AMERICAN LOOKBACK OPTIONS: EARLY EXERCISE

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by

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ABSTRACT

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Lookback options are path dependent contingent claims whose payoff depend on the extrema of a given security's price over a given period. Some of these options are already traded on specialized markets (such as foreign exchange) and mostly in over-the-counter market alongside with other path dependent options (knock-ins, knock-outs, etc.). This thesis examines the existing pricing models of conventional options as well as standard European lookback options and provides some results about early exercise of their American counterparts with the use of notions from the theory of optimal stopping.

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Chapter I

INTRODUCTION

During the 1980s the financial markets were witness to a host of new financial instruments, products and strategies. These products were created to meet specific financial needs and to obtain certain financial results, and most are still serving their purpose in the financial markets of 1990s. Interest rate and currency swaps, forward rate agreements, over-the-counter options, high-yield bonds, asset-backed commercial paper and medium term notes are employed by small and large corporations, investors, and financial intermediaries on a continuous basis.

Being ever-vigilant to both the requirements of the customers and the need for new sources of revenue and profitability, major financial institutions with strong capabilities in the above products have been very quick in the 1990s to create a series of new instruments and strategies designed to provide additional (or tailor-made) types of risk protection or investment opportunity.

Many of these products have their foundation in the basic instruments mentioned above; in certain cases they represent 'derivatives' of now standard products like interest rate and currency swaps. Leveraged swaps, inverse floater swaps, differential swaps, and index principal swaps are all examples of derivatives on the standard swap structure. In certain other cases, the products are entirely new to the markets. Though they may parallel certain concepts found in existing instruments such as swaps, these products tend to be unique in most other aspects, primarily because they allow for very specific results to be attained in markets which have not previously been tapped. Many of the equity derivative swaps and exotic options structures form this cadre of new products. In addition to creating new products to meet new or specific needs, many financial institutions have been active in recent years in recommending alternative risk management and/or profit-making strategies which rely on use of over-the-counter (OTC) options, in linked combinations which yield very specific results: unlimited profit opportunities with unlimited downside risk; limited profit opportunities with limited downside risk; effective risk hedging with potential upside profit; and so on. Put and call spreads, straddles and strangles are all examples of such strategies.

In addition to these derivative products, certain financial institutions have been pioneering other structures which yet have to make their debuts, or to be utilized in a significant and widespread fashion by market participants. Examples of these include property, tax, macroeconomic, credit, insurance and inflation derivatives.

Chapter II

DERIVATIVE INSTRUMENTS

A. OPTION PRODUCTS

1. Interest rate options

Caps/floors - the most common interest rate options (swaptions, yield curve options and bond options are all considered part of the interest rate option category as well). Caps and floors are barriers which, when exceeded, will provide an inflow to the purchaser and an outflow to the seller, in an amount equal to:

$Payout_C = (CMP - SP) * N$

$Payout_F = (SP - CMP) * N$, where CMP is the current price of the rate index, SP is the strike rate of the cap or floor, N is the notional.

Collars - a combination of caps and floors (e.g. selling a cap and buying a floor).

Corridors - a combination of caps and floors (e.g. buying a cap and selling a higher strike cap).

When-in-the-money option - an option which allows the buyer to defer payment of premium until such time as the option moves in-the-money.

Compound, or nested, options (sometimes also known as *captions* or *floortions*) are options on other options, namely caps and floors. A compound option gives the purchaser the right, but not the obligation, to enter into a cap or floor agreement with a compound option seller in exchange for a premium payment.

Power caps - a relatively recent creation, the same as a standard cap, except instead of receiving the difference between CMP and SP, the buyer receives the difference raised to some exponent or 'power'.

2. Equity derivative options

Conventional options - Puts and calls on individual equities and on equity indexes are widely used by participants in the equity derivatives markets acting as either investor, issuer or hedger. While many of equity options are listed, much of the activity in these instruments takes place in the OTC market, which allows for greater flexibility in setting strike prices and maturity dates. Puts and calls on equity may be struck in-, at-, or out-of-the-money, and they may be for European or American exercise. Maturities in the OTC market may range from several weeks to several years.

Path Dependent Barrier Options - These are the options with a payout directly related to the movement of the underlying security. A barrier option is an option which either comes into existence, or is eliminated, when a market price reaches a pre-determined strike price.

