42	0.016
S=0.0673472				

Table A.4: Minitab output for Vasicek model calibration 2008-2009

Date	Simulated Value
1/2/08	3.17000
1/3/08	3.10096
1/4/08	3.14830
⋮	⋮
12/31/09	0.05282

Table A.5: Snapshot of Excel simulation for Vasicek model 2008-2009

A.2.2 2010-2011

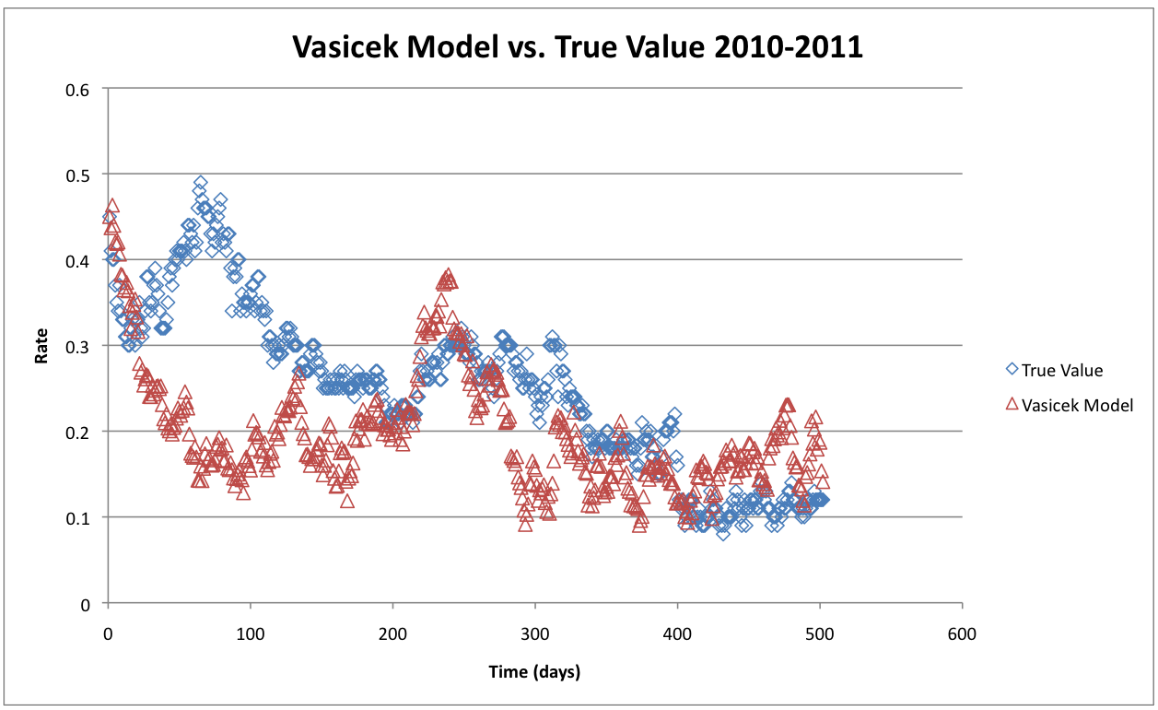

Figure A.5: Vasicek model vs. true value 2010-2011

Predictor	Coef	SE Coef	T	P
Constant	0.002779	0.001796	1.55	0.122
Lag 1	-0.013776	0.006704	-2.05	0.040
S=0.0144931				

Table A.6: Minitab output for Vasicek model calibration 2010-2011

Date	Simulated Value
1/4/10	0.45000
1/5/10	0.43698
1/6/10	0.46342
⋮	⋮
12/30/11	0.14116

Table A.7: Snapshot of Excel simulation for Vasicek model 2010-2011

A.2.3 2012-2013

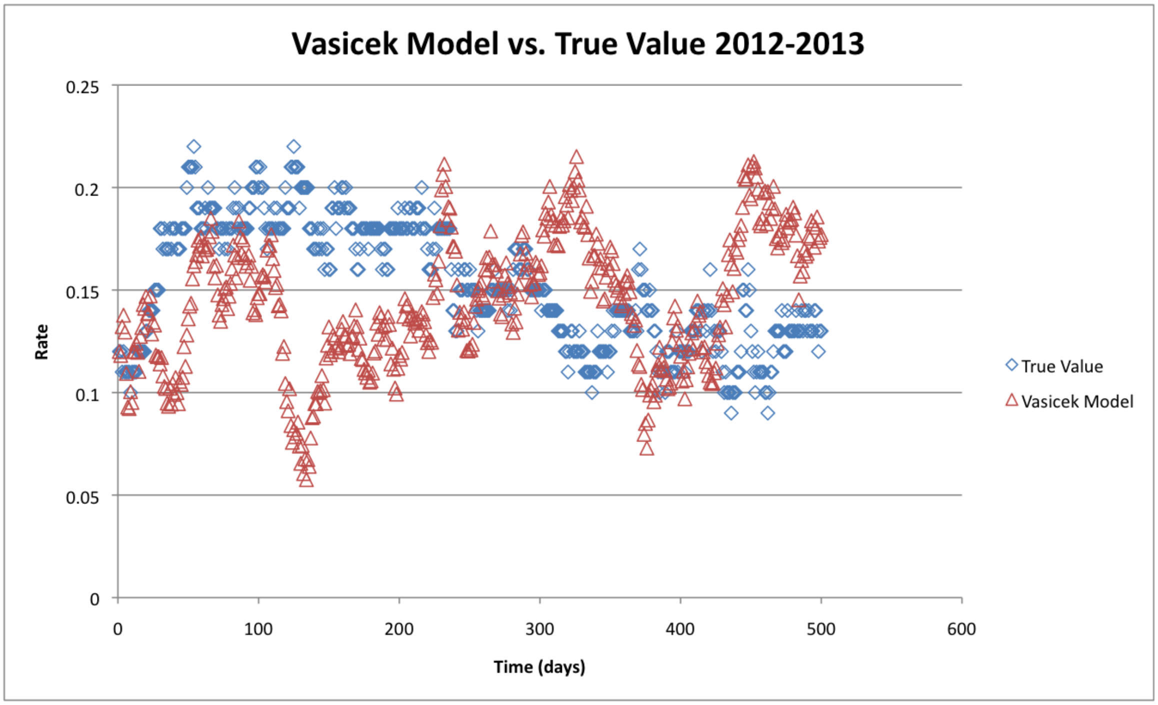

Figure A.6: Vasicek model vs. true value 2012-2013

Predictor	Coef	SE Coef	T	P
Constant	0.006165	0.001947	3.17	0.002
Lag 1	-0.04015	0.01248	-3.22	0.001

S=0.00838844

Table A.8: Minitab output for Vasicek model calibration 2012-2013

Date	Simulated Value
1/3/12	0.12000
1/4/12	0.11807
1/5/12	0.13212
⋮	⋮
12/31/13	0.17703