Knock-outs

Up-and-out - an option is canceled out if an index rises above a certain level.

Down-and-out - an option is canceled if an index falls to a certain level.

Knock-ins

Up-and-in - an option comes into existence if an index reaches a certain level (up).

Down-and-in - an option comes into existence if the index falls below a certain level (down).

Lookback - the buyer of the put or call will receive from the seller the greatest economic value achieved during the life of the option, regardless of the level at expiry.

Asian - an average strike option, which allows the buyer to set the strike as average over a given period.

Step-lock - an option will lock in gains at pre-specified levels, once those levels have been achieved (additional pre-specified levels which are breached add to the value of the option).

Cliquet (rachet) - an option which locks in any gains based on time, as opposed to price, thresholds (i.e. locks in the gains each quarter or semi-annual period).

Binary options - Binary or digital options are options which provide discontinuous payoff or protection. Under this type of option structure, the buyer will pay or receive one of two different flows if a particular level is reached.

All-or-nothing - will provide the option buyer a payout of 'all' or 'nothing' depending on the value of the option at expiration.

One-touch - identical to all-or-nothing except they are exercisable as soon as the strike is reached, evaluation need not wait till maturity.

Time dependent options

Preference (chooser) option - gives the buyer great flexibility in selecting specific characteristics of the option within a given time frame. During this time frame the buyer may elect whether the option should be a put or a call, where a put and a call may each have their own distinct strike and maturity parameters (complex chooser). At the conclusion of the time frame ('choice date') the option seller will enter into transaction with the details as specified by the buyer. Certain preference options restrict the buyer to deciding whether an option will be a put or a call, with no flexibility as to maturity or strike price (regular choosers).

Deferred payment American option

DPA option - under the terms of this option, the buyer will enter into an American exercise option with a final maturity at time *t*. If the options moves in-the-money at some point during the transaction, the buyer will exercise and be entitled to the intrinsic value. However, the option seller will simply defer payment until the original expiration of the option, denoted by time *t*.

Outperformance/Basket options

Outperformance and basket options provide the call or put purchaser with the ability to benefit from the comparative upward or downward performance between two markets, or between the better (worse) performing of several markets against predetermined strike level.

Outperformance (relative performance option, RPO) option - an option on the differential between two markets, individual stocks, certain group of stocks in one industry against a group of stocks in a second industry.

Basket option - allows the purchaser to combine a series of indexes (or individual stocks) on a weighted basis, and to receive the appreciation (for a call) of that group of indexes (stocks) over and above some strike level.

Quantos

Quanto (guaranteed exchange rate contract) - forward exchange contract incorporated into an underlying equity option or equity swap structure which allows the investor to lock in a known foreign exchange flow and eliminate currency risk.

B.OPTION STRATEGIES

Except for issuing or buying different derivative products, a bank can take different positions in regular options to create the same payoff, or risk, or profitability. These strategies take advantage of either price movements of the instruments, of their volatility or of their time to maturity.

Price driven positions

Long call - taken when a market is expected to rise.

Short call - taken when a market is expected to fall.

Long put - taken when a market is expected to fall.

Short put - taken when a market is expected to rise.

Vertical Spreads

Bullish Vertical call spread - a combination of a long call with a low strike and a short call with a higher strike.

Bullish Vertical put spread - a combination of a long put with a low strike and a short put at a higher strike.

Bearish Vertical call spread - reverse of the bullish vertical call spread.

Bearish Vertical put spread - reverse of the bullish vertical put spread.

Volatility Driven Strategies - Volatility driven strategies are those which seek to take advantage of movements in volatility, as opposed to absolute prices or price direction.

Delta - the rate of change (or percentage change) in the price of the option for a unit change in the price of the underlying instrument.

Gamma - the rate of change (or percentage change) in delta of the option for a unit change in the price of the underlying instrument.

Theta - the rate at which an option loses value with the passage of time (also known as time to decay).

Lambda (zeta, vega or kappa) - the rate at which the value of the option changes for a unit percentage change in volatility.

Straddles - formed by a combination of puts and calls, with equal strike levels and identical maturity dates.

Strangles - a combination of puts and calls with the same expiration dates but different strike levels (guts - strangles where there is an in-the-money portion at the outset).

Butterflies - a combination of 4 separate puts and calls, designed to provide results which are similar to straddles or ratio verticals. A long butterfly is created by buying the low and high strikes and selling the middle strikes (all with the same exercise date).