Table A.9: Snapshot of Excel simulation for Vasicek model 2012-2013

A.3 COX–INGERSOLL–ROSS MODEL

A.3.1 2008-2009

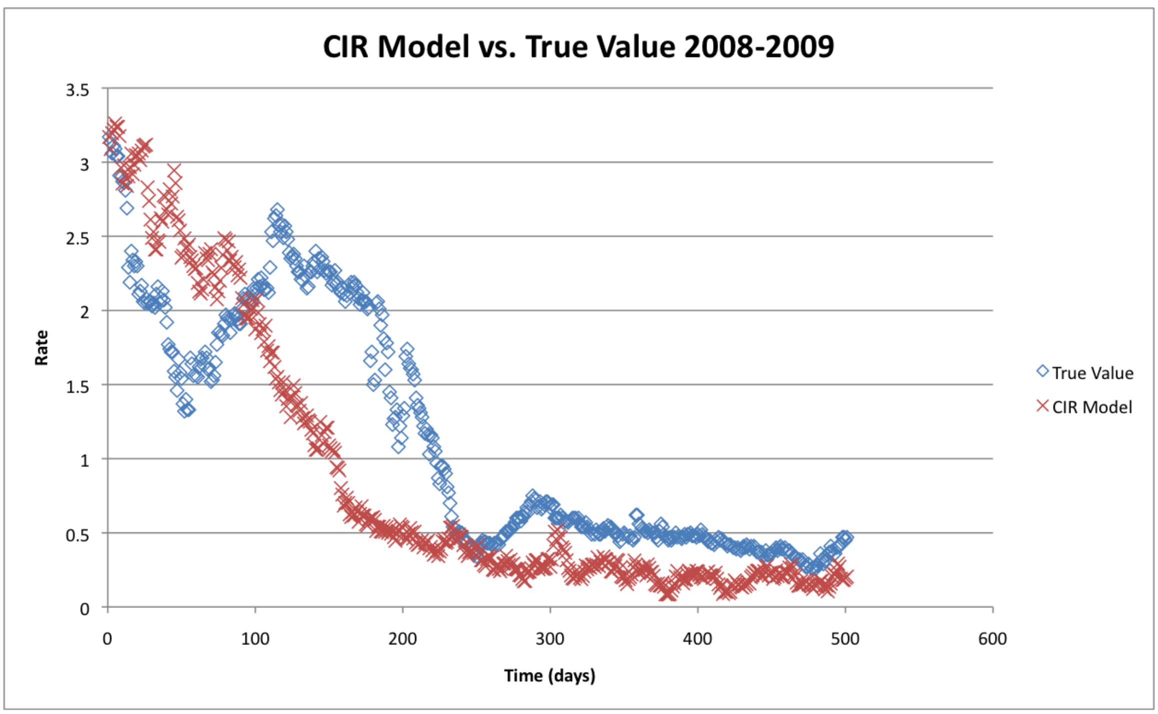

Figure A.7: CIR model vs. true value 2008-2009

Predictor	Coef	SE Coef	T	P
Sqrt(Lag 1)	-0.008144	0.003682	-2.21	0.027
Sqrt(Lag 1) Inverse	0.003989	0.003283	1.22	0.225
S=0.0560611				

Table A.10: Minitab output for CIR model calibration 2008-2009

Date	Simulated Value
1/2/08	3.17000
1/3/08	3.08757
1/4/08	3.19475
⋮	⋮
12/31/09	0.20494

Table A.11: Snapshot of Excel simulation for CIR model 2008-2009

A.3.2 2010-2011

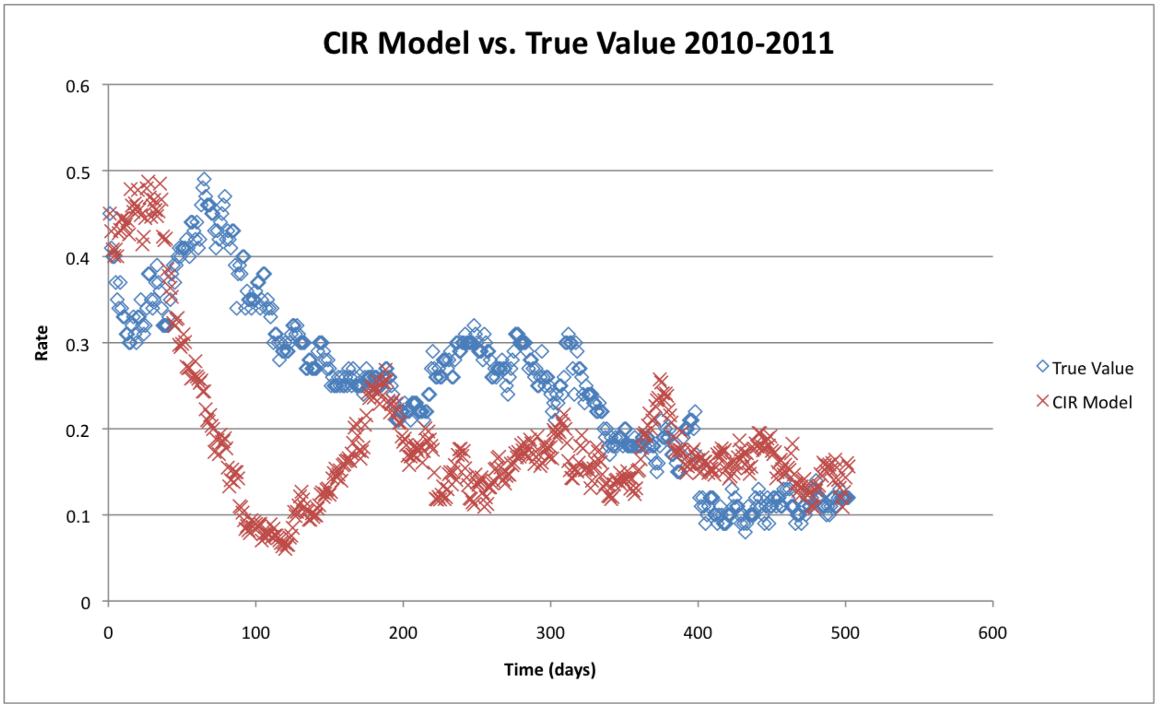

Figure A.8: CIR model vs. true value 2010-2011

Predictor	Coef	SE Coef	T	P
Sqrt(Lag 1)	-0.013591	0.006341	-2.14	0.033
Sqrt(Lag 1) Inverse	0.002733	0.001438	1.90	0.058

S=0.0296151

Table A.12: Minitab output for CIR model calibration 2010-2011

Date	Simulated Value
1/4/10	0.45000
1/5/10	0.42951
1/6/10	0.40830
⋮	⋮
12/30/11	0.15628