Condors - similar to butterfly spreads, except strikes of the various options are further apart.

Ratio Horizontal Spreads (time or calendar spreads) - are created when a bank buys or sells the option closer to expiry while doing the opposite with the option further to expiry. Spreads are established by utilizing the same type of option (puts or calls, but not both) together with the same strike price.

Backspreads - are created when an institution buys more contracts than it sells, whether puts or calls. All contracts have the same expiry date but, to remain delta neutral, a call backspread would require buying calls with higher strikes and selling the lower strike ones, or buying puts with lower strikes and selling the higher strike ones.

Ratio Vertical Spreads - occur when a bank establishes a greater sold than bought position in puts or calls with the same expiry date (the opposite of backspread).

Long strap - long one put and long two calls with the same strike.

Short strap - short one put and short two calls with the same strike.

Long strip - long one call and long two puts with the same strike.

Short strip - short one call and short two puts with the same strike.

Box spread - long one call and short one put with the same low strike and short one call and long one put with the same high strike; or long one put and short one call with the same low strike and long one call and short one put with the same high strike.

C. SYNTHETICS

By taking positions in the underlying security and its option at the same time a bank is able to create 'synthetic options' (i.e. these positions will replicate the option's payoff and risk).

Synthetic options

Synthetic	=	Underlying	+	Option
Long call	=	Long underlying	+	Long put
Long put	=	Short underlying	+	Long call
Short put	=	Long underlying	+	Short call
Short call	=	Short underlying	+	Short put

Synthetic underlying

Synthetic	=	Option	+	Option
Long underlying	=	Long call	+	Short put
Short underlying	=	Short call	+	Long put

D. SWAP PRODUCTS

Swaps took off during 1980s. The first swaps to emerge were currency swaps, which began in mid-1960s. But the market remained a fairly small one during the 1970s. The emergence of the interest rate swap in 1980/81 transformed the situation. The market now has a huge range of counterparties, using swaps for a range of different reasons. Banks use swaps for asset and liability management; corporations use them for similar purposes and also to raise finance through the bond market, notably the Eurobond market, at rates which are better than those their bankers can offer. While standard or 'plain vanilla' swap structures (i.e. those which involve the periodic exchange of interest or currency flows between two parties, on a par, or 'on-market' basis) are by now commonplace, there are additional swap structures created by banks which are attracting increased, or renewed, use and attention.

A swap is a contractual agreement evidenced by a single document in which two parties agree to make periodic payments to each other. Contained in the swap agreement is a specification of the currencies to be exchanged (which may or may not be the same), the rate of interest applicable to each (which may be fixed or floating), the timetable by which the payments are to be made, and any other provisions bearing on the relationship between the parties.

The underlying assets may or may not be exchanged and are referred to as notionals. The swap commences on its effective (value) date and terminates on its termination (maturity) date. The period of time between these two dates is called the swap's tenor or maturity. The periodic payments made by the parties are called service payments. The service payments of a party with fixed rate are called the swap coupon. The floating rate of the second party is pegged to some specific spot market rate called the reference rate , which is observed on specific dates (reset dates). The actual dates on which the exchanges of payments occur are called the payments dates.

1. Interest rate swaps

Basis swap (floating-for-floating) - a swap on which both legs are floating but tied to two different indexes.

Yield curve swap - a swap similar to a basis swap except the floating legs are tied to long-term rates.

Forward (deferred) swaps - a swap in which the swap coupon is set on the transaction date but the swap does not commence until a later date.

Zero coupon swap - a fixed-for-floating swap on which a fixed rate is a zero coupon.

Roller coaster swap - a swap which provides for a period of accretion followed by a period of amortization.

Delayed-rate (deferred-rate) swap - a swap that commences immediately but on which a swap coupon is not set until a later date.

Callable/putable/extendable swap - a swap on which one party has the right but not the obligation, to either extend or shorten the maturity of the swap. In a callable swap - the fixed rate payer has this right, in putable - the floating rate payer, in extendable one party has the right to extend the tenor of the swap beyond its scheduled termination date.

Rate-capped swap - a swap on which a floating rate is capped.

Reversible swaps - a swap on which a fixed and floating rates payers can reverse roles one or more time during the life of the swap.

2. Equity derivative swaps

Equity derivative swaps allow one or both swap parties to achieve financial gains based on the financial appreciation or depreciation of an equity, a basket of equities, or an equity index; maturities usually range from one to five years, though transactions can be structured with longer or shorter terms. In a standard two-sided swap the parties may exchange payments periodically. In an equity call swap, one of the parties is not payer of depreciation of the index, in a put swap index appreciation would be irrelevant.

Inverse floater swaps (sometimes known as *reverse swaps*) have been in existence for several years, and seek to take advantage of a steep yield curve. A typical structure calls for paying or receiving a floating rate index (say Libor) versus receiving or paying a fixed rate less the same floating rate index. The general form of the flows is:

Pay or rec:(x% fixed-floating index) vs Rec or pay: floating index

Leveraged swaps are closely related to inverse floater swaps, except they involve far greater amounts of leverage.

Differential swaps (often referred to as *'diff' swaps*) have gained substantial popularity in recent years with institutions and investors who are attempting to capitalize on a view of foreign markets, without having to be exposed to currency risk. In a typical non-leveraged differential swap, an institution may wish to receive 6-month Libor in exchange for paying US dollar Libor, all payable in dollars. Since the flows are payable in a single currency (e.g. dollars), the institution is taking no currency risk. In a diff swap there is typically a spread subtracted from the institution's receive flow (or added to its pay flow), which reflects an interest differential between the two currencies. Another popular structure is to combine a differential swap with a leveraged and/or inverse floater structure: our bank may wish to pay 15%-(2*Yen Libor), in exchange for receipt of US dollar Libor-.25%, all payable or receivable in dollars.

Amortizing swaps, which have been in use for several years, are swaps which are based on an ever changing principal balance. Any type of principal reduction can be considered and swapped, though an even flow of reductions tends to be most common.

Mortgage swaps are a relatively recent creation, but are drawing increased interest from a broad range of institutional investors. Mortgage swaps are, in essence, swaps which are structured to replicate the flows in a certain pool or package of mortgage backed securities (MBS); since the structure is synthetic, it is generally true that no physical mortgage backed securities are held by the investor. The structure of the swap, however, allows the investor's return to precisely equal the return derived from a specific pool of MBS. An example of a MB swap would be one where the investor pays a monthly floating rate (typically Libor), in exchange for a fixed rate, 'decompounded' and paid monthly. The notional principle of the swap would amortize down at a rate which is equivalent to the amortization level of a certain FNMA, for instance. There are a variety of other forms by which a mortgage swap can be created: the flows may seek not to duplicate a specific class of securities but a specific security, or a pool of mortgage backed securities (such as a CMO). In this case the investor would pay Libor and receive a fixed coupon, while the swap notional would amortize down on a schedule determined by the Public Securities Association (PSA) rates, tied to a monthly amortization factor. Another structure might involve the replication of a FNMA (adjustable rate mortgage (ARM)) passthrough based on the COF Index. Under such a swap, the investor would pay Libor and receive COF (a floating index), on a notional amount which amortizes at a rate equal to the FNMA COF ARM passthroughs.

Index principal swaps (IPSs), sometimes also referred to as *index amortizing swaps* (IARs), are a recent creation. A typical IPS is one where the notional principal amortizes when rates fall. In a standard structure, as Libor falls, the notional principle of the swap amortizes on a set schedule. While the notional will amortize if the floating rate falls , it will not amortize if rates rise; in this case the swap will take on characteristics of a plain vanilla transaction. In IPSs, the bank paying the floating rate is called the 'purchaser' of the swap; in exchange for paying the floating rate it receives a fixed rate which is typically above market.

Chapter III

OPTION PRICING MODELS

The long history of option pricing began in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula on the assumption that stock prices follow a Brownian motion with zero drift. Since that time, numerous researchers have contributed to the theory. The approaches taken range from sophisticated general equilibrium models to ad hoc statistical fits. The amount of time and energy spent on the option pricing theory may be questioned because options are relatively unimportant financial securities. One of the justifications may be the fact that a theory of option pricing leads to a more general theory of contingent-claims pricing. Some have argued that all such securities can be expressed as combinations of basic options, as such a theory of option pricing constitutes a theory of contingent-claims pricing. Moreover, there exist large quantities of the data available.