Table A.13: Snapshot of Excel simulation for CIR model 2010-2011

A.3.3 2012-2013

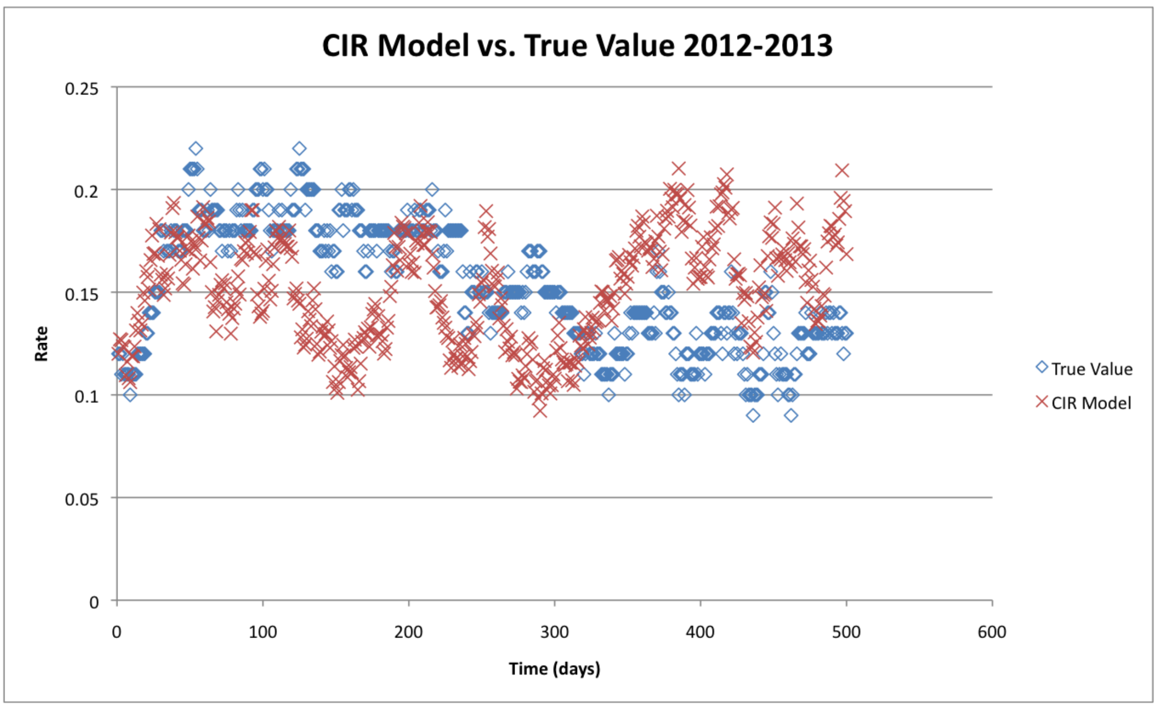

Figure A.9: CIR model vs. true value 2012-2013

Predictor	Coef	SE Coef	T	P
Sqrt(Lag 1)	-0.04128	0.01253	-3.29	0.001
Sqrt(Lag 1) Inverse	0.006337	0.001879	3.37	0.001
S=0.0218529				

Table A.14: Minitab output for CIR model calibration 2012-2013

Date	Simulated Value
1/3/12	0.12000
1/4/12	0.12711
1/5/12	0.12663
⋮	⋮
12/31/13	0.16854