Along with the publication of the seminal Black-Scholes paper, the spring of 1973 also marked the creation of the first organized markets for options on common stocks. In April of that year, the Chicago Board Options Exchange

(CBOE) began trading call options on 12 companies' shares. A significant part of the academic empirical research would not have been possible without the transactions data generated by these markets. However, the commercial success of these specialized markets is not the reason that option pricing analysis has become one of the cornerstones of general finance theory. Instead, the central role for option analysis evolves from the fat that option-like structures permeate virtually every part of the field. The continuous-time theory of option pricing forms the methodological foundation for the general theory of contingent-claims pricing which can be applied to a wide range of problems in corporate finance, financial intermediation, and capital markets, including the pricing of corporate liabilities, the determination of the term structure of interest rates, and the evaluation of complex capital budgeting decisions.

A. Black-Scholes Option Pricing Formula

An American call option is a security that gives its owner a right to purchase a share of stock at a given ('exercise') price on or before a given date. An American put option gives its owner the right to sell a share of stock on or before a given date. A European option has the same terms as its American counterpart except that it cannot be exercised before the last date of the contract.

In their classic paper on the theory of option pricing, Black and Scholes (1973) present a mode of analysis that has revolutionized the theory of corporate liability pricing. In part their approach was a breakthrough because they use only observable variables.

To derive their option pricing formula Black and Scholes assume 'ideal conditions' in the market for the stock and option. These conditions are:

1. 'Frictionless' markets: there are no transactions costs or differential taxes. Trading takes place continuously in time. Borrowing and short-selling are allowed without restriction and with full proceeds available. The borrowing and lending rates are equal.

2. The short term interest rate is known and constant through time.

3. The stock pays no dividends or other distributions during the life of the option.

4. The option is European in that it can only be exercised at the expiration date.

5. The stock price follows a 'geometric' Brownian motion through time which produces a log-normal distribution for the stock price between any two points in time.

To derive the formula, they assume that the option price is a function of the stock price and time to expiration. Over short time intervals, a hedge portfolio of the common stock, the option and short term riskless security is formed where the portfolio weights are chosen to eliminate all 'market' risk. By the assumption of the Capital Asset Pricing Model, any portfolio with a zero market risk must have an expected return equal to the risk free rate. Hence, an equilibrium condition is established between the expected return on the option, the expected return on the stock, and the riskless rate. From the equilibrium condition on the option yield, a partial differential equation expressing this return in terms of the option price function and its partial derivatives is derived. The solution to this equation for a European call is

$$f(S,\tau;K)=SN(d_1)-Ke^{-r\tau}N(d_2) \tag{1}$$

where N(x) is the cumulative standard normal distribution function, S is the stock price, K - exercise price, r - risk free interest rate, τ is the time before exercise date (T-t), σ^2 is the instantaneous variance of the return on the common stock,

$$d_1 \equiv \frac{\log(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{2}$$

and $d_2 \equiv d_1 - \sigma\sqrt{\tau}$.

The manifest characteristic of the formula is the number of variables that it does not depend on. The option price does not depend on the expected return on the common stock, risk preferences of the investors, or the aggregate supply of assets. It does depend on the rate of interest which is observable and the total variance of the return on the common stock which is often a stable number and hence accurate estimates are possible from the time series data.

B. Merton's Option Pricing Formula

In a subsequent, alternative derivation of the Black Scholes formula, Merton (1973a) demonstrated that their basic mode of analysis obtains even when the interest rate is stochastic; the stock pays dividends; and the option is exercisable prior to expiration.

Thorp (1973) and Ingersoll (1976) have shown that neither dividends nor differential taxes for capital gains change the analysis as long as the stock-price dynamics can be described by a continuous-time diffusion process whose sample path is continuous with probability one.

Merton (1973a) points out that the critical assumption in Black-Scholes derivation is that trading takes place continuously in time and that the price dynamics of the stock have a continuous sample path with probability one. However, this solution is invalid, even in the continuous limit, when the stock-price dynamics cannot be represented by a stochastic process with a continuous sample path.

To take into consideration these discontinuities ('jumps') Merton writes the stock-price return process as a combination of a diffusion and a Poisson processes:

$$\frac{dS}{S} = (\alpha - \lambda k)\,dt + \sigma\,dZ + dq \qquad (3)$$

where α is the instantaneous expected return on the stock; σ^2 is the instantaneous variance of the return, conditional on no arrivals of important new information; dZ is a standard Gauss-Wiener process; q(t) is the independent Poisson process; dq and dZ are assumed to be independent; λ is the mean number of arrivals per unit time; $k \equiv \epsilon\{Y-1\}$ where Y-1 is the random variable percentage change in the stock price if the Poisson event occurs; and ϵ is the expectation operator over the random variable Y.

If $\lambda=0$ the return dynamics are identical with those of Black-Scholes. The solution for this dynamics is given by:

$$F(S,\tau) = \sum_{n=0}^{\infty} \frac{\exp(-\lambda\tau)\,(\lambda\tau)^n}{n!} \epsilon_n\{W[SX_n \exp(-\lambda k\tau), \tau; K, \sigma^2, r]\} \qquad (4)$$

where W(S, τ; K, σ^2, r) is the Black-Scholes option pricing formula for the no-jump case, ϵ_n expectation operator over the distribution of X_n which has the same distribution as the product of n independent identically distributed random variables each identically distributed to the random variable Y defined by (3). While (4) is not a closed-form solution, it does admit to reasonable computational approximation provided the density functions for the $\{X_n\}$ are not too complicated.

C. Further Developments In The Theory

Scholes (1976) modifies the basic Black-Scholes model to take account of the effects of taxes on both option prices and the prescribed mix of stocks and bonds in the replicating-portfolio strategy. Leland (1985) and Hodges and Neuberger (1989) examine option pricing and the risks of imperfect hedging when there are transactions costs. Hull and White (1987), Johnson and Shanno (1987), Scott (1987), and Wiggins (1987) investigate option pricing with price dynamics following a diffusion process with a stochastic variance rate.

Cox and Ross (1976) and Cox, Ross, and Rubenstein (1979) derive the equations of option pricing for other types of processes. To be more exact, they derive a simple discrete-time model which as a special limiting case contains Black-Scholes. Their binomial option pricing formula is:

$$C=S\Phi[a;n,p'] - Kr^{-n}\Phi[a;n,p], \quad (5)$$

where $p\equiv(r-d)/(u-d)$ and $p'\equiv(u/r)p$, $a\equiv$the smallest non-negative integer greater than $\log(K/Sd^n)/\log(u/d)$, u is a return on stock if there is an upward move in the stock's price, d - return on a downward move, r - risk-free interest rate.

$\Phi[a; n,p]$ is a complementary binomial distribution function of the form:

$$\Phi[a;n,p]=[\sum_{j=a}^{n}(\frac{n!}{j!(n-j)!})p^{j}(1-p)^{n-j}(\frac{u^{j}d^{n-j}}{r^{n}})] \tag{6}$$

If $a>n$, $C=0$.

One of the limiting cases of this formula is the Black-Scholes formula and the other is Jump Process Option Pricing Formula:

$$C=S\Psi[x;y]-Kr^{-t}\Psi[x;y/u], \tag{7}$$

where $y\equiv(\log r-\zeta)ut/(u-1)$, and $x\equiv$the smallest non-negative integer greater than $(\log(K/S)-\zeta t)/\log u$. $\Psi[x;y]$ is defined to be:

$$\Psi[x;y]=\sum_{i=x}^{\infty}\frac{e^{-y}y^{i}}{i!}. \tag{8}$$

Despite a large number of closed-form formulas for option prices such solutions to the fundamental partial differential equation of option pricing are rare. Extensive research efforts have been undertaken to develop numerical methods for approximate solutions to this partial differential equation. The books by Ames (1977) and Smith (1978) are general mathematical references on the finite-difference method of solution. Geske and Shastri (1985) compare and contrast the various numerical methods used in option price valuations.

Chapter IV

PATH DEPENDENT OPTION PRICING

Lookback options are state contingent claims whose payoff depend on the minimum or maximum of a given security's price over a certain period of time. From Chapter I we know that a buyer of the put or call lookback option will receive from the seller the greatest economic value achieved during the life of the option, regardless of the level at expiry.

Some of these options are already traded on specialized markets (such as foreign exchange) and mostly in over-the-counter market alongside with other path dependent options (knock-ins, knock-outs, etc.). The existing literature on this type of options is quite scarce compared to the literature on conventional options.

Goldman, Sosin, and Gatto (1979) and Goldman , Sosin, and Shepp (1979) did some work on what may be called 'a standard lookback option' (an option to buy at the historical lowest price over a certain period). Merton (1973a) mentioned a knock-out option ('down and out').