Table A.15: Snapshot of Excel simulation for CIR model 2012-2013

A.4 HULL–WHITE ONE-FACTOR MODEL

A.4.1 2008-2009

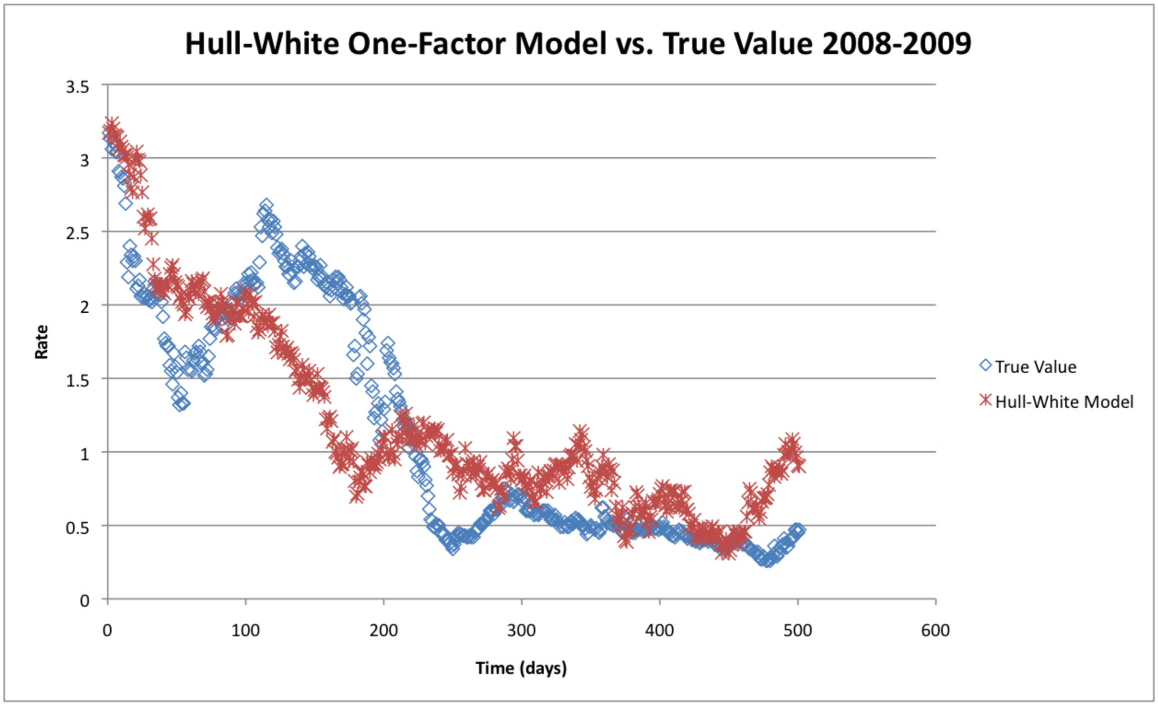

Figure A.10: Hull-White one-factor model vs. true value 2008-2009

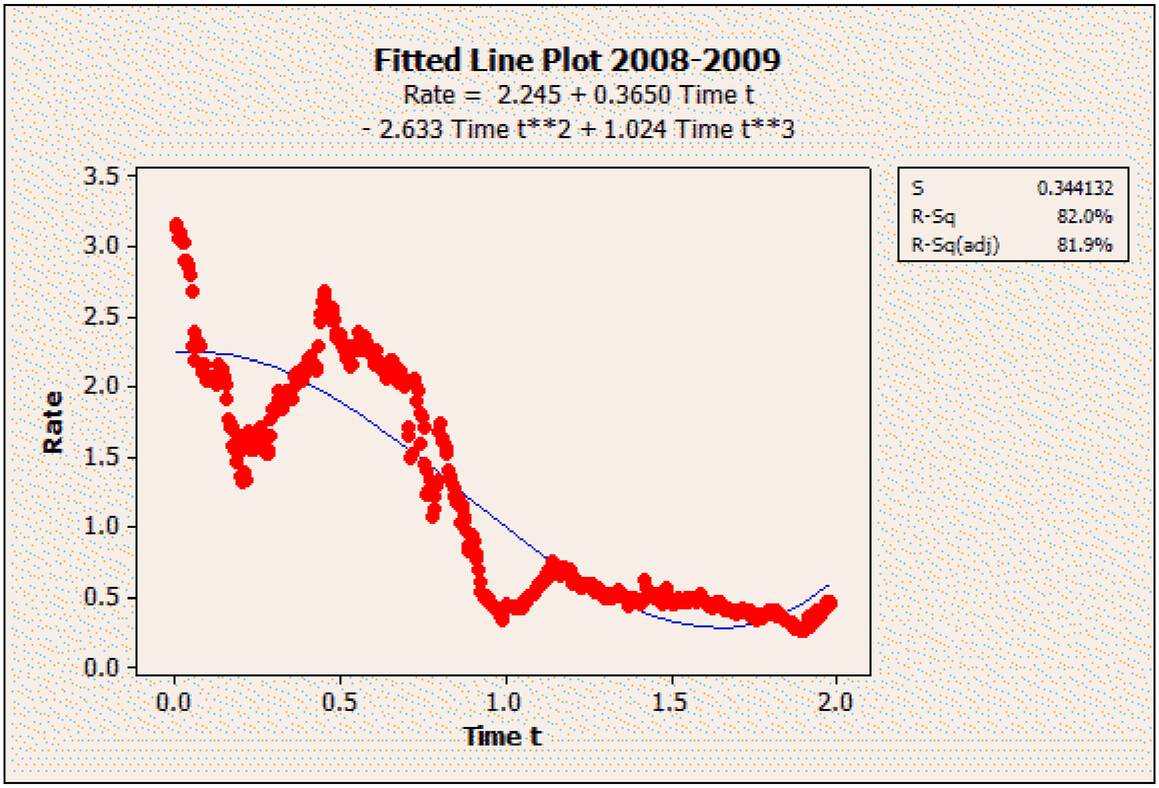

Figure A.11: Fitted line plot used in Hull-White one-factor model calibration 2008-2009

Predictor	Coef	SE Coef	T	P
C6-C3	0.026526	0.008763	3.03	0.003
S=0.0672121				

Table A.16: Minitab output for Hull-White one-factor model calibration 2008-2009

Date	Simulated Value
1/2/08	3.17000
1/3/08	3.18284
1/4/08	3.23546
⋮	⋮
12/31/09	0.90115