Antoine Conze and Viswanathan (1991) define in their article the following options: 'a standard lookback call', 'a call on maximum', 'a limited risk call' , and 'a partial lookback call' and develop their respective pricing models. They show that in the framework of the Black and Scholes model of option valuation, it is possible to obtain closed form solutions for the value of the defined European lookback options. They also present some results for American option pricing.

A. Lookback Options Pricing

In their paper they consider a horizon date T. As in the Black Scholes model, the economy consists of two assets, a security of price S_t , and a zero-coupon bond maturing at T, of constant rate *r*. As usual S_t is assumed to be a stochastic process on a probability space (Ω, F, *p*) adapted to the filtration $F=\{F_t, 0\leq t\leq T)$ and *p* a probability measure. The technical conditions collectively known as "usual requirements or usual conditions" insure that the flow of information is continuous (see Duffie (1988), pp 130-135). The security price S_t satisfies the usual stochastic differential equation

$$dS_t=\alpha S_t dt+\sigma S_t dW_t, S_0>0, \tag{9}$$

where W_t is a standard Brownian motion on (Ω, F, *p*), and α and σ are two constants. The market is said to be arbitrage free if exists a probability *p** equivalent to *p* under which the process $S_t e^{-rt}$ is a martingale. The market is said to be complete if this probability is unique.

It is known that if the market is complete, every contingent claim may be replicated with the stock and the zero-coupon bond self-financing strategy. The price of any option in this case is the discounted expectation of its payoff under *p**. The value of the option at time *t* can be written as

$$V_t=e^{-r(T-t)}E_{p*}[\Pi | F_t], \quad \textbf{(10)}$$

where E_{p*} is the expectation under $p*$, V_t the value of the option, and Π the payoff of the option at time T.

Harrison and Pliska (1981) show that the Black and Scholes market is complete, and that under $p*$, the security's price follows the stochastic differential equation

$$dS_t=rS_t dt+\sigma S_t d\hat{W}_t, S_0>0, \quad \textbf{(11)}$$

where $\hat{W}_t$ a new standard Brownian motion under $p*$. Throughout the paper N(x) denotes the cumulative distribution function of a standard Gaussian variable, $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$

From (11), we have:

$$S_t=S_0\exp\{(r-\frac{1}{2}\sigma^2)t+\sigma W_t\}. \quad (12)$$

Through the rest of the paper:

$$X_t=\ln\frac{S_t}{S_0}=(r-\frac{1}{2}\sigma^2)t+\sigma W_t, \quad (13)$$

$$M_{t_1}^{t_2}=\max\{S_s/s\in[t_1,t_2]\}, \quad (14)$$

$$m_{t_1}^{t_2}=\min\{S_s/s\in[t_1,t_2]\}, \quad (15)$$

$$Y_t=\ln\frac{M_0^t}{S_0}=\max\{X_s/s\in[0,t]\}, \quad (16)$$

$$y_t=\ln\frac{m_0^t}{S_0}=\min\{X_s/s\in[0,t]\}. \quad (17)$$

Option prices will be computed at time 0, and it is assumed that the contracts have been initiated at time $T_0\le 0$. I set $\mu=r-\frac{1}{2}\sigma^2$. The following lemmas follow from Harrison

(1985):

Lemma 1: Take (x, y) such that y≥0 and y≥x. Then

$$p(X_t\le x, Y_t\le y)=N\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right)-e^{\frac{2\mu y}{\sigma^2}}N\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right). \quad (18)$$

Lemma 2: Take (x, y) such that y≤0 and y≤x. Then

$$p(X_t\ge x, y_t\ge y)=N\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right)-e^{\frac{2\mu y}{\sigma^2}}N\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right). \quad (19)$$

Lemma 3: Take $y \geq 0$. Then

$$p(Y_t \leq y) = N\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right). \quad (20)$$

Lemma 4: Take $y \leq 0$. Then

$$p(y_t \geq y) = N\left(\frac{-y+\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right). \quad (21)$$

Note $m_{T_0}^T = \min(m_{T_0}^0, m_0^T)$ (resp. $M_{T_0}^T = \max(M_{T_0}^0, M_0^T)$) where, at time 0,

m^0_{T0} and M^0_{T0} are known constants.