Table A.17: Snapshot of Excel simulation for Hull-White one-factor model 2008-2009

A.4.2 2010-2011

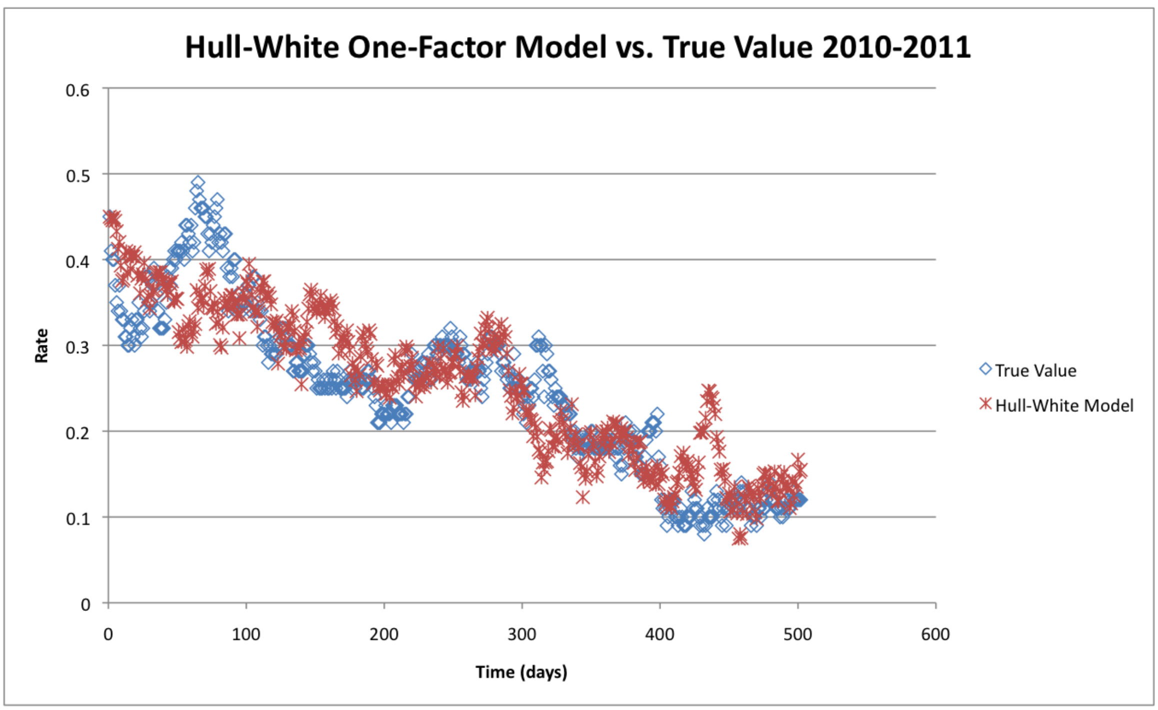

Figure A.12: Hull-White one-factor model vs. true value 2010-2011

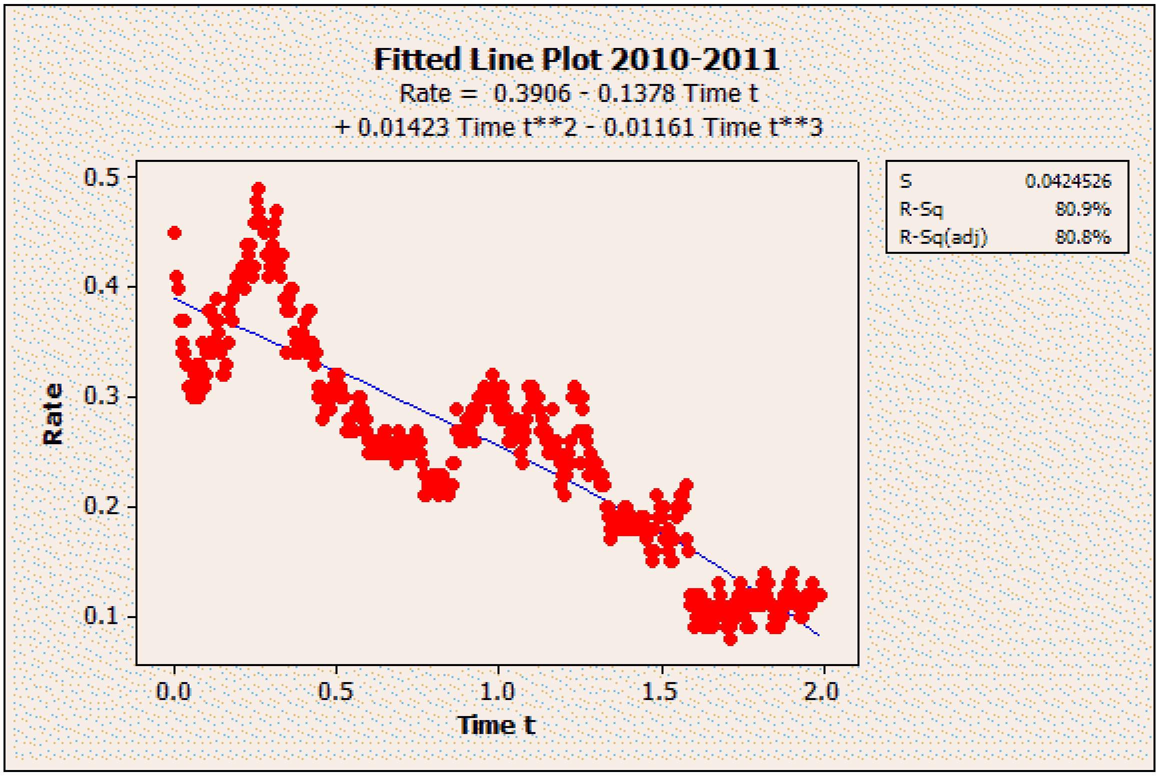

Figure A.13: Fitted line plot used in Hull-White one-factor model calibration 2010-2011

Predictor	Coef	SE Coef	T	P
C6-C3	0.06047	0.01522	3.97	0.000
S=0.0143302				

Table A.18: Minitab output for Hull-White one-factor model calibration 2010-2011

Date	Simulated Value
1/4/10	0.45000
1/5/10	0.44801
1/6/10	0.44583
⋮	⋮
12/30/11	0.15494

Table A.19: Snapshot of Excel simulation for Hull-White one-factor model 2010-2011