We can also set:

$$d = \frac{(\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}, \quad (22)$$

$$d' = \frac{(\ln\frac{S_0}{M_{T_0}^0} + rT + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}, \quad (23)$$

$$d'' = \frac{(\ln\frac{S_0}{m_{t_0}^0} + rT + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}. \quad (24)$$

The payoff of the call is $S_T - m^T_{T0}$. The value of the call at time 0 is

$$C = e^{-rT} E[S_T - \min(m^0_{T_0}, m^T_0)] = e^{-rT} E[S_T] - e^{-rT} E[\min(m^0_{T_0}, m^T_0)]$$

$$= S_0 - e^{-rT} E[\min(m^0_{T_0}, m^T_0)],$$
(25)

which gives

$$C = S_0 N(d'') - e^{-rT} m^0_{T_0} N(d'' - \sigma\sqrt{T}) +$$

$$+ e^{-rT} \frac{\sigma^2}{2r} S_0 [(\frac{S_0}{m^0_{T_0}})^{-\frac{2r}{\sigma^2}} N(-d'' + \frac{2r}{\sigma}\sqrt{T}) - e^{rT} N(-d'')]. \quad (26)$$

Note, that we don't need to know the joint distribution of the extrema and the terminal value of the Brownian motion. All that is required is the knowledge of the conditional distribution of the extrema. To order a lookback call it is sufficient to subtract from the current stock price the value of a security that pays off the realized minimum. For the put it is sufficient to value a security that pays off realized maximum and to subtract current stock price:

$$P = -S_0 N(-d') + e^{-rT} M^0_{T_0} N(-d' + \sigma\sqrt{T}) +$$

$$+ e^{-rT} \frac{\sigma^2}{2r} S_0 [-(\frac{S_0}{M^0_{T_0}})^{-\frac{2r}{\sigma^2}} N(d' - \frac{2r}{\sigma}\sqrt{T}) + e^{rT} N(d')]. \quad (27)$$

By making the Cox-Ross transformation and then computing the discounted expected value of the terminal maximum and minimum Goldman, Sosin and Gatto (1979) arrive at the same formula (pp.1116-1117).

B. Early Exercise of American Lookbacks

From Karatzas (1990) the value of an American option can be given by:

$$C(t)=ess\ sup_{\tau\in S_{t,T}} E[f(\tau)\, e^{-\int_t^{\tau} r(s)\,ds} \mid F_t]\, ,a.s. \quad (28)$$

for every $t\in[0,T]$, where ess sup is essential supremum, f(t) is a payoff on exercise, and τ is a stopping time.

To see when an American option is not worth exercising before maturity define:

$$u(t)\equiv sup_{\tau\in S_{t,T}} EQ(\tau)\,;\ Q(t)=f(t)\, e^{-\int_0^t r(s)\,ds}\, ,0\le t\le T \quad (29)$$

From the theory of optimal stopping there exist a supermartingale $\{Y_t,\ F_t\}$ such that the function u() is given as $u(t)=EY(t)$ and $Y(t)=ess\ sup_{\tau\in S_{t,T}} E[Q(\tau) \mid F_t]$ a.s. Y is a Snell envelope for Q.

The stopping time τ_t can be written:

$$\tau_t=inf\{s\in[t,T]\,;C(s)=f(s)\},a.s.; \quad (30)$$

and τ_0 provides the optimal exercise time for the American option.

If the process Q is a submartingale under p^*; then $u(t)=EQ(T)$, $\tau_t=T$ for every $0\leq t\leq T$, and the American option should not be exercised.

For an American lookback $f(t)=(S_t-m^T_{T0})^+$. If it was written on a stock which pays no dividends: $\mu(t)=0$, $r(t)\geq 0$, then

$$Q(t)=(S_t e^{-\int_0^t r(s)\,ds}-m^t_{T_0}e^{-\int_0^t r(s)\,ds})^+ \quad (31)$$

is a submartingale, and I get the same result as Conze and Viswanathan that the value of an American lookback call on a stock that pays no dividends is equal to the value of a European lookback call on the same stock.

The aforementioned lemmas on joint distribution of the extremum and terminal values of Brownian motion can be further applied to find pricing models of such options as asians, cliquets, and step-locks. Another area of research deserving interest is to relax no dividends assumption and explore early exercise of the path dependent options in this case.

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