A.4.3 2012-2013

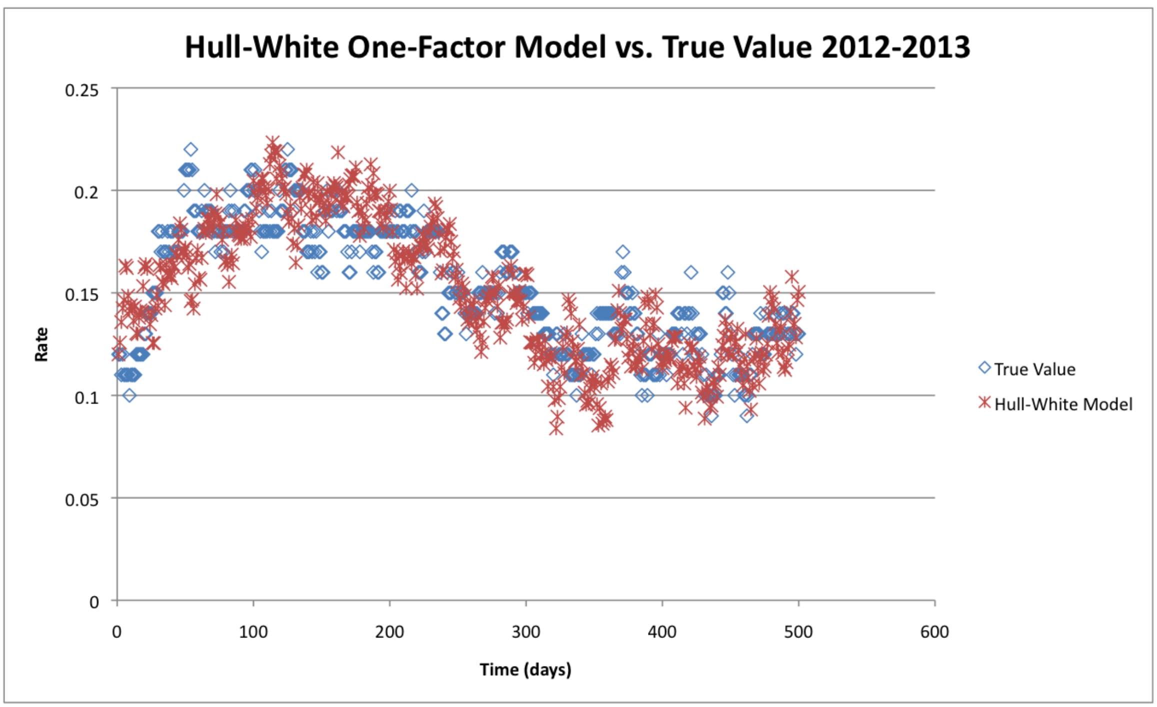

Figure A.14: Hull-White one-factor model vs. true value 2012-2013

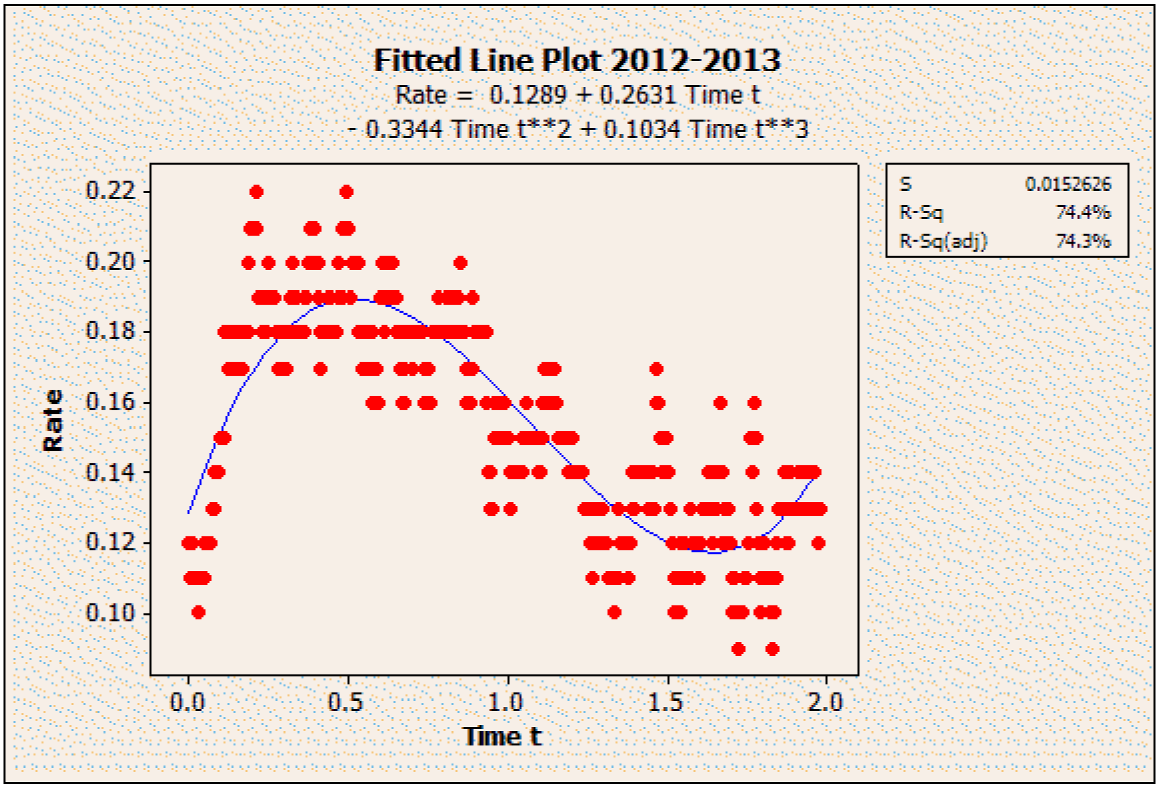

Figure A.15: Fitted line plot used in Hull-White one-factor model calibration 2012-2013

Predictor	Coef	SE Coef	T	P
C6-C3	0.15429	0.02395	6.44	0.000
S=0.00813339				

Table A.20: Minitab output for Hull-White one-factor model calibration 2012-2013

Date	Simulated Value
1/3/12	0.12000
1/4/12	0.12577
1/5/12	0.13572
⋮	⋮
12/31/13	0.15048

Table A.21: Snapshot of Excel simulation for Hull-White one-factor model 2012-2013

BIBLIOGRAPHY

[1] Amin, H. N. (2012). *Calibration of different interest rate models for a good fit of yield curves.* Retrieved October 25, 2013, from `http://repository.tudelft.nl/assets/`.

[2] Cescato, C. D. & Lemgruber, E. F. (2011). Valuation of American interest rate options by the least-squares Monte Carlo method. *Pesquisa Operacional, 31*(3), 521-541.

[3] Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica, 53*(2), 385-407.

[4] Hull, J. C. (2009). *Options, futures, and other derivatives.* Upper Saddle River, NJ: Prentice Hall.

[5] Hull, J. C. & White, A. (1990). Pricing interest-rate-derivative securities. *The Review of Financial Studies, 3*(4), 573-592.

[6] Øksendal, B. (2007). *Stochastic differential equations*. New York: Springer.

[7] Park, F. C. (2004). *Implementing interest rate models: A practical guide*. Retrieved January 25, 2014, from `http://www.cmpr.co.kr/asset/`.

[8] Rendleman, R. J. & Bartter, B. J. (1980). The pricing of options on debt securities. *Journal of Financial and Quantitative Analysis, 15*(1), 11-24.

[9] Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics, 5*, 177-188.